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The obstruction to excision in K-theory and in cyclic homology

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Abstract. Let $f : A \to B$ be a ring homomorphism of not necessarily unital rings and $I \lhd A$ an ideal which is mapped by f isomorphically to an ideal of B. The obstruction to excision in K-theory is the failure of the map between relative K-groups $K_*(A : I) \to K_*(B : f(I))$ to be an isomorphism; it is measured by the birelative groups $K_*(A, B : I)$. Similarly the groups $HN_*(A, B : I)$ measure the obstruction to excision in negative cyclic homology. We show that the rational Jones-Goodwillie Chern character induces an isomorphism

$$ch_* : K_*(A, B : I) \otimes \mathbb{Q} \xrightarrow{\sim} HN_*(A \otimes \mathbb{Q}, B \otimes \mathbb{Q} : I \otimes \mathbb{Q}).$$

0. Introduction

Algebraic K-theory does not satisfy excision. This means that if $f : A \to B$ is a ring homomorphism and $I \lhd A$ is an ideal carried isomorphically to an ideal of B, then the map of relative K-groups $K_*(A : I) \to K_*(B : I) := K_*(B : f(I))$ is not an isomorphism in general. The obstruction is measured by birelative groups $K_*(A, B : I)$ which are defined so as to fit in a long exact sequence

$$K_{n+1}(B : I) \to K_n(A, B : I) \to K_n(A : I) \to K_n(B : I).$$

Similarly the obstruction to excision in negative cyclic homology is measured by birelative groups $HN_*(A, B : I)$. K-theory and negative cyclic homology are related by a character $K_n A \to HN_n A$, the *Jones-Goodwillie*

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Chern character ([17, Chap. II]; see also [21, §8.4]). Tensoring with $\mathbb{Q}$ and composing with the natural map $HN_n(A) \otimes \mathbb{Q} \to HN_n(A \otimes \mathbb{Q})$ we obtain a rational Chern character

$$(1) \qquad ch_n : K_n^{\mathbb{Q}}(A) := K_n(A) \otimes \mathbb{Q} \to HN_n(A \otimes \mathbb{Q}) =: HN_n^{\mathbb{Q}}(A).$$

The main theorem of this paper is the following.

Main theorem 0.1. *Let $f : A \to B$ be a homomorphism of not necessarily unital rings and $I \triangleleft A$ an ideal which is carried by f isomorphically onto an ideal of B. Then* (1) *induces an isomorphism*

$$ch_* : K_*^{\mathbb{Q}}(A, B : I) \xrightarrow{\sim} HN_*^{\mathbb{Q}}(A, B : I).$$

If moreover A and B are $\mathbb{Q}$-algebras, then $K_*(A, B : I)$ is a $\mathbb{Q}$-vectorspace, and the exponent $\mathbb{Q}$ is not needed.

The theorem above can also be stated in terms of *cyclic homology* (denoted HC), as we shall see presently. We recall that, unlike HN, HC commutes with $\otimes\mathbb{Q}$, so that $HC_*^{\mathbb{Q}}(A) := HC_*(A \otimes \mathbb{Q}) = HC_*(A) \otimes \mathbb{Q}$.

Corollary 0.2. *There is a natural isomorphism*

$$\nu_* : K_*^{\mathbb{Q}}(A, B : I) \cong HC_{*-1}^{\mathbb{Q}}(A, B : I).$$

Proof. Write HP for periodic cyclic homology. There is a long exact sequence ([21, 5.1.5])

$$(2) \qquad HC_{n-1}(A, B : I) \to HN_n(A, B : I) \\ \to HP_n(A, B : I) \to HC_{n-2}(A, B : I).$$

Cuntz-Quillen's excision theorem [9] establishes that

$$(3) \qquad HP_*^{\mathbb{Q}}(A, B : I) = 0.$$

Here $HP_*^{\mathbb{Q}}(\ ) := HP_*(\ \otimes\mathbb{Q})$. From (3) and from (2) applied to $A \otimes \mathbb{Q}$, $B \otimes \mathbb{Q}$ and $I \otimes \mathbb{Q}$, it follows that $HC_{*-1}^{\mathbb{Q}}(A, B : I) = HN_*^{\mathbb{Q}}(A, B : I)$. $\square$

Next we shall review some related results in the literature, so as to put ours in perspective. Bass ([2, Thm. XII.8.3]; see also [19, pp. 295–298]) proved K-theory satisfies excision in nonpositive degrees; in our setting this means $K_n(A, B : I) = 0$ for $n \leq 0$. The analogous result for the negative degrees of cyclic homology is also true. In fact, by definition ([21, 2.1.15]) if R is a unital ring and $J \triangleleft R$ an ideal then $HC_n(R : J) = 0$ for $n < 0$ and $HC_0(R, J) = J/[R, J]$, the quotient by the subgroup generated by the commutators $[r, j] = rj - jr$. Hence in the situation of 0.1

$$HC_{-1}(A, B : I) = \operatorname{coker}(I/[A, I] \to I/[B, I]) = 0.$$

Thus 0.2 is true for $*$ nonpositive, as both birelative groups vanish. The particular case of 0.2 when $* = 1$ was proved in [11]. In [10] the statement

of Corollary 0.2 was conjectured to hold when A and B are commutative unital $\mathbb{Q}$-algebras and B is a finite integral extension of A, ($KABI$ conjecture) and it was shown its validity permits computation of the K-theory of singular curves in terms of their cyclic homology and of the K-theory of nonsingular curves. In [12] 0.2 was conjectured for unital $\mathbb{Q}$-algebras and it was shown that for $* = 2$ the left hand side maps surjectively onto the right hand side.

A special case of the main theorem concerns the birelative groups $K_*(A : I, J)$ associated to any pair of ideals $I, J \triangleleft A$; they are defined so as to fit in a long exact sequence

$$K_{n+1}(A/J : I + J/J) \to K_n(A : I, J) \to K_n(A : I) \to K_n(A/J : I + J/J).$$

Note that if $I \cap J = 0$ then $A \to A/J$ maps I isomorphically onto $I + J/J = I/I \cap J$, whence 0.2 applies, and we have an a rational isomorphism

$$K_*^{\mathbb{Q}}(A : I, J) \overset{\sim}{\to} HC_{*-1}^{\mathbb{Q}}(A : I, J). \tag{4}$$

For $* \leq 1$ it is well-known that both birelative K- and cyclic homology groups vanish, whence (4) follows. The case $* = 2$ was proved independently in [18] and [20]. One can further generalize this to the case when $I \cap J$ is nilpotent, using a theorem of Goodwillie's ([17, Main Thm.] see also [21, §11.3]), which says that if $I \triangleleft R$ is a nilpotent ideal of a ring R, then there is a natural isomorphism

$$K_*^{\mathbb{Q}}(R : I) \cong HC_{*-1}^{\mathbb{Q}}(R : I). \tag{5}$$

We point out that, even if Goodwillie states his theorem for R unital, the nonunital case follows from the unital ([5, §4.2]; see also Lemma 1.1 below). As in the case of 0.2, (5) can also be stated in terms of negative cyclic homology. Actually Goodwillie shows (see [17, 0.3]) that ch_* induces an isomorphism

$$ch_* : K_*^{\mathbb{Q}}(R : I) \overset{\sim}{\to} HN_*^{\mathbb{Q}}(R : I) \tag{6}$$

and then uses the singly relative version of (2) in combination with another theorem of his, ([16, II.5.1]; see also [9, 3.5]), which says that $HP_*^{\mathbb{Q}}(R, I) = 0$ if I is nilpotent. Now we use (5) to generalize (4).

Corollary 0.3. *Let A be a ring and $I, J \triangleleft A$ ideals such that $I \cap J$ is nilpotent. Then there is a rational isomorphism of birelative groups*

$$K_*^{\mathbb{Q}}(A : I, J) \overset{\sim}{\to} HC_{*-1}^{\mathbb{Q}}(A : I, J).$$

Proof. The case $I \cap J = 0$ is explained above. To prove the general case, consider the intermediate groups K_n^{inf} fitting in the long exact sequence

$$HN_{n+1}^{\mathbb{Q}}(R) \longrightarrow K_n^{inf}(R) \longrightarrow K_n^{\mathbb{Q}}(R) \xrightarrow{ch_n} HN_n^{\mathbb{Q}}(R). \tag{7}$$

(This notation will be justified below, see (11)). By 0.1, K^{inf} satisfies excision; by (6), it is invariant under nilpotent extensions, or *nilinvariant*. But a diagram chase shows that any homology theory of rings H satisfying both excision and nilinvariance verifies $H_*(A : I, J) = 0$ if $I \cap J$ is nilpotent. Applying this to both K^{inf} and $HP^{\mathbb{Q}}$, we obtain

$$K_*^{\mathbb{Q}}(A : I, J) \cong HN_*^{\mathbb{Q}}(A : I, J) \cong HC_{*-1}^{\mathbb{Q}}(A : I, J).$$

□

The result of the corollary above for A unital was announced in [23]; however the proof in *loc.cit.* turned out to have a gap (see [13, p. 591, line 1]). An application of 0.3 to the computation of the K-theory of particular rings – other than coordinate rings of curves – in terms of their cyclic homology was given in [13, Thm. 3.1]; see also [10, Thm. 7.3].

As another precedent of the main theorem of this paper, we must cite the work of Suslin and Wodzicki. To state their theorems we introduce some notation. We say that a ring I is *excisive* for a homology theory H if $H_*(A, B; I) = 0$ for every homomorphism $A \to B$ as in 0.1. In [28], M. Wodzicki characterized those rings which are excisive for cyclic homology as those whose bar homology vanishes

$$A \text{ is } HC^{\mathbb{Q}}\text{-excisive} \iff H_*^{bar} A \otimes \mathbb{Q} = 0. \tag{8}$$

In fact (8) is a particular case of [28, (3)]. He also showed that if A is excisive for rational K-theory then it is excisive for rational cyclic homology ([28, (4)]) and conjectured that the converse also holds. The latter was proved by Suslin and Wodzicki; they showed ([24, Thm. A])

$$H_*^{bar}(A) \otimes \mathbb{Q} = 0 \Rightarrow A \text{ is excisive for } K^{\mathbb{Q}}\text{-theory.} \tag{9}$$

They proved further that if A is a $\mathbb{Q}$-algebra then (9) still holds even if we do not tensor with $\mathbb{Q}$ ([24, Thm. B]). Note that (9) is a formal consequence of 0.2 and (8). Actually our proof of 0.1 involves proving a version of Suslin-Wodzicki's theorem for a certain type of pro-rings (Theorem 3.16).

Sketch of proof of 0.1. The assertion concerning $\mathbb{Q}$-algebras follows (see 4.1 below) from Weibel's result [26] (see also [24, 1.9]) that for all $n \geq 2$, K-theory with $\mathbb{Z}/n$ coefficients satisfies excision for such algebras. The rest of the proof has four parts:

a) *An abstraction of arguments of Cuntz and Quillen and its combination with the tautological characters of* [5]. (This is done in Sect. 1 below). We consider functors from the category $Ass_1 = Ass_1(k)$ of unital rings over a commutative ring k to fibrant spectra which preserve products up to homotopy. If X is such a functor and $I \lhd A \in Ass_1$ an ideal, we put

$$X(A : I) := \text{hofiber}(X(A) \to X(A/I)), \quad \hat{X}(A/I^\infty) := \text{holim}_n X(A/I^n),$$
$$\hat{X}(A : I^\infty) := \text{holim}_n X(A : I^n).$$

If $f : A \to B \in Ass_1$ is as in 0.1, we set

$$X(A, B : I) = \text{hofiber}(X(A : I) \to X(B : I)),$$
$$\hat{X}(A, B : I^\infty) = \text{hofiber}(\hat{X}(A : I^\infty) \to \hat{X}(B : I^\infty)).$$

We call I ∞-excisive if $\hat{X}(A, B : I^\infty)$ is weakly contractible for every homomorphism $A \to B$ as above. We say that X is *nilinvariant* if $X(A : I)$ is contractible for every nilpotent ideal $I \triangleleft A$. We show that if X is nilinvariant and every ideal I of every free unital algebra is ∞-excisive then X is excisive (Proposition 1.6). This gives a criterion for proving excision which generalizes that used by Cuntz and Quillen in the particular case of HP in [9]. Next, given any not necessarily excisive or nilinvariant functor X as above, we consider its *noncommutative infinitesimal hypercohomology* ([5, §5]) $\mathbb{H}(A_{inf}, X)$. This is a nilinvariant functor and is equipped with a natural map $\mathbb{H}(A_{inf}, X) \to X(A)$. Write $\tau X(A)$ for the delooping of the fiber of the latter map. We have a homotopy fibration

$$\mathbb{H}(A_{inf}, X) \to X(A) \xrightarrow{c^\tau} \tau X(A). \tag{10}$$

We call c^τ the *tautological character*. Applying the Cuntz-Quillen criterion to $\mathbb{H}(A_{inf}, X)$, we obtain (Theorem 1.8) that if every ideal of every free unital algebra $F \in Ass_1$ is ∞-excisive for both X and τX then the map $X(A, B : I) \to \tau X(A, B : I)$ is a weak equivalence for every algebra homomorphism as in 0.1.

b) *Agreement of the tautological and rational Jones-Goodwillie characters.* (Sect. 2). We apply (10) to $X = \mathbf{K}^{\mathbb{Q}}$, the nonconnective rational K-theory spectrum. We show that there is a natural isomorphism $HN_n^{\mathbb{Q}}(A) \cong \tau K_n^{\mathbb{Q}}(A)$ under which the tautological character c_n^τ is identified with ch_n (Theorem 2.1). In particular, the *infintesimal K-theory groups*

$$K_n^{inf} A := \pi_n \mathbb{H}\big(A_{inf}, \mathbf{K}^{\mathbb{Q}}\big) \qquad (n \in \mathbb{Z}). \tag{11}$$

are the "intermediate groups" of (7). Theorem 2.1 is of independent interest as it shows that $HN^{\mathbb{Q}}$ can be functorially derived from $K^{\mathbb{Q}}$, and extends to all $n \in \mathbb{Z}$ previous results of the author ([4, 6.3]; [5, 6.2]; [6, 5.1]).

c) *Suslin-Wodzicki's theorem for A^∞.* (Sect. 3). We show that if A is a ring such that for all $r \geq 0$ the pro-vectorspace

$$H_r^{bar}(A^\infty) \otimes \mathbb{Q} := \left\{H_r^{bar}(A^n) \otimes \mathbb{Q}\right\}_n$$

is zero, then A is ∞-excisive for $K^{\mathbb{Q}}$ (Theorem 3.16). Note that in the particular case when $A = A^2$, the latter assertion and (9) coincide, since A^∞ is just the constant pro-ring A in this case. Our proof follows the strategy of Suslin and Wodzicki's proof of (9), (see the summary at the beginning of Sect. 3) and adapts it to the pro-setting. Some technical results on pro-spaces needed in this section are proved in the Appendix.

d) *Application of known results on bar and cyclic pro-homology*. (Sect. 4). By parts (a), (b) and (c), to finish the proof it is enough to show that if I is an ideal of a free unital ring, then both (i) and (ii) below hold.

(i) $H_r^{bar}(I^\infty) \otimes \mathbb{Q} = 0$ $(r \geq 0)$.
(ii) I is ∞-excisive for $HN^{\mathbb{Q}}$.

By (2) and (3) the latter property is equivalent to

(ii)' I is ∞-excisive for $HC^{\mathbb{Q}}$.

As $\otimes\mathbb{Q}$ commutes with both H^{bar} and HC and sends free unital rings to free unital $\mathbb{Q}$-algebras, it suffices to verify (i) and (ii)' for ideals of free unital $\mathbb{Q}$-algebras. Both of these are well-known and are straightforward from results in the literature; see 4.2 for details. □

The rest of this paper is organized as follows. In Sect. 1 we carry out part (a) of the sketch above. We generalize Cuntz-Quillen's excision principle (Proposition 1.6), recall the construction of the tautological character (1.7) and obtain a criterion for proving that the latter computes the obstruction to excision (Theorem 1.8). Part (b) corresponds to Sect. 2, where we show that the tautological character for rational K-theory is the rational Jones-Goodwillie character (Theorem 2.1). Section 3 is devoted to part (c); we prove a version of (9) for pro-rings of the form A^∞ (Theorem 3.16). The proof of 0.1 is completed in Sect. 4, where we carry out part (d) of the sketch and show that if in the situation of the Main theorem, A and B are $\mathbb{Q}$-algebras, then the groups $K_*(A, B : I)$ are $\mathbb{Q}$-vectorspaces (Lemma 4.1). Notation for pro-spaces as well as some technical results on them which are used in Sect. 3 are the subject of the Appendix.

Note on notation. In this paper space = simplicial set.

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1. Cuntz-Quillen excision principle

Summary. In this section we formulate in an abstract setting the method used by Cuntz and Quillen to prove excision for periodic cyclic homology of algebras over a field of characteristic zero. In their proof they first show that HP satisfies excision for pro-ideals of the form I^∞ where I is an ideal of a free algebra and then combine this with the invariance of HP under nilpotent extensions to prove excision holds for all ideals. Our setting is that

of functors X from the category of algebras over a commutative ring k to that of fibrant spectra. We assume that X preserves products up to homotopy. We show that if X is invariant (up to weak equivalence) under nilpotent extensions and satisfies excision for pro-rings of the form I^∞ whenever I is an ideal of a free unital algebra, then X satisfies excision for all algebras (Prop. 1.6). Then we apply this to give a criterion for proving that the tautological character $c^\tau : X \to \tau X$ of [5] induces an equivalence at the level of the groups which measure the obstruction to excision (Theorem 1.8). □

We consider associative, not necessarily unital algebras over a fixed ground ring k. We write $Ass := Ass(k)$ for the category of algebras, Ass_1 for the subcategory of unital algebras and unit preserving maps, and

$$A \mapsto \tilde{A} \tag{12}$$

for the left adjoint functor to the inclusion $Ass_1 \subset Ass$. By definition,

$$\tilde{A} = A \oplus k, \quad (a, \lambda)(b, \mu) := (ab + \mu a + \lambda b, \lambda\mu).$$

Throughout this section X will be a fixed functor from Ass_1 to fibrant spectra (terminology for spectra is as in [25]). We shall assume that X preserves finite products up to homotopy; this means that if $A, B \in Ass_1$ then the canonical map is a weak equivalence:

$$X(A \times B) \xrightarrow{\sim} X(A) \times X(B). \tag{13}$$

Let $A \in Ass_1$, $I \triangleleft A$ an ideal, $\pi : A \to A/I$ the projection. Write $X(A : I)$ for the homotopy fiber of $X(\pi)$. We say that $A \in Ass$ is *excisive* (with respect to our fixed functor X) if for every commutative diagram

$$\begin{tikzcd} & A \arrow[dl, hook'] \arrow[dr, hook] & \\ B \arrow[rr] & & C \end{tikzcd} \tag{14}$$

such that $B \to C$ is a map in Ass_1 and both arrows out of A embed it as an ideal, the map

$$X(B : A) \longrightarrow X(C : A)$$

is an equivalence. Equivalently, A is excisive if

$$X(B, C : A) := \mathrm{hofiber}(X(B : A) \to X(C : A))$$

is contractible for every diagram (14). We note it is enough to check this condition for $B = \tilde{A}$. The functor X is called *excisive* if every $A \in Ass$ is.

The rule

$$A \to Y(A) := X(\tilde{A} : A) \tag{15}$$

defines a functorial fibrant spectrum on Ass. If $A \in Ass_1$, then $\tilde{A} \cong A \times k$, and because the homotopy category of spectra is additive, we have

$$X(\tilde{A}) \overset{\sim}{\to} Y(A) \times X(k).$$

Since we are assuming that X preserves finite products up to homotopy (13), it follows that for $A \in Ass_1$, there is a natural homotopy equivalence

$$Y(A) \overset{\sim}{\to} X(A).$$

Thus Y extends X up to homotopy. Note further that, by definition,

$$Y(0) = \mathrm{hofiber}(X(k) \xrightarrow{1} X(k))$$

is contractible. If $A \triangleleft B$ is an ideal we put

$$Y(B : A) := \mathrm{hofiber}(Y(B) \to Y(B/A)).$$

The natural map $Y(R) \to X(\tilde{R})$ induces

$$Y(B : A) \to X(\tilde{B} : A). \tag{16}$$

Lemma 1.1. *The map* (16) *is a weak equivalence. In particular,* $Y(A : A) \overset{\sim}{\to} YA$.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
Y(B : A) \arrow[r] \arrow[d] & YB \arrow[r] \arrow[d] & Y(B/A) \arrow[d] \\
X(\tilde{B} : A) \arrow[r] \arrow[d] & X(\tilde{B}) \arrow[r] \arrow[d] & X(\widetilde{B/A}) \arrow[d] \\
* \arrow[r] & X(k) \arrow[r, "1"] & X(k).
\end{tikzcd}$$

Here 1 denotes the identity map. By definition, all rows as well as the middle and right hand columns are homotopy fibrations; it follows that also the column on the left is one. □

Corollary 1.2. *Let $B \to C$ be a homomorphism in Ass carrying an ideal $A \triangleleft B$ isomorphically onto an ideal of C. Then the natural map*

$$Y(B, C : A) := \mathrm{hofiber}(Y(B : A) \longrightarrow Y(C : A)) \to X(\tilde{B}, \tilde{C} : A)$$

is an equivalence.

Corollary 1.3. *If A is excisive and $A \triangleleft B$, then the map $Y(A) \to Y(B : A)$ is a weak equivalence.*

Proof. By 1.1, it suffices to show that

$$Y(A:A) \to Y(B:A) \tag{17}$$

is a weak equivalence. But by 1.2, the homotopy fiber of (17) is weakly equivalent to $X(\tilde{A}, \tilde{B} : A)$, which is contractible because A is excisive. □

Lemma 1.4. *If every ideal of every free unital algebra is excisive, then X is excisive.*

Proof. Let $F : Sets \to Ass_1$ be the free unital algebra functor. For $R \in Ass_1$ let $JR = \ker(FR \to R)$ and $\rho : R \to FR$ be respectively the kernel of the projection and its natural set-theoretic section. The ideal JR is generated by $\rho(1) - 1$, by $\rho(0) - 0$ and by the elements of the form $\rho(a+b) - \rho(a) - \rho(b)$ and of the form $\rho(ab) - \rho(a)\rho(b)$ $(a, b \in R)$. In particular J preserves surjections. It follows from this and the snake lemma that if $B \in Ass_1$ and $A \triangleleft B$ is an ideal, then $I = \ker(F(B) \to F(B/A))$ maps onto A. The kernel of the surjection $\tilde{I} \to \tilde{A}$ is $L := JB \cap I$. We have a commutative diagram

$$\begin{tikzcd}
 & JB \arrow[rr] \arrow[dd] & & J(B/A) \arrow[dd] \\
L \arrow[ur] \arrow[rr] \arrow[dd] & & 0 \arrow[ur] \arrow[dd] & \\
 & FB \arrow[rr] \arrow[dd] & & F(B/A) \arrow[dd] \\
\tilde{I} \arrow[ur] \arrow[rr] \arrow[dd] & & k \arrow[ur] \arrow[dd] & \\
 & B \arrow[rr] & & B/A \\
\tilde{A} \arrow[ur] \arrow[rr] & & k \arrow[ur] &
\end{tikzcd} \tag{18}$$

Let Y be as in (15). By 1.3, applying Y to the diagram above yields a diagram whose columns are homotopy fibrations, and of which the two top squares are homotopy cartesian. It follows that also the bottom square is homotopy cartesian. □

We say that X is *nilinvariant* if $X(B : A)$ is contractible for every nilpotent ideal $A \triangleleft B$. If $A = \{A_n\}$ is a pro- unital algebra, and $I = \{I_n\}$ a (levelwise) pro-ideal we put

$$\hat{X}(A) = \mathrm{holim}_n X(A_n), \quad \hat{X}(A : I) = \mathrm{holim}_n X(A_n : I_n).$$

We say that A is ∞-excisive if for every diagram (14) the induced map

$$\hat{X}(B : A^\infty) \longrightarrow \hat{X}(C : A^\infty)$$

is an equivalence. Here as in [9], $A^\infty = \{A^n\}_n$ with the natural inclusions as transition maps.

Lemma 1.5. *If X is nilinvariant and A is ∞-excisive, then A is excisive.*

Proof. Let $B \to C$ be as in (14). We have a commutative diagram of level pro-maps

$$\begin{array}{ccccc} B & \longrightarrow & B/A^\infty & \longrightarrow & B/A \\ \downarrow & & \downarrow & & \downarrow \\ C & \longrightarrow & C/A^\infty & \longrightarrow & C/A. \end{array}$$

Apply $\hat{X}$ to this diagram, noting that $\hat{X}(R) \xrightarrow{\sim} X(R)$ if R is a constant pro-algebra. □

Proposition 1.6. *Assume that X is nilinvariant and that every ideal of every free unital algebra is ∞-excisive. Then X is excisive.*

Proof. Immediate from 1.5 and 1.4. □

Definition 1.7. The *infinitesimal hypercohomology* of an object $A \in Ass_1$ with coefficients in X is the fibrant spectrum

$$\mathbb{H}(A_{inf}, X) := \operatorname*{holim}_{A_{inf}} X. \tag{19}$$

Here A_{inf} is the category of all surjections $B \twoheadrightarrow A \in Ass$ with nilpotent kernel; X is viewed as a functor on A_{inf} by $(B \to A) \mapsto X(B)$. Although the category A_{inf} is large, it is proved in [5, 5.1] that it has a left cofinal small subcategory in the sense of [3, Chap. IX§9], which by [5, 2.2.1] can be chosen to depend functorially on A. The homotopy limit is taken over this small subcategory. There is a natural map $\mathbb{H}(A_{inf}, X) \to X(A)$; we write τX for the delooping of its homotopy fiber. We have a map of spectra

$$c^\tau : X(A) \to \tau X(A)$$

of which the homotopy fiber is weakly equivalent to $\mathbb{H}(A_{inf}, X)$. We call c^τ the *tautological character* of X.

Theorem 1.8. *Assume every ideal of every free unital algebra is ∞-excisive for both X and τX. Let $A \to B \in Ass_1$ be a homomorphism carrying an ideal $I \triangleleft A$ isomorphically onto an ideal of B. Then the tautological character induces a weak equivalence*

$$c^\tau : X(A, B : I) \xrightarrow{\sim} \tau X(A, B : I).$$

In particular the functor $A \mapsto \mathbb{H}(A_{inf}, X)$ is excisive.

Proof. Immediate from 1.6 and the nilinvariance of $\mathbb{H}(A_{inf}, X)$ (see [5, 5.2]). □

Remark 1.9. For $A \in Ass$, the tautological character $c^\tau : X \to \tau X$ induces a map

$$c^{\tau'} : Y(A) \to \tau' Y(A) := \tau X(\tilde{A} : A). \tag{20}$$

If A happens to be unital, we have a commutative diagram with vertical weak equivalences

$$\begin{tikzcd} X(A) \arrow[r, "c^{\tau'}"] \arrow[d, "\wr"'] & \tau X(A) \arrow[d, "\wr"] \\ Y(A) \arrow[r, "c^{\tau'}"'] & \tau' Y(A). \end{tikzcd} \tag{21}$$

Thus c^τ and $c^{\tau'}$ agree up to homotopy for $A \in Ass_1$. By 1.2, under the hypothesis of 1.8,

$$c^{\tau'} : Y(A, B : I) \to \tau' Y(A, B : I)$$

is an equivalence. On the other hand, if in formula (19) we substitute for A_{inf} the category $inf(Ass \downarrow A)$ of all nilpotent extensions of A in Ass, and define τYA as the delooping of the homotopy fiber of the canonical map, we obtain a homotopy fibration sequence

$$\mathbb{H}(inf(Ass \downarrow A), Y) \to Y(A) \xrightarrow{c^\tau} \tau Y(A). \tag{22}$$

The following lemma says that the characters of (22) and (21) agree up to homotopy; this will be used in Sect. 2 (Proof of 2.1 and Remarks 2.2 and 2.3).

Lemma 1.10. *Let $A \in Ass$. Then there is a weak equivalence $\tau' Y(A) \xrightarrow{\sim} \tau Y(A)$ such that the diagram*

$$\begin{tikzcd} & Y(A) \arrow[dl, "c^{\tau'}"'] \arrow[dr, "c^\tau"] & \\ \tau' Y(A) \arrow[rr, "\sim"] & & \tau Y(A) \end{tikzcd}$$

is homotopy commutative.

Proof. We shall abuse notation and write A_{inf} for $inf(Ass \downarrow A)$ and $\tilde{A}_{inf}$ for $inf(Ass_1 \downarrow \tilde{A})$. Let $FA := \mathrm{hofiber}(\mathbb{H}(\tilde{A}_{inf}, X) \to \mathbb{H}(k_{inf}, X))$; there is a homotopy fibration sequence

$$FA \to Y(A) \to \tau' Y(A).$$

The functor

$$\tilde{}: A_{inf} \to \tilde{A}_{inf} \tag{23}$$

is left adjoint to the pullback over $A \to \tilde{A}$. Hence (23) is left cofinal in the sense of [3], whence

$$\mathbb{H}\big(\tilde{A}_{inf}, X\big) \to \mathbb{H}\big(A_{inf}, X \circ \tilde{}\,\big)$$

is a weak equivalence. Thus in the following diagram with homotopy fibrations as rows,

$$\begin{tikzcd}
FA \arrow[r] \arrow[d] & \mathbb{H}\big(\tilde{A}_{inf}, X\big) \arrow[r] \arrow[d] & \mathbb{H}(k_{inf}, X) \arrow[d] \\
F'A \arrow[r] & \mathbb{H}\big(A_{inf}, X \circ \tilde{}\,\big) \arrow[r] & \mathbb{H}\big(0_{inf}, X \circ \tilde{}\,\big)
\end{tikzcd}$$

the columns are weak equivalences. On the other hand we have a commutative diagram

$$\begin{tikzcd}
\mathbb{H}(A_{inf}, Y) \arrow[r] \arrow[d] & \mathbb{H}\big(A_{inf}, X \circ \tilde{}\,\big) \arrow[r] \arrow[d] & \mathbb{H}(A_{inf}, X(k)) \arrow[d] \\
\mathbb{H}(0_{inf}, Y) \arrow[r] & \mathbb{H}\big(0_{inf}, X \circ \tilde{}\,\big) \arrow[r] & \mathbb{H}(0_{inf}, X(k)).
\end{tikzcd}$$

Because $\mathbb{H}(A_{inf}, \ \ ) = \operatorname{holim}_{A_{inf}}$ preserves homotopy fibrations, both rows above are homotopy fibrations. Note $F'A$ is the homotopy fiber of the vertical map in the middle. Let $F''A$ be the homotopy fiber of the vertical map on the left; I claim that both $F''A \to \mathbb{H}(A_{inf}, Y)$ and $F''A \to F'A$ are weak equivalences. In fact because 0_{inf} has an initial object, the leftmost spectrum in the bottom row is weak equivalent to $Y(0)$, and therefore contractible, while the other two spectra in the same row are weak equivalent to each other and to $X(k)$. To prove that the rightmost vertical arrow is a weak equivalence, it suffices to show that the natural map $\mathbb{H}(A_{inf}, X(k)) \to X(k)$ is one. But this is clear from the fact that there is a left cofinal functor $\mathbb{N}^{op} \times \Delta \to A_{inf}$ ([5, 5.1]). The claim is proved. We have thus constructed a diagram

$$\begin{tikzcd}
FA \arrow[r, "\sim"] \arrow[drr] & F'A & & F''A \arrow[ll, "\sim"'] \arrow[r, "\sim"] & \mathbb{H}(A_{inf}, Y) \arrow[dll] \\
& & Y(A) & &
\end{tikzcd}$$

in which all horizontal arrows are weak equivalences. Inverting $F''A \to F'A$ up to homotopy, taking homotopy fibers of the slanted arrows and delooping them, we get the homotopy commutative diagram of the lemma. □

2. The Chern character is the tautological character

Summary. In this section we recall the construction of the rational Chern character of Jones and Goodwillie

$$ch_n : K_n^{\mathbb{Q}}(A) \to HN_n^{\mathbb{Q}}(A) \tag{24}$$

for nonnegative n and extend it to all $n \in \mathbb{Z}$. On the other hand the general construction of the previous section applied to the nonconnective rational K-theory spectrum $\mathbf{K}^{\mathbb{Q}}(A)$ gives a tautological character $c_n^\tau : K_n^{\mathbb{Q}}(A) \to \tau K_n^{\mathbb{Q}}(A)$. We prove that the characters ch_n and c_n^τ agree up to a canonical isomorphism $HN_n^{\mathbb{Q}}(A) \cong \tau K_n^{\mathbb{Q}}(A)$ $(n \in \mathbb{Z})$ (see 2.1). In particular this shows that (24) comes from a map of spectra which is functorially derived from $K^{\mathbb{Q}}$. $\square$

Let A be a unital ring. We write BG for the nerve of the group G; thus for us BG is a pointed simplicial set. The rational plus construction of the general linear group is the Bousfield-Kan $\mathbb{Q}$-completion

$$K^{\mathbb{Q}}(A) := \mathbb{Q}_\infty BGL(A).$$

There is a nonconnective spectrum $\mathbf{K}^{\mathbb{Q}}A$ of which the n-th space is

$${}_n\mathbf{K}^{\mathbb{Q}}A := \Omega K^{\mathbb{Q}}(\Sigma^{n+1}A) \tag{25}$$

where Σ is Karoubi's suspension functor ([14]). Next we consider the complex for negative cyclic homology, which we call CN_*; this is the normalized version of that denoted $\mathrm{Tot}_*\mathfrak{B}C^-$ in [21, 5.1.7]; normalization is as in [21, 2.1.9]. For $n \geq 1$ the character (24) is induced by a map of spaces

$$K^{\mathbb{Q}}(A) \overset{ch}{\to} SCN_{\geq 1}^{\mathbb{Q}}(A). \tag{26}$$

Here $SCN_{\geq 1}$ is the simplicial abelian group the Dold-Kan correspondence associates to the truncation of CN_* which is 0 in degree 0, the kernel of the boundary operator in degree one, and CN_n in degrees $n \geq 2$. The superscript $\mathbb{Q}$ means $SCN_{\geq 1}$ is applied to $A \otimes \mathbb{Q}$. The map ch_0 can be obtained from ch_1, using the inclusion $A \subset A[t, t^{-1}]$ and naturality; see [21, 8.4.10]. Consider the sequence

$$0 \to M_\infty A \to \Gamma A \to \Sigma A \to 0 \tag{27}$$

where Γ is Karoubi's cone functor. As A is unital, $M_\infty A$ is excisive for both $K^{\mathbb{Q}}$-theory and $HN^{\mathbb{Q}}$ (cf. [28]) and has the same $K^{\mathbb{Q}}$- and $HN^{\mathbb{Q}}$-groups as A. Moreover $K_*^{\mathbb{Q}}(\Gamma A) = HN_*^{\mathbb{Q}}(\Gamma A) = 0$. Thus by naturality we get a commutative diagram with vertical isomorphisms

$$\begin{array}{ccc} K_n^{\mathbb{Q}}(A) & \xrightarrow{ch_n} & HN_n^{\mathbb{Q}}(A) \\ \cong\uparrow & & \cong\uparrow \\ K_{n+1}^{\mathbb{Q}}(\Sigma A) & \xrightarrow[ch_{n+1}\Sigma]{} & HN_{n+1}^{\mathbb{Q}}(\Sigma A). \end{array} \tag{28}$$

This says that the map

$$ch_n := ch_{n+r}\Sigma^r : K_n^{\mathbb{Q}}(A) \to HN_n^{\mathbb{Q}}(A) \tag{29}$$

does not depend on $r \geq 1 - n$. On the other hand the general framework of the previous section produces a map of spectra

$$c^\tau : \mathbf{K}^{\mathbb{Q}}(A) \to \boldsymbol{\tau}\mathbf{K}^{\mathbb{Q}}(A)$$

which at the level of homotopy groups gives a character

$$c_n^\tau : K_n^{\mathbb{Q}}(A) \to \tau K_n^{\mathbb{Q}}(A) \qquad (n \in \mathbb{Z}).$$

We shall show that this map agrees with the rational Jones-Goodwillie Chern character up to canonical isomorphism.

Theorem 2.1. *There is a natural isomorphism* $HN_n^{\mathbb{Q}}(A) \cong \tau K_n^{\mathbb{Q}}(A)$ *($n \in \mathbb{Z}$) which makes the following diagram commute*

$$\begin{tikzcd} K_n^{\mathbb{Q}}(A) \arrow[r, "c_n^\tau"] \arrow[dr, "ch_n"'] & \tau K_n^{\mathbb{Q}}(A) \\ & HN_n^{\mathbb{Q}}(A). \arrow[u, "\wr"] \end{tikzcd}$$

Proof. By [6, 5.1 and 5.5], for every $R \in Rings := Ass(\mathbb{Z})$ there is a homotopy fibration sequence of spaces

$$\mathbb{H}\big(R_{inf}, K^{\mathbb{Q}}\big) \longrightarrow K^{\mathbb{Q}}(R) \xrightarrow{ch} SCN_{\geq 1}^{\mathbb{Q}}(R). \tag{30}$$

Here $R_{inf} = inf(Rings \downarrow R)$ is the nonunital version of the infinitesimal site, and infinitesimal hypercohmology is taken with coefficients in a space, rather than a spectrum; this is defined by the same formula (19) (see [5, §5]). By definition, $K^{\mathbb{Q}}R :=$ hofiber$(K^{\mathbb{Q}}\tilde{R} \to K^{\mathbb{Q}}\mathbb{Z})$. Note that $CN_{\geq 1}^{\mathbb{Q}}\mathbb{Z} = CN_{\geq 1}\mathbb{Q} = 0$, so that $SCN_{\geq 1}^{\mathbb{Q}}R :=$ hofiber$(SCN_{\geq 1}^{\mathbb{Q}}\tilde{R} \to SCN_{\geq 1}^{\mathbb{Q}}\mathbb{Z}) \cong SCN_{\geq 1}^{\mathbb{Q}}\tilde{R}$. Using (30) we will prove that for A unital, ch_n agrees with the map induced by the nonunital version (22) of c^τ; by (21) and Lemma 1.10, the latter map is the same as that induced by the unital version. Applying (30) to $R = \Sigma^{n+1}A$ we get the homotopy fibration sequence in the right column of the following diagram

$$\begin{tikzcd} \mathbb{H}\big(A_{inf}, \mathbb{H}\big(\Sigma^{n+1}(\ )_{inf}, K^{\mathbb{Q}}\big)\big) \arrow[r] \arrow[d] & \mathbb{H}\big(\Sigma^{n+1}A_{inf}, K^{\mathbb{Q}}\big) \arrow[d] \\ \mathbb{H}\big(A_{inf}, K^{\mathbb{Q}}\Sigma^{n+1}\big) \arrow[r] \arrow[d] & K^{\mathbb{Q}}(\Sigma^{n+1}A) \arrow[d, "ch\Sigma^{n+1}"] \\ \mathbb{H}\big(A_{inf}, SCN_{\geq 1}^{\mathbb{Q}}\Sigma^{n+1}\big) \arrow[r] & SCN_{\geq 1}^{\mathbb{Q}}(\Sigma^{n+1}A). \end{tikzcd} \tag{31}$$

The column on the left results from applying $\mathbb{H}(A_{inf},\ \ )$ to the homotopy fibration sequence on the right; because holim preserves such sequences, it is again one. It follows from [5, §5] and from the fact that Σ preserves nilpotent extensions that the horizontal map at the top is an equivalence. Thus the map between loopspaces of the fibers of the first two horizontal maps from the bottom is an equivalence. Because loopspaces commute with homotopy limits, and in view of the identity (25), we get that the vertical map to the left of the following diagram is an equivalence

$$\begin{tikzcd} {}_{n-1}\tau\mathbf{K}^{\mathbb{Q}}(A) \arrow[r] \arrow[d, "\wr"] & \mathbb{H}\big(A_{inf}, {}_n\mathbf{K}^{\mathbb{Q}}\big) \arrow[r] \arrow[d] & {}_n\mathbf{K}^{\mathbb{Q}}(A) \arrow[d] \\ F \arrow[r] & \mathbb{H}\big(A_{inf}, \Omega SCN^{\mathbb{Q}}_{\geq 1}\Sigma^{n+1}\big) \arrow[r] & \Omega SCN^{\mathbb{Q}}_{\geq 1}(\Sigma^{n+1}A). \end{tikzcd}$$

Here both rows are homotopy fibrations and ${}_{n-1}\tau\mathbf{K}^{\mathbb{Q}}$ is the $n-1$ space of the spectrum $\tau\mathbf{K}^{\mathbb{Q}}$. We shall show that $\mathbb{H}(A_{inf}, \Omega SCN^{\mathbb{Q}}_{\geq 1}\Sigma^{n+1})$ is contractible; the lemma follows from this. Because HN is Morita invariant and satisfies excision for sequences of the form (27), there is an equivalence

$$\mathbb{H}\big(A_{inf}, \Omega SCN^{\mathbb{Q}}_{\geq 1}(\Sigma^{n+1}(\ \ ))\big) \xrightarrow{\sim} \mathbb{H}\big(A_{inf}, SCN^{\mathbb{Q}}_{\geq -n}[-n]\big). \tag{32}$$

Here S is again the Dold-Kan correspondence and $CN_{\geq -n}[-n]$ is the negative cyclic complex which we have truncated below n in the same way as explained above for $n=1$, and then shifted upwards to a nonnegative chain complex. By [25, 5.32] it suffices to show that the infinitesimal hypercohomology of the truncated complex $CN_{\geq -n}[-n]$ vanishes in nonnegative degrees. By [6, 5.4], we can replace CN_* by the complex NX_* for the negative cyclic homology of Cuntz-Quillen's mixed complex X_*. We have a second quadrant spectral sequence

$$E^1_{p,q} = H^{-p}\big(A_{inf}, \big(NX^{\mathbb{Q}}_{\geq -n}\big)_{q-n}\big) \Rightarrow \mathbb{H}^{-p-q}\big(A_{inf}, NX^{\mathbb{Q}}_{\geq -n}[-n]\big)$$

By the proof of [6, 5.1], $E^1_{p,q} = 0$ for $q \neq 0$ and also for $q=0$ and $p \geq -2$. □

Remark 2.2. If A is a nonunital ring, the Chern character is defined as the map

$$K^{\mathbb{Q}}_*(A) := K^{\mathbb{Q}}_*(\tilde{A} : A) \to HN^{\mathbb{Q}}_*(\tilde{A} : A) =: HN^{\mathbb{Q}}_*(A) \tag{33}$$

induced by $ch_* : K^{\mathbb{Q}}_*(\tilde{A}) \to HN^{\mathbb{Q}}_*(\tilde{A})$. By the theorem above, (21), and Lemma 1.10, (33) coincides with the map induced by the tautological character (22).

Remark 2.3. If $A \in Ass(\mathbb{Q})$, by [6, 5.1], one has a homotopy fibration sequence

$$\mathbb{H}(A_{inf}, K) \longrightarrow K(A) \xrightarrow{ch^{\mathbb{Z}}} SCN_{\geq 1}(A) \tag{34}$$

where $K = \mathbb{Z}_\infty GL$ is the integral plus construction and $ch^{\mathbb{Z}}$ is the integral Jones-Goodwillie character. By the same method as described above in this section for $ch_* = ch_*^{\mathbb{Q}}$, one can define characters $ch_*^{\mathbb{Z}}$ for all $* \in \mathbb{Z}$. If further A is unital, using (34) and the same argument as that of the proof of 2.1, one gets that $ch_*^{\mathbb{Z}}$ agrees with the tautological character for the integral nonconnective K-theory spectrum $\mathbf{K}$, obtained by delooping the fiber of $\mathbb{H}(inf(Ass(\mathbb{Q}) \downarrow A), \mathbf{K}) \to \mathbf{K}$, as well as with that obtained using $inf(Ass_1(\mathbb{Q}) \downarrow A)$ instead. This can be further extended to nonunital A by the same argument as in 2.2.

3. Suslin-Wodzicki theorem for A^∞

Summary. In this section we show (Theorem 3.16) that if A is a ring such that the pro-vectorspace

$$H_r^{bar}((A \otimes \mathbb{Q})^\infty) = H_r^{bar}(A^\infty) \otimes \mathbb{Q} = \left\{H_r^{bar}(A^n) \otimes \mathbb{Q}\right\}_n \tag{35}$$

is zero for all $r \geq 0$, then A is ∞-excisive for rational K-theory in the sense of Sect. 1. As explained in the introduction, for the particular case when $A = A^2$, this says the same as Suslin-Wodzicki's theorem (9). The strategy of proof imitates that used by Suslin and Wodzicki in [24]. The three main steps are Propositions 3.4, 3.6 and 3.14. Next we outline some of the contents of these propositions, and explain how the main result of the section (3.16) follows from them. In this summary, as well as in most of the section, only homology with rational coefficients is considered, and the $\mathbb{Q}$ is dropped from our notation; thus here $H_*(\ ) = H_*(\ , \mathbb{Q})$. In 3.4, we consider the affine group $\widetilde{GL}(A) = GL(A) \ltimes M_{1\infty}A$. We show that if the map

$$H_r(GL(A^\infty)) \to H_r(\widetilde{GL}(A^\infty)) \tag{36}$$

is an isomorphism for all r and the same is true with A^{op} substituted for A, then A is ∞-excisive for rational K-theory. In 3.6 it is shown that the condition that (36) be an isomorphism is equivalent to two other conditions. One of these (3.6 (b)) involves the space $\cup_{\tau,n} BT_n^\tau \subset BGL$, union over all n and all finitely supported partial orders τ of the classifying spaces of the triangular groups $T_n^\tau \subset GL_n$ defined by the orders. The condition is that

$$\left\{H_r\left(\bigcup_{n,\tau} BT_n^\tau(A^m)\right)\right\}_m \to \left\{H_r\left(\bigcup_{n,\tau} B\widetilde{T}_n^\tau(A^m)\right)\right\}_m \tag{37}$$

be an isomorphism for all r. In particular (36) is an isomorphism if (37) is. In 3.14(b) we show that the condition that

$$H_r^{bar}(A^\infty)\otimes\mathbb{Q}=0 \tag{38}$$

for all r implies (37) is an isomorphism. Thus (38) implies (36) is an isomorphism. But we prove in 3.15 that H^{bar} does not change if we replace an algebra by its opposite. Hence (38) implies that (36) is an isomorphism not only for A but also for A^{op}, whence (38) implies excision in rational K-theory, by 3.4. □

Recall that if $A\in Rings$, its general linear group is defined as

$$GL(A):=\ker(GL(\tilde{A})\to GL(\mathbb{Z})).$$

Here the map $GL(\tilde{A})\to GL(\mathbb{Z})$ is that induced by $\tilde{A}\to\mathbb{Z}$, $(a,n)\mapsto n$. We remark that if A happens to be unital, then $GL(A)$ as defined above is naturally isomorphic to the usual $GL(A)$. More generally, if $G:Ass_1\to$ ((Groups)) is any functor which preserves finite products and A is unital then

$$\ker(G(\tilde{A})\to G(\mathbb{Z}))=\ker(G(A)\times G(\mathbb{Z})\to G(\mathbb{Z}))=G(A).$$

We consider the affine group

$$\widetilde{GL}(A)=GL(A)\ltimes M_{1\infty}A. \tag{39}$$

Lemma 3.1. *Let $A\in$ Rings, and assume that for every diagram* (14) *the natural map of pro-vectorspaces*

$$\left\{K_n^{\mathbb{Q}}(B:A^m)\right\}_m\longrightarrow\left\{K_n^{\mathbb{Q}}(C:A^m)\right\}_m$$

is an isomorphism for $n\geq 1$. Then A is ∞-excisive for $\mathbf{K}^{\mathbb{Q}}$.

Proof. There is a map of exact sequences

$$\begin{array}{ccccc}
\lim_m{}^1K_{n+1}^{\mathbb{Q}}(B:A^m) & \hookrightarrow & \pi_n\hat{\mathbf{K}}^{\mathbb{Q}}(B:A^\infty) & \twoheadrightarrow & \lim_m K_n^{\mathbb{Q}}(B:A^m)\\
\downarrow & & \downarrow & & \downarrow\\
\lim_m{}^1K_{n+1}^{\mathbb{Q}}(C:A^m) & \hookrightarrow & \pi_n\hat{\mathbf{K}}^{\mathbb{Q}}(C:A^\infty) & \twoheadrightarrow & \lim_m K_n^{\mathbb{Q}}(C:A^m).
\end{array}$$

The hypothesis implies that the vertical arrows on both extremes are isomorphisms for $n\geq 1$ and that the leftmost vertical arrow is an isomorphism for $n=0$. Because K-theory is excisive in dimension ≤ 0, the map

$$K_n^{\mathbb{Q}}(B:A^m)\longrightarrow K_n^{\mathbb{Q}}(C:A^m)$$

is an isomorphism for all m and all $n\leq 0$. We conclude that, under the hypothesis of the lemma, the vertical map in the middle of the diagram above is an isomorphism for all $n\in\mathbb{Z}$. □

Lemma 3.2. *Let R be a unital ring, $A \triangleleft R$ an ideal and $\overline{GL}(R/A)$ the image of $GL(R)$ in $GL(R/A)$. Write $F(R:A)$ for the fiber of the fibration $K^{\mathbb{Q}}(R) \to \mathbb{Q}_\infty B\overline{GL}(R/A)$, and $F'(R:A) :=$ hofiber$(K^{\mathbb{Q}}(A) \to K^{\mathbb{Q}}(R))$. Then the canonical map $F(R:A) \to F'(R:A)$ induces a weak equivalence onto the connected component of $F'(R:A)$.*

Proof. We have a map of homotopy fibration sequences

$$\begin{tikzcd} F(R:A) \arrow[r] \arrow[d] & K^{\mathbb{Q}}(R) \arrow[r] \arrow[d, "1"'] & \mathbb{Q}_\infty(B\overline{GL}(R/A)) \arrow[d] \\ F'(R:A) \arrow[r] & K^{\mathbb{Q}}(R) \arrow[r] & K^{\mathbb{Q}}(R/A). \end{tikzcd} \tag{40}$$

From the long exact sequence of homotopy groups of the top fibration, we get that $F(R:A)$ is connected. Hence it suffices to prove $\pi_n F(R:A) \to \pi_n F'(R:A)$ is an isomorphism for $n \geq 1$. In turn, by the five lemma applied to the map of long exact sequences of homotopy groups associated to (40), we are further reduced to proving that the right vertical arrow induces a isomorphism at the level of π_n for $n \geq 2$ and an injection in π_1. We have the identities

$$\begin{aligned}[GL(R/A), GL(R/A)] &= E(R/A) = \text{Image}(\text{E(R)} \to \overline{\text{GL}}\text{(R/A)}) \\ &= \text{Image}([\text{GL(R), GL(R)}] \to \text{GL(R/A)}) \\ &= [\overline{GL}(R/A), \overline{GL}(R/A)].\end{aligned}$$

Put $\overline{K}_1(R/A) :=$ Image(K_1(R) $\to$ K_1(R/A)). We have a map of fibrations

$$\begin{tikzcd} BE(R/A) \arrow[r] \arrow[d, "1"'] & B\overline{GL}(R/A) \arrow[r] \arrow[d, "\iota"'] & B\overline{K}_1(R/A) \arrow[d] \\ BE(R/A) \arrow[r] & BGL(R/A) \arrow[r] & BK_1(R/A). \end{tikzcd}$$

Apply $\mathbb{Q}_\infty$ to the diagram above. Because $K_1(R/A)$ is commutative,

$$\mathbb{Q}_\infty B\overline{K}_1(R/A) \overset{\sim}{\to} B(\mathbb{Q} \otimes \overline{K}_1(R/A)) \text{ and}$$
$$\mathbb{Q}_\infty BK_1(R/A) \overset{\sim}{\to} B(\mathbb{Q} \otimes K_1(R/A)).$$

Thus $\pi_n \mathbb{Q}_\infty(\iota)$ is an isomorphism for $n \geq 2$ and the inclusion $\mathbb{Q} \otimes \overline{K}_1(R/A) \subset \mathbb{Q} \otimes K_1(R/A)$ for $n = 1$. □

Notation 3.3. From now until the end of the current section, we shall mostly consider homology with rational coefficients, hence we shall drop the coefficients from our notation, except when both rational and integral homology appear, in which case the coefficients will be emphasized. Thus if X is a space, $H_*(X)$ will mean $H_*(X, \mathbb{Q})$, and if G is a group,

$$H_*(G) = H_*(BG) = H_*(G, \mathbb{Q}).$$

Proposition 3.4. *Let $A \in$ Rings and $\widetilde{GL}(A)$ be as in* (39). *Assume that the inclusion $GL(A) \subset \widetilde{GL}(A)$ induces an isomorphism of pro-abelian groups*

$$H_r(GL(A^\infty)) \xrightarrow{\sim} H_r(\widetilde{GL}(A^\infty)) \tag{41}$$

for all $r \geq 1$, and that the same is true with A^{op} substituted for A. Then A is ∞-excisive for $\mathbf{K}^{\mathbb{Q}}$.

Proof. Let $\iota : GL(A) \subset \widetilde{GL}(A)$ be the inclusion and $p : \widetilde{GL}(A) \to GL(A)$ the projection. Because $p\iota = 1$, ιp induces the identity in pro-homology. In terms of transition maps, this means that for every n and every r there exists $k = k(n,r) \geq n$ such that $\iota p \sigma_{k,n} = \sigma_{k,n}$ on $H_r(\widetilde{GL}(A^k))$ (notations are as in A.1). Since A^{op} also satisfies this by hypothesis, and since both p and ι are level maps, and moreover, by naturality, the σ commute with the action of $Gl(\mathbb{Z})$, the argument of [24, Prop. 1.5], shows that

$$\sigma_{k,n}\big(H_r(GL(A^k))\big) \subset \big(H_r(GL(A^n))\big)^{GL(\mathbb{Z})}. \tag{42}$$

Next the argument of [24, Cor. 1.6] proves that if $A \triangleleft B$ is an embedding as an ideal of a unital ring B then (42) holds with $GL(B)$ subsituted for $GL(\mathbb{Z})$

$$\sigma_{k,n}\big(H_r(GL(A^k))\big) \subset \big(H_r(GL(A^n))\big)^{GL(B)}. \tag{43}$$

Now proceed as in the proof of [24, 1.7], using the levelwise Serre spectral sequence and then A.2 and A.5, to conclude that the map $BGL(A^\infty) \to F(B : A^\infty)$ induces an isomorphism in rational pro-homology. In particular if $B \to C$ is as in (14), then the map $\alpha : F(B : A^\infty) \to F(C : A^\infty)$ is a rational pro-homology equivalence, by the argument of the proof of [24, 1.7]. Because by 3.2 both $F(B : A^\infty)$ and $F(C : A^\infty)$ are towers of connected spaces each of which is a homotopy loopspace whose homotopy groups are $\mathbb{Q}$-vectorspaces, it follows from Prop. A.7 that α is a pro-homotopy equivalence. By the previous lemmas, this shows that A is ∞-excisive. □

Notation 3.5. In the next proposition, $E(A) \subset GL(A)$ is the elementary subgroup, $T_n^\tau(A) \subset GL_n(A)$ is the triangular subgroup defined by the partial order τ of the set $\{1,\dots,n\}$, and $\cup_{\tau,n} BT_n^\tau \subset BGL$ is the union over all n and all finitely supported partial orders τ. The definition of each of the groups just mentioned for nonunital A can be found in [24]. If $G \subset GL(A)$ is a subgroup, we write

$$\widetilde{G} := G \ltimes M_{1\infty}A.$$

Proposition 3.6. *Let $A \in$ Rings, $\iota : GL(A) \to \widetilde{GL}(A)$ the inclusion. The following are equivalent.*

(a) The map ι induces a pro-isomorphism

$$H_r(GL(A^\infty)) \to H_r(\widetilde{GL}(A^\infty)) \qquad (\forall r \geq 0).$$

(b) The map ι induces a pro-isomorphism

$$H_r(E(A^\infty)) \to H_r(\widetilde{E}(A^\infty)) \qquad (\forall r \geq 0).$$

(c) The map ι induces a pro-isomorphism

$$\left\{H_r\Big(\bigcup_{n,\tau} BT_n^\tau(A^m)\Big)\right\}_m \to \left\{H_r\Big(\bigcup_{n,\tau} B\widetilde{T}_n^\tau(A^m)\Big)\right\}_m \qquad (\forall r \geq 0). \tag{44}$$

Proof. Let $C(A) = [GL(A), GL(A)]$. We have a natural map of short exact sequences

$$\begin{array}{ccccc} C(A) & \longrightarrow & GL(A) & \longrightarrow & GL_{ab}(A) \\ \downarrow & & \downarrow & & \downarrow\cong \\ \tilde{C}(A) & \longrightarrow & \widetilde{GL}(A) & \longrightarrow & \widetilde{GL}(A)/\tilde{C}(A). \end{array}$$

This map induces a level map of pro-Lyndon-Serre spectral sequences

$$\begin{array}{ccc} E^2_{p,q} = \left\{H_p\big(GL_{ab}(A^n), H_q(C(A^n))\big)\right\}_n & \Longrightarrow & H_{p+q}(GL(A^n)) \\ \downarrow & & \downarrow \\ E^2_{p,q} = \left\{H_p\big(GL_{ab}(A^n), H_q(\tilde{C}(A^n))\big)\right\}_n & \Longrightarrow & H_{p+q}(\widetilde{GL}(A^n)). \end{array}$$

By the proof of [24, Cor. 1.14], (see formula (12)), $E(A^{2^n})$ is closed under conjugation by elements of $GL(A^{2^{n+1}})$. The inclusions

$$C(A^{2^{n+1}}) \subset E(A^{2^n}), \quad E(A^{2^{n+1}}) \subset [E(A^{2^n}), E(A^{2^n})] \subset C(A^{2^n})$$

induce, for each $n \geq 2$, two homomorphisms of pairs (see the Appendix for a definition)

$$\begin{aligned} (GL(A^{2^{n+1}}), C(A^{2^{n+1}})) &\to (GL(A^{2^{n+1}}), E(A^{2^n})) \\ &\to (GL(A^{2^{n-1}}), C(A^{2^{n-1}})). \end{aligned}$$

These give mutually inverse pro-isomorphisms

$$\{(GL(A^n), C(A^n))\}_n \overset{\sim}{\longleftrightarrow} \{(GL(A^{2^{n+1}}), E(A^{2^n}))\}_n.$$

Taking homology we obtain a pro-isomorphism

$$\left\{H_p\big(GL_{ab}(A^n), H_q(C(A^n))\big)\right\}_n \cong \left\{H_p\big(GL_{ab}(A^{2^{n+1}}), H_q(E(A^{2^n}))\big)\right\}_n.$$

Similarly,

$$\left\{H_p\big(GL_{ab}(A^n), H_q(\tilde{C}(A^n))\big)\right\}_n \cong \left\{H_p\big(GL_{ab}(A^{2^{n+1}}), H_q(\tilde{E}(A^{2^n}))\big)\right\}_n.$$

The proof that (a) $\Longleftrightarrow$ (b) now follows as in the proof of [24, 2.10], using [24, 1.14] and A.2, A.4 and A.5. Similarly, that (b) $\Longleftrightarrow$ (c) follows from [24, 2.9 and 2.7], and A.2 and A.5. $\square$

Notation 3.7. In the following lemma and below, $\mathfrak{gl}_n(A)$ is the general linear Lie algebra, $\mathfrak{c}_n(A) = [\mathfrak{gl}_n(A), \mathfrak{gl}_n(A)]$, $\mathfrak{sl}_n(A) = \ker(\mathrm{Trace}\colon \mathfrak{gl}_n(A) \to A/[A, A])$, and $\mathfrak{e}_n(A) \subset \mathfrak{gl}_n(A)$ is the Lie subalgebra generated by the matrices $e_{i,j}(a)$ $(i \neq j, a \in A)$.

Lemma 3.8. *Let $A \in Ass(\mathbb{Q})$, $n \geq 3$. Then*

$$\mathfrak{e}_n(A) = \bigoplus_{1\leq i\neq j\leq n} e_{i,j}(A) \oplus \bigoplus_{i=1}^{n-1}(e_{i,i} - e_{i+1,i+1})(A^2) \oplus e_{1,1}([A, A])$$

$$\mathfrak{sl}_n(A) = \bigoplus_{1\leq i\neq j\leq n} e_{i,j}(A) \oplus \bigoplus_{i=1}^{n-1}(e_{i,i} - e_{i+1,i+1})(A) \oplus e_{1,1}([A, A])$$

$$\mathfrak{c}_n(A) = \bigoplus_{1\leq i\neq j\leq n} e_{i,j}(A^2) \oplus \bigoplus_{i=1}^{n-1}(e_{i,i} - e_{i+1,i+1})(A^2) \oplus e_{1,1}([A, A]).$$

Proof. Straightforward. □

Corollary 3.9.

$$\mathfrak{sl}_n(A^2) \subset \mathfrak{c}_n(A) \subset \mathfrak{sl}_n(A)$$

$$\mathfrak{sl}_n(A^2) \subset \mathfrak{e}_n(A) \subset \mathfrak{sl}_n(A).$$

□

Notation 3.10. In the following proposition, $\mathfrak{t}_n^\tau(A) \subset \mathfrak{gl}_n(A)$ is the triangular Lie subalgebra defined by the partial order τ of $\{1, \dots, n\}$ (see [24]). We put

$$\mathfrak{gl}_{n+1}(A) \supset \widetilde{\mathfrak{gl}}_n(A) = \left\{ \begin{pmatrix} g & v \\ 0 & 0 \end{pmatrix} : g \in \mathfrak{gl}_n(A), v \in M_{n,1}(A) \right\}$$

and for $\mathfrak{g} = \mathfrak{sl}, \mathfrak{c}, \mathfrak{e}, \dots$

$$\tilde{\mathfrak{g}}_n(A) = \mathfrak{g}_{n+1}(A) \cap \widetilde{\mathfrak{gl}}_n(A).$$

If $\mathfrak{g}_n(A)$ is either $\mathfrak{gl}_n(A)$ or one of the Lie algebras above, then $\mathfrak{g}(A)$ with no subscript stands for $\cup_n \mathfrak{g}_n(A)$. In the proposition below, A is a $\mathbb{Q}$-algebra and Lie algebra homology is taken with coefficients in $\mathbb{Q}$. We write

$$H_*(\mathfrak{g}) := H_*(\mathfrak{g}, \mathbb{Q})$$

for all Lie algebras $\mathfrak{g}$ appearing below. The groups $H_*(\mathfrak{g})$ are the homology of the the Chevalley-Eilenberg complex $C_*(\mathfrak{g})$ for Lie algebra homology over $\mathbb{Q}$ [21, 10.1.3]. This complex appears in item (c) of the proposition below.

Proposition 3.11. *Let $A \in Ass(\mathbb{Q})$, $\iota : \mathfrak{gl}(A) \to \widetilde{\mathfrak{gl}}(A)$ the inclusion. The following are equivalent.*

(a) The map ι induces a pro-isomorphism

$$H_r(\mathfrak{gl}(A^\infty)) \to H_r(\widetilde{\mathfrak{gl}}(A^\infty)) \qquad (r \geq 0).$$

(b) The map ι induces a pro-isomorphism

$$H_r(\mathfrak{e}(A^\infty)) \to H_r(\widetilde{\mathfrak{e}}(A^\infty)) \qquad (r \geq 0).$$

(c) The map ι induces a pro-isomorphism

$$\left\{H_r\Big(\sum_{n,\tau} C_*\big(\mathfrak{t}_n^\tau(A^m)\big)\Big)\right\}_m \to \left\{H_r\Big(\sum_{n,\tau} C_*\big(\widetilde{\mathfrak{t}}_n^\tau(A^m)\big)\Big)\right\}_m \qquad (r \geq 0).$$

Proof. As in [24, Formula (34)], we shall write $\nu(A)$ and $\tilde{\nu}(A)$ for the Volodin chain complexes. To start, mimic the proof of [24, 2.9], to show that the actions of $\mathfrak{e}(A^2)$ on $H_*(\nu(A))$ and of $\tilde{\mathfrak{e}}(A^2)$ on $H_*(\tilde{\nu}(A))$ are trivial. Then use 3.8 and imitate the proof of [24, 1.14], to get that the actions of $\mathfrak{gl}(A^2)$ on $H_*(\mathfrak{sl}(A))$, of $\mathfrak{gl}(A^2)$ on $H_*(\mathfrak{e}(A))$, of $\widetilde{\mathfrak{gl}}(A^2)$ on $H_*(\widetilde{\mathfrak{sl}}(A))$ and of $\widetilde{\mathfrak{gl}}(A^2)$ on $H_*(\tilde{\mathfrak{e}}(A))$ are trivial. Then follow as in the proof of 3.6, substituting the Hochschild-Serre spectral sequence for that of Lyndon-Serre, and [24, 4.9] for [24, 2.7]. □

Notation 3.12. If R is a ring, we write R_1 for the ring of 2×2 matrices

$$M_2R \supset R_1 := \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in R \right\}.$$

A straightforward calculation shows that the construction $R \mapsto R_1$ commutes with powers

$$(R^n)_1 = (R_1)^n.$$

In particular if A is any ring, then the pro-ring $A_1^\infty = \{A_1^n\}_n$ is unambiguously defined.

In the proof of the lemma below, the *bar complex* $C_*^{bar}(A)$ of a ring A is considered. Recall from [28] that $C_n^{bar}(A) = 0$ if $n \leq 0$ and $C_n^{bar} = A^{\otimes n}$ if $n \geq 1$, with boundary operator given by

$$b'(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n-1} (-1)^{i+1} a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n.$$

By definition, the bar homology of A is the homology of $C_*^{bar}(A)$.

Lemma 3.13. *Let $A \in Ass(\mathbb{Q})$. Assume $H_r^{bar}(A^\infty) = 0$ $(r \geq 0)$. Then*

(a)

(45) $$H_r^{bar}\big(A_1^\infty\big) = 0 \qquad (r \geq 0).$$

(b) The inclusion $A \subset A_1$ induces a pro-isomorphism

(46) $$HC_r(A^\infty) \cong HC_r\big(A_1^\infty\big) \qquad (r \geq 0).$$

Proof. (a) This is straightforward from the proof of a particular case of [28, 9.1]. We recall the argument for the reader's convenience. If R is any $\mathbb{Q}$-algebra, we write $N(R)$ for the following ideal of R_1

$$N(R) := \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in R \right\}.$$

We may regard $N(R)$ as an R-bimodule; note that R_1 is isomorphic to the semidirect product $R \ltimes N(R)$. Hence the proof of [28, 9.1] applies to R_1, and shows in particular (see the first displayed formula on p. 622 of *loc.cit.*) that the bar complex has a direct sum decomposition

$$C^{bar}(R_1) = C^{bar}(R) \oplus \bigoplus_{l=1}^{\infty} C^{bar}(R_1, l).$$

By [28, 9.2], for each $l \geq 1$ there is a first quadrant spectral sequence converging to the homology of $C^{bar}(R_1, l)$ of which the zero term is of the form

$$\left(E^0_{p*}(R_1, l), d^0\right) = C^{bar}(R) \otimes V(R, l)[l-1] \tag{47}$$

where $V(R, l)$ is some vectorspace depending on R and l. Applying all this levelwise to A_1^∞, and noting that the hypothesis that the bar pro-homology of A^∞ vanishes implies

$$\left\{H_r^{bar}(A^n) \otimes V_n\right\} = 0 \qquad (r \geq 0). \tag{48}$$

for every pro-vectorspace V, proves (a).

(b) It suffices to show that the inclusion $A \subset A_1$ induces a pro-isomorphism $HH_r(A^\infty) \cong HH_r(A_1^\infty)$. To prove this we use the argument of the proof of [28, 11.1]. It is shown in *loc.cit.* that for $R \in Ass(\mathbb{Q})$ the Hochschild complex decomposes as $C(R_1) = C(R) \oplus \bigoplus_{l=1}^{\infty} C(R, l)$ (see formula (50) in *loc.cit.*) and that again for each $l \geq 1$ there is a first quadrant spectral sequence converging to $H_*(C(R, l))$ of the form (47) (see [28, 11.2]). As in (a), applying this levelwise to A_1^∞, we obtain (b). □

Proposition 3.14. *(a) Let $A \in Ass(\mathbb{Q})$. Assume*

$$H_r^{bar}(A^\infty) = 0 \qquad (r \geq 0). \tag{49}$$

Then the natural inclusions $T_n(A) \subset \tilde{T}_n(A)$ $(n \geq 1)$ induce a pro-isomorphism

$$\left\{H_r\left(\bigcup_{n,\tau} BT_n^\tau(A^m), \mathbb{Z}\right)\right\}_m \xrightarrow{\sim} \left\{H_r\left(\bigcup_{n,\tau} B\tilde{T}_n^\tau(A^m), \mathbb{Z}\right)\right\}_m \qquad (r \geq 0).$$

(b) Let $A \in Rings$. Assume

$$H_r^{bar}(A^\infty) \otimes \mathbb{Q} = 0 \quad (r \geq 0).$$

Then the natural inclusions $T_n(A) \subset \tilde{T}_n(A)$ $(n \geq 1)$ induce a pro-isomorphism

$$\left\{H_r\left(\bigcup_{n,\tau} BT_n^\tau(A^m), \mathbb{Q}\right)\right\}_m \xrightarrow{\sim} \left\{H_r\left(\bigcup_{n,\tau} B\tilde{T}_n^\tau(A^m), \mathbb{Q}\right)\right\}_m \quad (r \geq 0).$$

Proof. (a) According to [24, Cor. 5.14], for any $m > 0$ we have canonical isomorphisms

$$\begin{aligned} \tilde{H}_*\left(\bigcup_{n,\tau} BT_n^\tau(A^m), \mathbb{Z}\right) &\xrightarrow{\sim} \tilde{H}_*\left(\sum_{n,\tau} C_*\left(t_n^\tau(A^m)\right)\right) \\ \tilde{H}_*\left(\bigcup_{n,\tau} B\tilde{T}_n^\tau(A^m), \mathbb{Z}\right) &\xrightarrow{\sim} \tilde{H}_*\left(\sum_{n,\tau} C_*\left(\tilde{t}_n^\tau(A^m)\right)\right). \end{aligned} \tag{50}$$

Thus to establish the required pro-isomorphism it suffices to show that the canonical map

$$\left\{H_*\left(\sum_{n,\tau} C_*\left(t_n^\tau(A^m)\right)\right)\right\}_m \to \left\{H_*\left(\sum_{n,\tau} C_*\left(\tilde{t}_n^\tau(A^m)\right)\right)\right\}_m$$

is a pro-isomorphism, which according to Proposition 3.11 is equivalent to verifying that the canonical map $H_*(\mathfrak{gl}(A^\infty)) \to H_*(\widetilde{\mathfrak{gl}}(A^\infty))$ is a pro-isomorphism. Since the pro-Lie algebra $\widetilde{\mathfrak{gl}}(A^\infty)$ is a retract in $\mathfrak{gl}(A_1^\infty)$, it is enough to show that the embedding

$$\mathfrak{gl}(A^\infty) \to \mathfrak{gl}\left(A_1^\infty\right) \tag{51}$$

induces an isomorphism in pro-homology. By (49) and 3.13, (45) holds for A. But according to [7, Thm. 4.2] if P is any pro-algebra such that $H_l^{bar}(P) = 0$ for all l, then

$$H_r(\mathfrak{gl}(P)) \cong (\wedge(HC(P)[-1]))_r$$

for all r. Here $HC(P) = \{\bigoplus_{n\geq 0} HC_n(P_m)\}_m$; the $[-1]$ indicates a levelwise degree shift. Thus

$$\begin{aligned} H_r(\mathfrak{gl}(A^\infty)) &\cong (\wedge(HC(A^\infty)[-1]))_r \\ H_r\left(\mathfrak{gl}\left(A_1^\infty\right)\right) &\cong \left(\wedge\left(HC\left(A_1^\infty\right)[-1]\right)\right)_r. \end{aligned} \tag{52}$$

By virtue of (46), this implies that (51) induces an isomorphism in pro-homology, as wanted. Part (b) follows from part (a) and [24, Cor. 5.19]. □

Lemma 3.15. *The map*

$$y : C_*^{bar}(A) \to C_*^{bar}(A^{op}), \quad y(a_1 \otimes \cdots \otimes a_n) = (-1)^{\frac{n(n-1)}{2}} a_n \otimes \cdots \otimes a_1$$

is an isomorphism of chain complexes.

Proof. Straightforward. □

Theorem 3.16. *Let A be a ring. Assume that $H_r^{bar}(A^\infty) \otimes \mathbb{Q} = 0$ ($r \geq 0$). Then A is ∞-excisive (in the sense of Sect. 1) for rational K-theory.*

Proof. By 3.4 and 3.15, to prove A is ∞-excisive for $\mathbf{K}^{\mathbb{Q}}$ it suffices to show that for all $r \geq 0$, the map (41) is an isomorphism. By 3.6 this is equivalent to showing that (44) is an isomorphism, which is in fact the case, as proved by Proposition 3.14. □

4. Proof of the Main theorem

Summary. Here we carry out part (d) of the sketch of the introduction and complete the proof of the Main theorem (4.2). We prove first (Lemma 4.1) that for $\mathbb{Q}$-algebras A and B the K-theory obstruction groups $K_*(A, B : I)$ are $\mathbb{Q}$-vectorspaces. As explained in the introduction, after this lemma and Theorems 1.8, 2.1 and 3.16, to finish the proof it remains only to show that if A is an ideal of a unital free algebra then $H_r^{bar}(A^\infty) = 0$ ($r \geq 0$) and A is ∞-excisive for HC. We explain how to derive this from what is proved in [8], [9] and [27]. □

Lemma 4.1. *Let $A \to B$ be a homomorphism of $\mathbb{Q}$-algebras carrying an ideal $I \triangleleft A$ isomorphically onto an ideal of B. Then the map*

$$K_*(A, B : I) \longrightarrow K_*^{\mathbb{Q}}(A, B : I)$$

is an isomorphism.

Proof. By 1.2, we may assume that $A \to B$ is a homomorphism of unital algebras. It suffices to show that the group $K_*(A, B : I)$ is uniquely divisible. Let $m \in \mathbb{Z}$; there is a long exact sequence

$$(53)\quad K_{n+1}(A, B : I, \mathbb{Z}/m) \to K_n(A, B : I) \stackrel{\cdot m}{\to} K_n(A, B : I) \to K_n(A, B : I, \mathbb{Z}/m).$$

By [26] the groups in both extremes are zero. □

4.2. Proof of the Main theorem. By the previous lemma, it suffices to consider rational K-theory. By Corollary 1.2, Theorems 1.8 and 2.1, and Remarks 1.9 and 2.2, this will follow if we show that if A is an ideal of a unital free ring, then A is ∞-excisive for both $K^{\mathbb{Q}}$ and $HN^{\mathbb{Q}}$. But if $H_r^{bar}(A^\infty) \otimes \mathbb{Q} = 0$ for $r \geq 0$, then A is ∞-excisive for $K^{\mathbb{Q}}$ by Theorem 3.16. As explained in the sketch of the proof of the Main theorem in the introduction (part (d)), it suffices to prove that if A is an ideal of a free unital $\mathbb{Q}$-algebra then

$$(54)\qquad H_r^{bar}(A^\infty) = 0 \qquad (r \geq 0)$$

and that A is ∞-excisive for HC. Both the latter property and (54) are well-known, and, when properly restated in terms of cofibrations and weak equivalences, hold with much greater generality than needed here; for example in [8, Thms. 5.2 and 6.3], they are proved for pro-algebra objects in a linear category with a tensor product. We shall explain presently how a more direct proof of (54) in our specific situation can be extracted from [9] and [27]. First of all we observe that even if [27] is written for the topological algebra setting the algebraic case works in exactly the same way. From [9, Prop. 4.2], A is quasi-unital in the sense of [27, Def. 4.6]. Proposition 4.7 of [27] states that, for quasi-unital A, the bar pro-complex $C^{bar}_*(A^\infty)$ is weakly contractible. Actually the proof of [27, 4.7] shows that for each n the map

$$C^{bar}_*(A^{2^{n+1}}) \to C^{bar}_*(A^{2^n}) \tag{55}$$

is null-homotopic; (54) is immediate from this. Next, to prove A is ∞-excisive for HC, it suffices, by virtue of the version of the SBI-sequence involving HC and Hochschild homology HH ([21, 2.5.8]), to show that A is ∞-excisive for HH. This means that if B is a unital algebra and $A \triangleleft B$, then for the Hochschild complex Ω^* and the Dold-Kan functor S, the map

$$\mathrm{holim}_n S\Omega^*(\tilde{A} : A^n) \to \mathrm{holim}_n S\Omega^*(B : A^n)$$

is an equivalence. In view of the exact sequence

$$0 \to \lim_n{}^1 \pi_{r+1} X^n \to \pi_r \mathrm{holim}_n X^n \to \lim_n \pi_r X^n \to 0$$

it suffices to show that for each r the map

$$HH_r(\tilde{A} : A^\infty) \to HH_r(B : A^\infty) \tag{56}$$

is an isomorphism of pro-vectorspaces. One can deduce this directly from (54) by mimicking a proof of Wodzicki's theorem, such as for example the brief proof given in [15]. Alternatively one can deduce it from the results of [27] and [8] as follows. We have proved above that $C^{bar}(A^\infty)$ is weakly contractible, which by [8, Thm. 5.2], (or [27, 4.4]) implies that $\Omega^*(\tilde{A} : A^\infty) \to \Omega^*(B : A^\infty)$ is a weak equivalence (a finite weak equivalence in the notation of [27]) of pro-complexes. Hence (56) is an isomorphism by [8, Lem. 2.1.1]. $\square$

Remark 4.2. If $A \in Ass(\mathbb{Q})$ then by the Main theorem and Remark 2.3,

$$A \mapsto \mathbb{H}(\mathit{inf}(Ass(\mathbb{Q}) \downarrow A), \mathbf{K})$$

is excisive.

Appendix A. Pro-spaces

Summary. We introduce the notation for pro-objects used throughout the paper, and prove some technical results on pro-spaces which are used in Sect. 3. Two of these concern the Serre spectral sequence (A.2 and A.5), and another rational pro-homotopy and pro-homology (A.7). □

Notation A.1. If $\mathcal{C}$ is a category, we write pro-$\mathcal{C}$ for the category of all inverse systems

$$\{\sigma_{n+1} : C_{n+1} \to C_n : n \in \mathbb{Z}_{\geq 1}\}$$

of objects of $\mathcal{C}$. The set of homomorphisms of two pro-objects C and D in pro-$\mathcal{C}$ is by definition

$$\hom_{\text{pro-}\mathcal{C}}(C, D) = \lim_n \operatorname*{colim}_m \hom_{\mathcal{C}}(C_m, D_n).$$

Thus a map $C \to D \in$ pro-$\mathcal{C}$ is an equivalence class of maps of inverse systems

$$f : \{C_{k(n)}\}_n \to \{D_n\}_n$$

where $k : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ satisfies $k(n) \geq n$ for all n. In the particular case when k is the identity we say that f (and also its equivalence class) is a *level map*. We use the letter σ (often with no subscript) to mean the transition maps of any pro-objects appearing in this paper. If $k \geq n$, we write $\sigma_{k,n}$ for the composite

$$\sigma_{k,n} := \sigma_{n+1} \circ \cdots \circ \sigma_k.$$

Note that our pro-$\mathcal{C}$ is equivalent to the category of countably indexed pro-objects considered by Cuntz and Quillen ([9]); both are much smaller than that of Artin-Mazur ([1]).

If (G, L) is a pair consisting of a group G and a left G-module L then by a morphism from (G, L) to another such pair (H, M) we mean a group homomorphism $f : G \to H$ together with a homomorphism of abelian groups $L \to M$ – which we shall also call f – such that

$$f(gl) = f(g)f(l) \qquad (\forall g \in G, l \in L).$$

We write $C_*(G, M)$ for the bar complex of G with coefficients in M.

Lemma A.2. *Let* $(G, M) = \{\sigma_{n+1} : (G_{n+1}, M_{n+1}) \to (G_n, M_n) : n \geq 1\}$ *be a sequence of pairs consisting at each level n of a group* G_n *and a left* G_n*-module* M_n*, together with transition maps* σ_n *which are morphisms of such pairs in the sense of A.1. Assume that*

$$(\forall n)(\exists k = k(n) > n) \text{ such that } (\forall g \in G_k, x \in M_k)\ \ \sigma_{k,n}(gx) = \sigma_{k,n}(x).$$

Then there is a natural short exact sequence pro-graded abelian groups

$$\begin{aligned} 0 \to \{H_*(G_n, \mathbb{Z}) \otimes_{\mathbb{Z}} M_n\} &\to \{H_*(G_n, M_n)\} \\ &\to \{\operatorname{Tor}(H_{*-1}(G_n, \mathbb{Z}), M_n)\} \to 0. \end{aligned} \tag{57}$$

Proof. In the particular case when each G_n acts trivially on each M_n, we have an inverse system of levelwise exact sequences

$$0 \to H_*(G_n, \mathbb{Z}) \otimes M_n \to H_*(G_n, M_n) \to \mathrm{Tor}(H_{*-1}(G_n, \mathbb{Z}), M_n) \to 0.$$

This gives (57) in this particular case. Next we observe that applying the functor $(G, M) \to C_*(G, M)$ maps isomorphic pro-pairs to isomorphic pro-complexes, which lead to isomorphic pro-homology. Hence it suffices to show that any inverse system $\{(G_n, M_n)\}$ satisfying the hypothesis of the lemma is isomorphic to one as in the particular case considered above. The inverse systems formed by the maps $\sigma_{k^n(1)} : (G_{k^n(1)}, M_{k^n(1)}) \to (G_n, M_n)$ and $1 : (G_{k^n(1)}, M_{k^n(1)}) \to (G_{k^n(1)}, M_{k^n(1)})$ represent two mutually inverse isomorphisms

$$\{(G_n, M_n)\} \cong \{(G_{k^n(1)}, M_{k^n(1)})\}$$

We may therefore assume that for all n, $k(n) = n + 1$. For each n consider σM_n as a G_n-module via σ, and let $\iota : \sigma M_n \subset M_{n-1}$ be the inclusion. Then the inverse systems of maps $(1, \sigma) : (G_n, M_n) \to (G_n, \sigma M_n)$ and $(\sigma_n, \iota) : (G_n, \sigma M_n) \to (G_{n-1}, M_{n-1})$ represent mutually inverse isomorphisms

$$\{(G_n, M_n)\} \cong \{(G_n, \sigma M_n)\}$$

By hypothesis, the action of G_n on σM_n is trivial. □

Notation A.3. Let $\mathcal{A}$ be an abelian category, $(E^2_{p,q}, d_2)$ a first quadrant spectral sequence in $\mathcal{A}$ of homological type, and $F, G : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ two functors. We require that both functors be *biadditive*, by which we mean that they be additive separately on each variable. We say that the spectral sequence is (F, G)*-dominated* if there are exact sequences

$$0 \to F\big(E^2_{p,0}, E^2_{0,q}\big) \to E^2_{p,q} \to G\big(E^2_{p-1,0}, E^2_{0,q}\big) \to 0. \tag{58}$$

If E'^2 is another (F, G)-dominated spectral sequence, then by an (F, G)-homomorphism $E^2 \to E'^2$ we understand a family of homomorphisms $f_{p,q} : E^2_{p,q} \to E'^2_{p,q}$ compatible with differentials and such that for all p and q, the obvious diagram involving $f_{p,q}$, $F(f_{p,0}, f_{0,q})$ and $G(f_{p-1,0}, f_{0,q})$, commutes.

Example A.4. Let $f : G \to G''$ be a homomorphism of inverse systems of groups such that each $f_n : G_n \to G''_n$ is surjective; put $G' = \{\ker f_n\}$. We have an inverse system of spectral sequences

$${}_nE^2_{p,q} = H_p\big(G''_n, H_q\big(G'_n, \mathbb{Z}\big)\big) \Rightarrow H_{p+q}(G_n, \mathbb{Z}). \tag{59}$$

We regard (59) as a spectral sequence E^* in the category $\mathcal{A} = \text{pro-}Ab$ of pro-abelian groups. Because each ${}_nE^*$ is located in the first quadrant, E^* is convergent. Assume further that the action of G'' on $H_*(G'; \mathbb{Z})$ satisfies the hypothesis of A.2. Then by A.2 we have an exact sequence

$$0 \to \big\{{}_nE^2_{p,0} \otimes {}_nE^2_{0,q}\big\} \to E^2_{p,q} \to \big\{\mathrm{Tor}\big({}_nE^2_{p-1,0}, {}_nE^2_{0,q}\big)\big\} \to 0 \tag{60}$$

Thus E^2 satisfies the requirements of A.3 with

$$F(M,N) = \{M_n \otimes N_n\} \text{ and } G(M,N) = \{\mathrm{Tor}(M_n, N_n)\}.$$

Proposition A.5. *Let $E^2 \to E'^2$ be a map of (F,G)-dominated first quadrant spectral sequences in an abelian category $\mathcal{A}$. Let H and H' be filtered graded objects such that E^* converges to H and E'^* to H'. Consider the maps*

$$E^2_{*,0} \to E'^2_{*,0}, \quad E^2_{0,*} \to E'^2_{0,*}, \quad H_* \to H'_* \tag{61}$$

If any two of the maps in (61) *is an isomorphism, then so is the third.*

Proof. Zeeman's proof of his comparison theorem [29] proves this statement. □

Recall that if Z is a space and k a field, then the diagonal map $\Delta : Z \to Z \times Z$ induces a k-coalgebra structure on $H(Z,k) = \bigoplus_n H_n(Z,k)$. Recall further that the *primitive part* of $H(Z,k)$ is the graded vectorspace given in degree n by

$$\mathrm{Prim}_n H(Z,k) = \ker(\tilde{\Delta} : H_n(Z,k) \to \bigoplus_{0 \le p \le n} H_p(Z,k) \otimes H_{n-p}(Z,k)) \tag{62}$$

where $\tilde{\Delta}(\xi) = \Delta(\xi) - \xi \otimes 1 - 1 \otimes \xi$. In the next lemma we consider the pro-extension of the functor $\mathrm{Prim}_n H(\ \ ,k)$.

Lemma A.6. *Let k be a field and $f : X \to Y$ a morphism of pro-spaces. Assume that for each n the map $H_n(f) \otimes k : H_n(X,k) \to H_n(Y,k)$ is an isomorphism of pro-vectorspaces. Then the induced map* $\mathrm{Prim}_n H(X,k) \to \mathrm{Prim}_n H(Y,k)$ *is an isomorphism for each n.*

Proof. Immediate from the observation that for each fixed n, $\mathrm{Prim}_n H(X,k)$ depends only on finitely many homology pro-vectorspaces $H_p(X,k)$, namely on those with $p \le n$ (see (62)). □

Proposition A.7. *Let $f : X \to Y$ be a pro-map of towers of connected fibrant pointed spaces. Assume that for each m the spaces X_m, Y_m are homotopy loopspaces and that $H_n(f) \otimes \mathbb{Q} : H_n(X,\mathbb{Q}) \to H_n(Y,\mathbb{Q})$ is an isomorphism of pro-vectorspaces for each n. Then $\pi_n(f) \otimes \mathbb{Q}$ is a isomorphism for each n.*

Proof. We have a commutative diagram

$$\begin{tikzcd} \pi_n(X) \otimes \mathbb{Q} \arrow[r] \arrow[d] & \pi_n(Y) \otimes \mathbb{Q} \arrow[d] \\ \mathrm{Prim}_n H(X,\mathbb{Q}) \arrow[r] & \mathrm{Prim}_n H(Y,\mathbb{Q}) \end{tikzcd}$$

where the horizontal maps are induced by f, and the vertical ones by the levelwise Hurewicz maps. In view of the hypothesis on the X_m and Y_m, the vertical maps are levelwise isomorphisms. Because by hypothesis $H_p(f) \otimes \mathbb{Q}$ is an isomorphism for all p, the bottom row is an isomorphism as well, by Lemma A.6. It follows that the top row is an isomorphism. □

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