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PERIODIC SOLUTIONS OF RESONANT SYSTEMS WITH RAPIDLY ROTATING NONLINEARITIES

Pablo Amster

Departamento de Matemática, Universidad de Buenos Aires and CONICET
Ciudad Universitaria, Pabellón I, (1428) Buenos Aires, Argentina

Mónica Clapp

Instituto de Matemáticas, Universidad Nacional Autónoma de México
Circuito Exterior, C.U., 04510 México D.F., Mexico

Abstract. We obtain existence of T-periodic solutions to a second order system of ordinary differential equations of the form

$$u'' + cu' + g(u) = p$$

where $c \in \mathbb{R}$, $p \in C(\mathbb{R}, \mathbb{R}^N)$ is T-periodic and has mean value zero, and $g \in C(\mathbb{R}^N, \mathbb{R}^N)$ is e.g. sublinear. In contrast with a well known result by Nirenberg [6], where it is assumed that the nonlinearity g has non-zero uniform radial limits at infinity, our main result allows rapid rotations in g.

1. Introduction.

In [4] Lazer considered the periodic problem for the scalar differential equation

$$x'' + cx' + g(x) = p(t), \tag{1}$$

where c is any constant and $p(t)$ is a continuous T-periodic function with zero average. As a particular case of his main result, existence of a T-periodic solution of equation (1) follows when $g : \mathbb{R} \to \mathbb{R}$ is bounded, continuous, and satisfies

$$g(x) > 0 > g(-x) \tag{2}$$

for $x > 0$ sufficiently large.

When one interprets the equation as an oscillator, condition (2) means that outside a compact set the force $-g(x)$ points everywhere toward the origin. The boundedness condition is assumed in order to avoid the linear resonance occurring at $c = 0$ and $g(x) = \lambda_n x$, $n = 1, 2, \ldots$, where $\lambda_n = \left(\frac{2\pi n}{T}\right)^2$ is the n-th eigenvalue of the T-periodic problem for the linear operator $Lx = -x''$.

The preceding result admits an immediate generalization to systems. Indeed, if we consider (1) as a system in $\mathbb{R}^N$, where the continuous T-periodic function $p(t)$ is vector valued with zero average and $g = (g_1, \ldots, g_N)$ is a bounded continuous map of $\mathbb{R}^N$, then condition (2) may be replaced by

$$g_k(x_1, \ldots, x_k, \ldots, x_N) > 0 > g_k(x_1, \ldots, -x_k, \ldots, x_N) \tag{3}$$

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for $x_k > 0$ sufficiently large and $k = 1, \dots, N$. The existence of a T-periodic solution follows from the main theorem in [5], which extends Lazer's result to $N > 1$ and applies, in particular, to the case of weakly coupled systems. Many other extensions of (2) were discussed in the literature around the seventies.

From a topological point of view condition (2) says two different things: firstly, that g does not vanish outside a compact set; secondly, that its Brouwer degree over the interval $(-R, R)$ is different from zero when R is large. Thus, one might believe that a natural extension of (2) to $\mathbb{R}^N$ could be to require that

$$g(x) \neq 0 \qquad \text{for } |x| \geq R \tag{4}$$

and

$$\deg(g, B_R(0), 0) \neq 0, \tag{5}$$

where deg refers to the Brouwer degree and $B_R(0)$ is the open ball of radius R centered at the origin.

This possible extension was analyzed by Ortega and Sánchez in [8], where they constructed an example showing that (4) and (5) are not sufficient to guarantee the existence of a periodic solution. The pathological g rotates very fast as x moves in some specific directions.

Motivated by this observation, the following result, which follows from the main theorem in the work of Ruiz and Ward [10], can be regarded as an extension of the preceding results.

We write $B_r(v) := \{x \in \mathbb{R}^N : |x - v| < r\}$ and $\overline{B}_r(v)$ for its closure, and $\text{co}(A)$ for the convex hull of a subset A of $\mathbb{R}^N$. We denote by $C_T(\mathbb{R}, \mathbb{R}^N)$ the space of T-periodic functions $u : \mathbb{R} \to \mathbb{R}^N$ with the uniform norm $\|\cdot\|_\infty$, and the mean value of u by

$$\overline{u} := \frac{1}{T} \int_0^T u(t)\, dt.$$

Theorem 1.1. *Let $c \in \mathbb{R}$ and assume that $g \in C(\mathbb{R}^N, \mathbb{R}^N)$ is bounded and satisfies the following condition:*
For each $r > 0$ there exists $R > r$ such that

$$0 \notin \text{co}(g(\overline{B}_r(v))) \qquad \text{if } \; v \in \mathbb{R}^N \text{ and } |v| = R \tag{6}$$

and

$$\deg(g, B_R(0), 0) \neq 0.$$

Then, for each $p \in C_T(\mathbb{R}, \mathbb{R}^N)$ with $\overline{p} = 0$, there exists a T-periodic solution of problem (1).

The role of condition (6) is easily understood when one attempts to solve problem (1) using Leray-Schauder degree methods. Indeed, the key step for proving Theorem 1.1 consists in showing that, for $0 < \lambda \leq 1$, equation

$$u'' + cu' + \lambda g(u) = \lambda p(t) \tag{7}$$

has no T-periodic solution on $\partial\Omega$, where

$$\Omega := \{u \in C_T(\mathbb{R}, \mathbb{R}^N) : \|u - \overline{u}\|_\infty < r, \;\; |\overline{u}| < R\}$$

for some accurate r and the corresponding R given by condition (6).

An appropriate value of r is obtained after observing that, if u is T-periodic and satisfies (7), then

$$\|u'\|_\infty \leq k \left(\|p\|_{L^1(0,T)} + T\|g\|_\infty \right)$$

for some constant k, independent of the data. Thus, the choice of any value $r > kT(\|p\|_{L^1(0,T)}+T\|g\|_\infty)$ provides an a priori bound for $\|u-\overline{u}\|_\infty$. Then, if we assume that $|\overline{u}| = R$, a contradiction is obtained in the following way: since the convex hull of $g(\overline{B}_r(\overline{u}))$ is compact, the geometric version of the Hahn-Banach theorem asserts that there exists a hyperplane H passing through the origin such that $g(\overline{B}_r(\overline{u})) \subset \mathbb{R}^N\backslash H$. As $\|u-\overline{u}\|_\infty < r$, we conclude that $g(u(t))$ remains on the same side of H for every value of t, which contradicts the fact that $\int_0^T g(u(t))\,dt = \int_0^T p(t)\,dt = 0$.

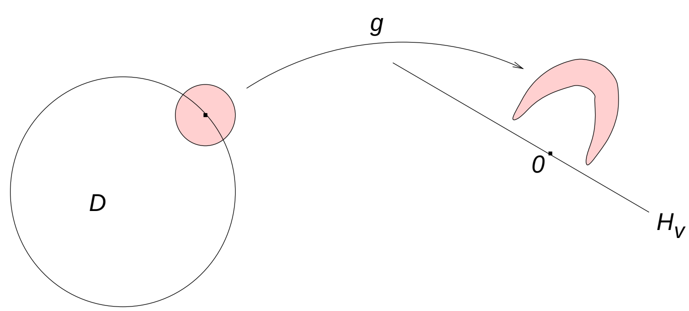
Note that, if one fixes p and chooses r as before, the role of $B_R(0)$ may be assumed by a more general domain $D \subset \mathbb{R}^N$ if conditions (6) and (5) are replaced respectively by (8) and (9), as follows:

There exists a bounded open subset D of $\mathbb{R}^N$ with the following properties: for each $v \in \partial D$ there exists a hyperplane H_v passing through the origin such that

$$g(\overline{B}_r(v)) \subset \mathbb{R}^N\backslash H_v, \tag{8}$$

and

$$\deg(g, D, 0) \neq 0. \tag{9}$$



Moreover, as in Lazer's original result in [4], Theorem 1.1 still holds if g is unbounded but *sublinear*, that is,

$$\frac{g(u)}{|u|} \to 0 \qquad \text{as } |u| \to \infty. \tag{10}$$

Indeed, sublinearity implies that for any given $\varepsilon > 0$ there exists a constant M_ε such that

$$|g(u)| \le \varepsilon\,|u| + M_\varepsilon \quad \text{for every } u \in \mathbb{R}^N.$$

Thus, if u is a T-periodic solution of (7) for some $\lambda \in (0,1]$, then

$$\begin{aligned}\|u'\|_\infty &\le k(\|p\|_{L^1(0,T)} + \varepsilon\|u\|_{L^1(0,T)} + M_\varepsilon T)\\ &\le k\left[\|p\|_{L^1(0,T)} + M_\varepsilon T + \varepsilon T(\|u-\overline{u}\|_\infty + |\overline{u}|)\right].\end{aligned}$$

Assume that $|\overline{u}| = R < \alpha K T$ for some constants $\alpha > 1$, $K > 0$. Then, if $\|u'\|_\infty \ge K$, the previous inequality yields

$$K(1 - k\varepsilon T^2(1+\alpha)) \le k(\|p\|_{L^1(0,T)} + M_\varepsilon T).$$

Consequently, taking $\alpha > 1$,

$$0 < \varepsilon < \frac{1}{kT^2(1+\alpha)}, \qquad K > \frac{k(\|p\|_{L^1(0,T)} + M_\varepsilon T)}{1 - k\varepsilon T(T+\alpha)}, \qquad r := KT, \tag{11}$$

we conclude that any T-periodic solution of (7) for $\lambda \in (0,1]$ such that $|\overline{u}| = R < \alpha r$ must satisfy

$$\|u'\|_\infty < K \qquad \text{and} \qquad \|u - \overline{u}\|_\infty < r.$$

The aim of this paper is to prove the existence of T-periodic solutions to equation (1) in some situations where condition (8) is not satisfied. More specifically, we shall allow $g(B_r(v))$ to intersect H_v, provided that g maps an appropriate subset of $B_r(v)$ sufficiently far away from H_v.

A subset of $B_r(v)$ of the form

$$\mathcal{S}(v) = \{u \in B_r(v) : |\langle u - v, \xi_v\rangle| < \delta\},$$

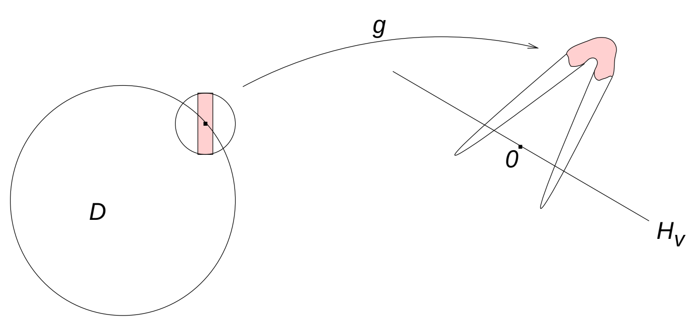
for some $\xi_v \in \mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ and $\delta > 0$, will be called a *strip of width* 2δ.

Our main theorem reads as follows.

Theorem 1.2. *Let $c \in \mathbb{R}$ and assume that $g \in C(\mathbb{R}^N, \mathbb{R}^N)$ satisfies (10) and $p \in C_T(\mathbb{R}, \mathbb{R}^N)$ satisfies $\overline{p} = 0$. Further, assume that for some $\alpha > 1$, and ε, K and r satisfying (11), there exists a domain $D \subset B_{\alpha r}(0)$ with the following properties:*
$(\mathbf{D}_1)$ *For every $v \in \partial D$ there exist a hyperplane H_v passing through the origin and a strip $\mathcal{S}(v)$ of width 2δ such that $g(\mathcal{S}(v)) \subset \mathbb{R}^N \backslash H_v$ and*

$$\mathrm{dist}(g(\mathcal{S}(v)), H_v) > \left(\frac{r}{2\delta} - 1\right) \mathrm{dist}(g(u), H_v)$$

for every $u \in B_r(v)$ with $g(u) \in H_v^-$, where H_v^- denotes the closure of the connected component of $\mathbb{R}^N \backslash H_v$ not containing $g(\mathcal{S}(v))$.
$(\mathbf{D}_2)$ $\deg(g, D, 0) \neq 0$.
Then there exists a T-periodic solution u of equation (1) such that $\overline{u} \in D$ and $\|u - \overline{u}\|_\infty < r$.



In particular, if (6) holds then $g(B_r(v)) \cap H_v^- = \emptyset$, and condition $(\mathbf{D}_1)$ is trivially satisfied. Observe also that, if (6) does not hold and $\delta \geq \frac{r}{2}$, then condition $(\mathbf{D}_1)$ simply says that $\mathrm{dist}(g(\mathcal{S}(v), H_v) > 0$.

Condition $(\mathbf{D}_1)$ is motivated by some results in the scalar case involving rapidly oscillating nonlinearities. In the following section we discuss the effect of *rapidly rotating* nonlinearities and give some examples where condition $(\mathbf{D}_1)$ allows to obtain existence results in situations where condition (6) fails. The proof of Theorem 1.2 is given in section 3. Finally, in section 4 we give further sufficient conditions on g which provide a priori bounds on the solutions for a given p, and we present an example for which the assumptions of our main theorem are satisfied.

2. **The effect of rotation.** We begin with a simple remark concerning the scalar case. From the discussion following Theorem 1.1 it is immediately seen that, for a given p, the condition

$$g < 0 \quad \text{in } I^- \qquad \text{and} \qquad g > 0 \quad \text{in } I^+,$$

where $I^\pm$ are large enough bounded intervals, is sufficient for the existence of a solution. Indeed, when $N = 1$ condition (8) for a general domain $D = (a, b)$ simply reads

$$g \neq 0 \qquad \text{in} \quad (a - r, a + r) \cup (b - r, b + r),$$

and if the signs of g over $(a - r, a + r)$ and $(b - r, b + r)$ are different the degree condition is also satisfied.

This means that, in contrast with (2), oscillations of g around 0 at $\pm\infty$ are, in fact, allowed, but the length of the intervals $I^\pm$ where g does not change sign is determined by g and p, and cannot be arbitrarily small. For instance, when $g(u) = \sin u$, there are well known examples of forcing terms p with zero average and $c \neq 0$ such that the problem has no solutions (see [1], [7], [9]).

There are, however, some particular situations in which g is oscillatory, but solvability for arbitrary p can still be ensured. This is the case of the so-called expansive nonlinearities, like

$$g(u) = \sin(u^{1/3}).$$

Indeed, here the gap between consecutive zeros of g becomes arbitrarily large as $|u|$ tends to infinity. Thus, for any choice of p, the existence of appropriate intervals $I^\pm$ is verified. Furthermore, since g changes sign infinitely many times, we deduce the existence of infinitely many solutions.

The preceding argument obviously fails in the case of non-expansive nonlinearities. Despite this fact, some existence results are known when g presents rapid oscillations (see e.g. [2], [3] and the references therein). For example, assume that g is bounded from below and that $g < 0$ over some large interval I^-, but no interval of positivity of g is long enough to satisfy (8). Then it is possible to compensate this ‘smallness’ by assuming that g is larger than an appropriate constant C over some subset of one of these positivity intervals. Again, the value of the constant depends on p, and also on the length of the interval: faster oscillations require larger values of g. If we expect to prove solvability for arbitrary p using this approach, then g must necessarily be unbounded. For example, we may consider a function g bounded from below with expansive nonlinearities for $u < 0$, and that behaves as

$$u^2[\sin u^2 + 1] + \sin u^2$$

for $u > 0$. It can be proved that, even if the length of the positivity intervals of g tends to 0 as $u \to +\infty$, the large factor u^2 guarantees solvability for any p.

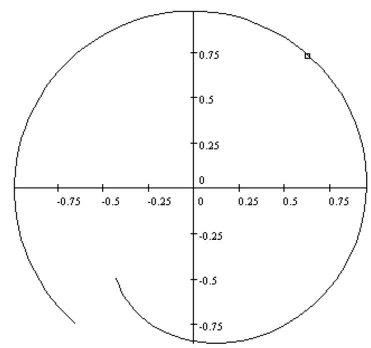
Our main theorem may be considered as an extension of this idea for rapid oscillations to the case $N > 1$. Although an extension of some of the results in [2] and [3] is rather straightforward for weakly coupled systems, there seem to be no results which extend the results in [10] for *rapidly rotating* nonlinearities.

We first observe that Theorem 1.1 provides a better understanding of the non-existence result given in [8]. Indeed, conditions (4) and (6) are equivalent for $N = 1$, but when $N = 2$ the function

$$g_\rho(z) = e^{i\frac{\mathrm{Re}(z)}{\rho}} \frac{z}{\sqrt{1 + |z|^2}}, \qquad \rho > 0,$$

(in complex notation) considered in [8], satisfies (4) but not (6).

It is worth taking a closer look at this function g_ρ in order to understand why condition (6) is violated for some choices of r. If $r \geq \rho\pi$ and $R \gg 0$, then for $z_0 \in \partial B_R(0)$ it suffices to consider the curve $z(t) = z_0 - t$ with $t \in [-\rho\pi, \rho\pi]$. Since $R \gg 0$, the variation of $|g_\rho(z(t))|$ is small, but $g_\rho(z(t))$ rotates around the origin and contains points belonging to antipodal rays in each direction.



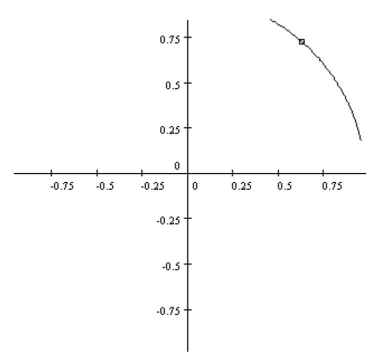
$g_1(4+t), \qquad t \in [-\pi, \pi]$

Note that $|g_\rho(z)|$ does not depend on ρ, so the choice of the appropriate r depends only on p. For fixed p, condition (6) is satisfied for large values of ρ. An approximate lower bound for ρ would be $\frac{2r}{\pi}$. But (6) fails to hold for values of ρ which are smaller than some $\rho(p)$, i.e. for those nonlinearities g_ρ which rotate too fast.

Note, however, that the effect of rotation appears only when we consider the image of the whole ball $B_r(z)$ under the function g, whereas the image of a vertical strip

$$\mathcal{S}(z) := \{u \in B_r(z) : |\mathrm{Re}(u) - \mathrm{Re}(z)| < \delta\}$$

under g remains in the same half-plane for δ small enough.



$g_1(4+it), \ \ t \in [-\pi, \pi]$

According to our main theorem, when (6) fails, existence of solutions can still be proved if the distance between $g_\rho(\mathcal{S}(z))$ and some line through the origin is large enough. In this sense, our result can be regarded as a generalization of the above mentioned results for rapid oscillations in the scalar case. In particular, for a given

p, existence of solutions can be proved for nonlinearities g_ρ in a range of values of ρ which is larger than the one given by condition (6).

3. **Proof of the main theorem. Proof of Theorem 1.2.** Set

$$\Omega = \{u \in C_T(\mathbb{R}, \mathbb{R}^N) : \|u - \overline{u}\|_\infty < r,\ \overline{u} \in D\}.$$

By standard continuation methods, it suffices to prove that the equation

$$u'' + cu' + \lambda g(u) = \lambda p \tag{12}$$

has no solutions in $\partial\Omega$ for $\lambda \in (0, 1]$.

If $u \in \overline{\Omega}$ is a solution of (12) for some $\lambda \in (0, 1]$ then, since we are assuming that $D \subset B_{\alpha r}(0)$, our choice of K and $r := KT$ yields

$$\|u'\|_\infty < K \qquad \text{and} \qquad \|u - \overline{u}\|_\infty < r. \tag{13}$$

Thus, it only remains to prove that $\overline{u} \notin \partial D$.

Note that, if we take w_v to be the unit normal vector of H_v satisfying $\langle g(v), w_v\rangle > 0$, then condition $(\mathbf{D}_1)$ is equivalent to the following one:

$(\mathbf{D}_1')$ For every $v \in \partial D$ there exist a vector $w_v \in \mathbb{S}^{N-1}$ and a strip $\mathcal{S}(v)$ of width 2δ such that

$$\inf_{u_1 \in \mathcal{S}(v)} \langle g(u_1), w_v\rangle + \left(\frac{r}{2\delta} - 1\right) \langle g(u), w_v\rangle > 0 \tag{14}$$

for every $u \in B_r(v)$ with $\langle g(u), w_v\rangle \le 0$.

Arguing by contradiction, assume that $\overline{u} \in \partial D$ and take $w_{\overline{u}} \in \mathbb{S}^{N-1}$ and a strip $\mathcal{S}(\overline{u}) = \{u \in B_r(\overline{u}) : |\langle u - \overline{u}, \xi_{\overline{u}}\rangle| < \delta\}$ with $\xi_{\overline{u}} \in \mathbb{S}^{N-1}$ which satisfy (14). Since u solves (12), we have that

$$0 = \int_0^T \langle g(u(t)), w_{\overline{u}}\rangle\, dt = \int_0^T \langle g(u(t)) - \tilde{\Delta} w_{\overline{u}}, w_{\overline{u}}\rangle\, dt + \tilde{\Delta} T,$$

where

$$\tilde{\Delta} := \inf_{t \in [0,T]} \langle g(u(t)), w_{\overline{u}}\rangle.$$

This implies that $\tilde{\Delta} \le 0$.

Set $\varphi(u) := \langle u, \xi_{\overline{u}}\rangle$. From the mean value theorem for vector-valued integrals we deduce that $\overline{u} \in \mathrm{co}(\mathrm{im}(u))$, where $\mathrm{im}(u)$ stands for the image of the periodic function u. Hence $\varphi(\overline{u}) \in \varphi(\mathrm{im}(u))$. Thus, setting $\overline{t} \in [0, T]$ such that $\varphi(u(\overline{t})) = \varphi(\overline{u})$ and using (13) we obtain

$$|\varphi(u(t)) - \varphi(\overline{u})| \le \left|u(t) - u(\overline{t})\right| \le K \left|t - \overline{t}\right|.$$

It follows that $u(t) \in \mathcal{S}(\overline{u})$ if $\left|t - \overline{t}\right| < \frac{\delta}{K}$. Using the periodicity of u we conclude that $\mathrm{meas}(A) \ge \frac{2\delta}{K}$, where $A = \{t \in [0, T] : u(t) \in \mathcal{S}(\overline{u})\}$. Moreover, since $[0, T]$ is compact, there exists $t_0 \in [0, T]$ such that $\langle g(u(t_0)), w_{\overline{u}}\rangle = \tilde{\Delta} \le 0$. Therefore,

$$\begin{aligned}
0 &\ge \int_A \langle g(u(t)) - \tilde{\Delta} w_{\overline{u}}, w_{\overline{u}}\rangle\, dt + T\tilde{\Delta} \\
&\ge \frac{2\delta}{K} \inf_{v \in \mathcal{S}(\overline{u})} \langle g(v), w_{\overline{u}}\rangle + \left(T - \frac{2\delta}{K}\right) \tilde{\Delta} \\
&= \frac{2\delta}{K} \left[\inf_{v \in \mathcal{S}(\overline{u})} \langle g(u), w_{\overline{u}}\rangle + \left(\frac{r}{2\delta} - 1\right) \langle g(u(t_0)), w_{\overline{u}}\rangle\right],
\end{aligned}$$

contradicting (14). □

Note that the result is still valid if one considers a more general strip defined by

$$\mathcal{S}(v) = \{u \in B_r(v) : |\varphi(u) - \varphi(v)| < \delta\},$$

where $\delta > 0$ and $\varphi : B_r(v) \to \mathbb{R}$ is Lipschitz continuous with constant 1 and satisfies that $\varphi(v) \in \varphi(U)$ for every connected $U \subset B_r(v)$ such that $v \in \mathrm{co}(U)$.

This last condition is quite restrictive. It is an open question whether a similar result holds, for example, for a lower dimensional strip, i.e. for a δ-neighborhood of a subspace of codimension > 1 in $B_r(v)$.

4. **Conditions on the nonlinearity.** In this section we obtain other versions of our main theorem assuming other conditions on g instead of sublinearity.

In first place, it is easy to prove that in the scalar case no restrictions on the growth of g have to be imposed if the inequalities in (2) are reversed, that is to say, if $g(u)u < 0$ for $|u|$ large enough. This fact suggests to consider the assumption

$$\langle g(u), u \rangle < \kappa \qquad \text{for all } u \in \mathbb{R}^N. \tag{15}$$

It is readily seen that the case $\kappa < 0$ is contained in Theorem 1.1.

For $\kappa \geq 0$, we consider in fact a weaker assumption, namely, we require that

$$\langle g(u), u \rangle < \kappa + \mu |u|^\theta \qquad \text{for all } u \in \mathbb{R}^N, \tag{16}$$

where $\theta < 2$ and $\mu \geq 0$.

Then, if u is a T-periodic solution of the equation

$$u'' + cu' = \lambda(p - g(u)), \tag{17}$$

equality

$$-\int_0^T \langle u'', u - \overline{u} \rangle = \lambda \left(\int_0^T \langle g(u), u \rangle - \int_0^T \langle p, u - \overline{u} \rangle \right)$$

holds, and therefore

$$\|u'\|_{L^2}^2 \leq \|p\|_{L^2} \|u - \overline{u}\|_{L^2} + \kappa T + \mu \|u\|_{L^\theta}^\theta.$$

Now we may proceed as in the introduction in order to get bounds K, depending on some fixed $\alpha > 1$, and $r := KT$ such that any T-periodic solution of (17) for $\lambda \in (0, 1]$ with $|\overline{u}| < \alpha r$ satisfies

$$\|u'\|_\infty < K \qquad \text{and} \qquad \|u - \overline{u}\|_\infty < r. \tag{18}$$

For example, if $g(z) = e^{i|z|} \frac{z}{\sqrt{1+|z|^2}}$, $z \in \mathbb{C}$, then

$$\langle g(z), z \rangle = \frac{|z|^2}{\sqrt{1 + |z|^2}} \cos(|z|).$$

So condition (15) is not satisfied. However, (16) holds.

Finally, let us point out that there is still another way of obtaining a priori bounds (18). Again, we recall the case $N = 1$, and note that condition (10) in Lazer's result can also be dropped if we assume instead that g is bounded from one side, i.e. that either

$$g(u) \leq M \quad \text{for all } u \in \mathbb{R}, \quad \text{or} \quad g(u) \geq M \quad \text{all } u \in \mathbb{R}.$$

This condition can be generalized to $N > 1$ by assuming that

$$g(\mathbb{R}^N) \subset \xi + \left(\mathbb{R}^N \setminus \bigcup_{j=1}^N H_j \right), \tag{19}$$

where $\xi \in \mathbb{R}^N$, and $H_j \subset \mathbb{R}^N$ are linearly independent hyperplanes through the origin. In other words, (19) says that the range of g is contained in an 'angular sector' of $\mathbb{R}^N$.

If (19) holds, a priori bounds (18) can be obtained as follows. Let u satisfy $u'' + cu' + \lambda g(u) = \lambda p$ for some $0 < \lambda \le 1$, and set

$$d_j := \inf_{u \in \mathbb{R}^N} \langle g(u), w_j \rangle, \qquad v_j := d_j w_j,$$

where $\{w_1, \ldots, w_N\}$ is a basis of unit vectors of $\mathbb{R}^N$ chosen in such a way that $\langle g(u) - \xi, w_j \rangle \ge 0$ for every $u \in \mathbb{R}^N$. Then

$$\langle g(u) - v_j, w_j \rangle \ge d_j - \langle v_j, w_j \rangle = 0.$$

Thus,

$$|\langle u''(t), w_j \rangle| \le \langle g(u) - v_j, w_j \rangle + |\langle v_j - p, w_j \rangle|$$

and, in consequence,

$$\int_0^T |\langle u''(t), w_j \rangle| \, dt \le \int_0^T |\langle v_j - p, w_j \rangle| \, dt - T\langle \xi_j, w_j \rangle := K_j.$$

Hence, for each $t \in [0, T]$ we have

$$|\langle u'(t), w_j \rangle| \le K_j$$

and

$$|\langle u(t) - \overline{u}, w_j \rangle| \le K_j T.$$

Although sharper results could be obtained by taking $r_j > K_j T$ and modifying the definition of Ω accordingly, for simplicity we shall consider a value K such that $\|u'\|_\infty < K$. Then $\|u - \overline{u}\|_\infty < KT := r$. In this case R can be arbitrarily chosen.

Corollary 1. Theorem 1.2 *remains true if* (10) *is replaced by* (16) *or* (19)*, and* K, r *and* R *are defined as previously shown in this section.*

We end this paper with a simple example of a radial nonlinearity $g(u) = \gamma(|u|)u$ to which our theorem applies for arbitrary p.

Let $\gamma : [0, +\infty) \to \mathbb{R}$ satisfy

$$\gamma(s) \le \mu s^{-\sigma}$$

for some $\mu, \sigma > 0$. Thus, condition (16) holds, although γ is allowed to take arbitrarily large negative values.

Let $R = \alpha r$ with $\alpha > 1$. Regarding condition $(\mathbf{D}_1)$, it proves convenient to choose $D = B_R(0)$ and, for $|v| = R$, to take $w_v = -\frac{v}{R}$ and

$$\mathcal{S}(v) = \{u \in B_r(v) : |\langle u - v, v \rangle| < \delta R\}.$$

Then, $\langle g(u), w_v \rangle = -\frac{\gamma(|u|)}{R} \langle u, v \rangle$.

Let us assume that $\gamma(R) < 0$ and that $\gamma \not< 0$ in $[R - r, R + r]$, since otherwise Theorem 1.1 applies. Then $\langle g(v), w_v \rangle > 0$ and condition (14) reads

$$-\sup_{u \in \mathcal{S}(v)} \gamma(|u|)\langle u, v \rangle > \left(\frac{r}{2\delta} - 1\right) \gamma(|u|)\langle u, v \rangle \quad \text{for all } u \in B_r(v),$$

or equivalently

$$-\sup_{u \in \mathcal{S}(v)} |u| \, \gamma(\, |u| \,) \cos(\beta_{u,v}) > \left(\frac{r}{2\delta} - 1\right) t\gamma(t) \quad \text{for all } t \in (R - r, R + r),$$

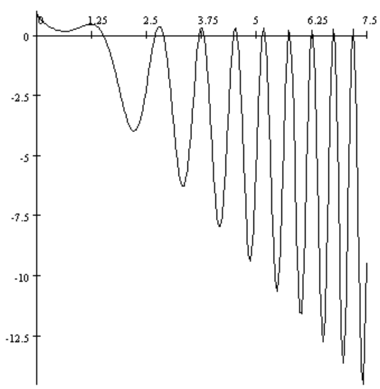
where $\beta_{u,v}$ denotes the angle between u and v. As $R = \alpha r$, a simple computation shows that $\cos(\beta_{u,v}) \ge \frac{\sqrt{\alpha^2 - 1}}{\alpha}$ for every $u \in B_r(v)$.

Assume that $\delta > 0$ is chosen so that $\gamma(t) < 0$ for every $t \in (R-\delta, \sqrt{R^2 + 2\delta R + r^2})$. Then $\gamma(\,|u|\,) < 0$ for every $u \in \mathcal{S}(v)$ and a sufficient condition for the above inequality to hold is

$$-\frac{\sqrt{\alpha^2-1}}{\alpha} \sup_{R-\delta<t<\sqrt{R^2+2\delta R+r^2}} t\gamma(t) > \left(\frac{r}{2\delta} - 1\right) \sup_{R-r<t<R+r} t\gamma(t). \qquad (20)$$

For example, we may consider

$$\gamma(t) = t(\sin(t^2) - 1) + \frac{\mu}{t^\sigma + 1}.$$



Set $R_n := \sqrt{\left(2n - \frac{1}{2}\right)\pi}$ and $\delta_n := 1/2\sqrt{n}$. Since for n large enough

$$t\gamma(t) < -t^2 \qquad \text{if} \quad |t - R_n| < \delta_n$$

and

$$\sup_{R-r<t<R+r} t\gamma(t) = O(t^{1-\sigma}),$$

then for any fixed p we may choose n large enough and $\delta \in (0, \delta_n)$ so that

$$\sqrt{R_n^2 + 2\delta R_n + r^2} < R_n + \delta_n$$

and (20) holds.

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E-mail address: pamster@dm.uba.ar
E-mail address: mclapp@matem.unam.mx