



Research article

Sensitivity equations for measure-valued solutions to transport equations

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Abstract: We consider the following transport equation in the space of bounded, nonnegative Radon measures $\mathcal{M}^+(\mathbb{R}^d)$:

$$\partial_t \mu_t + \partial_x(v(x)\mu_t) = 0.$$

We study the sensitivity of the solution μ_t with respect to a perturbation in the vector field, $v(x)$. In particular, we replace the vector field v with a perturbation of the form $v^h = v_0(x) + hv_1(x)$ and let μ_t^h be the solution of

$$\partial_t \mu_t^h + \partial_x(v^h(x)\mu_t^h) = 0.$$

We derive a partial differential equation that is satisfied by the derivative of μ_t^h with respect to h , $\partial_h(\mu_t^h)$. We show that this equation has a unique very weak solution on the space Z , being the closure of $\mathcal{M}(\mathbb{R}^d)$ endowed with the dual norm $(C^{1,\alpha}(\mathbb{R}^d))^*$. We also extend the result to the nonlinear case where the vector field depends on μ_t , i.e., $v = v[\mu_t](x)$.

Keywords: transport equations; space of Radon measures; differentiability of solutions; very weak solutions

1. Introduction

Transport type equations arise ubiquitously in the physical, biological and social sciences (e.g., see [1–3]). They were, for example, recently used to approximate the dynamics of opinion formation [3] (see also [4] and [5]), to describe flow on networks (see [6–8]) and to model the dynamics of structured populations [9]. Because of the natural setting of the space of measures for these equations, as it

allows for unifying discrete and continuous dynamics under the same framework, researchers have recently focused their efforts to study well-posedness of such equations on this space [10–12]; hence generalizing previous results that treated these equations in the space of integrable functions (e.g., [1]).

The importance of understanding differentiability of solutions of differential equation models with respect to parameters is crucial for many applications including optimal control (e.g., [13, 14]), parameter estimation and least-square problems of fitting models to data [15, 16], and sensitivity analysis of solutions to model parameters that can be used to obtain information on parameter uncertainty including confidence intervals for estimated model parameters (e.g., [17–19]). Such applications require the minimization of a functional that depends on the model solution and hence (numerically) solving for the critical points of the equation that represents the derivative of the solutions with respect to parameter often becomes necessary.

In this paper, we focus on deriving an equation that represents the derivative of a transport equation with respect to the vector field. To this end, consider the following transport equation in the space of bounded, nonnegative Radon measures $\mathcal{M}^+(\mathbb{R}^d)$:

$$\partial_t \mu_t + \partial_x(v(x)\mu_t) = 0 \quad (1.1)$$

where $\mu_t : [0, T] \rightarrow \mathcal{M}^+(\mathbb{R}^d)$ and $v : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a given vector field. Equation (1.1) is equipped with the initial condition $\mu_{t=0} = \mu_0$. It is well-known that if $v \in W^{1,\infty}(\mathbb{R}^d)$, this equation has a unique solution in $C([0, +\infty), \mathcal{M}^+(\mathbb{R}^d))$ given by $\mu_t = T_t^\# \mu_0$ where T_t is the flow of v (defined in (2.2)) and $T_t^\#$ denotes the push - forward along the map T_t (see Eq (2.5)). Here, the space of measures is endowed with the so-called *bounded Lipschitz* norm $\|\cdot\|_{BL^*}$ (see Eq (2.1)).

Here, we focus on the regularity of μ_t with respect to v , i.e., if v is slightly perturbed, how will μ_t change? To be more precise, suppose $v(x)$ is replaced with the new vector field $v_h(x) := v_0(x) + hv_p(x)$ where v_0 and v_p are fixed vector fields and h can vary. The perturbed equation is then

$$\partial_t \mu_t^h + \partial_x(v_h(x)\mu_t^h) = 0 \quad (1.2)$$

which has the unique solution $\mu_t^h = (T_t^h)^\# \mu_0$ where T_t^h is the flow of the vector field v_h . It is easy to see, using the representation formula for solutions to (1.2) presented in [20] or [21] (Eq 1.3) and estimates similar to the ones used to prove Lemma 3.8 in [22], that the map $h \mapsto \mu^h$ is Lipschitz continuous in $C([0, T], \mathcal{M}^+(\mathbb{R}^d))$ so that in particular $\lim_{h \rightarrow 0} \mu^h = \mu$ in $C([0, T], \mathcal{M}^+(\mathbb{R}^d))$ for any $T > 0$ (see also Eq (2.7)).

The next step in understanding the regularity of $h \mapsto \mu_t^h$ consists in studying the existence of the derivative $\partial_h \mu^h$. This type of questions has been recently addressed in [23] for linear transport equation and for general nonlinear structured population models (including transport equation) in [24]. Briefly, denoting by $\rho_{t,h}^{\Delta h} := (\mu_t^{h+\Delta h} - \mu_t^h) / \Delta h$, a difference quotient, the question is to give a precise mathematical meaning to the limit $\lim_{\Delta h \rightarrow 0} \rho_{t,h}^{\Delta h}$. It turns out that this type of problems cannot be answered in the framework of bounded Lipschitz norm (see Example 3.5 in [24]). Indeed it is necessary to move to the bigger space Z defined as the closure of $\mathcal{M}(\mathbb{R}^d)$ endowed with the dual norm $(C^{1,\alpha}(\mathbb{R}^d))^*$ (see Section 2.3 for a brief introduction). Then, according to Theorem 1.1 in [23], one can prove that there exists $\rho_{t,h} \in Z$ such that $\lim_{\Delta h \rightarrow 0} \rho_{t,h}^{\Delta h} = \rho_{t,h}$ (see also Theorem 2.1 below).

In this paper we want to characterize ρ as the unique solution to some equation. In fact, one of the main results in this work (see, Theorem 4.1 below) states that ρ is the unique solution to the equation

$$\partial_t \rho_t + \partial_x(v_0(x)\rho_t) = -\partial_x(v_p(x)\mu_t).$$

This equation can then be thought of as the sensitivity equation satisfied by the directional derivative of μ under the perturbation $v_0 + hv_p$. We will also prove an analogous result in the non-linear case when the vector-field v depends on μ (see Theorem 5.1 below).

The proofs of these results require the detailed study of a linear transport equation in Z of the form

$$\partial_t \mu_t + \partial_x(v(x)\mu_t) = v_t, \quad \mu_{t=0} = \mu_0. \quad (1.3)$$

While the existence of a solution to (1.3) can be established by extending standard techniques to the current setting on the space Z , the uniqueness issue presents some unexpected difficulties which led to a new notion of solution. With this new concept of solution, we are able to prove in Theorem 3.1 that this equation is well-posed.

The paper is organized as follows. In Section 2 we briefly recall some known facts concerning transport equations in the space of measures and the space Z . We also establish new properties of the space Z . For a smooth flow of the paper we provide the details of long proofs of these new properties in the Appendix. In Section 3, we prove the existence and uniqueness of a solution to linear equation of type (1.3) in Z . This allows to formulate sensitivity equations in the the linear (Section 4) and the nonlinear (Section 5) cases. In Section 6, we discuss possible applications of our results.

2. Preliminaries

2.1. Transport equation in the space of measures

We briefly review here the formulation of the transport equation

$$\partial_t \mu_t + \partial_x(v(x)\mu_t) = 0$$

on the space nonnegative Radon measures $\mathcal{M}^+(\mathbb{R}^d)$. This space is equipped with the bounded Lipschitz norm defined for $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ as

$$\|\mu\|_{BL^*} = \sup_{\|\psi\|_{W^{1,\infty}(\mathbb{R}^d)} \leq 1} \int_{\mathbb{R}^d} \psi(x) d\mu(x), \quad (2.1)$$

as the total variation norm is too strong. Here, $W^{1,\infty}(\mathbb{R}^d)$ is the space of bounded and globally Lipschitz functions.

Let v be a vector field with $v \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$. Then, the flow of v denoted by $T_t v : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is defined as the solution to the ODE:

$$\frac{d}{dt}(T_t v)(x) = v((T_t v)(x)), \quad (T_0 v)(x) = x. \quad (2.2)$$

Notice that $(T_t v)(x)$ is defined for all $t \in \mathbb{R}$. If there is no risk of confusion, we write T_t instead of $T_t v$. Now, the classical method of characteristics allows to solve the transport equation

$$\partial_t \mu_t + \partial_x(v(x)\mu_t) = v_t, \quad \mu_t|_{t=0} = \mu_0, \quad (2.3)$$

where $v_t \in C([0, T], \mathcal{M}^+(\mathbb{R}^d))$. More precisely, the unique measure solution in $C([0, T], \mathcal{M}(\mathbb{R}^d))$ to (2.3) is given by propagating the initial condition μ_0 along the flow of v , namely

$$\mu_t = T_t^\# \mu_0 + \int_0^t T_{t-s}^\# v_s ds, \quad (2.4)$$

where for $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and measure $\mu \in \mathcal{M}^+(\mathbb{R}^d)$, $f^\# \mu$ is the *push-forward measure* defined as

$$f^\# \mu(A) = \mu(f^{-1}(A)) \quad \text{for any measurable } A \subset \mathbb{R}^d. \quad (2.5)$$

We remark here that the definition of the push-forward measure yields the following change of variables formula: for all measurable maps $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$,

$$\int_{\mathbb{R}^d} \phi(x) d(T^\# \mu)(x) = \int_{\mathbb{R}^d} \phi \circ T(x) d\mu(x). \quad (2.6)$$

For the proof, see [25] for the case $v = 0$ and Proposition 3.6 in [21]. Let us also note that formula (2.4) is true also in the setting of bounded Radon measures $\mathcal{M}(\mathbb{R}^d)$: as the equation is linear, one can apply the Hahn-Jordan decomposition (see Section 4.2 in [26]) and solve the equations for the positive and the negative parts of the measure separately.

Now, let v_1 and v_2 be two bounded and globally Lipschitz vector fields. Let $\mu_t^{(1)}$ and $\mu_t^{(2)}$ be the solutions to (2.3) with vector fields v_1 and v_2 , respectively. Then, there is a constant $C = C(T, \|v_1\|_{W^{1,\infty}}, \|v_2\|_{W^{1,\infty}}, \mu_0)$ such that

$$\|\mu_t^{(1)} - \mu_t^{(2)}\|_{BL^*} \leq C \|v_1 - v_2\|_\infty, \quad \text{for any } t \in [0, T]. \quad (2.7)$$

For the proof, one simply applies the triangle and Gronwall inequalities as in the proof of Lemma 3.8 in [22]. The solution to (2.3) thus depends continuously on v .

The transport equation (2.3) can also be studied in a nonlinear setting where the vector field depends on the measure solution itself. Then, the nonlinear transport equation takes the form

$$\partial_t \mu_t + \partial_x (v[\mu_t](x) \mu_t) = 0. \quad (2.8)$$

where $v : \mathcal{M}^+(\mathbb{R}^d) \rightarrow W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$. It is common in application that v depends on μ through some weighted mean of μ of the form

$$v[\mu](x) = V \left(x, \int_{\mathbb{R}^d} K_V(x, y) d\mu(y) \right) \quad (2.9)$$

for given maps $V : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ and $K_V : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$.

2.2. The Hölder space $C^{1,\alpha}(\mathbb{R}^d)$.

Given $\alpha \in (0, 1)$, we consider the space $C^{1,\alpha}(\mathbb{R}^d)$ of bounded continuous functions with bounded and α -Hölder derivative endowed with the norm

$$\|u\|_{C^{1,\alpha}} := \|u\|_\infty + \|Du\|_\infty + \sup_{x \neq y} \frac{|Du(x) - Du(y)|}{|x - y|^\alpha}.$$

Lemma 2.1. 1. For any $u \in C^{1,\alpha}(\mathbb{R}^d)$,

$$|u(x + y) - u(y) - \nabla u(x)y| \leq \|\nabla u\|_{C^{0,\alpha}} |y|^{1+\alpha} \quad \text{for any } x, y \in \mathbb{R}^d. \quad (2.10)$$

2. If $\phi \in C^{1,\alpha}(\mathbb{R}^d)$ and $T \in C^{1,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$ then $\phi \circ T \in C^{1,\alpha}(\mathbb{R}^d)$ with norm bounded by a constant depending only on a bound of $\|\phi\|_{C^{1,\alpha}}$ and $\|T\|_{C^{1,\alpha}}$.

Proof. The first assertion follows from

$$\begin{aligned} u(x+y) - u(y) - \nabla u(x)y &= \int_0^1 \frac{d}{dt} u(x+ty) - \nabla u(x)y \, dt \\ &= \int_0^1 (\nabla u(x+ty) - \nabla u(x))y \, dt. \end{aligned}$$

For the second one we only need to estimate $|D(\phi \circ T)(x) - D(\phi \circ T)(y)|$. We have

$$\begin{aligned} &|D\phi(T(x))DT(x) - D\phi(T(y))DT(y)| \\ &\leq |D\phi(T(x))(DT(x) - DT(y)) + (D\phi(T(x)) - D\phi(T(y)))DT(y)| \\ &\leq \|\phi\|_{C^{1,\alpha}} \|T\|_{C^{1,\alpha}} |x-y|^\alpha + \|D\phi\|_{C^{1,\alpha}} |T(x) - T(y)|^\alpha \|T\|_{C^{1,\alpha}} \\ &\leq C|x-y|^\alpha, \end{aligned}$$

where $C = \|\phi\|_{C^{1,\alpha}} \|T\|_{C^{1,\alpha}} + \|D\phi\|_{C^{1,\alpha}} \|T\|_{C^{1,\alpha}}^{1+\alpha}$. □

We also recall the following result from Cor. 3.16 in [24] regarding the regularity of the flow $T_t v$ defined in (2.2):

Proposition 2.1. *Assume that $v \in C^{1,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$. Then there exists a constant $C_T > 0$ depending only on T and $\|v\|_{C^{1,\alpha}}$ such that $\|D(T_t v)\|_{C^{0,\alpha}} \leq C_T$ for any $t \in [0, T]$. Moreover it can be checked upon inspection of the proof that $C_T \rightarrow 1$ as $T \rightarrow 0$.*

2.3. The space Z

We consider the space Z defined as the closure of $\mathcal{M}(\mathbb{R}^d)$ endowed with the dual norm $(C^{1,\alpha}(\mathbb{R}^d))^*$ for some α (see Remark 2.1 on the choice of α). This space was first introduced in [23] where the authors demonstrated that Z has a lot of convenient topological properties. In particular, Z is a separable Banach space with its dual being isometrically isomorphic to $C^{1,\alpha}(\mathbb{R}^d)$. Indeed it was proved in [23][Prop. 5.1] that $\text{span}\{\delta_x, x \in \mathbb{Q}^d\}$ is dense in Z . In particular this implies that any element of Z can be approximated by bounded measures.

Notice that using duality we have for any $\mu \in Z$,

$$\|\mu\|_Z = \sup_{\|\psi\|_{C^{1,\alpha}} \leq 1} (\mu, \psi).$$

The main advantage of space Z is its applicability to studying differentiation problems with respect to perturbation of transport equations. More precisely, let us consider Eq (2.3) with $v_t = 0$ and vector field $v_0(x) + hv_p(x)$ where $h \in [-M, M]$ for some $M > 0$, and denote by μ_t^h its solution, namely

$$\partial_t \mu_t^h + \partial_x \left((v_0 + hv_p) \mu_t^h \right) = 0, \quad \mu_t^h|_{t=0} = \mu_0.$$

One is then interested in the limit $\frac{\mu_t^{h+\Delta h} - \mu_t^h}{\Delta h}$ as $\Delta h \rightarrow 0$. The following result was obtained in [23]:

Theorem 2.1. *Let $v_0, v_p \in C^{1,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$. Then, $\frac{\mu_t^{h+\Delta h} - \mu_t^h}{\Delta h}$ converges in $C([0, T], Z)$ as $\Delta h \rightarrow 0$.*

Remark 2.1. Let $Z_\alpha = \overline{\mathcal{M}(\mathbb{R}^d)}^{(C^{1,\alpha}(\mathbb{R}^d))^*}$. Notice that if $0 < \alpha < \alpha' < 1$ then $C^{1,\alpha'} \subset C^{1,\alpha}$ from which we deduce that $Z_\alpha \subset Z_{\alpha'}$ with continuous injection. Therefore, if incremental quotient $\frac{\mu_t^{h+\Delta h} - \mu_t^h}{\Delta h}$ converges in Z_α , it also does so in $Z_{\alpha'}$ for any $\alpha' < \alpha$. Moreover, since $Z_\alpha \subset Z_{\alpha'}$ continuously, both limits coincide. So there is no ambiguity and we simply write Z instead of Z_α .

Such a perturbation problem can be also studied for the nonlinear transport equation (2.8) with a vector-field $v_0[\mu]$ like (2.9). We perturb $v_0[\mu]$ considering $v^h[\mu](x)$ defined as

$$\begin{aligned} v^h[\mu](x) &= v_0[\mu](x) + hv_p[\mu](x) \\ &= V_0\left(x, \int_{\mathbb{R}^d} K_{V_0}(x, y)d\mu(y)\right) + hV_p\left(x, \int_{\mathbb{R}^d} K_{V_p}(x, y)d\mu(y)\right). \end{aligned} \tag{2.11}$$

Then, we have the following result:

Theorem 2.2. Let $\alpha > \frac{1}{2}$ and $v^h[\mu]$ be given by (2.11), where $V_0, V_p \in C^{1+\alpha}(\mathbb{R}^d \times \mathbb{R}, \mathbb{R}^d)$ and $K_{V_0}, K_{V_p} \in C^{2+\alpha}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$. Let μ_t^h be the unique solution to (2.8) with the vector field $v^h[\mu]$. Then, $\frac{\mu_t^{h+\Delta h} - \mu_t^h}{\Delta h}$ converges in $C([0, T], Z)$ as $\Delta h \rightarrow 0$.

Remark 2.2. The proof of existence and uniqueness of solutions as well as of a differentiability result was actually given only for the case of \mathbb{R}^+ in [22] and [24] respectively. However, the proof can be easily extended to \mathbb{R}^d . Indeed, the main idea is to construct approximating sequence as follows. The interval of time $[0, T]$ is divided into 2^k subintervals of the form $[l\frac{T}{2^k}, (l+1)\frac{T}{2^k}]$ where $l = 0, 1, \dots, 2^k - 1$. Then, the following approximation is defined recursively: for $t \in (l\frac{T}{2^k}, (l+1)\frac{T}{2^k}]$, let μ_t be the solution to

$$\partial_t \mu_t + \partial_x(v[\mu_{l\frac{T}{2^k}}](x)\mu_t) = 0.$$

with initial condition $\mu_{l\frac{T}{2^k}}$. One then uses the formula for the solution of the linear problem (2.4) to conclude the proof. See [22] and [24] for more details.

2.4. New facts about the space Z

The following Propositions discuss the distributional derivatives of bounded Radon measures as elements of space Z . For easier flow of this section long proofs are provided in the Appendix.

We can see a Radon measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ as a distribution by $(\mu, \phi) = \int \phi d\mu$, $\phi \in C_c^\infty(\mathbb{R}^d)$. We denote by $\partial_x \phi := \nabla \phi \cdot x$ the derivative of ϕ in direction $x \in \mathbb{R}^d$. We then define a distribution $\partial_x \mu$ by duality letting $(\partial_x \mu, \phi) = -(\mu, \partial_x \phi)$. The next result shows that in fact $\partial_x \mu$ belongs to Z when μ is bounded.

Proposition 2.2. For any bounded $\mu \in \mathcal{M}(\mathbb{R}^d)$, the distributional derivative $\partial_x \mu$ of μ in direction $x \in \mathbb{R}^d$ belongs to Z .

Proof. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be bounded. To prove that the distributional derivative $\partial_x \mu$ belongs to Z , we need to find a sequence $\nu_h \in \mathcal{M}(\mathbb{R}^d)$ such that $\nu_h \rightarrow \partial_x \mu$ as $h \rightarrow 0$ in Z . Let τ_h be the translation operator defined by $\tau_h \phi(y) := \phi(y + hx)$ for any ϕ . Take $\nu_h := (\tau_h^\# \mu - \mu)/h \in \mathcal{M}(\mathbb{R}^d)$. Then for any $\phi \in C^{1,\alpha}(\mathbb{R}^d)$ with $\|\phi\|_{C^{1,\alpha}(\mathbb{R}^d)} \leq 1$ we have using (2.10) that

$$|(\nu_h, \phi) - (-\partial_x \mu, \phi)| = \int_{\mathbb{R}^d} \left| \frac{\phi(y + hx) - \phi(y)}{h} - \partial_x \phi(y) \right| d\mu(y) \leq |h|^\alpha \|\mu\|_{TV}$$

so that $\nu_h \rightarrow -\partial_x \mu$ in Z as $h \rightarrow 0$. □

Proposition 2.3. Consider $\mu_n, \mu \in \mathcal{M}_b(\mathbb{R}^d)$ such that $\mu_n \rightarrow \mu$ narrowly (i.e. in duality with bounded and continuous functions $C_b(\mathbb{R}^d)$). Then $\partial_x \mu_n \rightarrow \partial_x \mu$ in Z .

Proof. See Appendix. □

Proposition 2.4. For a bounded vector field v on \mathbb{R}^d and $\mu \in \mathcal{M}_b(\mathbb{R}^d)$ we have

$$\|\partial_x(v\mu)\|_Z \leq \|\mu\|_{TV} \|v\|_\infty. \quad (2.12)$$

Moreover, consider measures $\mu_n, \mu \in \mathcal{M}_b(\mathbb{R}^d)$ such that $\mu_n \rightarrow \mu$ narrowly and vector fields $v_n, v \in C_b(\mathbb{R}^d, \mathbb{R}^d)$ such that $v_n \rightarrow v$ uniformly. Then $\partial_x(v_n \mu_n) \rightarrow \partial_x(v\mu)$ in Z .

Proof. For any ϕ such that $\|\phi\|_{C^{1,\alpha}} \leq 1$ we have

$$|(\partial_x(v\mu), \phi)| = |(\mu, v \partial_x \phi)| \leq \|\mu\|_{TV} \|v \partial_x \phi\|_\infty \leq \|\mu\|_{TV} \|v\|_\infty.$$

Then, in view of Proposition 2.3, to verify the second assertion, it is sufficient to prove that $v_n \mu_n \rightarrow v\mu$ narrowly. For $\phi \in C_b(\mathbb{R}^d)$, we have

$$(v_n \mu_n - v\mu, \phi) = (\mu_n, (v_n - v)\phi) + (\mu_n - \mu, v\phi)$$

where (\cdot, \cdot) denotes the dual pairing. The first term can be bounded by $\|\mu_n\|_{TV} \|(v_n - v)\phi\|_\infty \leq C \|(v_n - v)\|_\infty \rightarrow 0$ while the second tends to 0 since $v\phi \in C_b(\mathbb{R}^d)$. □

We deduce that

Corollary 2.1. Let $[0, T] \ni t \mapsto \mu_t \in \mathcal{M}_b(\mathbb{R}^d)$ be a narrowly continuous map and $v \in C_b(\mathbb{R}^d, \mathbb{R}^d)$. Then $\partial_x(v\mu_t) \in C([0, T], Z)$.

It will also be useful to define the push-forward of an element of Z . The idea is quite simple. In fact, since this is well-defined on the space of measures, we can extend its definition for elements of Z by means of Cauchy sequences.

Proposition 2.5. Let $T \in C^{1,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$. Then for any $\mu \in Z$ we can define $T^\# \mu \in Z$ by

$$T^\# \mu := \lim_{n \rightarrow \infty} T^\# \mu_n$$

where $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathcal{M}(\mathbb{R}^d)$ is any sequence such that $\mu_n \rightarrow \mu$ in Z . Then, for any $\phi \in C^{1,\alpha}(\mathbb{R}^d)$ we have the following analogue of the change of variables formula (2.6):

$$(T^\# \mu, \phi) = (\mu, \phi \circ T)$$

where $\phi \circ T$ denotes composition of the maps ϕ and T .

Proof. See Appendix. □

We conclude this section with the following classical observation. By definition, if $\mu \in Z$, there is a sequence of bounded measures $\{\mu_n\}_{n \in \mathbb{N}}$ such that $\mu_n \rightarrow \mu$ in Z . Now, if $\mu \in C([0, T], Z)$, for each $t \in [0, T]$, one can choose an approximating sequence for each μ_t , $t \in [0, T]$. However, it is possible to construct an approximating sequence that is continuous in time and so, that approximates the whole curve $t \mapsto \mu_t$, $t \in [0, T]$. This is the content of the following lemma.

Lemma 2.2. Let $\mu \in C([0, T], Z)$. There is a sequence $\{\mu^{(n)}\}_{n \in \mathbb{N}} \subset C([0, T], \mathcal{M}_b(\mathbb{R}^d))$ such that $\mu^{(n)} \rightarrow \mu$ in $C([0, T], Z)$ as $n \rightarrow \infty$.

Proof. See Appendix. □

Corollary 2.2. Let $v \in C([0, T], Z)$ and $v \in C^{1,\alpha}(\mathbb{R}^d)$. Then the map $t \rightarrow T_t v^\# v_t$ is continuous from $[0, T]$ to Z .

Proof. See Appendix. □

3. Transport equation in Z

In this section, we study the following transport equation in the space Z :

$$\partial_t \mu_t + \partial_x(v(x)\mu_t) = v_t, \quad \mu_{t=0} = \mu_0, \quad (3.1)$$

where $v \in C^{1,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$, $v \in C([0, T], Z)$ and $\mu_0 \in Z$. We begin with a concept of a very weak solution.

Definition 3.1. We say that $\mu \in C([0, T], Z)$ is a very weak solution to (3.1) in Z if for any $\varphi \in C([0, T] \times \mathbb{R}^d)$ with $\varphi \in C([0, T], C^{2+\alpha}(\mathbb{R}^d))$ and $\varphi_t \in C([0, T], C^{1+\alpha}(\mathbb{R}^d))$ we have:

$$(\mu_T, \varphi(x, T)) = (\mu_0, \varphi(x, 0)) + \int_0^T (\mu_t, \varphi_t(\cdot, t) + v \cdot \nabla \varphi(\cdot, t)) dt + \int_0^T (v_t, \phi(\cdot, t)) dt. \quad (3.2)$$

Note that we have to use test functions of regularity at least $C^{2+\alpha}$ in space variable x so that function $\varphi_t + v(x) \cdot \nabla_x \varphi$ lies in Z , the domain of the functional μ_t .

Proposition 3.1. Equation (3.1) has at least one very weak solution in $C([0, T], Z)$ given by

$$\mu_t = T_t^\# \mu_0 + \int_0^t T_{t-s}^\# v_s ds \quad (3.3)$$

where the integral is a Bochner integral in Z .

Moreover, if $\mu_0 = 0$ and $v_t = 0$, then for any weak solution μ_t we have

$$(\mu_t, \eta) = 0 \quad (3.4)$$

for all $\eta \in C^{2+\alpha}(\mathbb{R}^d)$ and $t \in [0, T]$.

Proof. We first verify that the integral appearing on the right-hand side of (3.3) is a Bochner integral in Z . According to Corollary 2.2 the map $f : s \in [0, t] \rightarrow T_{t-s}^\# v_s \in Z$ is continuous. Thus for any $z^* \in Z^*$, $z^* \circ f$ is also continuous. Since Z is separable, we conclude using Pettis theorem that f is measurable. Moreover for any $\phi \in C^{1,\alpha}(\mathbb{R}^d)$, $\|\phi\|_{C^{1,\alpha}} \leq 1$, we have

$$|(f(s), \phi)| = |(v_s, \phi \circ T_{t-s})| \leq \|v_s\|_Z \|\phi \circ T_{t-s}\|_{C^{1,\alpha}} \leq C_T$$

since $v \in C([0, T], Z)$ and in view of Lemma 2.1 and Proposition 2.1. It follows that $\max_{0 \leq s \leq t} \|f(s)\|_Z \leq C_T$ and thus that f is Bochner-integrable. It is also easily seen that $\int_0^t T_{t-s}^\# v_s ds$ is continuous in t .

Let μ_t be defined by (3.3). Clearly $\mu \in C([0, T], Z)$. We now verify that μ_t is a solution in the sense of Definition 3.1. According to Lemma 2.2, we we can find sequences $\{\nu^{(n)}\}_{n \in \mathbb{N}} \subset C([0, T], \mathcal{M}_b(\mathbb{R}^d))$

and $\{\mu_0^n\}_{n \in \mathbb{N}} \subset \mathcal{M}_b(\mathbb{R}^d)$ such that $\|\mu_0^n - \mu_0\|_Z \rightarrow 0$ and $\|v_t^{(n)} - v_t\|_Z \rightarrow 0$ uniformly in $t \in [0, T]$. Then the transport equation

$$\partial_t \mu_t + \partial_x(v(x)\mu_t) = v_t^{(n)}, \quad \mu_{|t=0} = \mu_0^{(n)} \quad (3.5)$$

has a unique solution $\mu^{(n)} \in C([0, T], \mathcal{M}_b(\mathbb{R}^d))$ given by

$$\mu_t^{(n)} = T_t^\# \mu_0^{(n)} + \int_0^t T_{t-s}^\# v_s^{(n)} ds. \quad (3.6)$$

According to Proposition 2.5, $T_t^\# \mu_0^{(n)} \rightarrow T_t^\# \mu_0$ in Z and, for any s , $T_{t-s}^\# v_s^{(n)} \rightarrow T_{t-s}^\# v_s$ in Z . Since $\|T_{t-s}^\# v_s^{(n)}\|_Z \leq C_T$ we have applying the Dominated Convergence Theorem that $\int_0^t T_{t-s}^\# v_s^{(n)} ds \rightarrow \int_0^t T_{t-s}^\# v_s ds$ in Z . Thus for any $t \in [0, T]$, $\mu_t^{(n)}$ converges in Z to μ_t given by

$$\mu_t := \lim_{n \rightarrow +\infty} \mu_t^{(n)} = T_t^\# \mu_0 + \int_0^t T_{t-s}^\# v_s ds.$$

Clearly, $\mu \in C([0, T], Z)$. On the other hand, weak formulation for (3.5) is valid for test functions of class $C^1([0, T] \times \mathbb{R}^d) \cap W^{1,\infty}([0, T] \times \mathbb{R}^d)$. In particular, taking test functions as in Definition 3.1, we send $n \rightarrow \infty$ in the weak formulation for (3.5) to deduce that μ_t is a very weak solution to (3.1).

To prove (3.4), we use the so-called dual problem (cf. Remark 8.1.5 and Proposition 8.1.7 in [27] or Proposition 5.34 in [25]). More precisely, given some function $\psi(x, t)$, let φ be the solution of

$$\partial_t \varphi + v(x) \cdot \nabla_x \varphi = \psi, \quad \varphi(x, T) = 0. \quad (3.7)$$

which is explicitly given by $\varphi(x, t) = -\int_t^T \psi(T_{s-t}(x), s) ds$. We consider ϕ of the form $\psi(x, t) = \xi(t)\eta(x)$ where $\xi \in C_c^\infty([0, T])$ and $\eta \in C^{2,\alpha}(\mathbb{R}^d)$. We then use the corresponding solution ϕ of (3.7) as a test function in (3.2) to conclude

$$\int_0^T \xi(t)(\mu_t, \eta) dt = 0.$$

Since the map $t \mapsto (\mu_t, \eta(x))$ is continuous for $t \in [0, T]$ and since ξ is arbitrary, we deduce that $(\mu_t, \eta) = 0$ for any $\eta \in C^{2,\alpha}(\mathbb{R}^d)$ and $t \in [0, T]$. \square

Unfortunately, condition (3.4) does not imply that $\mu_t = 0$ so that we cannot deduce the uniqueness of a solution to (3.1). The problem here is that $C^{2+\alpha}(\mathbb{R}^d)$ is not dense in $C^{1+\alpha}(\mathbb{R}^d)$. The following two examples shows the typical problem with approximation of Hölder functions.

Example 3.1. One can easily check that $f(x) = \sqrt{x} \in C^{1/2}([0, 1])$. Suppose there is a sequence $\{f_n\}_{n \in \mathbb{N}} \subset C^1([0, 1])$ such that $\|f_n - f\|_{C^{1/2}} \rightarrow 0$. Then

$$0 \leftarrow \|f_n - f\|_{C^{1/2}} \geq \sup_{x \in (0,1]} \left| 1 - \frac{f_n(x) - f_n(0)}{\sqrt{x}} \right| \geq \sup_{x \in (0,1]} \frac{|f_n(x) - f_n(0)|}{\sqrt{x}} - 1,$$

contradicting $\{f_n\}_{n \in \mathbb{N}} \subset C^1([0, 1])$.

Example 3.2. We construct a nontrivial functional on $C^{1/2}([0, 1])$ which vanishes on $C^{1,1/2}([0, 1])$. In particular, this shows that functionals on $C^{1/2}([0, 1])$ cannot be uniquely characterized by their values on $C^{1,1/2}([0, 1])$. Let $X = C^1([0, 1]) \oplus \text{lin}(\sqrt{x})$ be a linear subspace of $C^{1/2}([0, 1])$. On X , we can define a functional $\varphi : X \rightarrow \mathbb{R}$ with

$$\varphi(f) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{\sqrt{x}}.$$

Notice that ϕ is continuous since $|\varphi(f)| \leq \|f\|_{C^{1/2}}$. By the analytic version of Hahn-Banach Theorem (Theorem 1.1 in [28]), we can then extend φ to a continuous functional on $C^{1/2}([0, 1])$. It is easily seen that $\phi(f) = 0$ for any $f \in C^1([0, 1])$ by Taylor's estimate but that $\varphi(\sqrt{x}) = 1$.

There is also characterization of subset in C^α consisting of functions that can be approximated by smooth functions:

Remark 3.1. Let $\Omega \subset \mathbb{R}^d$. Then, $f \in C^\alpha(\Omega)$ can be approximated by smooth functions if and only if f is an element of the set

$$\mathcal{F}^\alpha(\Omega) = \left\{ f \in C^\alpha(\Omega) : \lim_{t \rightarrow 0^+} \sup_{|x-y| \leq t} \frac{|f(x) - f(y)|}{|x-y|^\alpha} = 0 \right\}.$$

One easily checks that for $\Omega = [0, 1]$, $\sqrt{x} \notin \mathcal{F}^{1/2}(\Omega)$. Moreover, for any $\beta > \alpha$, $C^\beta(\Omega) \subset \mathcal{F}^\alpha(\Omega)$.

Therefore, we realize that the space of test functions is too small to deduce uniqueness of weak solutions. This is the case for many PDEs formulated in the weak sense. Probably one of the most famous is Euler's equation where one can construct infinitely many distributional solutions with prescribed energy profile (thus contradicting conservation of energy), see [29] and references therein. The standard procedure in such situation for many evolutionary problems is to require some additional conditions to be satisfied by a weak solution (like entropy condition for conservation laws, see [30], section 3.4).

To establish additional conditions required from weak solutions, we should get some insight about which solutions we would like to extract. First, note that if $\nu \in Z$, there is an approximating sequence of measures $\nu_n \in \mathcal{M}(\mathbb{R}^d)$ such that $\nu_n \rightarrow \nu$ in Z . Now, recall that we want to find an equation that is satisfied by the derivative of the solution to (3.1) with respect to perturbation parameter h . Therefore, in our case, such approximating sequence is of the form $\frac{\mu_t^{h+\Delta h} - \mu_t^h}{\Delta h}$. We will see in the proof of Theorem 4.1 below that $\|\frac{\mu_t^{h+\Delta h} - \mu_t^h}{\Delta h}\|_{BL^*} \leq C_T$ for some constant C independent of h , Δh and t . This suggests to define the following admissibility class:

$$\mathcal{A} = \{\nu \in Z : \exists \{\nu_n\}_{n \in \mathbb{N}} \subset \mathcal{M}(\mathbb{R}^d) \text{ s.t. } \nu_n \rightarrow \nu \text{ in } Z \text{ and } \|\nu_n\|_{BL^*} \leq C\}. \quad (3.8)$$

Notice that \mathcal{A} is a subspace of Z containing the bounded measures $\mathcal{M}_b(\mathbb{R}^d)$ so that \mathcal{A} is dense in Z . In view of the proof of Proposition 2.2 we also have that $\partial_x \mu \in \mathcal{A}$ for any $\mu \in \mathcal{M}_b(\mathbb{R}^d)$. In fact we have the following stronger result:

Proposition 3.2. Let $\mu : [0, T] \rightarrow \mathcal{M}_b(\mathbb{R}^d)$ be continuous and TV-bounded, and let $x \in \mathbb{R}^d$. Then $\partial_x \mu \in C([0, T], Z)$ with values in \mathcal{A} and in fact there exists $\rho^h \in C([0, T], \mathcal{M}_b(\mathbb{R}^d))$, $h \in (0, 1)$, such that

$$\lim_{t \rightarrow 0} \max_{0 \leq t \leq T} \|\rho_t^h - \partial_x(\mu_t)\|_Z = 0 \quad \text{and} \quad \sup_{h \in (0, 1), t \in [0, T]} \|\rho_t^h\|_{BL^*} \leq C.$$

Proof. According to Proposition 2.3, $\partial_x \mu \in C([0, T], Z)$. Let τ_h be the translation defined by $\tau_h \phi(y) = \phi(y + hx)$. It is then easy to verify using the same arguments as in the proof of Proposition 2.2 that $\rho_t^h := (\tau_h^\# \mu_t - \mu_t)/h$ satisfies the requirements. \square

We can now define a weak solutions as follows.

Definition 3.2. We say that $\mu \in C([0, T], Z)$ is a weak solution to (3.1) in Z if μ is a very weak solution (see Definition 3.1) and for all $t \in [0, T]$, $\mu_t \in \mathcal{A}$.

With this definition we are now able to establish the following existence and uniqueness result:

Theorem 3.1. Let $\mu_0 \in \mathcal{A}$ and $v \in C([0, T], Z)$ with values in \mathcal{A} . Assume that there exists a sequence $v^n \in C([0, T], \mathcal{M}_b(\mathbb{R}^d))$, $n \in \mathbb{N}$, such that

$$\lim_{n \rightarrow +\infty} \max_{0 \leq t \leq T} \|v_t^n - v_t\|_Z = 0 \quad \text{and} \quad \sup_{n \in \mathbb{N}, t \in [0, T]} \|v_t^n\|_{BL^*} \leq C. \quad (3.9)$$

Then, equation (3.1) has a unique weak solution in the sense of Definition 3.2 which is given by

$$\mu_t = T_t^\# \mu_0 + \int_0^t T_{t-s}^\# v_s ds. \quad (3.10)$$

Note that according to Proposition 3.2, the Theorem applies in particular when $v_t = \partial_x(\mu_t)$ with $\mu : [0, T] \rightarrow \mathcal{M}_b(\mathbb{R}^d)$ continuous and TV-bounded.

Proof. To prove the uniqueness statement, since the equation is linear, it is sufficient to prove that if $\mu_0 = 0$ and $v_t = 0$ for all $t \in [0, T]$, then $\mu_t = 0$ for all $t \in [0, T]$. This is equivalent to $(\mu_t, \eta) = 0$ for any $\eta \in C^{1,\alpha}(\mathbb{R}^d)$. Fix $\eta \in C^{1,\alpha}(\mathbb{R}^d)$ and for $\epsilon > 0$ denote by η^ϵ the standard mollification of η . Since η and its derivatives are uniformly continuous, we have $\|\eta^\epsilon - \eta\|_{W^{1,\infty}} \rightarrow 0$ as $\epsilon \rightarrow 0$. Moreover, for fixed $\epsilon > 0$, $\eta^\epsilon \in C^{2,\alpha}(\mathbb{R}^d)$ so that $(\mu_t, \eta^\epsilon) = 0$ by (3.4). Since $\mu_t \in \mathcal{A}$ there exists a BL-bounded sequence $\mu_t^{(n)} \in \mathcal{M}_b(\mathbb{R}^d)$ converging in Z to μ_t . For a fixed $\epsilon > 0$ we then write

$$(\mu_t, \eta) = (\mu_t, \eta^\epsilon) + (\mu_t, \eta - \eta^\epsilon) = \lim_{n \rightarrow \infty} (\mu_t^{(n)}, \eta - \eta^\epsilon)$$

with

$$(\mu_t^{(n)}, \eta - \eta^\epsilon) \leq \|\mu_t^{(n)}\|_{BL^*} \|\eta - \eta^\epsilon\|_{W^{1,\infty}} \leq C \|\eta - \eta^\epsilon\|_{W^{1,\infty}}$$

for some constant C independent of n . Thus

$$|(\mu_t, \eta)| \leq C \|\eta - \eta^\epsilon\|_{W^{1,\infty}}.$$

Since $\epsilon > 0$ is arbitrary, we conclude $(\mu_t, \eta) = 0$.

Concerning the existence, we already know from Proposition 3.1 that $\mu_t = T_t^\# \mu_0 + \int_0^t T_{t-s}^\# v_s ds$ belongs to $C([0, T], Z)$ and solves the equation. It remains to prove that $\mu_t \in \mathcal{A}$ for any $t \in [0, T]$. Since $\mu_0 \in \mathcal{A}$ there exists a BL-bounded sequence $\mu_0^{(n)} \in \mathcal{M}_b(\mathbb{R}^d)$ converging in Z to μ_0 . Let

$$\mu_t^{(n)} := T_t^\# \mu_0^{(n)} + \int_0^t T_{t-s}^\# v_s^{(n)} ds$$

where v_n satisfies (3.9). We verify as in the proof of Proposition 3.1 that $\mu_t^{(n)} \rightarrow \mu_t$ in Z for any given t . Moreover for any bounded Lipschitz ϕ we have

$$\begin{aligned}(\mu_t^{(n)}, \phi) &= (\mu_0^{(n)}, \phi \circ T_t) + \int_0^t (v_s^{(n)}, \phi \circ T_{t-s}) ds \\ &\leq \|\mu_0^{(n)}\|_{BL^*} \|\phi \circ T_t\|_{BL} + \int_0^t \|v_s^{(n)}\|_{BL^*} \|\phi \circ T_{t-s}\|_{BL} ds\end{aligned}$$

Since $Lip(T_t) \leq e^{tLip(v)}$ we have $\|\phi \circ T_t\|_{BL} \leq e^{tLip(v)}$. Thus, choosing $C_T = e^{tLip(v)} \left(\sup_n \|\mu_0^{(n)}\|_{BL^*} + T \sup_{n, 0 \leq s \leq T} \|v_s^{(n)}\|_{BL^*} \right)$ we see that

$$(\mu_t^{(n)}, \phi) \leq C_T.$$

Hence, $\sup_{n \in \mathbb{N}, t \in [0, T]} \|\mu_t^{(n)}\|_{BL^*} \leq C_T$. □

4. Sensitivity equation for a linear transport equation

In this section we formulate an equation that is satisfied by the derivative of the solutions μ_t with respect to h , i.e., $\rho_{t,h} = \lim_{\Delta h \rightarrow 0} \frac{\mu_t^{h+\Delta h} - \mu_t^h}{\Delta h}$, where μ_t^h solves

$$\partial_t \mu_t^h + \partial_x(v^h(x)\mu_t^h) = 0 \tag{4.1}$$

with initial condition $\mu_{|t=0}^h = \mu_0$ and $v^h = v_0 + hv_p$ where $v_0, v_p \in C^{1+\alpha}(\mathbb{R}^d, \mathbb{R}^d)$ are given vector fields. The derivative $\rho_{t,h}$ exists according to Theorem 2.1.

To obtain the equation $\rho_{t,h}$ should solve, we subtract the equations satisfied by μ_t^h and $\mu_t^{h+\Delta h}$, namely

$$\partial_t \mu_t^h + \partial_x(v^h(x)\mu_t^h) = 0$$

$$\partial_t \mu_t^{h+\Delta h} + \partial_x((v^h(x) + \Delta h v_p(x))\mu_t^{h+\Delta h}) = 0$$

to obtain that $\rho_{t,h}^{\Delta h} := \frac{\mu_t^{h+\Delta h} - \mu_t^h}{\Delta h}$ satisfies

$$\partial_t \rho_{t,h}^{\Delta h} + \partial_x(v^h(x)\rho_{t,h}^{\Delta h}) = -\partial_x(v_p(x)\mu_t^{h+\Delta h}).$$

Thus, intuitively the limit $\rho_{t,h} = \lim_{\Delta h \rightarrow 0} \rho_{t,h}^{\Delta h}$ should satisfy

$$\partial_t \rho_{t,h} + \partial_x(v^h(x)\rho_{t,h}) = -\partial_x(v_p(x)\mu_t^h). \tag{4.2}$$

Since the right-hand side belongs to Z in view of Proposition 2.4, we are naturally led to study this equation in Z . The following Theorem asserts that this intuition is correct and can be rigorously justified.

Theorem 4.1. *The derivative $\rho_{t,h} = \lim_{\Delta h \rightarrow 0} \frac{\mu_t^{h+\Delta h} - \mu_t^h}{\Delta h}$ where μ_t^h and $\mu_t^{h+\Delta h}$ solve (4.1) is the unique weak solution (cf. Definition 3.2) of*

$$\partial_t \rho_{t,h} + \partial_x(v^h(x)\rho_{t,h}) = -\partial_x(v_p(x)\mu_t^h) \tag{4.3}$$

with initial condition $\rho_{0,h} = 0$.

Proof. Let $\rho_{t,h}^{\Delta h} := (\mu_t^{h+\Delta h} - \mu_t^h)/\Delta h$. Since $\mu_t^{h+\Delta h}$ and μ_t^h are solutions to (4.1), we have that for any $\varphi \in C^1([0, T] \times \mathbb{R}^d) \cap W^{1,\infty}([0, T] \times \mathbb{R}^d)$:

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x, t) d\mu_t^h(x) - \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x) \\ = \int_0^t \int_{\mathbb{R}^d} \partial_t \varphi(x, s) d\mu_s^h(x) ds + \int_0^t \int_{\mathbb{R}^d} (v_0(x) + hv_p(x)) \cdot \nabla \varphi(x, s) d\mu_s^h ds \end{aligned}$$

and similarly

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x, t) d\mu_t^{h+\Delta h}(x) - \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x) \\ = \int_0^t \int_{\mathbb{R}^d} \partial_t \varphi(x, s) d\mu_s^{h+\Delta h}(x) ds + \int_0^t \int_{\mathbb{R}^d} (v_0(x) + (h + \Delta h)v_p(x)) \cdot \nabla \varphi(x, s) d\mu_s^{h+\Delta h} ds. \end{aligned}$$

Subtracting these equations and dividing by Δh , we obtain that

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x, t) d\rho_{t,h}^{\Delta h} &= \int_0^t \int_{\mathbb{R}^d} \partial_t \varphi(x, s) d\rho_{s,h}^{\Delta h} ds + \int_0^t \int_{\mathbb{R}^d} v^h(x) \cdot \nabla \varphi(x) d\rho_{s,h}^{\Delta h}(x) ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} v_p(x) \cdot \nabla \varphi(x) d\mu_s^{h+\Delta h} ds. \end{aligned}$$

Since $\mu^{h+\Delta h} \rightarrow \mu^h$ in $C([0, T], \mathcal{M}(\mathbb{R}^d))$ as $\Delta h \rightarrow 0$, We can pass to the limit $\Delta h \rightarrow 0$ in the last term on the right-hand side using the Dominated Convergence Theorem to obtain

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x, t) d\rho_{t,h}^{\Delta h} &= \int_0^t \int_{\mathbb{R}^d} \partial_t \varphi(x, s) d\rho_{s,h}^{\Delta h} ds + \int_0^t \int_{\mathbb{R}^d} v^h(x) \cdot \nabla \varphi(x) d\rho_{s,h}^{\Delta h}(x) ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} v_p(x) \cdot \nabla \varphi(x) d\mu_s^h ds. \end{aligned} \tag{4.4}$$

Recall that we know from Theorem 2.1 that the limit $\rho_h = \lim_{h \rightarrow 0} \rho_h^{\Delta h}$ exists in $C([0, T], Z)$ - in particular $\|\rho_{t,h}^{\Delta h}\|_Z \leq C_T$ for any $t \in [0, T]$ and any Δh small. Now, if φ satisfies $\varphi \in C([0, T], C^{2+\alpha}(\mathbb{R}^d))$ and $\partial_t \varphi \in C([0, T], C^{1+\alpha}(\mathbb{R}^d))$, using that $v_0, v_p \in C^{1+\alpha}(\mathbb{R}^d, \mathbb{R}^d)$, we deduce that as $\Delta h \rightarrow 0$,

$$(\rho_{s,h}^{\Delta h}, \partial_t \varphi(\cdot, s) + v^h \cdot \nabla \varphi(\cdot, s)) \rightarrow (\rho_{s,h}, \partial_t \varphi(\cdot, s) + v^h \cdot \nabla \varphi(\cdot, s)).$$

Moreover for any $s \in [0, T]$ and Δh small,

$$|(\rho_{s,h}^{\Delta h}, \partial_t \varphi(\cdot, s) + v^h \cdot \nabla \varphi(\cdot, s))| \leq \|\rho_{t,h}^{\Delta h}\|_Z \|\partial_t \varphi(\cdot, s) + v^h \cdot \nabla \varphi(\cdot, s)\|_{C^{1+\alpha}} \leq C_T.$$

Using the Dominated Convergence Theorem, we can thus send $\Delta h \rightarrow 0$ in (4.4) to deduce:

$$\begin{aligned} (\rho_{t,h}, \varphi(\cdot, t)) &= \int_0^t \int_{\mathbb{R}^d} v_p(x) \cdot \nabla \varphi(x, s) d\mu_s^h ds \\ &\quad + \int_0^t (\rho_{s,h}, \partial_t \varphi(\cdot, s) + v^h(\cdot) \cdot \nabla \varphi(\cdot, s)) ds. \end{aligned} \tag{4.5}$$

Thus, $\rho_{t,h}$ is a very weak solution of (4.3) with initial condition $\rho_{0,h} = 0$.

Let us prove that $\rho_{t,h}$ is the unique weak solution to (4.3). First note that due to Corollary 2.1, $\partial_x(v_p(x)\mu_t^h) \in C([0, T], Z)$. Moreover, we claim that $\rho_{t,h} \in \mathcal{A}$ for all $t \in [0, T]$, where \mathcal{A} is the admissibility class defined in (3.8). Indeed since $\rho_{t,h}^{\Delta h} \rightarrow \rho_{t,h}$ in Z , it suffices to verify that $\|\rho_{t,h}^{\Delta h}\|_{BL^*} \leq C$ with C independent of Δh . Recall that $\mu_t^h = (T_t^h)^\# \mu_0$ where T_t^h is the flow of v^h . Using Gronwall inequality it is easy to see that

$$\|T_t^h - T_t^{h+\Delta h}\|_\infty \leq \Delta h \|v_p\|_\infty \exp(\text{Lip}(v^h)t).$$

Thus for any $\phi \in W^{1,\infty}(\mathbb{R}^d)$, $\|\phi\|_{W^{1,\infty}} \leq 1$,

$$\begin{aligned} (\mu_t^h - \mu_t^{h+\Delta h}, \phi) &= (\mu_0, \phi \circ T_t^h - \phi \circ T_t^{h+\Delta h}) \leq \|\mu_0\|_\infty \|T_t^h - T_t^{h+\Delta h}\|_\infty \\ &\leq \|\mu_0\|_\infty \Delta h \|v_p\|_\infty \exp(\text{Lip}(v^h)t) =: C_{T,h} \Delta h \end{aligned}$$

Taking the supremum over such ϕ , we deduce that $\|\mu_t^h - \mu_t^{h+\Delta h}\|_{BL^*} \leq C_{T,h} \Delta h$. Therefore, in view of Theorem 3.1, we conclude that $\rho_{t,h}$ is the unique weak solution to (4.3). \square

Notice that in the previous proof we exploited the fact that we already knew that the derivative $\rho_h = \lim_{h \rightarrow 0} \rho_h^{\Delta h}$ exists due to [23]. But the well-posedness theory we established in the previous section and the fact ρ_t^h is characterized as the unique solution to equation (4.3) allow us to give an alternative short proof of the existence of ρ_h . Indeed let us *define* $\rho_{t,h}$ as the unique solution to (4.3). We then need to prove that

$$\lim_{\Delta h \rightarrow 0} \max_{0 \leq t \leq T} \|\rho_{t,h}^{\Delta h} - \rho_{t,h}\|_Z = 0. \quad (4.6)$$

In view of (4.4), $\rho_{t,h}^{\Delta h}$ satisfies

$$\partial_t \rho_{t,h}^{\Delta h} + \partial_x(v^h(x)\rho_{t,h}^{\Delta h}) = -\partial_x(v_p(x)\mu_t^{h+\Delta h}).$$

Since $\rho_{t,h}, \rho_{t,h}^{\Delta h} \in \mathcal{A}$, Theorem 3.1 yields

$$\begin{aligned} \rho_{t,h} &= \int_0^t (T_{t-s}^h)^\# v_s^h ds, & v_s^h &= -\partial_x(v_p(x)\mu_s^h), \\ \rho_{t,h}^{\Delta h} &= \int_0^t (T_{t-s}^h)^\# v_s^{h+\Delta h} ds, & v_s^{h+\Delta h} &= -\partial_x(v_p(x)\mu_s^{h+\Delta h}), \end{aligned}$$

where T_t^h is the flow of v^h . Then

$$\|\rho_{t,h}^{\Delta h} - \rho_{t,h}\| \leq \int_0^t \|(T_{t-s}^h)^\# v_s^{h+\Delta h} - (T_{t-s}^h)^\# v_s^h\|_Z ds \leq C_{T,h} \int_0^t \|v_s^{h+\Delta h} - v_s^h\|_Z ds$$

where we used in the last equality that $\|\phi \circ T_t^h\|_{C^{1+\alpha}} \leq C_{T,h}$ for any $\|\phi\|_{C^{1+\alpha}} \leq 1$. We deduce (4.6) using Lemma 4.1 below.

Lemma 4.1. *There holds*

$$\lim_{\Delta h \rightarrow 0} \max_{0 \leq t \leq T} \|v_s^{h+\Delta h} - v_s^h\|_Z = 0. \quad (4.7)$$

Proof. The proof follows the line of the proof of Proposition 2.3. Suppose that (4.7) is not true so that there exist $\varepsilon > 0$ and sequences $\{t_{\Delta h}\} \subset [0, T]$, $\{\phi_{\Delta h}\} \subset C^{1+\alpha}(\mathbb{R}^d)$, $\|\phi_{\Delta h}\|_{C^{1+\alpha}} \leq 1$ such that

$$(\mu_{t_{\Delta h}}^{h+\Delta h} - \mu_{t_{\Delta h}}^h, v_p \nabla \phi_{\Delta h}) \geq \varepsilon > 0. \quad (4.8)$$

As in the proof of Proposition 2.3 there exists $\phi \in C^{1+\alpha}(\mathbb{R}^d)$, $\|\phi\|_{C^{1+\alpha}} \leq 1$, such that up to a subsequence $\phi_{\Delta h} \rightarrow \phi$ in $C_{loc}^1(\mathbb{R}^d)$. Moreover there exists $t_0 = \lim_{\Delta h \rightarrow 0} t_{\Delta h}$ up to a subsequence. Independently recall that $\mu_t^h = (T_t^h)^\# \mu_0$ and $\mu_t^{h+\Delta h} = (T_t^{h+\Delta h})^\# \mu_0$. It follows that $\|\mu_t^h\|_{TV}, \|\mu_t^{h+\Delta h}\|_{TV} \leq \|\mu_0\|_{TV}$ and also that for any $\delta > 0$ there exists a compact set $K \subset \mathbb{R}^d$ such that

$$|\mu_t^{h+\Delta h}|(\mathbb{R}^d \setminus K), |\mu_t^h|(\mathbb{R}^d \setminus K) \leq \delta \quad \text{for any } |\Delta h| \leq 1 \text{ and } t \in [0, T].$$

Since $v_p \nabla \phi_{\Delta h} \rightarrow v_p \nabla \phi$ in $C_{loc}(\mathbb{R}^d)$ it follows that

$$(\mu_{t_{\Delta h}}^{h+\Delta h}, v_p \nabla \phi_{\Delta h}) - (\mu_{t_{\Delta h}}^{h+\Delta h}, v_p \nabla \phi) \rightarrow 0. \quad (4.9)$$

Eventually letting $\psi := v_p \nabla \phi$ we have

$$(\mu_{t_{\Delta h}}^{h+\Delta h} - \mu_{t_{\Delta h}}^h, v_p \nabla \phi) = \int_{\mathbb{R}^d} \psi(T_{t_{\Delta h}}^{h+\Delta h}(x)) - \psi(T_{t_{\Delta h}}^h(x)) d\mu_0(x).$$

Since ψ is bounded and $T_{t_{\Delta h}}^{h+\Delta h}(x) \rightarrow T_{t_0}^h(x)$, $T_{t_{\Delta h}}^h(x) \rightarrow T_{t_0}^h(x)$ for any $x \in \mathbb{R}^d$, the Dominated Convergence Theorem gives $(\mu_{t_{\Delta h}}^{h+\Delta h} - \mu_{t_{\Delta h}}^h, v_p \nabla \phi) \rightarrow 0$. This and (4.9) contradicts (4.8). \square

5. Sensitivity equation for a nonlinear transport equation

In this Section we formulate an equation satisfied by the derivative

$$\rho_{t,h} = \lim_{\Delta h \rightarrow 0} \frac{\mu_t^{h+\Delta h} - \mu_t^h}{\Delta h}$$

where μ_t^h solves

$$\partial_t \mu_t^h + \partial_x(v^h[\mu_t^h](x)\mu_t^h) = 0 \quad (5.1)$$

with the initial condition $\mu_{t=0}^h = \mu_0$ and $v^h[\mu_t^h]$ is a vector field which depends a priori in a non-linear way of μ_t^h .

Let us first present some heuristic computations to determine the equation $\rho_{t,h}$ should satisfy. Let $\rho_{t,h}^{\Delta h} := (\mu_t^{h+\Delta h} - \mu_t^h)/\Delta h$. Since $\mu_t^{h+\Delta h}$ and μ_t^h are solutions to (5.1) we have that for any $\varphi \in C^1([0, T] \times \mathbb{R}^d) \cap W^{1,\infty}([0, T] \times \mathbb{R}^d)$:

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x, t) d\mu_t^h(x) - \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x) &= \\ &= \int_0^t \int_{\mathbb{R}^d} \partial_t \varphi(x, s) d\mu_s^h(x) ds + \int_0^t \int_{\mathbb{R}^d} v^h[\mu_s^h](x) \cdot \nabla \varphi(x, s) d\mu_s^h(x) ds \end{aligned} \quad (5.2)$$

and similarly

$$\int_{\mathbb{R}^d} \varphi(x, t) d\mu_t^{h+\Delta h}(x) - \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x) = \int_0^t \int_{\mathbb{R}^d} \partial_t \varphi(x, s) d\mu_s^{h+\Delta h}(x) ds + \int_0^t \int_{\mathbb{R}^d} (v^h[\mu_s^{h+\Delta h}](x) + \Delta h \cdot v_p[\mu_s^{h+\Delta h}](x)) \cdot \nabla \varphi(x, s) d\mu_s^{h+\Delta h} ds. \quad (5.3)$$

Notice that $\mu_s^{h+\Delta h} = \mu_s^h + \Delta h \rho_{t,h}^{\Delta h}$. Then performing formally a first order Taylor expansion,

$$\begin{aligned} v^h[\mu_s^{h+\Delta h}](x) &= v^h[\mu_s^h + \Delta h \rho_{t,h}^{\Delta h}] \\ &= v^h[\mu_s^h] + \Delta h \cdot Dv^h[\mu_s^h] \cdot \rho_{t,h}^{\Delta h} + \Delta h \cdot o(1). \end{aligned} \quad (5.4)$$

Subtracting (5.9) from (5.10) and dividing by Δh , we then obtain

$$\begin{aligned} &\int_{\mathbb{R}^d} \varphi(x, t) d\rho_{t,h}^{\Delta h}(x) - \int_0^t \int_{\mathbb{R}^d} \partial_t \varphi(x, s) d\rho_{s,h}^{\Delta h}(x) ds - \int_0^t \int_{\mathbb{R}^d} v^h[\mu_s^h](x) \nabla \phi(x, s) d\rho_{s,h}^{\Delta h} ds \\ &= \int_0^t \int_{\mathbb{R}^d} (Dv^h[\mu_s^h] \rho_{s,h}^{\Delta h} + v_p[\mu_s^{h+\Delta h}] + o(1)) \nabla \phi(x, s) d\mu_s^{h+\Delta h} ds. \end{aligned}$$

Thus $\rho_{t,h}^{\Delta h}$ solves the linear equation

$$\partial_t \rho_{t,h}^{\Delta h} + \partial_x (v^h[\mu_t^h] \rho_{t,h}^{\Delta h}) = -\partial_x ((Dv^h[\mu_t^h] \rho_{s,h}^{\Delta h} + v_p[\mu_t^{h+\Delta h}] + o(1)) \mu_t^{h+\Delta h}) \quad (5.5)$$

with initial condition $\rho_{t=0,h}^{\Delta h} = 0$. We thus expect the limit $\rho_{t,h}$ to solve

$$\partial_t \rho_{t,h} + \partial_x (v^h[\mu_t^h] \rho_{t,h}) = -\partial_x ((Dv^h[\mu_t^h] \rho_{t,h} + v_p[\mu_t^h] \mu_t^h). \quad (5.6)$$

Comparing with the linear case studied in the previous section where we obtained the sensitivity equation (4.3), the situation now is more complicated because even if (5.6) is linear in $\rho_{t,h}$, the right-hand side depends on $\rho_{t,h}$ and the existence and uniqueness theory developed so far does not apply directly.

It turns out however that the previous formal reasoning (in particular the formal Taylor expansion (5.4)) can be justified when $v^h[\mu]$ is of the form (2.11), namely

$$\begin{aligned} v^h[\mu](x) &= v_0[\mu](x) + h v_p[\mu](x) \\ &= V_0 \left(x, \int_{\mathbb{R}^d} K_{V_0}(x, y) d\mu(y) \right) + h V_p \left(x, \int_{\mathbb{R}^d} K_{V_p}(x, y) d\mu(y) \right) \end{aligned}$$

with $V_0, V_p \in C^{1+\alpha}(\mathbb{R}^d \times \mathbb{R}, \mathbb{R}^d)$ and $K_{V_0}, K_{V_p} \in C^{2+\alpha}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$ for some $\alpha > \frac{1}{2}$. In that case the derivative $\rho_{t,h}$ exists according to Theorem 2.2 and we have the following result from [24] (Lemma 4.6):

Lemma 5.1. *Let $V, K_V \in C^{1+\alpha}(\mathbb{R}^d \times \mathbb{R}^d)$ and the map $h \mapsto \mu_t^h$ be differentiable in Z . Then, for every $x \in \mathbb{R}^d$, the map $h \mapsto V \left(x, \int_{\mathbb{R}^d} K_V(x, y) d\mu_t^h \right)$ is $C^{1+\alpha}(\mathbb{R}, \mathbb{R}^d)$ with norms bounded by some constant depending on the $C^{1+\alpha}$ norms of V and K_V as well as Z norm of derivative of μ_t^h . Moreover, if $\rho_{t,h} = \lim_{\Delta h \rightarrow 0} \frac{\mu_t^{h+\Delta h} - \mu_t^h}{\Delta h}$, we have the following chain rule:*

$$\frac{\partial}{\partial h} V \left(x, \int_{\mathbb{R}^d} K_V(x, y) d\mu_t^h(y) \right) = \nabla_y V \left(x, \int_{\mathbb{R}^d} K_V(x, y) d\mu_t^h(y) \right) (\rho_{t,h}, K_V(x, \cdot)).$$

where $\nabla_y V$ denotes the gradient of V with respect to the second variable.

Then Lemma 5.1 and Lemma 2.1 gives the following rigorous Taylor expansion:

Corollary 5.1. *In the framework of Lemma 5.1,*

$$\begin{aligned} & V\left(x, \int_{\mathbb{R}^d} K_V(x, y) d\mu_t^{h+\Delta h}\right) - V\left(x, \int_{\mathbb{R}^d} K_V(x, y) d\mu_t^h\right) \\ &= C[V, \mu_t^h](x) (\rho_{t,h}, K_V(x, \cdot)) + O(|h|^{1+\alpha}) \end{aligned} \quad (5.7)$$

where

$$C[V, \mu](x) = \nabla_y V\left(x, \int_{\mathbb{R}^d} K_V(x, y) d\mu\right)$$

and the $O(|h|^{1+\alpha})$ is uniform in $x \in \mathbb{R}^d$.

The following theorem asserts that the sensitivity equation (5.6) we obtained formally is the correct one:

Theorem 5.1. *The derivative $\rho_{t,h} = \lim_{\Delta h \rightarrow 0} \frac{\mu_t^{h+\Delta h} - \mu_t^h}{\Delta h}$ where μ_t^h and $\mu_t^{h+\Delta h}$ solve (5.1) is the unique weak solution of*

$$\begin{aligned} \partial_t \rho_{t,h} + \partial_x (v^h[\mu_t^h](x) \rho_{t,h}) &= -\partial_x (v_p(x) \mu_t^h) \\ &\quad - \partial_x [C[V_0, \mu_t^h](x) (\rho_{s,h}, K_{V_0}(x, \cdot)) \mu_t^h] - \partial_x [C[V_p, \mu_t^h](x) (\rho_{s,h}, K_{V_p}(x, \cdot)) \mu_t^h] \end{aligned} \quad (5.8)$$

with initial condition $\rho_{0,h} = 0$. More precisely, the weak formulation is satisfied for all test functions $\varphi(x, t)$ of regularity $\varphi \in C([0, T], C^{2+\alpha}(\mathbb{R}^d))$, $\varphi_t \in C([0, T], C^{1+\alpha}(\mathbb{R}^d))$, and $\rho_{t,h} \in \mathcal{A}$ for all $t \in [0, T]$ where \mathcal{A} is defined in (3.8).

Proof. Let $\rho_{t,h}^{\Delta h} := (\mu_t^{h+\Delta h} - \mu_t^h)/\Delta h$. Since $\mu_t^{h+\Delta h}$ and μ_t^h are solutions to (5.1) we have that for any $\varphi \in C^1([0, T] \times \mathbb{R}^d) \cap W^{1,\infty}([0, T] \times \mathbb{R}^d)$:

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x, t) d\mu_t^h(x) - \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x) &= \int_0^t \int_{\mathbb{R}^d} \partial_t \varphi(x, s) d\mu_s^h(x) ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} (v_0[\mu_s^h](x) + hv_p[\mu_s^h](x)) \cdot \nabla \varphi(x, s) d\mu_s^h ds \end{aligned} \quad (5.9)$$

and similarly

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x, t) d\mu_t^{h+\Delta h}(x) - \int_{\mathbb{R}^d} \varphi(x, 0) d\mu_0(x) &= \int_0^t \int_{\mathbb{R}^d} \partial_t \varphi(x, s) d\mu_s^{h+\Delta h}(x) ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} (v_0[\mu_s^{h+\Delta h}](x) + (h + \Delta h)v_p[\mu_s^{h+\Delta h}](x)) \cdot \nabla \varphi(x, s) d\mu_s^{h+\Delta h} ds. \end{aligned} \quad (5.10)$$

The plan is to subtract these equations, divide by Δh and pass to the limit $\Delta h \rightarrow 0$. First, in view of (5.7),

$$\begin{aligned} v_0[\mu_s^{h+\Delta h}](x) + hv_p[\mu_s^{h+\Delta h}](x) &= v_0[\mu_s^h](x) + hv_p[\mu_s^h](x) \\ &\quad + C[V_0, \mu_s^h](x) (\rho_{s,h}, K_{V_0}(x, \cdot)) + C[V_p, \mu_s^h](x) (\rho_{s,h}, K_{V_p}(x, \cdot)) + O(|h|^{1+\alpha}). \end{aligned}$$

Therefore, for $\varphi(x, t)$ of regularity $\varphi \in C([0, T], C^{2+\alpha}(\mathbb{R}^d))$ and $\varphi_t \in C([0, T], C^{1+\alpha}(\mathbb{R}^d))$, we subtract (5.9) from (5.10), divide by Δh and send $\Delta h \rightarrow 0$. Recalling that $\rho_{t,h}^{\Delta h} \rightarrow \rho_{t,h}$ in Z uniformly in $t \in [0, T]$, we obtain

$$\begin{aligned} (\rho_{t,h}, \varphi(\cdot, t)) &= \int_0^t \int_{\mathbb{R}^d} v_p[\mu_s^h](x) \cdot \nabla \varphi(x, s) d\mu_s^h(x) ds \\ &+ \int_0^t (\rho_{s,h}, \partial_t \varphi(\cdot, s) + v^h[\mu_s^h](\cdot) \cdot \nabla \varphi(\cdot, s)) ds \\ &+ \int_0^t \int_{\mathbb{R}^d} [C[V_0, \mu_s^h](x) (\rho_{s,h}, K_{V_0}(x, \cdot))] \cdot \nabla \varphi(x, s) d\mu_s^h(x) ds \\ &+ h \int_0^t \int_{\mathbb{R}^d} [C[V_p, \mu_s^h](x) (\rho_{s,h}, K_{V_p}(x, \cdot))] \cdot \nabla \varphi(x, s) d\mu_s^h(x) ds. \end{aligned}$$

Thus, $\rho_{t,h}$ is a weak solution of (5.8). It is also in the admissible class \mathcal{A} due to the Lipschitz continuity of solutions with respect to the vector field.

To obtain uniqueness, suppose that $\rho_{t,h}^{(1)}$ and $\rho_{t,h}^{(2)}$ are solutions to (5.8) with values in \mathcal{A} . Then, their difference $\rho_{t,h} = \rho_{t,h}^{(1)} - \rho_{t,h}^{(2)} \in \mathcal{A}$ satisfies

$$\begin{aligned} (\rho_{t,h}, \varphi(\cdot, t)) &= \int_0^t (\rho_{s,h}, \partial_t \varphi(\cdot, s) + v^h[\mu_s^h](\cdot) \cdot \nabla \varphi(\cdot, s)) ds \\ &+ \int_0^t \int_{\mathbb{R}^d} [C[V_0, \mu_s^h](x) (\rho_{s,h}, K_{V_0}(x, \cdot))] \cdot \nabla \varphi(x, s) d\mu_s^h(x) ds \\ &+ h \int_0^t \int_{\mathbb{R}^d} [C[V_p, \mu_s^h](x) (\rho_{s,h}, K_{V_p}(x, \cdot))] \cdot \nabla \varphi(x, s) d\mu_s^h(x) ds. \end{aligned} \quad (5.11)$$

Fix $\psi \in C^{2+\alpha}(\mathbb{R}^d)$. As in the proof of Proposition 3.1, we again use the duality method to find a test function $\varphi_\psi(x, t)$ such that

$$\partial_t \varphi_\psi(\cdot, s) + v^h[\mu_s^h](x) \cdot \nabla \varphi_\psi(x, s) = 0 \quad \varphi_\psi(x, t) = \psi(x).$$

Actually, it can be given explicitly as $\varphi_\psi(x, s) = \psi(T(x, t, s))$ where T is the flow of the non-autonomous vector field $v^h[\mu_s^h]$ which solves the ODE:

$$\partial_s T(x, s, t) = v^h[\mu_s^h](T(x, s, t)), \quad T(x, t, t) = x,$$

see Remark 8.1.5 and Proposition 8.1.7 in [27]. Using the test-function φ_ψ in (5.11) we deduce

$$\begin{aligned} (\rho_{t,h}, \psi) &= \int_0^t \int_{\mathbb{R}^d} [C[V_0, \mu_s^h](x) (\rho_{s,h}, K_{V_0}(x, \cdot))] \cdot \nabla \varphi_\psi(x, s) d\mu_s^h(x) ds \\ &+ h \int_0^t \int_{\mathbb{R}^d} [C[V_p, \mu_s^h](x) (\rho_{s,h}, K_{V_p}(x, \cdot))] \cdot \nabla \varphi_\psi(x, s) d\mu_s^h(x) ds \end{aligned} \quad (5.12)$$

for any $\psi \in C^{2+\alpha}(\mathbb{R}^d)$. Since the kernels K_{V_0} and K_{V_p} are both assumed to be $C^{2+\alpha}(\mathbb{R}^d \times \mathbb{R}^d)$, there is a constant C such that

$$(\rho_{s,h}, K_{V_0}(x, \cdot)), (\rho_{s,h}, K_{V_p}(x, \cdot)) \leq C \sup_{\|\psi\|_{C^{2+\alpha}} \leq 1} (\rho_{s,h}, \psi).$$

Moreover, for $\psi \in C^{2+\alpha}(\mathbb{R}^d)$ with $\|\psi\|_{C^{2+\alpha}} \leq 1$ we see from the explicit formula that there is another constant C such that $\|\nabla\varphi_\psi\|_\infty \leq C$. Therefore, from (5.12), we conclude

$$\sup_{\|\psi\|_{C^{2+\alpha}} \leq 1} (\rho_{t,h}, \psi) \leq C \int_0^t \sup_{\|\psi\|_{C^{2+\alpha}} \leq 1} (\rho_{s,h}, \psi) ds$$

for some possibly bigger constant C . Now, Gronwall inequality implies

$$(\rho_{s,h}, \psi) = 0$$

for all $s \in [0, t]$ and all $\psi \in C^{2+\alpha}(\mathbb{R}^d)$. As $\rho_{s,h}$ is in the admissible class \mathcal{A} , we can repeat the uniqueness proof from Theorem 3.1 to deduce that $\rho_{s,h} = 0$ as desired. \square

6. Applications

As mentioned above, transport-type equations like (1.1) represent a big variety of phenomena occurring in physics, biology and social sciences. In this section we present applications that the theory developed here is of use.

6.1. Optimal control

Here we are interested in functionals of the form

$$\mathcal{J}(h) = \int_{\mathbb{R}^d} F(x) d\mu_t^h(x),$$

where μ_t^h is a measure solution to the perturbed transport equation (1.2) on the space of nonnegative Radon measure, while $F \in C^{1+\alpha}(\mathbb{R}^d)$. Such functionals can describe various quantities of practical importance. For example, for $F(x) = 1$ this functional provides the total number of individuals in a population, since $\mu^h \in C([0, T], \mathcal{M}^+(\mathbb{R}^d))$.

Now, let $\partial_h \mu_t^h \in C([0, T], \mathcal{Z})$ be the derivative of μ_t^h with respect to h . Then, $h \mapsto \mathcal{J}(h)$ is differentiable and

$$\partial_h \mathcal{J}(h) = (\partial_h \mu_t^h, F),$$

value of this derivative can be used in the optimization of the functional \mathcal{J} , i.e., finding value of h for which \mathcal{J} is the smallest. Our work characterizes the derivative as the solutions of some PDE, thus allowing to work on appropriate approximating schemes for the quantity $(\partial_h \mu_t^h, F)$.

6.2. Parameter estimation

Another application of paramount importance is parameter estimation and fitting models to data, as this allows for model validation. To this end, let $\int_{\mathbb{R}^d} d\mu_t^h$, represents the total number of individuals in a population at time t provided by the perturbed transport equation model considered on the space of nonnegative Radon measures. Suppose that D_k represents data on the number of individuals in the population at time t_k , $k = 1, \dots, K$ (a time series of the total population). Consider the following minimization problem involving a least-squares functional that measures the distance between the model solution and data:

$$\min_h \mathcal{J}(h) = \min_h \sum_{k=1}^K \left| \int_{\mathbb{R}^d} d\mu_{t_k}^h - D_k \right|^2,$$

subject to

$$\partial_t \mu_t^h + \partial_x(v^h[\mu](x)\mu_t^h) = 0, \quad \mu_t^h|_{t=0} = \mu_0 \in \mathcal{M}^+(\mathbb{R}^d).$$

The derivative $\partial_h \mathcal{J}(h)$ which depends on the derivative of $\partial_h(\mu_t^h)$, the solution to (4.3), can be used to minimize the least-squares distance $\mathcal{J}(h)$. The value \bar{h} that minimizes $\mathcal{J}(h)$, also provides an estimate for the vector field given by $v^{\bar{h}}$.

We conclude by pointing out that the above two applications demonstrate the need for the development of numerical approximation schemes for computing solutions to transport equations of the type (4.1) or (4.3). There has been some efforts in the direction of solving transport equations in the space of nonnegative Radon measures endowed with the Bounded Lipschitz norm (e.g. [31, 32]), but to our knowledge, no such numerical schemes exist for solving transport equations in the space Z . Furthermore, because minimization problems generally involve computing the solution multiple times until a minimizer is reached, it is important that for any scheme developed to be efficient and fast.

Appendix

6.3. Proof of Proposition 2.3.

Proof. We want to prove that if $\mu_n \rightarrow \mu$ narrowly, then $\partial_x \mu_n \rightarrow \partial_x \mu$ in Z i.e.

$$\lim_{n \rightarrow +\infty} \|\partial_x \mu_n - \partial_x \mu\|_Z = \lim_{n \rightarrow +\infty} \sup_{\|\phi\|_{C^{1,\alpha}} \leq 1} |(\mu_n - \mu, \partial_x \phi)| = 0.$$

Assume that this is not true. Then there exist $\varepsilon > 0$, a subsequence $(\mu_{n_k})_k$ that we still denote by $(\mu_n)_n$ for simplicity, and functions ϕ_n , $\|\phi_n\|_{C^{1,\alpha}} \leq 1$, such that

$$|(\mu_n - \mu, \partial_x \phi_n)| \geq \varepsilon > 0. \quad (6.1)$$

By Arzela-Ascoli theorem, up to a subsequence, $\phi_n \rightarrow \phi$ in $C^1(K)$ for any compact set $K \subset \mathbb{R}^d$. Passing to the limit in $|\phi_n(x)| \leq 1$, $|\phi'_n(x)| \leq 1$, and $|\phi'_n(x) - \phi'_n(y)| \leq |x - y|^\alpha$, we obtain that $\|\phi\|_{C^{1,\alpha}} \leq 1$. From Theorem 5 in [33], we deduce that

$$(\mu_n, \partial_x \phi_n) \rightarrow (\mu, \partial_x \phi).$$

Moreover, from Theorem 4 in [33], we know that the sequence $\{\mu_n\}_{n \in \mathbb{N}}$ is tight and TV-bounded. It follows that μ is bounded and thus tight. We deduce that

$$(\mu, \partial_x \phi_n) \rightarrow (\mu, \partial_x \phi).$$

These two facts contradict (6.1). □

6.4. Proof of Proposition 2.5.

Proof. Let $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathcal{M}(\mathbb{R}^d)$ be such that $\mu_n \rightarrow \mu$ in Z for $\mu \in Z$. Let $\phi \in C^{1,\alpha}(\mathbb{R}^d)$ with $\|\phi\|_{C^{1,\alpha}} \leq 1$. Since $T \in C^{1,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$ we have $\phi \circ T \in C^{1,\alpha}(\mathbb{R}^d)$ with $\|\phi \circ T\|_{C^{1,\alpha}} \leq C$, independently of ϕ . Then

$$|(T^\# \mu_n - T^\# \mu_m, \phi)| = |(\mu_n - \mu_m, \phi \circ T)| \leq \|\mu_n - \mu_m\|_Z \|\phi \circ T\|_{C^{1,\alpha}} \leq C \|\mu_n - \mu_m\|_Z.$$

Thus, $\|T^\# \mu_n - T^\# \mu_m\|_Z \leq C \|\mu_n - \mu_m\|_Z$ and so the sequence $\{T^\# \mu_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in Z . By completeness of Z , it converges to some element we denote by $T^\# \mu$. This is independent of the choice of the approximating sequence μ_n because if $\{\tilde{\mu}_n\}_{n \in \mathbb{N}} \subset \mathcal{M}(\mathbb{R}^d)$ is another sequence such that $\tilde{\mu}_n \rightarrow \mu$ in Z then

$$|(T \circ \tilde{\mu}_n - T \circ \mu_n, \phi)| = |(\mu_n - \tilde{\mu}_n, \phi \circ T)| \leq C \|\mu_n - \mu\|_Z + C \|\tilde{\mu}_n - \mu\|_Z$$

so that $\|T \circ \tilde{\mu}_n - T \circ \mu_n\|_Z \rightarrow 0$. Moreover, for any $\phi \in C^{1,\alpha}(\mathbb{R}^d)$,

$$(T^\# \mu, \phi) = \lim_{n \rightarrow \infty} (T^\# \mu_n, \phi) = \lim_{n \rightarrow \infty} (\mu_n, \phi \circ T) = (\mu, \phi \circ T).$$

□

6.5. Proof of Lemma 2.2

Proof. First note that map $t \in [0, T] \mapsto \mu_t$ is uniformly continuous so there is a nondecreasing function $\omega : [0, \infty] \rightarrow [0, \infty]$ with $\lim_{t \rightarrow 0^+} \omega(t) = \omega(0) = 0$ (it is usually called modulus of continuity) such that

$$\|\mu_t - \mu_s\|_Z \leq \omega(|t - s|) \quad s, t \in [0, T].$$

Given $n \in \mathbb{N}$, let $\delta_n = T/n$. We consider the partition $\{t_0^{(n)}, \dots, t_n^{(n)}\}$ of $[0, T]$ with mesh points $t_k^{(n)} = k\delta_n$ for $k = 0, \dots, n$. For each such k , consider a bounded measure $\mu_k^{(n)}$ such that

$$\|\mu_{t_k^{(n)}}^{(n)} - \mu_k^{(n)}\|_Z \leq 1/n.$$

Then, we define $\mu^{(n)} \in C([0, T], Z)$ as the polygonal curve passing through the points $(t_k^{(n)}, \mu_k^{(n)})$, $k = 0, \dots, n$, namely

$$\mu_t^{(n)} = \begin{cases} \mu_k^{(n)} & \text{if } t = t_k^{(n)} \text{ for some } k = 0, \dots, n. \\ \frac{t - t_k^{(n)}}{\delta_n} \mu_{k+1}^{(n)} + \frac{t_{k+1}^{(n)} - t}{\delta_n} \mu_k^{(n)} & \text{if } t \in (t_k^{(n)}, t_{k+1}^{(n)}) \text{ for some } k = 0, \dots, n-1. \end{cases}$$

Clearly, $\mu^{(n)} \in C([0, T], Z)$ and for any n , $\max_{0 \leq t \leq T} \|\mu_t^{(n)}\|_{TV} \leq C_n$.

Now, for $t \in [0, T]$, let \hat{t} and \check{t} be the closest mesh points from left and right respectively. Then,

$$\begin{aligned} \left\| \mu_t^{(n)} - \mu_{\hat{t}}^{(n)} \right\|_Z &= \frac{t_{k+1}^{(n)} - t}{\delta_n} \left\| \mu_{\check{t}}^{(n)} - \mu_{\hat{t}}^{(n)} \right\|_Z \leq \left\| \mu_{\check{t}}^{(n)} - \mu_{\hat{t}}^{(n)} \right\|_Z \\ &\leq 2/n + \left\| \mu_{\check{t}} - \mu_{\hat{t}} \right\|_Z \leq 2/n + \omega(|\check{t} - \hat{t}|) \leq 2/n + \omega(\delta_n) \end{aligned}$$

Therefore, for any $t \in [0, T]$:

$$\begin{aligned} \left\| \mu_t^{(n)} - \mu_t \right\|_Z &\leq \left\| \mu_t^{(n)} - \mu_{\hat{t}}^{(n)} \right\|_Z + \left\| \mu_{\hat{t}}^{(n)} - \mu_{\hat{t}} \right\|_Z + \left\| \mu_{\hat{t}} - \mu_t \right\|_Z \\ &\leq (2/n + \omega(\delta_n)) + 1/n + \omega(\delta_n) \end{aligned}$$

Thus $\lim_{n \rightarrow +\infty} \max_{0 \leq t \leq T} \left\| \mu_t^{(n)} - \mu_t \right\|_Z = 0$. □

6.6. Proof of Corollary 2.2.

Proof. In view of Lemma 2.2 there exists $(v^n)_n \subset C([0, T], \mathcal{M}_b(\mathbb{R}^d))$ such that $\lim_{n \rightarrow +\infty} \|v_t^n - v_t\|_Z = 0$ uniformly in $t \in [0, T]$. For any $\phi \in C^{1,\alpha}(\mathbb{R}^d)$, $\|\phi\|_{C^{1,\alpha}} \leq 1$, and any $s, t \in [0, T]$, we write

$$\begin{aligned} |(T_s^\# v_s - T_t^\# v_t, \phi)| &\leq |(T_s^\# v_s - T_s^\# v_s^n, \phi)| + |(T_s^\# v_s^n - T_t^\# v_t^n, \phi)| + |(T_t^\# v_t^n - T_t^\# v_t, \phi)| \\ &\leq \|v_s - v_s^n\|_Z \|\phi \circ T_s\|_{C^{1,\alpha}} + |(T_s^\# v_s^n - T_t^\# v_t^n, \phi)| + \|v_t - v_t^n\|_Z \|\phi \circ T_t\|_{C^{1,\alpha}} \end{aligned}$$

In view of Lemma 2.1 and Proposition 2.1 we have $\|\phi \circ T_\tau\|_{C^{1,\alpha}} \leq C_T$ for any $\tau \in [0, T]$. Thus

$$|(T_s^\# v_s - T_t^\# v_t, \phi)| \leq |(T_s^\# v_s^n - T_t^\# v_t^n, \phi)| + 2C_T \max_{0 \leq t \leq T} \|v_t - v_t^n\|_Z$$

Now, we handle the first term on the right-hand side as follows

$$\begin{aligned} |(T_s^\# v_s^n - T_t^\# v_t^n, \phi)| &\leq |(T_s^\# v_s^n - T_s^\# v_t^n, \phi)| + |(T_s^\# v_t^n - T_t^\# v_t^n, \phi)| \\ &\leq \|v_s^n - v_t^n\|_Z \|\phi \circ T_s\|_{C^{1,\alpha}} + \|v_t^n\|_{TV} \|\phi \circ T_s - \phi \circ T_t\|_\infty \\ &\leq C_T \|v_s^n - v_t^n\|_Z + \|v_t^n\|_{TV} \|v\|_\infty |s - t|. \end{aligned}$$

Thus,

$$|(T_s^\# v_s - T_t^\# v_t, \phi)| \leq C_T \|v_s^n - v_t^n\|_Z + \|v_t^n\|_{TV} \|v\|_\infty |s - t| + 2C_T \max_{0 \leq t \leq T} \|v_t - v_t^n\|_Z.$$

We conclude recalling that for a fixed n , v_t^n is continuous in t for the Z -norm and TV-bounded uniformly in $t \in [0, T]$. \square

Acknowledgements

Nicolas Saintier is supported by the University of Buenos Aires through the grant UBACYT 20020170200256BA. Jakub Skrzeczkowski is supported by National Science Center, Poland through project no. 2017/27/B/ST1/01569.

Conflict of interest

The authors declare there is no conflicts of interest.

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