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The reconstruction formula for Banach frames and duality[☆]

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Abstract

We study conditions on a Banach frame that ensures the validity of a reconstruction formula. In particular, we show that any Banach frames for (a subspace of) L_p or $L_{p,q}$ $(1 \le p < \infty)$ with respect to a solid sequence space always satisfies an unconditional reconstruction formula. The existence of reconstruction formulas allows us to prove some James-type results for atomic decompositions: an unconditional atomic decomposition (or unconditional Schauder frame) for X is shrinking (respectively, boundedly complete) if and only if X does not contain an isomorphic copy of ℓ_1 (respectively, c_0).

Keywords: Banach frames; Atomic decompositions; Duality; Reflexivity

Introduction

Banach frames emerged in the theory of frames related to Gabor and Wavelet analysis and were formally introduced in 1991 by Gröchenig [15] as an extension of the notion of frames for Hilbert spaces to the Banach space setting. Before the concept of Banach frames was formalized, it appeared in the foundational work of Feichtinger and Gröchenig [11,12] related

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to atomic decompositions. Loosely speaking, atomic decompositions allow a representation of every element of the space via a series expansion in terms of a fixed sequence of elements, *the atoms*. Banach frames, on the other hand, ensure reconstruction via a bounded *synthesis operator* and, many times, to find an explicit formula for this operator presents additional difficulties. One of our main results (Theorem 3.1) shows that the synthesis operator associated to a wide class of Banach frames, is given by a series expansion with unconditional convergence, whose coefficients depend linearly and continuously on the entry.

Banach frames and atomic decompositions appeared in the field of applied mathematics providing applications to signal processing, image processing and sampling theory among other areas. In the wavelet context, Frazier and Jawerth presented decompositions for Besov spaces in their early work [13], and later for distribution spaces in [14], where a new approach to the traditional atomic decomposition of Hardy spaces can be found. Feichtinger characterized Gabor atomic decomposition for modulation spaces [10] and, at the same time, the general theory was developed in his joint work with Gröchenig [11,12]. Here, the authors show that reconstruction through atomic decompositions are not limited to Hilbert spaces. Indeed, they construct frames for a large class of Banach spaces, namely the *coorbit spaces*. Thereafter, a vast literature was dedicated to the subject (see the references in [6]).

We focus our discussion within the framework of abstract approximation theory in Banach spaces. This allows us to relate the concepts of Banach frames and atomic decomposition to properties of Banach spaces such as separability and reflexivity.

We show that a Banach frame for a Banach space X with respect to a solid space Z (in our terminology, an unconditional Banach frame) admits a reconstruction formula whenever X does not contain a copy of c_0. In this case, the Banach frame automatically defines an unconditional atomic decomposition. This holds for reflexive Banach spaces or spaces with finite cotype. As a consequence, any Banach frame for L_p ($1 \leq p < \infty$) and Lorentz function space $L_{p,q}$ ($1 < p, q < \infty$), or any of their subspaces, with respect to a solid sequence space admits a reconstruction formula. The reconstruction formula for Banach frames is applied to obtain some James-type results (for bases in Banach spaces, see [17]): an unconditional atomic decomposition or Schauder frame for X is shrinking if and only if X does not contain a copy of ℓ_1, and is boundedly complete if and only if X does not contain a copy of c_0. This improves some results of [3,21].

The paper is organized as follows. In the first section, we introduce the basic definitions that will be used throughout. In Section 2, we recall the definitions of shrinking and boundedly complete atomic decompositions, and present some basic duality results. Section 3 is devoted to the main results of the article.

For further information on atomic decompositions and Banach frames see, for example, [6] and the references therein. For an historical survey on some aspects of frame theory for Hilbert spaces see [16] and the references therein. We refer to [9,19,20] for a background in Banach spaces and Banach lattices.

1. Background and generalities

By a Banach sequence space we mean a Banach space of scalar sequences, indexed by $\mathbb{N}$, for which the coordinate functionals are continuous. We say that the space is a Schauder sequence space if, in addition, the unit vectors $\{e_i\}$ given by $(e_i)_j = \delta_{i,j}$ form a basis for it. In this case, a sequence $a = (a_i)$ can be written as $a = \sum_{i=1}^{\infty} a_i e_i$.

We start by recalling the definition of a Banach frame:

Definition 1.1. Let X be a Banach space and Z be a Banach sequence space. Let (x_i') be a sequence in X' and let $S: Z \to X$ be a continuous operator. The pair $((x_i'), S)$ is said to be a Banach frame for X with respect to Z if for all $x \in X$:

(a) $(\langle x_i', x\rangle) \in Z$,
(b) $A\|x\| \leq \|(\langle x_i', x\rangle)\|_Z \leq B\|x\|$, with A and B positive constants,
(c) $x = S(\langle x_i', x\rangle)$.

The operator S is said to be the *synthesis operator*. Conditions (a) and (b) allow the definition of the *analysis operator* $J: X \to Z$, $Jx := (\langle x_i', x\rangle)_i$. The synthesis and analysis operators determine the Banach frame in the following sense: if $((x_i'), S)$ is a Banach frame then, $SJ = id_X$ and $x_i' = J'e_i'$, where $(e_i') \subset Z'$ is the dual basic sequence of (e_i) and $J': Z' \to X'$ is the transpose of J. On the other hand, if $S: Z \to X$ and $J: X \to Z$ are continuous operators such that $SJ = id_X$ then, $((J'e_i'), S)$ is a Banach frame for X with respect to Z (see [3, Remark 1.2] and [4, Page 712]).

Whenever Z is a Schauder sequence space, the continuity of S gives the reconstruction formula for the Banach frame:

$$x = \sum_{i=1}^{\infty} \langle x_i', x\rangle Se_i \tag{1}$$

for all $x \in X$. If the canonical sequence (e_i) does not span Z, the reconstruction formula does not necessarily hold, even for separable Banach spaces X. Let us see a simple example of this:

Example 1.2. Let $X = c_0$ (the space of null sequences) and $Z = c$ (the space of convergent sequences). We consider the following operators:

$$S: Z \to X, \quad Sa := (\ell, a_1 - \ell, a_2 - \ell, \ldots), \text{ where } \ell = \lim_i a_i,$$

and

$$J: X \to Z, \quad Jx := (x_1, x_2 + x_1, x_3 + x_1, \ldots).$$

Note that S and J are bounded with $\|S\| = \|J\| = 2$. If we set $x_1' := e_1'$ and $x_i' := e_1' + e_i'$ for $i \geq 2$, then $Jx = (\langle x_i', x\rangle)$ and $SJ = id_X$, so $((x_i'), S)$ is a Banach frame for c_0 with respect to c. Let us see that the reconstruction formula does not hold for $x = e_1$. Since $Se_i = e_{i+1}$, we have for each n

$$\begin{aligned} \sum_{i=1}^{n} \langle x_i', e_1\rangle Se_i &= \langle x_1', e_1\rangle e_2 + \sum_{i=2}^{n} \langle x_i', e_1\rangle e_{i+1} \\ &= e_2 + e_3 + \cdots + e_n. \end{aligned}$$

Then, $\sum_{i=1}^{\infty} \langle x_i', e_1\rangle Se_i$ does not converge.

One of the purposes of this work is to establish conditions that ensures that the reconstruction formula is satisfied by a Banach frame. A similar but subtly different structure is that of atomic decomposition. The reconstruction formula is imposed as part of the definition, and in return we give up the existence of a linear operator S defined on the whole space Z:

Definition 1.3. Let X be a Banach space and Z be a Banach sequence space. Let (x_i') and (x_i) be sequences in X' and X respectively. We say that $((x_i'), (x_i))$ is an atomic decomposition of X with respect to Z if for all $x \in X$:

(a) $(\langle x'_i, x\rangle) \in Z$,
(b) $A\|x\| \le \|(\langle x'_i, x\rangle)\|_Z \le B\|x\|$, with A and B positive constants,
(c) $x = \sum_{i=1}^{\infty} \langle x'_i, x\rangle x_i$.

The comments above say that a Banach frame with respect to a Schauder sequence space automatically defines an atomic decomposition. Moreover, any Banach frame satisfying a reconstruction formula defines an atomic decomposition.

Let us describe a sort of converse of this statement. A separable Banach space admits an atomic decomposition if and only if it has the bounded approximation property (see [18,22] and also [6, Theorem 2.10]). Moreover, if $((x'_i), (x_i))$ is an atomic decomposition of X with respect to some Banach sequence space Z, it is always possible to find a Schauder sequence space X_d and an operator $S: X_d \to X$ such that $Se_i = x_i$ and $((x'_i), (x_i))$ is also an atomic decomposition of X with respect to X_d. In this case, $((x'_i), S)$ turns out to be a Banach frame for X with respect to X_d. Therefore, we might say that an atomic decomposition defines a Banach frame, as long as we are allowed to change the sequence space involved. Note that the natural inclusion from c_0 into ℓ_∞ defines an atomic decomposition for c_0 with respect to ℓ_∞, but there is no Banach frame for c_0 with respect to ℓ_∞. Therefore, it is sometimes really necessary to change the sequence space.

On the other hand, the Banach frame of Example 1.2 does not define an atomic decomposition, since the reconstruction formula does not hold.

Let $((x'_i), (x_i))$ be an atomic decomposition for X with respect to a Banach sequence space Z. There is a natural procedure that allows us to replace Z by a Schauder sequence space X_d so that $((x'_i), (x_i))$ is also an atomic decomposition of X with respect to X_d (see [6, Theorem 2.6]). For the sake of completeness, we sketch the construction of X_d under the assumption that x_i is nonzero, for all i, since this assumption avoids some technicalities. Consider c_{00} the linear space of scalar finite support sequences with unit vectors (e_i) endowed with the norm:

$$\left\| \sum_i a_i e_i \right\| = \sup_n \left\| \sum_{i=1}^n a_i x_i \right\|_X .$$

Now, define X_d as the completion of c_{00} with the norm given above. In fact,

$$X_d = \left\{ (a_i) \Big/ \sum_{i=1}^{\infty} a_i x_i \text{ converges} \right\} \tag{2}$$

and $((x'_i), (x_i))$ turns out to be an atomic decomposition of X with respect to X_d. We will call this Schauder sequence space *the canonical associated Schauder space* to the corresponding atomic decomposition. We also remark that Theorem 2.6 of [6] (or the existence of X_d) implies that a Banach space admits an atomic decomposition if and only if it is complemented in a Banach space with an basis.

One of the advantages of working with Banach frames or atomic decomposition is that these structures have a nicer behavior than that of bases with respect to subspaces. First, note that if $((x'_i), S)$ is a Banach frame for X with respect to Z then X is isomorphic to a complemented subspace of Z. This property relies on the simple fact that $SJ = id_X$ and therefore JS is the desired projection. We also have:

Remark 1.4. Let X be a Banach space and Z be a Banach sequence space. Suppose $(x'_i) \subset X'$ satisfies properties (a) and (b) of Definition 1.3 and let $P: X \to X$ be a projection then,

(i) if $((x_i'), S)$ is a Banach frame for X with respect to Z then, $((P'x_i'), PS)$ is a Banach frame for the space PX with respect to Z.
(ii) if $((x_i'), (x_i))$ is an atomic decomposition for X with respect to Z then, $((P'x_i'), (Px_i))$ is an atomic decomposition for the space PX with respect to Z.

Proof. Let $((x_i'), S)$ be a Banach frame. Since $\langle P'x_i', x\rangle = \langle x_i', Px\rangle = \langle x_i', x\rangle$ for all $x \in PX$ and for all i, the sequence $(\langle P'x_i', x\rangle)$ belongs to Z, for all $x \in PX$.

Also, if $x \in PX$, we have that $\|(\langle P'x_i', x\rangle)\|_Z = \|(\langle x_i', x\rangle)\|_Z$ and $S(\langle P'x_i', x\rangle) = S(\langle x_i', x\rangle) = x$. Thus, statement (i) is proved. Now, take $((x_i'), (x_i))$ an atomic decomposition. To prove (ii) it only remains to show that the reconstruction formula holds. Indeed, we have

$$Px = P(Px) = P\left(\sum_{i=1}^{\infty}\langle x_i', Px\rangle x_i\right) = \sum_{i=1}^{\infty}\langle P'x_i', Px\rangle Px_i. \quad \square$$

In most applications, sequence spaces associated to a Banach frame are solid. Recall that a Banach sequence space Z is called solid if for any pair of sequences $a = (a_i)$ and $b = (b_i)$ with $a \in Z$ and such that $|b_i| \leq |a_i|$ for all i, we have that $b \in Z$ and $\|b\| \leq \|a\|$. Classical examples of solid sequence spaces are c_0, ℓ_p and Lorentz and Orlicz sequence spaces. Solid sequence spaces are Banach lattices modeled over the natural numbers, with the coordinatewise order, and are also called Köthe sequence spaces, or normal Banach sequence spaces. For the theory of Banach lattices we refer to [2,20]. Any Banach space with a 1-unconditional basis is a solid space and those with arbitrary unconditional basis can be renormed to be solid. On the other hand, if Z is a solid sequence space, then the canonical unit vectors form a 1-unconditional basic sequence. This motivates the following:

Definition 1.5. A Banach frame with respect to a solid sequence space is said to be an unconditional Banach frame.

Note that in our definition of unconditional Banach frame we do not require the solid sequence space to have a basis. Unconditional Banach frames have a natural counterpart in the atomic decomposition framework: unconditional atomic decompositions. This last concept introduced and studied in [3] is equivalent to that of "framing for Banach spaces" given in [6] and of "unconditional frame" given in [5,21] (see the comments at the end of this section).

Definition 1.6. An atomic decomposition $((x_i'), (x_i))$ for X with respect to a Banach sequence space Z is said to be unconditional if for any $x \in X$, its series expansion $\sum_{i=1}^{\infty}\langle x_i', x\rangle x_i$ converges unconditionally, that is

$$x = \sum_i \langle x_i', x\rangle x_i,$$

with unconditional convergence.

It is known that if a series $\sum_i x_i$ converges unconditionally, then, for every bounded sequence of scalars $\{a_i\}$, the series $\sum_i a_i x_i$ converges and the operator $\ell_\infty \to X$ defined by $(a_i) \mapsto \sum_i a_i x_i$ is a bounded linear operator (see [19], page 16). Thus, a repeated use of the uniform boundedness principle (or a single application of the bilinear Banach–Steinhaus theorem [8, Ex 1.11(a)]) shows:

Remark 1.7. Let $((x_i'), (x_i))$ be an unconditional atomic decomposition for X. Then the mapping

$$B: X \times \ell_\infty \to X, \quad B(x, a) := \sum_i a_i \langle x_i', x \rangle x_i$$

is a well defined bounded bilinear operator.

The norm of the bilinear operator B defined above can be seen as an unconditional constant for the atomic decomposition $((x_i'), (x_i))$. Equivalent constants have been introduced in [6,5].

If $((x_i'), (x_i))$ is an unconditional atomic decomposition for X, it is always possible to find a solid sequence space with Schauder basis X_d and an operator $S: X_d \to X$ such that $Se_i = x_i$ and $((x_i'), (x_i))$ is also an unconditional atomic decomposition of X with respect to X_d. The construction is similar to (2) and can be found in [6, Theorem 3.6]. Assuming again that $x_i \neq 0$ for all i, the solid sequence space with Schauder basis X_d is

$$X_d := \left\{ (a_i) \Big/ \sum_i a_i x_i \text{ converges unconditionally} \right\}, \tag{3}$$

endowed with the norm $\|(a_i)\|_{X_d} := \sup_{b \in B_{\ell_\infty}} \|\sum_i b_i a_i x_i\|$. We will refer to this space as *the canonical solid Schauder space* associated to the corresponding atomic decomposition. Therefore, an unconditional atomic decomposition defines a Banach frame with respect to some solid Schauder sequence space. Conversely, a Banach frame with respect to a solid Schauder sequence space defines an unconditional atomic decomposition. Moreover, [6, Theorem 3.6] says that a Banach space admits an unconditional atomic decomposition if and only if it is complemented in a Banach space with an unconditional basis.

Most of the properties of atomic decompositions we will study are independent of the associated sequence space. Also, the construction of the canonical Schauder spaces (2) and (3) associated to an atomic decomposition, *only* involve the reconstruction formulas and not the original sequence space. Therefore, unless specific properties of the associated space are required, we will talk about atomic decompositions without reference to any sequence space (having in mind, if necessary, the canonical sequence spaces associated to the decomposition). When the Banach sequence space is disregarded, the concept of (unconditional) atomic decomposition is equivalent to that of (unconditional) Schauder frame in the sense of [5].

2. Some remarks on duality for atomic decompositions

In order to relate atomic decomposition to duality properties of Banach spaces, the notion of shrinking atomic decomposition was introduced in [3]. Before we recall the definition, consider the linear operators $T_N: X \to X$ by $T_N(x) = \sum_{i \geq N} \langle x_i', x \rangle x_i$. These operators are uniformly bounded by the uniform boundedness principle. With this notation we have:

Definition 2.1. Let $((x_i'), (x_i))$ be an atomic decomposition of X. We say that $((x_i'), (x_i))$ is shrinking if for all $x' \in X'$

$$\|x' \circ T_N\| \longrightarrow 0. \tag{4}$$

As a matter of fact, the definition in [3] was referred to as an atomic decomposition with respect to a concrete Banach sequence space Z. However, it must be noted that the condition (4) is independent of the associated sequence space. In general, $T_N \circ T_M$ may be different from

$T_{\min(N,M)}$, which means that T_N is not a projection on the closure of $[x_i : i \geq N]$. This shows one of the main differences between atomic decompositions and bases. Indeed, the definition above is not equivalent to $\|x'|_{[x_i:i\geq N]}\|$ going to 0 for every $x' \in X'$ (see [3] for details). For some particular atomic decompositions, which in fact are simultaneously Banach frames, it is shown in [3, Theorem 1.4] that shrinking atomic decompositions behave as shrinking Schauder bases in the following sense: suppose X_d is a Schauder sequence space with basis (e_i) and there exists a synthesis operator $S: X_d \to X$ such that $Se_i = x_i$ and $((x_i'), (Se_i))$ is an atomic decomposition of X with respect to X_d. Then the dual pair $((Se_i), (x_i'))$ is an atomic decomposition for X' with respect to $(X_d)'$ if and only if $((x_i'), (Se_i))$ is shrinking. Let us see how we can extend this result to arbitrary atomic decompositions:

Proposition 2.2. *Let X be a Banach space and $((x_i'), (x_i))$ be an atomic decomposition for X. The following are equivalent:*

(i) *the pair $((x_i'), (x_i))$ is shrinking,*
(ii) *the pair $((x_i), (x_i'))$ is an atomic decomposition for X',*
(iii) *for all $x' \in X'$, $\sum_{i=1}^{\infty} \langle x', x_i\rangle x_i'$ is convergent.*

Proof. For (i) $\Rightarrow$ (ii), consider X_d the canonical Schauder space associated to $((x_i'), (x_i))$ presented in (2). As we have mentioned, $((x_i'), (x_i))$ is shrinking as an atomic decomposition with respect to X_d, since the definition of shrinking atomic decomposition is independent of the sequence space. Then the result follows from [3, Theorem 1.4]. The implication (ii) $\Rightarrow$ (iii) is immediate and (iii) $\Rightarrow$ (i) follows directly from the equality $\|x' \circ T_N\| = \left\|\sum_{i=N}^{\infty} \langle x', x_i\rangle x_i'\right\|$. □

As a consequence of this result, a Banach space admitting a shrinking atomic decomposition has necessarily a separable dual. In particular, ℓ_1 does not admit such a decomposition. Also, Proposition 2.2 shows the equivalence between the notion of shrinking atomic decomposition and the concept of pre-shrinking atomic decomposition given in [21].

Another concept related to duality is that of boundedly complete atomic decomposition, which was introduced in [3] and is a natural extension of the definition of boundedly complete Schauder basis (in [21], this last concept is defined as "pre-boundedly complete").

Definition 2.3. Let X be a Banach space and let $((x_i'), (x_i))$ be an atomic decomposition of X. The atomic decomposition is said to be boundedly complete if for each $x'' \in X''$, the series $\sum_{i=1}^{\infty} \langle x'', x_i'\rangle x_i$ converges in X.

We have already mentioned that "admitting an atomic decomposition" is a property that is inherited by complemented subspaces (Remark 1.4). The same happens if we require the atomic decomposition to be shrinking or boundedly complete, this fact will be used later:

Remark 2.4. Let $((x_i'), (x_i))$ be an atomic decomposition of X and let $P: X \to X$ be a continuous linear projection. If $((x_i'), (x_i))$ is boundedly complete (shrinking), then $((P'x_i'), (Px_i))$ is also a boundedly complete (shrinking) atomic decomposition for PX.

Proof. Put $Y = PX$ and let us show the statement for complete boundedness. Given $y'' \in Y''$, consider $\sum_{i=1}^{\infty} \langle P''y'', x_i'\rangle x_i$ which converges since $((x_i'), (x_i))$ is boundedly complete. Then, $\sum_{i=1}^{\infty} \langle y'', P'x_i'\rangle Px_i = P\left(\sum_{i=1}^{\infty} \langle P''y'', x_i'\rangle x_i\right)$ is also convergent. The arguments for shrinking atomic decompositions are similar. □

If an atomic decomposition is modeled on a Schauder sequence space X_d with a boundedly complete basis, then the atomic decomposition is boundedly complete. It is easy to find an example to show that the converse of this result is false. Indeed, take X a reflexive Banach space with basis (f_i). Consider $X_d = X \oplus c_0$ with the basis which alternates the elements f_i with the elements of the canonical basis of c_0. The natural inclusion $J : X \hookrightarrow X_d$ and projection $S : X_d \to X$ define a boundedly complete atomic decomposition, but clearly the basis (e_i) of X_d is not boundedly complete.

The following remark shows that not every separable Banach space admits a boundedly complete atomic decomposition (take, for instance, $X = c_0$).

Remark 2.5. Let X be a Banach space with a boundedly complete atomic decomposition. Then, X is complemented in its bidual X''.

Proof. Let $((x_i'), (x_i))$ be a boundedly complete atomic decomposition for X. By the Banach-Steinhaus theorem, the following mapping is well defined and bounded:

$$P: X'' \to X, \quad Px'' := \sum_{i=1}^{\infty} \langle x'', x_i' \rangle x_i.$$

Now, the reconstruction formula says that P is the desired projection. $\square$

A kind of converse of the previous result holds for unconditional atomic decompositions (see Corollary 3.5).

3. The reconstruction formula and the James-type results

The following result provides us with a sufficient condition to ensure reconstruction formulas for unconditional Banach frames. The proof is based on that of a result for Banach lattices given in [20, Proposition 1.c.6].

Theorem 3.1. *Let $((x_i'), S)$ be an unconditional Banach frame for X. If X does not contain an isomorphic copy of c_0, then we have reconstruction formula for $((x_i'), S)$. More precisely, if (e_i) denotes the sequence of the canonical unit vectors of the solid sequence space Z associated to the frame, then for all $x \in X$ we have*

$$x = \sum_i \langle x_i', x \rangle S e_i$$

unconditionally. Equivalently, $((x_i'), (Se_i))$ is an unconditional atomic decomposition for X.

Proof. Taking a subspace of Z if necessary, we may assume that $Se_i \neq 0$ for all i. Now, we follow the ideas of the proof of [20, Proposition 1.c.6]. We consider in Z the following semi-norm

$$|||a||| := \sup_{|b| \le |a|} \|Sb\|_X.$$

Note that if $|||a||| = 0$, since $|a_i e_i| \le |a|$ we have that $\|S(a_i e_i)\| = 0$ and then $a_i = 0$ for all i. Thus, $|||\cdot|||$ is indeed a norm. Let $\widetilde{Z}$ be the completion of Z with this norm and let $\iota: (Z, \|\cdot\|_Z) \to (\widetilde{Z}, |||\cdot|||)$ be the natural inclusion. Note that if $|b| \le |a|$ then $\|Sb\|_X \le \|S\| \, \|b\| \le \|S\| \, \|a\|$ and $\iota: Z \to \widetilde{Z}$ is bounded with $\|\iota\| \le \|S\|$. It is easy to see that $\tilde{Z}$ is a solid sequence space, its canonical unit vectors being $\tilde{e}_i := \iota(e_i)$.

The subspace $J(X)$ is complemented in Z and isomorphic to X, then $J(X)$ does not contain a subspace isomorphic to c_0. Since our construction of $\widetilde{Z}$ coincides with that of [20, Proposition 1.c.6], we are in conditions to ensure that $\widetilde{Z}$ is order continuous. In this case, the unit vectors form a basis for $\widetilde{Z}$. Indeed, since $\widetilde{Z}$ is solid, it is enough to show that every $a \in \widetilde{Z}$ with $a \geq 0$ belongs to $\overline{\mathrm{gen}}\{\tilde{e}_i\}$. But for such a, the sequence $a - \sum_{i=1}^{N} a_i \tilde{e}_i$ decreases to 0 in order and, by order continuity, in norm. We have then seen that $\widetilde{Z}$ is a Schauder sequence space with an unconditional basis.

Now, we may consider θ the restriction of ι to $J(X)$ and put $\widetilde{Y} = \theta J(X)$ obtaining a subspace of $\widetilde{Z}$ isomorphic to X. We have the following commutative diagram:

$$\begin{tikzcd}
& Z \arrow[dl, "S"'] \arrow[r, "\iota"] & \widetilde{Z} \arrow[d, "\widetilde{S}"] \\
X \arrow[r, "J"] & JX \arrow[u, hook] \arrow[r, "\theta"] & \widetilde{Y}
\end{tikzcd}$$

where $\widetilde{S}$ is defined on $\iota(Z)$ by $\widetilde{S}\iota = \theta J S$ and is then extended by continuity and density to $\widetilde{Z}$ (the continuity of $\widetilde{S}$ on $\iota(Z)$ follows from the definition of $|||\cdot|||$).

We claim that $((x_i'), S\theta^{-1}\widetilde{S})$ is a Banach frame for X with respect to $\widetilde{Z}$. If that is the case, since $S\theta^{-1}\widetilde{S}(e_i) = S(e_i)$ and $\widetilde{Z}$ is a Schauder sequence space, we would have the desired result. As $(\langle x_i', x\rangle) \in Z$ and every sequence in Z belongs to $\widetilde{Z}$, condition (a) of the definition of Banach frame holds. Also we have

$$\|x\| = \|S((\langle x_i', x\rangle))\| \leq |||(\langle x_i', x\rangle)||| = |||\iota(Jx)||| \leq \|\iota\|\,\|J\|\,\|x\| \leq \|S\|\,\|J\|\,\|x\|$$

and the second condition is also satisfied. Finally, $S\theta^{-1}\widetilde{S}(\langle x_i', x\rangle) = S\theta^{-1}\widetilde{S}(\iota(Jx)) = x$ gives the third condition. $\square$

Note that Theorem 3.1 applies, for example, to reflexive Banach spaces, or Banach spaces with finite cotype. In particular, any unconditional Banach frame for a subspace of L_p or of a Lorentz function space $L_{p,q}$ $(1 \leq p < \infty)$ has automatically a reconstruction formula (see [7] for cotype of Lorentz function spaces $L_{p,q}$). Analogous results can be obtained for many Lorentz or Orlicz functions spaces, the cotype of which are widely studied.

Theorem 1.c.7 in [20] asserts that given Y a complemented subspace of a Banach lattice, Y is reflexive if and only if no subspace of Y is isomorphic to c_0 or to ℓ_1. From this result and the comments previous to Remark 1.4 we have, in particular, that if X has an unconditional Banach frame, then X is reflexive if and only if X does not contain a copy of c_0 or ℓ_1. On the other hand, if a Banach space admits an atomic decomposition which is both shrinking and boundedly complete, then it is reflexive [3, Proposition 2.4]. The converse holds under the additional assumption that the reflexive space admits an unconditional atomic decomposition [3, Theorem 2.5]. We combine and rephrase these results as:

Remark 3.2. Let X be a Banach space which admits an unconditional atomic decomposition $((x_i'), (x_i))$. Then, the following are equivalent:

(i) $((x_i'), (x_i))$ is shrinking and boundedly complete,
(ii) X does not contain a copy of c_0 or ℓ_1,
(iii) X is reflexive.

Our goal now is to show that, just as in the Schauder basis context, we can split the equivalence (i) $\Leftrightarrow$ (ii) in the previous remark into two independent results. Note that in the previous remark and also the following results, the words "atomic decomposition" can be readily replaced by "Schauder frame". As a consequence, we improve some results in [21]. First we have:

Theorem 3.3. *Let X be a Banach space which admits an unconditional atomic decomposition $((x_i'), (x_i))$. Then, $((x_i'), (x_i))$ is shrinking if and only if X does not contain a copy of ℓ_1.*

Proof. Suppose $((x_i'), (x_i))$ is shrinking, by Proposition 2.2 $((x_i), (x_i'))$ is an atomic decomposition for X' with respect to some Banach sequence space, in particular, X' is separable. Then, X contains no subspace isomorphic to ℓ_1.

Conversely, suppose X does not admit a copy of ℓ_1. Let X_d be the canonical associated solid space respect to $((x_i'), (x_i))$ with synthesis operator S. Note that $(X_d)' = X_d^{\times}$, the dual of Köthe, then it is a sequence space and we may consider the coordinate functions (e_i''). Let $J: X \to X_d$ be the analysis operator. Since $J'S' = id_{X'}$, $((S''e_i''), J')$ is a Banach frame for X' with respect to $(X_d)'$. Now, $S''e_i'' = Se_i$. Indeed, for all $x' \in X'$ we have that

$$\langle S''e_i'', x'\rangle = \langle e_i'', S'x'\rangle = \langle S'x', e_i\rangle = \langle x', Se_i\rangle.$$

Thus, $((Se_i), J')$ is a Banach frame for X' respect to some solid space. Since X contains no copy of ℓ_1, X' contains no copy of c_0 ([19, Proposition 2.e.8]). Therefore, by Theorem 3.1, $((Se_i), (J'e_i'))$ is an unconditional atomic decomposition for X'. Moreover, we have $x_i = Se_i$ and $x_i' = J'e_i'$, then we obtain the reconstruction formula $x' = \sum_i \langle x', x_i\rangle x_i'$ for all $x' \in X'$. Finally, by Proposition 2.2, $((x_i'), (x_i))$ is shrinking. □

Regarding the containment of c_0, we obtain:

Theorem 3.4. *Let X be a Banach space which admits an unconditional atomic decomposition $((x_i'), (x_i))$. Then, $((x_i'), (x_i))$ is boundedly complete if and only if X does not contain a copy of c_0.*

Proof. Suppose that X contains a copy of c_0. Then, X being separable, by Sobczyc's theorem ([1, Theorem 2.5.8]), there exists a projection $P: X \to X$ such that $P(X)$ is isomorphic to c_0. If $((x_i'), (x_i))$ were boundedly complete, then, by Remark 2.4, $((P'x_i'), (Px_i))$ should be a boundedly complete atomic decomposition for $P(X)$. This fact contradicts Remark 2.5.

Conversely, suppose that $((x_i'), (x_i))$ is not boundedly complete. Then, there exists $x'' \in X''$ such that $\sum_{i=1}^{\infty} \langle x'', x_i'\rangle x_i$ is nonconvergent. In other words, we can find $\delta > 0$ and two sequences of positive integers (p_i), (q_i), so that $p_1 < q_1 < p_2 < q_2 < p_3 < q_3 < \cdots$ and

$$\left\| \sum_{i=p_j}^{q_j} \langle x'', x_i'\rangle x_i \right\| \geq \delta, \quad \text{for all } j.$$

Take $y_j = \sum_{i=p_j}^{q_j} \langle x'', x_i'\rangle x_i$. We will show that c_0 is embeddable in X. First, let us see that there exists $c > 0$ so that for any positive integer N and any choice of scalars $a_1, \ldots, a_N$, we have:

$$\left\| \sum_{j=1}^{N} a_j y_j \right\| \leq c \max_{1 \leq j \leq N} |a_j|. \tag{5}$$

Fix $\varepsilon > 0$, by Goldstine's lemma, given $N \in \mathbb{N}$, we can find $x_N \in X$ such that $\|x_N\| \le \|x''\|$ and

$$\left\| \sum_{j=1}^{N} a_j y_j \right\| = \left\| \sum_{i=1}^{M} b_i \langle x'', x_i' \rangle x_i \right\| \le \left\| \sum_{i=1}^{M} b_i \langle x_N, x_i' \rangle x_i \right\| + \varepsilon,$$

where b_i is a_j for some j or 0. Now, by Remark 1.7 we have that

$$\left\| \sum_{i=1}^{M} b_i \langle x_N, x_i' \rangle x_i \right\| \le \|b\|_\infty \|x_N\| \le \|a\|_\infty \|x''\|.$$

Thus, we obtain (5) for $c = \|x''\|$. Since $\|y_j\| > \delta$, by the Bessaga–Pelczynski theorem, it only remains to show that $y_j \stackrel{w}{\to} 0$. If this were not the case, passing to a subsequence if necessary, we may assume that there exists $x_0' \in X'$ so that $|\langle x_0', y_j \rangle| \ge 1$ for all j. Now, take $b_j = sign(\langle x_0', y_j \rangle)$,

$$N \le \sum_{i=1}^{N} |\langle x_0', y_j \rangle| = \left| \sum_{i=1}^{N} b_j \langle x_0', y_j \rangle \right| \le \|x_0'\| \left\| \sum_{i=1}^{N} b_j y_j \right\| \le c\|x_0'\| \quad \text{for all } N,$$

which is a contradiction. Then, $y_j \stackrel{w}{\to} 0$ and we have that X admits a copy of c_0 by a direct application of [2, Theorem 14.2]. □

As a consequence, we have the converse of Remark 2.5 for spaces with unconditional atomic decompositions. Indeed, if X is complemented in its bidual, it cannot contain c_0 (since, by Sobczyc theorem this copy would be complemented, and this would provide a projection from ℓ_∞ to c_0). So we have:

Corollary 3.5. *Let X be a Banach space with a unconditional atomic decomposition. Then, X is complemented in its bidual if and only if the atomic decomposition is boundedly complete (if and only if X does not contain a copy of c_0).*

For Banach frames, we have an analogous result, which follows from Corollary 3.5 and Theorem 3.1:

Corollary 3.6. *Suppose X admits an unconditional Banach frame. Then, X is complemented in its bidual if and only if X does not contain a copy of c_0.*

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References

[1] F. Albiac, N. Kalton, Topics in Banach Space Theory, in: Graduate Texts in Mathematics, vol. 233, Springer, New York, 2006.
[2] C. Aliprantis, O. Burkinshaw, Positive Operators, in: Pure and Applied Mathematics, vol. 119, Academic Press, Inc., Orlando, FL, 1985.
[3] D. Carando, S. Lassalle, Duality, reflexivity and atomic decompositions in Banach spaces, Studia Math. 191 (1) (2009).

[4] P. Casazza, O. Christensen, D. Stoeva, Frame expansions in separable Banach spaces, J. Math. Anal. Appl. 307 (2005) 710–723.
[5] P.G. Casazza, S.J. Dilworth, E. Odell, Th. Schlumprecht, A. Zsák, Coefficient quantization for frames in Banach spaces, J. Math. Anal. Appl. 348 (1) (2008) 66–86.
[6] P. Casazza, D. Han, D. Larson, Frames for Banach spaces, in: The Functional and Harmonic Analysis of Wavelets and Frames, San Antonio, TX, 1999, in: Contemp. Math., vol. 247, Amer. Math. Soc., Providence, RI, 1999, pp. 149–182.
[7] J. Creekmore, Type and cotype in Lorentz Lpq spaces, Indag. Math. Proc. A 84 (2) (1981) 145–152.
[8] A. Defant, K. Floret, Tensor Norms and Operator Ideals, in: North-Holland Mathematics Studies, vol. 176, North-Holland Publishing Co., Amsterdam, 1993.
[9] J. Diestel, Sequences and Series, in: Graduate Texts in Mathematics, vol. 92, Springer-Verlag, New York, 1984.
[10] H. Feichtinger, Atomic characterizations of modulation spaces through Gabor-type representations, Rocky Mountain J. Math. 19 (1) (1989) 113–125.
[11] H. Feichtinger, K. Gröchenig, Banach spaces related to integrable group representations and their atomic decompositions. I, J. Funct. Anal. 86 (2) (1989) 307–340.
[12] H. Feichtinger, K. Gröchenig, Banach spaces related to integrable group representations and their atomic decompositions. II, Monatsh. Math. 108 (2–3) (1989) 129–148.
[13] M. Frazier, B. Jawerth, Decomposition of Besov spaces, Indiana Univ. Math. J. 34 (4) (1985) 777–799.
[14] M. Frazier, B. Jawerth, A discrete transform and decompositions of distribution spaces, J. Funct. Anal. 93 (1) (1990) 34–170.
[15] K. Gröchenig, Describing functions: atomic decompositions versus frames, Monatsh. Math. 112 (1) (1991) 1–42.
[16] C. Heil, History and evolution of the density theorem for Gabor frames, J. Fourier Anal. Appl. 13 (2) (2007) 113–166.
[17] R. James, Bases and reflexivity of Banach spaces, Ann. of Math. 52 (2) (1950) 518–527.
[18] W.B. Johnson, H.P. Rosenthal, M. Zippin, On bases, finite dimensional decompositions and weaker structures in Banach spaces, Israel J. Math. 9 (1971) 488–506.
[19] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces I. Sequence Spaces, in: Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 92, Springer-Verlag, Berlin, New York, 1977.
[20] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces II. Function Spaces, in: Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 97, Springer-Verlag, Berlin, New York, 1979.
[21] R. Liu, On shrinking and boundedly complete schauder frames of Banach spaces, J. Math. Anal. Appl. 365 (1) (2010) 385–398.
[22] A. Pelczynski, Any separable Banach space with the bounded approximation property is a complemented subespace of a Banach space with a basis, Studia Math. 40 (1971) 239–243.