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On the existence of bounded solutions for a nonlinear elliptic system

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Abstract This work deals with the system $(-\Delta)^m u = a(x)\, v^p$, $(-\Delta)^m v = b(x)\, u^q$ with Dirichlet boundary condition in a domain $\Omega \subset \mathbb{R}^n$, where Ω is a ball if $n \geq 3$ or a smooth perturbation of a ball when $n = 2$. We prove that, under appropriate conditions on the parameters (a, b, p, q, m, n), any nonnegative solution (u, v) of the system is bounded by a constant independent of (u, v). Moreover, we prove that the conditions are sharp in the sense that, up to some border case, the relation on the parameters are also necessary. The case $m = 1$ was considered by Souplet (Nonlinear Partial Differ Equ Appl 20:464–479, 2004). Our paper generalize to $m \geq 1$ the results of that paper.

Keywords Elliptic systems · A priori estimates · Critical exponents · Weighted Sobolev spaces

Mathematics Subject Classification (2000) 35J48 · 35B45 · 35B33 · 46E35

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1 Introduction

In this paper, we consider the nonlinear problem

$$\begin{cases} (-\Delta)^m u = a(x)\, v^p & \text{in } \Omega \\ (-\Delta)^m v = b(x)\, u^q & \text{in } \Omega \\ \left(\frac{\partial}{\partial \nu}\right)^j u = \left(\frac{\partial}{\partial \nu}\right)^j v = 0 & \text{on } \partial\Omega \; 0 \le j \le m-1, \end{cases} \tag{1.1}$$

where Ω is the unit ball, namely, $\Omega = B = \{x \in \mathbb{R}^n : |x| < 1\}$ when $n \geq 3$, and B or some perturbations of B for the case $n = 2$ (see [7] for details of this perturbation), $\frac{\partial}{\partial \nu}$ is the normal derivative, $p, q > 0$, $pq > 1$, and a, b are nonnegative bounded functions. Let us remark that the restriction on the domains is due to the fact that we will use that the Green function of the corresponding linear problem is positive.

For the particular case $m = 1$, many authors have worked on the existence of different types of solutions, see for example [10,12,14,15]. In all these papers, the exponents

$$\alpha = \frac{2(p+1)}{pq-1} \quad \text{and} \quad \beta = \frac{2(q+1)}{pq-1}$$

play an important role.

On the other hand, in recent years, the weighted Lebesgue spaces $L^p_d(\Omega)$, where d is the distance to the boundary of Ω, have played an important role in the study of several questions in the theory of nonlinear elliptic problems (see for example [1,4,16–18]).

A priori bounds for nonnegative weak solutions of (1.1) with $m = 1$ in a C^2 bounded domain Ω were obtained by P. Souplet in [17]. He proved that if $\max\{\alpha, \beta\} > n - 1$, then

$$\|u\|_{L^\infty(\Omega)},\ \|v\|_{L^\infty(\Omega)} \leq C, \tag{1.2}$$

where the constant C depends only on p, q, a, b and Ω. Moreover, he proved that the result is sharp in the sense that if $\max\{\alpha, \beta\} < n - 1$, then there exist nonnegative bounded functions a and b for which nonnegative unbounded solutions of (1.1) exist.

Our goal is to obtain similar results for nonnegative weak solutions of (1.1) for general m. With this purpose, we will use the following generalization of the exponents α and β,

$$\alpha = \frac{2m(p+1)}{pq-1} \quad \text{and} \quad \beta = \frac{2m(q+1)}{pq-1}.$$

Let us mention that these exponents have appeared in different works (see for example [8,13,19]) where the authors studied existence of positive solutions of (1.1) in $\Omega = \mathbb{R}^n$.

An important part of the arguments used in [17] are some weighted a priori estimates for the associated linear problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, in order to generalize the results of [17] for the case $m \geq 2$, we will need to extend the weighted estimates to higher order linear problems. Nontrivial technical modifications are needed to prove those estimates. Moreover, since we need to use positivity of the Green function, we have to restrict the domain Ω as mentioned previously. Indeed, for $m \geq 2$ and general regions, the Green function is not necessarily positive.

2 Weighted a priori estimates for the linear problem

We will denote by $d(x)$ the distance from x to the boundary of Ω and we will work with the Banach space $L^p_{d^m}(\Omega)$ where the norm is given by

$$\|u\|_{L^p_{d^m}(\Omega)} = \left(\int_\Omega |u|^p\, d^m\, \mathrm{d}x\right)^{1/p}$$

for $1 \le p < \infty$ and $\|u\|_{L^\infty_{d^m}(\Omega)} := \|u\|_{L^\infty(\Omega)}$.

In our arguments, we will use some results given in [6] for the linear problem

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega \\ \left(\dfrac{\partial}{\partial \nu}\right)^j u = 0 & \text{on } \partial\Omega\ 0 \le j \le m-1. \end{cases} \tag{2.1}$$

We recall those results in the following lemma.

Definition 2.1 Let $f \in L^1_{d^m}(\Omega)$. A weak solution of (2.1) is a function $u \in L^1(\Omega)$ such that

$$\int_\Omega u\,(-\Delta)^m \varphi = \int_\Omega f\,\varphi$$

for all $\varphi \in C^{2m}(\overline{\Omega})$ with $\left(\frac{\partial}{\partial \nu}\right)^j \varphi = 0$ on $\partial\Omega$, $0 \le j \le m-1$.

We can see that the weak solution of this problem exists and is unique, in analogous way as in Lemma 1 in [1]. Furthermore, by using the density of $L^2(\Omega)$ in $L^1_{d^m}(\Omega)$, the solution is given by the representation formula

$$u(x) = \int_\Omega G_m(x, y)\, f(y)\, \mathrm{d}y$$

where $G_m(x, y)$ is the Green function of $(-\Delta)^m$.

By a weak solution (u, v) of (1.1), we understand a weak solution defined as in (2.1) assuming that v^p and u^q belong to $L^1_{d^m}(\Omega)$.

In what follows the letter C will denote a generic constant, not necessarily the same at each occurrence, whose dependence are made explicit when necessary.

Lemma 2.2 *Let $u \in C^{2m}(\overline{\Omega})$ and $f \in C(\overline{\Omega})$ satisfy (2.1).*

- *If $2m > n$, then there exists $C > 0$ such that for all $\theta \in [0, 1]$*

$$\|u\, d^{-m+\theta n}\|_{L^\infty(\Omega)} \le C\, \|f\, d^{m-(1-\theta)n}\|_{L^1(\Omega)}.$$

- *Let $1 \le p \le q \le \infty$. If $\frac{1}{p} - \frac{1}{q} < \min\{\frac{2m}{n}, 1\}$, then taking $\alpha \in (\frac{1}{p} - \frac{1}{q}, \min\{\frac{2m}{n}, 1\}]$ there exists $C > 0$ such that for all $\theta \in [0, 1]$*

$$\|u\, d^{-m+\theta n\alpha}\|_{L^q(\Omega)} \le C\, \|f\, d^{m-(1-\theta)n\alpha}\|_{L^p(\Omega)}.$$

Proof See Proposition 4.2 in [6]. □

Let us remark that these results, and consequently our proposition below, are valid in more general domains than those considered here. Indeed, the hypotheses used are that Ω is a bounded domain with C^{6m+4} boundary for $n = 2$ and C^{5m+2} boundary for $n \ge 3$.

Then, we have the following a priori estimates for solutions of problem (2.1).

Proposition 2.3 *Let* $1 \le p \le q \le \infty$. *Let* $f \in L^p_{d^m}(\Omega)$ *and let* u *be a weak solution of* (2.1).

We have

(1) *if* $n \le m$, *then* $u \in L^\infty(\Omega)$ *and there exists* $C > 0$ *such that*

$$\|u\|_{L^\infty(\Omega)} \le C \, \|f\|_{L^1_{d^m}(\Omega)}$$

(2) *if* $\frac{1}{p} - \frac{1}{q} < \frac{2m}{n+m}$, *then* $u \in L^q_{d^m}(\Omega)$ *and there exists* $C > 0$ *such that*

$$\|u\|_{L^q_{d^m}(\Omega)} \le C \, \|f\|_{L^p_{d^m}(\Omega)}.$$

Proof From Lemma 2.2, we have that, for $2m > n$ and $\theta \in [0, 1]$,

$$\|u \, d^{-m+\theta n}\|_{L^\infty(\Omega)} \le C \, \|f \, d^{m-(1-\theta)n}\|_{L^1(\Omega)}. \tag{2.2}$$

Then taking $\theta = 1$ and using that $-m + n < 0$ and $d(x) \le \operatorname{diam}(\Omega)$, we obtain

$$\|u\|_{L^\infty(\Omega)} \le C \, \|u \, d^{-m+n}\|_{L^\infty(\Omega)} \le C \, \|f \, d^m\|_{L^1(\Omega)}$$

and so (1) is proved.

On the other hand, using again Lemma 2.2, we have that, if there exists $\alpha \in \left(\frac{1}{p} - \frac{1}{q}, \min\{1, \frac{2m}{n}\}\right]$ and $\theta \in [0, 1]$ such that

$$\begin{cases} -m + \theta \, n \, \alpha = \frac{m}{q} \\ m - (1-\theta) \, n \, \alpha = \frac{m}{p} \end{cases} \tag{2.3}$$

we obtain

$$\|u\|_{L^q_{d^m}(\Omega)} \le C \, \|f\|_{L^p_{d^m}(\Omega)}$$

for $\frac{1}{p} - \frac{1}{q} < \min\{1, \frac{2m}{n}\}$.

Solving system (2.3) we obtain

$$\alpha = \left(2 + \frac{1}{q} - \frac{1}{p}\right) \frac{m}{n} \quad \text{and} \quad \theta = \left(\frac{1}{q} + 1\right) \left(2 - \frac{1}{p} + \frac{1}{q}\right)^{-1}.$$

We are going to show that α and θ satisfy the required conditions if $\frac{2m-n}{m} \le \frac{1}{p} - \frac{1}{q} < \frac{2m}{n+m}$.

Since $1 \le p$, we have $\theta \in [0, 1]$. On the other hand, from the definition of α, it is easy to see that the condition $\frac{1}{p} - \frac{1}{q} < \alpha$ is equivalent to $\frac{1}{p} - \frac{1}{q} < \frac{2m}{n+m}$, which is one of our hypothesis.

Finally, we have to see that $\alpha \le \min\{1, \frac{2m}{n}\}$. Since $p \le q$, we have $\alpha \le \frac{2m}{n}$. Therefore, it only remains to consider the case $\frac{2m}{n} > 1$. But $\alpha \le 1$ is equivalent to $\frac{2m-n}{m} \le \frac{1}{p} - \frac{1}{q}$, and so the proposition is proved under this restriction.

Suppose now that $\frac{1}{p} - \frac{1}{q} < \frac{2m-n}{m}$. In this case, for $2m > n$, using again the first part of Lemma 2.2, for all $\in [0, 1]$, we have

$$\|u \, d^{-m+\mathrm{n}}\|_{L^\infty(\Omega)} \le C \, \|f \, d^{m-(1-)\mathrm{n}}\|_{L^1(\Omega)}.$$

Moreover, if $\le \frac{\mathrm{m}}{\mathrm{nq}} + \frac{\mathrm{m}}{\mathrm{n}}$, it follows that

$$\|u\|_{L^q_{d^m}(\Omega)} \le \|u \, d^{-m+\mathrm{n}}\|_{L^\infty(\Omega)}.$$

Analogously, if $1 - \frac{m}{n} + \frac{m}{np} \leq$,

$$\|f\, d^{m-(1-)\mathrm{n}}\|_{L^1(\Omega)} \leq C\|f\|_{L^p_{d^m}(\Omega)}.$$

Therefore, if we can choose satisfying $1 - \frac{m}{n} + \frac{m}{np} \leq\leq \frac{\mathrm{m}}{\mathrm{nq}} + \frac{\mathrm{m}}{\mathrm{n}}$, we have

$$\|u\|_{L^q_{d^m}(\Omega)} \leq C\, \|f\|_{L^p_{d^m}(\Omega)},$$

but, such a exists because $\frac{1}{p} - \frac{1}{q} \leq \frac{2m-n}{m}$ and the proposition is proved. □

Remark 2.4 The condition in (2) is almost optimal, i.e. if $\frac{1}{p} - \frac{1}{q} > \frac{2m}{n+m}$, then the a priori estimate does not hold in general. We postpone the proof of this observation to the end of the paper because we will use the same technique as in the proof of our second main theorem.

In the proof of the following proposition, we will denote with $\lambda_{1,m}$, the first eigenvalue of the operator $(-\Delta)^m$ and with $\phi_{1,m} > 0$, a corresponding eigenfunction normalized by $\int_\Omega \phi_{1,m} = 1$. We will use that there exist two positive constants c_1 and c_2 such that, in Ω,

$$c_1\, d^m \leq \phi_{1,m} \leq c_2\, d^m, \tag{2.4}$$

see [5].

Proposition 2.5 *If u is a weak solution of (2.1) with $f \in L^1_{d^m}(\Omega)$ and $f \geq 0$, then there exists $C > 0$ such that*

(1) *If $n \leq m$ and $1 \leq k \leq \infty$,*

$$\|u\|_{L^k_{d^m}(\Omega)} \leq C\, \|u\|_{L^1_{d^m}(\Omega)}.$$

(2) *If $n > m$ and $1 \leq k < \frac{n+m}{n-m}$*

$$\|u\|_{L^k_{d^m}(\Omega)} \leq C\, \|u\|_{L^1_{d^m}(\Omega)}.$$

Proof By the definition of weak solution, we have that

$$\begin{aligned}\|f\|_{L^1_{d^m}} &= \int_\Omega f\, d^m\, \mathrm{d}x \leq C \int_\Omega f\, \phi_{1,m}\, \mathrm{d}x \\ &= C \int_\Omega u\, (-\Delta)^m \phi_{1,m}\, \mathrm{d}x = C\, \lambda_{1,m} \int_\Omega u\, \phi_{1,m}\, \mathrm{d}x \\ &\leq C \int_\Omega |u|\, d^m\, \mathrm{d}x \leq C\, \|u\|_{L^1_{d^m}}.\end{aligned}$$

Then, (1) follows directly from (1) in the previous proposition and (2) follows taking $p = 1$ in (2) of the same proposition. □

3 Main results

We consider problem (1.1) and define the exponents

$$\alpha = \frac{2m(p+1)}{pq-1} \quad \text{and} \quad \beta = \frac{2m(q+1)}{pq-1}.$$

Then, the natural extension of the results in [17] is given by the following

Theorem 3.1 *If*

$$\max(\alpha, \beta) > n - m, \tag{3.1}$$

then, any nonnegative weak solution of (1.1) *satisfies*

$$\|u\|_{L^\infty(\Omega)}, \ \|v\|_{L^\infty(\Omega)} \le C \tag{3.2}$$

where C is a positive constant which depends only on a, b, p, q, m and Ω.

We also prove, in the following theorem, that condition (3.1) is almost optimal. We cannot say optimal because we do not know what happens in the case $\max(\alpha, \beta) = n - m$.

Theorem 3.2 *If*

$$\max(\alpha, \beta) < n - m, \tag{3.3}$$

then, there exist nonnegative bounded functions a and b, such that (1.1) *have some nonnegative weak solution (u, v), with u and v unbounded functions.*

Remark 3.1 The analogues of Theorems 3.1 and 3.2 are also true for the problem

$$\begin{cases} (-\Delta)^m u = a(x)u^p & \text{in } \Omega \\ \left(\dfrac{\partial}{\partial \nu}\right)^j u = 0 & \text{on } \partial\Omega \ 0 \le j \le m-1. \end{cases}$$

In this case, the conditions 3.1 and 3.3 are replaced by $p < \frac{n+m}{n-m}$ and $p > \frac{n+m}{n-m}$, respectively. For the case $m = 1$, this exponent appears first in [2].

Once we have the results of the previous section, the proofs follow the lines of the case $m = 1$ proved in [17]. A key point in the arguments given in that paper are the estimates

$$\int_\Omega u\,\phi_{1,m}, \quad \int_\Omega v\,\phi_{1,m} \le C. \tag{3.4}$$

A straightforward extension of the arguments given in [18], to prove these estimates in the case $m = 1$, is not possible. Indeed, the proof given in that paper is based on Lemma 3.2 of [3], which uses the maximum principle in subsets of Ω. An analogous maximum principle is not valid in the case $m \ge 2$. We give first a different proof of an analogous lemma using pointwise estimates for the Green function G_m of problem (2.1) given later and conclude the proof of (3.4) using that new result.

Recall that we have taken Ω such that the Green function is positive there, i.e. we assume that $\Omega = B = \{x \in \mathbb{R}^n : |x| < 1\}$ when $n \ge 3$, and $\Omega = B$ or some perturbations of B for the case $n = 2$ (see [7] for details of this perturbation). We have: for $2m < n$,

$$G_m(x, y) \ge C\,|x - y|^{2m-n} \min\left\{1, \frac{d(x)^m\, d(y)^m}{|x - y|^{2m}}\right\}, \tag{3.5}$$

for $2m = n$,

$$G_m(x, y) \ge C \log\left(1 + \frac{d(x)^m\, d(y)^m}{|x - y|^{2m}}\right) \ge C \log\left(2 + \frac{d(y)}{|x - y|}\right) \min\left\{1, \frac{d(x)^m\, d(y)^m}{|x - y|^{2m}}\right\}, \tag{3.6}$$

and for $2m > n$,

$$G_m(x, y) \geq C\, d(x)^{m-n/2}\, d(y)^{m-n/2} \,\min\left\{1, \frac{d(x)^{n/2}\, d(y)^{n/2}}{|x-y|^n}\right\}. \tag{3.7}$$

The proofs of these estimates can be found in [7] for the case of $m = n = 2$ and in [11] for the rest of the cases.

Lemma 3.3 *Assume $h \geq 0$, $h \in L^1_{d^m}(\Omega)$ and v a weak solution of*

$$\begin{cases} (-\Delta)^m v = h & \text{in } \Omega \\ \left(\dfrac{\partial}{\partial \nu}\right)^j v = 0 & \text{on } \partial\Omega \; 0 \leq j \leq m-1. \end{cases} \tag{3.8}$$

Then there exists $C > 0$, depending only on Ω and m, such that for all $x \in \Omega$

$$\frac{v(x)}{d^m(x)} \geq C \int_\Omega h\, d^m. \tag{3.9}$$

Proof By the representation formula

$$v(x) = \int_\Omega G_m(x, y)\, h(y)\, \mathrm{d}y$$

it is enough to prove that

$$G_m(x, y) \geq C\, d(x)^m\, d(y)^m.$$

Consider, for example, the case $2m < n$ and suppose that $\frac{d(x)^m\, d(y)^m}{|x-y|^{2m}} \geq 1$. Then, it follows from (3.5) that

$$G_m(x, y) \geq C\, |x-y|^{2m-n} \geq d(x)^{m-n/2} d(y)^{m-n/2} \geq C d(x)^m d(y)^m$$

where in the last step, we have used that Ω is bounded. On the other hand, if the minimum on the right-hand side of (3.5) is attained in $\frac{d(x)^m\, d(y)^m}{|x-y|^{2m}}$, we have

$$G_m(x, y) \geq C\, |x-y|^{-n} d(x)^m d(y)^m \geq C d(x)^m d(y)^m.$$

The proofs for the cases $2m = n$ and $2m > n$ are analogous, using now (3.6) and (3.7), respectively. □

Proof of (3.4) For (u, v) nonnegative weak solution of (1.1), taking $f = av^p$, it follows from (2.4) and Lemma 3.3 that $u \geq C\left(\int_\Omega av^p\phi_{1,m}\right)\phi_{1,m}$and, for $f = bu^q$, $v \geq C\left(\int_\Omega bu^q\phi_{1,m}\right)\phi_{1,m}$. Then

$$\begin{aligned} \int_\Omega av^p\phi_{1,m} &\geq C\left(\int_\Omega a\phi_{1,m}^{p+1}\right)\left(\int_\Omega bu^q\phi_{1,m}\right)^p \\ &\geq C\left(\int_\Omega a\phi_{1,m}^{p+1}\right)\left(\int_\Omega b\phi_{1,m}^{q+1}\right)^p\left(\int_\Omega av^p\phi_{1,m}\right)^{qp}. \end{aligned}$$

And by the same way

$$\int_\Omega bu^q\phi_{1,m} \geq C\left(\int_\Omega b\phi_{1,m}^{q+1}\right)\left(\int_\Omega a\phi_{1,m}^{p+1}\right)^q\left(\int_\Omega bu^q\phi_{1,m}\right)^{qp}.$$

Since $pq > 1$, if we show that there exists a constant $C > 0$ such that

$$\int_\Omega a\phi_{1,m}^{p+1} \geq C \quad \text{and} \quad \int_\Omega b\phi_{1,m}^{q+1} \geq C, \tag{3.10}$$

we have

$$\int_\Omega av^p\phi_{1,m} \leq C \quad \text{and} \quad \int_\Omega bu^q\phi_{1,m} \leq C. \tag{3.11}$$

Taking $\phi_{1,m}$ as a test function in the problem (1.1), we have that (3.4) follows from (3.11).

To prove (3.10), let $\epsilon > 0$,

$$\begin{aligned}\int_\Omega a\phi_{1,m}^{p+1} &\geq \int_{\{\phi_{1,m}\geq\epsilon\}} a\phi_{1,m}^{p+1} \geq \epsilon^{p+1}\int_{\{\phi_{1,m}\geq\epsilon\}} a\\ &= \epsilon^{p+1}\left(\int_\Omega a - \int_{\{\phi_{1,m}<\epsilon\}} a\right)\\ &\geq \epsilon^{p+1}\left(\int_\Omega a - \|a\|_\infty\left|\{\phi_{1,m}<\epsilon\}\right|\right),\end{aligned}$$

and taking ϵ small enough, we have (3.10). □

Proof of Theorem 3.1

Step 1: Initialization.

From (2.4) and (3.4) it follows immediately that

$$\|u\|_{L^1_{d^m}} = \int_\Omega u\,d^m \leq C\int_\Omega u\,\phi_{1,m} \leq C$$

and

$$\|v\|_{L^1_{d^m}} = \int_\Omega v\,d^m \leq C\int_\Omega v\,\phi_{1,m} \leq C,$$

and therefore, for $n \leq m$,

$$\|u\|_{L^\infty(\Omega)},\ \|v\|_{L^\infty(\Omega)} \leq C$$

is an immediate consequence of (1) in Proposition2.3.

On the other hand, if $n > m$, it follows from Proposition 2.5 that

$$\|u\|_{L^k_{d^m}} + \|v\|_{L^k_{d^m}} \leq C(k) \tag{3.12}$$

for $1 \leq k < \frac{n+m}{n-m}$.

Without loss of generality, we may assume that $q \geq p$ and $\beta > n-m$ (the case $q < p$ is reduced to this case by interchanging u and v).

Then $(p-1)(q+1) \leq pq - 1 < \frac{2m(q+1)}{n-m}$ and follow that $(p-1) < \frac{2m}{n-m}$, i.e. $p < \frac{n+m}{n-m}$. So, there exists some k such that

$$k \geq p \quad \text{and} \quad k \geq \frac{n+m}{n-m} - \epsilon, \tag{3.13}$$

with ϵ to be chosen below, for which (3.12) holds.

Step 2: Bootstrap on the first equation of (1.1).

Let $k_1 \in (k, \infty]$ such that

$$\frac{1}{k_1} > \frac{p}{k} - \frac{2m}{n+m}. \tag{3.14}$$

Then, using Proposition 2.3, we have

$$\|u\|_{L^{k_1}_{dm}} \leq C\,\|(-\Delta)^m u\|_{L^{k/p}_{dm}} \leq C\,\|v^p\|_{L^{k/p}_{dm}} = C\,\|v\|^p_{L^k_{dm}}, \tag{3.15}$$

which is finite because $1 \leq k < \frac{n+m}{n-m}$.

Observe that, if $k > \frac{(n+m)pq}{2m(q+1)}$, we can take $k_1 > \frac{(n+m)q}{2m}$ satisfying (3.14).

Step 3: Bootstrap on the second equation of (1.1).

Assume

$$k_1 > q \tag{3.16}$$

and let $k_2 \in (k_1, \infty]$ be such that

$$\frac{1}{k_2} > \frac{q}{k_1} - \frac{2m}{n+m}. \tag{3.17}$$

From Proposition 2.3, we have

$$\|v\|_{L^{k_2}_{dm}} \leq C\,\|(-\Delta)^m v\|_{L^{k_1/q}_{dm}} \leq C\,\|u^q\|_{L^{k_1/q}_{dm}} = C\,\|u\|^q_{L^{k_1}_{dm}}$$

, which is finite by step 2.

Step 4: Conclusion.

We can choose $\rho \in (0, 1)$ such that (3.12) is true with k/ρ (see Remark below) with k satisfying (3.13) and $k \leq \frac{(n+m)pq}{2m(q+1)}$.

Iterating the procedure, we can reach, after a finite number of steps, some value $\bar{k} > \frac{(n+m)pq}{2m(q+1)}$. Then, it follows from the comment at the end of step 2 that there exists $\bar{k}_1 > \frac{(n+m)q}{2m} \geq \frac{(n+m)p}{2m}$ such that $\|u\|_{L^{\bar{k}_1}_{dm}} \leq C$.

Taking now $k_1 = \bar{k}_1$, we can take $k_2 = \infty$ in step 3 to conclude that $\|v\|_{L^\infty(\Omega)} \leq C$. Analogously, by step 2, we obtain $\|u\|_{L^\infty(\Omega)} \leq C$. □

Remark 3.4 Fulfilment of the bootstrap conditions. We can see that conditions (3.14), (3.16), (3.17) and $\min\{k_1, k_2\} > \frac{k}{\rho}$ for $\rho \in (0, 1)$, to be chosen below, are equivalent to

$$A := \frac{p}{k} - \frac{2m}{n+m} < \frac{1}{k_1} < \min\left\{\frac{\rho}{k}, \frac{1}{q}\right\} \tag{3.18}$$

and

$$\frac{q}{k_1} - \frac{2m}{n+m} < \frac{1}{k_2} < \frac{\rho}{k}. \tag{3.19}$$

Observe now that if

$$k \leq \frac{(n+m)\,pq}{2m(q+1)}, \tag{3.20}$$

we have $A > 0$. Therefore, (3.18) can be solved for $k_1 \in [1, +\infty)$ and with $\frac{1}{k_1}$ arbitrarily closed to A whenever

$$\frac{p-\rho}{k} < \frac{2m}{n+m} \tag{3.21}$$

and

$$\frac{p}{k} - \frac{2m}{n+m} < \frac{1}{q}. \tag{3.22}$$

But, (3.21) holds if ρ satisfies

$$\frac{n-m}{n+m}\,p < \rho < 1, \tag{3.23}$$

and such a ρ exists because $p < \frac{n+m}{n-m}$.

On the other hand, since $\beta = \frac{2m\,(q+1)}{pq-1} > n-m$, we have $\frac{1}{q} > \frac{p\,(n-m)}{n+m} - \frac{2m}{n+m}$. Then, since $k < \frac{n-m}{n+m}$, we can choose ϵ such that (3.22) holds.

Let us now see that condition (3.19) can be fulfilled. Indeed, it is enough to see that all our parameters can be chosen such that

$$\frac{q}{k_1} - \frac{2m}{n+m} < \frac{\rho}{k}. \tag{3.24}$$

Taking $\frac{1}{k_1}$ in (3.18) closed enough to A, we have that (3.24) is equivalent to

$$\rho > 1 - \eta, \tag{3.25}$$

where $\eta := \frac{2m}{n+m}\,(q+1)\,k - (pq-1)$.

Indeed, if $\frac{1}{k_1}$ is closed to $A = \frac{p}{k} - \frac{2m}{n+m}$, then $\frac{q}{k_1} - \frac{2m}{n+m}$ is closed to $\frac{qp}{k} - \frac{2mq}{n+m} - \frac{2m}{n+m}$.

Now, $\rho < 1$ is equivalent to

$$k > \frac{n+m}{\beta}, \tag{3.26}$$

but, since $\beta > n-m$, it is possible to take ϵ small enough in (3.13) such that (3.26) is satisfied.

Finally, we can take $\rho \in (0, 1)$ closed enough to one such that que (3.23) and (3.25) hold.

4 Existence of singular solutions

In order to prove Theorem 3.2, we follow the ideas of [17]. First, we will construct a function $f \in L^1_{d^m}(\Omega)$ such that the corresponding weak solution of the linear problem (2.1) is not bounded.

Recall that our domain Ω is a ball when $n \geq 3$ and smooth perturbations of a ball in the case $n = 2$. In any case, given $x_0 \in \partial\Omega$, there exist $r > 0$ and a revolution cone Σ_1 with vertex x_0 such that $\Sigma := \Sigma_1 \cap B_{2r}(x_0) \subset \Omega$. Now, for $0 < \alpha < n - m$, we define

$$f(x) = |x - x_0|^{-(\alpha+2m)} \chi_\Sigma,$$

where χ_Σ denotes the characteristic function of Σ. Then, it is easy to see that $f \in L^1_{d^m}(\Omega)$.

Let $u > 0$ be the solution of (2.1) with f as right-hand side. Then, we have

$$u(x) = \int_\Omega G_m(x, y)\, |y - x_0|^{-(\alpha+2m)} \chi_\Sigma(y)\, \mathrm{d}y.$$

Using this representation formula together with the estimates of the Green function (3.5), (3.6) and (3.7), it is not difficult to see that, for $x \in \Omega$,

$$u(x) \geq C\, |x - x_0|^{-\alpha} \chi_\Sigma(x). \tag{4.1}$$

Proof of Theorem 3.2 Recall that $\alpha = \frac{2m(p+1)}{pq-1}$ and $\beta = \frac{2m(q+1)}{pq-1}$, and we are assuming $0 < \alpha, \beta < n - m$. We define

$$\phi(x) = |x - x_0|^{-(\alpha+2m)} \chi_\Sigma(x) \quad \text{and} \quad \psi(x) = |x - x_0|^{-(\beta+2m)} \chi_\Sigma(x).$$

Let u and v be nonnegative and such that

$$\begin{cases} (-\Delta)^m u = \phi & \text{in } \Omega \\ (-\Delta)^m v = \psi & \text{in } \Omega \\ \left(\dfrac{\partial}{\partial \nu}\right)^j u = \left(\frac{\partial}{\partial \nu}\right)^j v = 0 & \text{on } \partial\Omega \quad 0 \leq j \leq m - 1. \end{cases}$$

Then, it follows from (4.1) that $u \notin L^\infty(\Omega)$, $v \notin L^\infty(\Omega)$,

$$v(x)^p \geq \left(C\, |x - x_0|^{-\beta} \chi_\Sigma(x)\right)^p = C\, |x - x_0|^{-(\alpha+2m)} \chi_\Sigma(x) = C\, \phi(x)$$

and

$$u(x)^q \geq \left(C\, |x - x_0|^{-\alpha} \chi_\Sigma(x)\right)^q = C\, |x - x_0|^{-(\beta+2m)} \chi_\Sigma(x) = C\, \psi(x).$$

Therefore, defining $a = \phi/v^p$ and $b = \psi/u^q$, we have that a and b are nonnegative bounded functions and (u, v) solves

$$(-\Delta)^m u = a(x)\, v^p \quad \text{and} \quad (-\Delta)^m v = b(x)\, u^q.$$

□

We end the paper by proving the observation given in Remark 2.4 concerning the optimality of condition (2) in Proposition 2.3.

Proposition 4.1 *Assume* $1 \leq p \leq q \leq \infty$ *and* $\frac{1}{p} - \frac{1}{q} > \frac{2m}{n-m}$. *Then, there exists* $f \in L^p_{d^m}(\Omega)$ *such that* $u \notin L^q_{d^m}(\Omega)$, *where* u *is the weak solution of* (2.1).

Proof Let $0 < \alpha < n - m$ and we define, as above, $f(x) = |x - x_0|^{-(\alpha+2m)} \chi_\Sigma(x)$. Then we have

$$\|f\|^p_{L^p_{d^m}(\Omega)} = \int_\Sigma |x - x_0|^{-(\alpha+2m)p}\, d(x)^m\, \mathrm{d}x \leq \int_\Sigma |x - x_0|^{-(\alpha+2m)p+m}\, \mathrm{d}x,$$

and then, since $p < \frac{n+m}{\alpha+2m}$, $f \in L^p_{d^m}(\Omega)$.

But, for $x \in \Sigma$, there exists a positive constant C such that $d(x) \geq C|x - x_0|$, and therefore, it follows from (4.1) that for $q \geq \frac{n+m}{\alpha}$, $u \notin L^q_{d^m}(\Omega)$. To conclude the proof, we observe that, since $\frac{1}{p} - \frac{1}{q} > \frac{2m}{n-m}$, we can choose $\alpha \in (0, n-m)$ such that $\frac{n+m}{q} < \alpha < \frac{n+m}{p-2m}$. □

Finally, let us mention that, to our knowledge, it is not known what happens in general in the limit case $\frac{1}{p} - \frac{1}{q} = \frac{2m}{n-m}$. In the case $p > m+1$, we have proved in [9] that

$$\|u\|_{L^q_{d^m}(\Omega)} \leq C \, \|f\|_{L^p_{d^m}(\Omega)}.$$

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References

1. Brezis, H., Cazenave, T., Martel, Y., Ramiandrisoa, A.: up for $u_t - \Delta u = g(u)$ revisited. Adv. Differ. Equ. **1**(1), 73–90 (1996)
2. Brézis, H., Turner, R.E.L.: On a class of superlinear elliptic problems. Comm. Partial Differ. Equ. **2**(6), 601–614 (1977)
3. Brezis, H., Cabré, X.: Some simple nonlinear PDE's without solutions. Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. 8, **1**(2), 223–262 (1998)
4. Cabré, X., Martel, Y.: Weak eigenfunctions for the linearization of extremal elliptic problems. J. Funct. Anal. **156**(1), 30–56 (1998)
5. Clément, Ph., Sweers, G.: Uniform anti-maximum principle for polyharmonic boundary value problems. Proc. Am. Math. Soc. **129**(2), 467–474 [Electronic] (2001)
6. Dall'Acqua, A., Sweers, G.: Estimates for Green function and Poisson kernels of higher-order Dirichlet boundary value problems. J. Differ. Equ. **205**(2), 466–487 (2004)
7. Dall'Acqua, A., Sweers, G.: On domains for which the clamped plate system is positivity preserving. In: Partial Differential Equations and Inverse Problems, vol. 362 of Contemp. Math. pp. 133–144. Amer. Math. Soc., Providence, RI (2004)
8. Domingos, A.R., Guo, Y.: A note on a Liouville-type result for a system of fourth-order equations in $\mathbb{R}^N$ Electron. J. Differ. Equ. No. 99, pp. 1–20 [electronic], (2002).
9. Durán, R.G. Sanmartino, M., Toschi, M.: Weighted a priori estimates for solutions of $(-\triangle)^m u = f$ with homogeneous dirichlet conditions poisson equation. Analysis in Theory and Applications **26**(4), (2010)
10. García-Huidobro, M., Manásevich, R., Mitidieri, E., Yarur, C.S.: Existence and nonexistence of positive singular solutions for a class of semilinear elliptic systems. Arch. Ration. Mech. Anal. **140**(3), 253–284 (1997)
11. Grunau, H.-Ch., Sweers, G.: Positivity for equations involving polyharmonic operators with Dirichlet boundary conditions. Math. Ann. **307**(4), 589–626 (1997)
12. Hulshof, J., Van der Vorst, R.: Differential systems with strongly indefinite variational structure. J. Funct. Anal. **114**(1), 32–58 (1993)
13. Liu, J., Guo, Y., Zhang, Y.: Liouville-type theorems for polyharmonic systems in $\mathbb{R}^N$. J. Differ. Equ. **225**(2), 685–709 (2006)
14. Mitidieri, E.: A Rellich type identity and applications. Comm. Partial Differ. Equ. **18**(1-2), 125–151 (1993)
15. Mitidieri, E.: Nonexistence of positive solutions of semilinear elliptic systems in $\mathbf{R}^N$. Differ. Integral Equ. **9**(3), 465–479 (1996)
16. Quittner, P., Souplet, Ph.: A priori estimates and existence for elliptic systems via bootstrap in weighted Lebesgue spaces. Arch. Ration. Mech. Anal. **174**(1), 49–81 (2004)
17. Souplet, Ph.: A survey on L^p_δ spaces and their applications to nonlinear elliptic and parabolic problems. Nonlinear Partial Differ. Equ. Appl. **20** GAKUTO Internat. Ser. Math. Sci. Appl. pp. 464–479. Gakkōtosho, Tokyo, (2004)
18. Souplet, Ph.: Optimal regularity conditions for elliptic problems via L^p_δ-spaces. Duke Math. J. **127**(1), 175–192 (2005)
19. Zhang, Y.: A Liouville type theorem for polyharmonic elliptic systems. J. Math. Anal. Appl. **326**(1), 677–690 (2007)