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Self-dual projective toric varieties

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Abstract

Let T be a torus over an algebraically closed field $\Bbbk$ of characteristic 0, and consider a projective T-module $\mathbb{P}(V)$. We determine when a projective toric subvariety $X \subset \mathbb{P}(V)$ is self-dual, in terms of the configuration of weights of V.

1. *Introduction*

The notion of duality of projective varieties, which appears in various branches of mathematics, has been a subject of study since the beginning of algebraic geometry [**12**, **16**]. Given an embedded projective variety $X \subset \mathbb{P}(V)$, its dual variety X^* is the closure in the dual projective space $\mathbb{P}(V^\vee)$ of the hyperplanes intersecting the regular points of X non-transversally.

A projective variety X is *self-dual* if it is isomorphic to its dual X^* as embedded projective varieties. The expected codimension of the dual variety is 1. If this is not the case, then X is said to be defective. Self-dual varieties other than hypersurfaces are defective varieties with 'maximal' defect.

Let $\Bbbk$ be an algebraically closed field of characteristic 0. Let T be an algebraic torus over $\Bbbk$ and V a finite-dimensional T-module. In this paper, we characterize self-dual projective toric varieties $X \subset \mathbb{P}(V)$ equivariantly embedded, in terms of the combinatorics of the associated configuration of weights A (cf. Theorems 4.4 and 4.16) and in terms of the interaction of the space of relations of these weights with the torus orbits (cf. Theorems 3.2 and 3.8). In particular, we show that X is self-dual if and only if $\dim X = \dim X^*$ and the smallest linear subspaces containing $X = X_A$ and X^* have the same dimension; see Theorems 3.3 and 3.7.

Given a basis of eigenvectors of V and the configuration of weights of the torus action on V, it is not difficult to check the equality of the dimensions of X and its dual (for instance, by means of the combinatorial characterization of the tropicalization given in [**6**]). But the complete classification of defective projective toric varieties in an equivariant embedding is open in full generality and involves a complicated combinatorial problem. For smooth toric varieties this characterization is obtained in [**8**]; the case of $\mathbb{Q}$-factorial toric varieties is studied in [**2**]. For non-necessarily normal projective toric varieties of codimension 2, a characterization is given in [**7**]. This has been extended for codimensions 3 and 4 in [**5**].

For smooth projective varieties, a full list of self-dual varieties is known [**9**, **10**, **16**]. This list is indeed short and reduces in the case of toric varieties to hypersurfaces or Segre embeddings of $\mathbb{P}^1 \times \mathbb{P}^{m-1}$, for any $m \geqslant 2$, under the assumption that $\dim X \leqslant 2 \dim \mathbb{P}(V)/3$. This was expected to be the whole classification under the validity of Hartshorne's conjecture [**10**]. We prove that this is indeed the whole list of self-dual smooth projective toric varieties in Theorem 5.8.

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There exist some classical examples of self-dual non-smooth varieties, as the quartic Kummer surface. Popov and Tevelev gave new families of non-smooth self-dual varieties that come from actions of isotropy groups of complex symmetric spaces on the projectivized nilpotent varieties of isotropy modules [**13**, **14**]. As a consequence of Theorem 4.4, it is easy to construct new families of self-dual projective toric varieties in terms of the Gale dual configuration (see Definition 4.1).

A big class of self-dual toric varieties are the toric varieties associated to Lawrence configurations (see Definition 5.1), which contain the configurations associated to the Segre embeddings. Lawrence constructions are well known in the domain of geometric combinatorics, where they are one of the prominent tools to visualize the geometry of higher dimensional polytopes (see [**18**, Chapter 6]); the commutative algebraic properties of the associated toric ideals are studied in [**1**]. We show in Section 5 other non-Lawrence concrete examples for any dimension bigger than 2 and any codimension bigger than 1.

We also introduce the notion of strongly self-dual toric varieties (see Definition 6.1), which is not only related to the geometry of the configuration of weights but also to number theoretic aspects. This concept is useful for the study of the existence of rational multivariate hypergeometric functions [**3**, **11**].

In Section 2, we gather some preliminary results about embedded projective toric varieties and duality of projective varieties. In Section 3, we characterize self-dual projective toric varieties in terms of the geometry of the action of the torus and we give precise assumptions under which self-dual projective varieties are precisely those with maximal defect. In Section 4, we give two (equivalent) combinatorial characterizations of self-duality. In Section 5, we collect several new examples of self-dual (non-smooth) projective toric varieties. Finally, in Section 6 we study strongly self-dual toric varieties.

2. *Preliminaries*

In this section, we collect some well-known results and useful observations on projective toric varieties and duality of projective varieties.

2.1. *Actions of tori*

Let T be an algebraic torus over an algebraically closed field $\Bbbk$ of characteristic 0. We denote by $\mathcal{X}(T)$ the lattice of characters of T; recall that $\Bbbk[T] = \bigoplus_{\lambda\in\mathcal{X}(T)} \Bbbk\lambda$. Any finite-dimensional rational T-module V, $\dim V = n$, decomposes as a direct sum of irreducible representations

$$V \cong \bigoplus_{i=1}^{n} \Bbbk v_i, \tag{2.1}$$

with $t\cdot v_i = \lambda_i(t)v_i$, $\lambda_i \in \mathcal{X}(T)$, for all $t \in T$.

The action of T on V canonically induces an action $T \times \mathbb{P}(V) \to \mathbb{P}(V)$ on the associated projective space, given by $t\cdot[v] = [t\cdot v]$, where $[v] \in \mathbb{P}(V)$ denotes the class of $v \in V\setminus\{0\}$. Recall that an irreducible T-variety X is called *toric* if there exists $x_0 \in X$ such that the orbit $\mathcal{O}(x_0)$ is open in X.

Let $A = \{\lambda_1,\ldots,\lambda_n\}$ (which may contain repeated elements) be the associated set of weights of a finite-dimensional T-module V; we call A the *configuration of weights* associated to the T-module V. To any basis $\mathcal{B} = \{v_1,\ldots,v_n\} \subset V$ of eigenvectors, we can associate a projective toric variety by

$$X_{V,\mathcal{B}} = \overline{\mathcal{O}\left(\left[\sum v_i\right]\right)} \subset \mathbb{P}(V).$$

Define by $\mathbb{T}^{n-1} = \{\sum p_i v_i \in \mathbb{P}(V) : \prod p_i \neq 0\}$. The dense orbit $\mathcal{O}([\sum v_i])$ in $X_{V,\mathcal{B}}$ coincides with the intersection $X_{V,\mathcal{B}} \cap \mathbb{T}^{n-1}$. Observe that since $\dim X_{V,\mathcal{B}}$ is equal to $\dim \mathcal{O}([\sum v_i])$, it follows that $\dim X_{V,\mathcal{B}}$ is maximal among the dimensions of the toric subvarieties of $\mathbb{P}(V)$, that is, those of the form $\overline{\mathcal{O}([v])}$ for some $[v] \in \mathbb{P}(V)$.

Based on the decomposition (2.1), in [**12**, Proposition II.5.1.5] it is proved that any projective toric variety in an equivariant embedding is of type $X_{V,\mathcal{B}}$ for some T-module V and a basis of eigenvectors $\mathcal{B} = \{v_1, \dots, v_n\}$ of V, in the following sense. Let U be a T-module and $Y \subset \mathbb{P}(U)$ be a toric subvariety; then there exist $A = \{\lambda_1, \dots, \lambda_n\} \subset \mathcal{X}(T)$ (with possible repetitions) and a T-equivariant linear injection $f : W := \bigoplus_{i=1}^n \Bbbk w_i \hookrightarrow U$, $t \cdot w_i = \lambda_i(t) w_i$, such that the induced equivariant morphism $\widehat{f} : \mathbb{P}(W) \hookrightarrow \mathbb{P}(U)$ gives an isomorphism $X_{W,\mathcal{B}} \cong Y$. Moreover, let W' be another T-module, $\mathcal{B}' = \{w_1', \dots, w_n'\} \subset W'$ a basis of eigenvectors of W' such that $t \cdot w_i' = \lambda_i(t) w_i'$, and consider $f \in \operatorname{Hom}_T(W, W')$ given by $f(w_i) = w_i'$. Clearly, f is an isomorphism of T-modules, and its induced morphism $\widehat{f} : \mathbb{P}(W) \to \mathbb{P}(W')$ is an isomorphism such that $\widehat{f}(X_{W,\mathcal{B}}) = X_{W',\mathcal{B}'}$.

In view of the preceding remark, the following notation makes sense.

Definition 2.1. The projective toric variety X_A associated to the configuration of weights A is defined as

$$X_A = X_{V,\mathcal{B}} = \overline{\mathcal{O}\left(\left[\sum v_i\right]\right)} \subset \mathbb{P}(V),$$

where V is a T-module with A as associated configuration of weights.

We can make a series of reductions on A and T, as in [**6**]. First, the following easy lemma allows to reduce our problem to the case of a faithful representation.

Lemma 2.2. *Given a T-module V of finite dimension and A the associated configuration of weights, consider the torus $T' = \operatorname{Hom}_{\mathbb{Z}}(\langle A\rangle_{\mathbb{Z}}, \Bbbk^*)$, where $\langle A\rangle_{\mathbb{Z}} \subset \mathcal{X}(T)$ denotes the $\mathbb{Z}$-submodule generated by A. The representation of T in $\mathrm{GL}(V)$ induces a faithful representation $T' \to \mathrm{GL}(V)$, which has the same set theoretical orbits in V.*

We can then replace T by the torus T'. It is easy to show that this is equivalent to the fact that $\langle A\rangle_{\mathbb{Z}} = \mathcal{X}(T)$, which we assume from now on without loss of generality.

Next, we enlarge the torus without affecting the action on $\mathbb{P}(V)$; this will allow us to easily translate affine relations to linear relations on the configuration of weights. If we let the algebraic torus $\Bbbk^* \times T$ act on V by $(t_0, t) \cdot v = t_0(t \cdot v)$, then the actions $T \times \mathbb{P}(V) \to \mathbb{P}(V)$ and $(\Bbbk^* \times T) \times \mathbb{P}(V) \to \mathbb{P}(V)$ have the same set theoretical orbits. More in general, let $\lambda \in \mathcal{X}(T)$ and $A' = \{\lambda + \lambda_1, \dots, \lambda + \lambda_n\}$. Consider the T-action on V given by $t \cdot_\lambda v_i = (\lambda + \lambda_i)(t) v_i$. The actions $\cdot$ and $\cdot_\lambda$ coincide on $\mathbb{P}(V)$, and the corresponding variety $X_{A'}$ coincides with X_A. Hence, we can assume that there is a splitting of $T = \Bbbk^* \times S$ in such a way that $(t_0, s) \cdot v = t_0(s \cdot v)$ for all $v \in V$, $t_0 \in \Bbbk^*$ and $s \in S$.

In fact, the previous reductions comprise the following more general setting.

Lemma 2.3 [**12**, Proposition II.5.1.2]. *Consider T, T' two tori and two finite configurations of n weights $A = \{\lambda_1, \dots, \lambda_n\} \subset \mathcal{X}(T)$, $A' = \{\lambda_1', \dots, \lambda_n'\} \subset \mathcal{X}(T')$. Assume that there exists a $\mathbb{Q}$-affine transformation $\psi : \mathcal{X}(T) \otimes \mathbb{Q} \to \mathcal{X}(T') \otimes \mathbb{Q}$ such that $\psi(\lambda_i) = \lambda_i'$ for all $i = 1, \dots, n$. Then $X_A = X_{A'}$.*

REMARK 2.4. (1) The dimension of the projective toric variety X_A equals the dimension of the affine span of A.

(2) Note that if $A = \{\lambda_1, \dots, \lambda_n\}$ is contained in a hyperplane off the origin, then $X_A = \mathbb{P}(V)$ precisely when $\dim T = n$ and the elements in A are a basis of $\mathcal{X}(T)$.

(3) If we denote by d the dimension of the affine span of A, then X_A is a hypersurface if and only if $n = d+2$. In this situation, either A coincides with the set of vertices of its convex hull $\mathrm{Conv}(A) \subset \mathcal{X}(T) \otimes \mathbb{R}$, or $\mathrm{Conv}(A)$ contains only one element $\lambda \in A$ in its relative interior, and $A \setminus \{\lambda\}$ is the set of vertices.

We end this paragraph by recalling some basic facts about the geometric structure of a toric variety X_A.

LEMMA 2.5 [4]. *Let $A = \{\lambda_1, \dots, \lambda_n\} \subset \mathcal{X}(T)$ be a configuration, where $\{\lambda_1, \dots, \lambda_s\}$ is the set of vertices of* $\mathrm{Conv}(A)$. *Set $X_i = \mathrm{Spec}(\Bbbk[\mathbb{Z}^+\langle(\lambda_j - \lambda_i) : \lambda_j \in A\rangle])$, $i = 1, \dots, s$. Then X_i is an affine toric T-variety, and there exist T-equivariant open immersions $\varphi_i : X_i \hookrightarrow X_A$, in such a way that*

$$X_A = \bigcup_{i=1}^{s} \varphi_i(X_i) = \bigcup_{i=1}^{s} \mathrm{Spec}(\Bbbk[\mathbb{Z}^+\langle \lambda_j - \lambda_i : \lambda_j \in A\rangle]).$$

In particular, X_A is a normal variety if and only if $\mathbb{Z}^+\langle \lambda_j - \lambda_i : \lambda_j \in A\rangle = (\mathbb{R}^+\langle \lambda_j - \lambda_i : \lambda_j \in A\rangle) \cap \mathcal{X}(T)$ for all $i = 1, \dots, s$.

Moreover, X_A is a smooth variety if, for all $i = 1, \dots, s$, there are exactly $\dim X_A$ *edges of* $\mathrm{Conv}(A)$ *from λ_i, and the subset $\{\lambda_{j_h} - \lambda_i : h = 1, \dots, \dim X_A\}$ is a basis of $\mathcal{X}(T)$, where λ_{j_h} is the 'first' point on an edge from λ_i.*

Proof. See, for example, [4, Appendix to Chapter 3]. □

2.2. *Configurations in lattices, pyramids and projective joins*

Let M' be a lattice of rank $d-1$. We let $M = \mathbb{Z} \times M'$ and consider the $\Bbbk$-vector space $M_\Bbbk = M \otimes_{\mathbb{Z}} \Bbbk$. Recall that, given a basis $\{\mu_1, \dots, \mu_d\}$ of M, we can identify M with $\mathbb{Z}^d$ and $M_\Bbbk$ with $\Bbbk^d$.

DEFINITION 2.6. A *lattice configuration* $A = \{\lambda_1, \dots, \lambda_n\} \subset M$ is a finite sequence of lattice points. We say that a configuration A is *regular* if it is contained in a hyperplane off the origin.

REMARK 2.7. Let T be an algebraic torus, and let $A = \{\lambda_1, \dots, \lambda_n\} \subset \mathcal{X}(T)$ be a configuration of weights. Then the following are equivalent.

(1) The configuration A is regular.

(2) Up to affine isomorphism, A has the form $\lambda_i = (1, \lambda_i')$ for all $i = 1, \dots, n$.

(3) There exists a splitting $T = \Bbbk^* \times S$, such that under the identification $\mathcal{X}(T) = \mathbb{Z} \times \mathcal{X}(S)$, the weights of A are of the form $\lambda_i = (1, \lambda_i')$, $i = 1, \dots, n$. See also the reductions made before Lemma 2.3.

DEFINITION 2.8. We denote by $\mathcal{R}_A \subset \mathbb{Z}^n$ the lattice of affine relations among the elements of A, that is, $(a_1, \dots, a_n)$ belongs to $\mathcal{R}_A$ if and only if $\sum_i a_i\lambda_i = 0$ and $\sum_i a_i = 0$.

If A is regular, then $\mathcal{R}_A$ coincides with the lattice of linear relations among the elements of A. Note that these (affine or linear) relations among the elements of A can be identified with the affine relations among the elements of the configuration $\{\lambda'_1, \dots, \lambda'_n\} \subset M'$. Thus, given any configuration $\{\lambda'_1, \dots, \lambda'_n\} \subset M'$, we can embed it in $M = \mathbb{Z} \times M'$ via $\lambda' \mapsto (1, \lambda')$ so that affine dependencies are translated to linear dependencies. In fact, the map $\lambda' \mapsto (1, \lambda')$ is an injective affine linear map. More in general, we have the following definition.

DEFINITION 2.9. We say that two configurations $A_i \subset \mathcal{X}(T_i), i = 1, 2$, are *affinely equivalent* if there exists an affine linear map $\varphi : \mathcal{X}(T_1) \otimes \mathbb{R} \to \mathcal{X}(T_2) \otimes \mathbb{R}$ (defined over $\mathbb{Q}$) such that φ sends A_1 bijectively to A_2 (in particular, φ defines an injective map from the affine span of A_1 to the affine span of A_2).

So, if A_1 and A_2 are affinely equivalent, then they have the same cardinal and, moreover, $\mathcal{R}_{A_1} = \mathcal{R}_{A_2}$. Any property of a configuration A shared by all its affinely equivalent configurations is called an *affine invariant* of A. In this terminology, Lemma 2.3 asserts that the projective toric variety X_A is an affine invariant of the configuration A.

DEFINITION 2.10. We say that $A = \{\lambda_1, \dots, \lambda_n\} \subset M$ is a *pyramid*/(or a *pyramidal configuration*) if there exists an affine hyperplane H such that $\#\{i \,/\, \lambda_i \notin H\} = 1$, that is, if all points in A but one lie in H, or equivalently, if there exist an index $i_0 \in \{1, \dots, n\}$ and an affine linear function $\ell : M_{\Bbbk} \to \Bbbk$ such that $\ell(\lambda_i) = 0$ for all $i \neq i_0$ and $\ell(\lambda_{i_0}) = 1$.

More precisely, we say that A is a *k-pyramidal configuration* if, after reordering, there exists a splitting of the lattice as a direct sum of lattices $M = M_1 \oplus M_2$, with $A_1 = \{\lambda_1, \dots, \lambda_r\}$ a basis of M_1 and $A_2 = \{\lambda_{r+1}, \dots, \lambda_n\} \subset M_2$, with A_2 not a pyramidal configuration of $M_2 \otimes_{\mathbb{Z}} \Bbbk$. In particular, the 0-pyramidal configurations are the *non-pyramidal configurations*.

REMARK 2.11. Being a pyramid is clearly an affine invariant of a configuration. It is straightforward to check that A is a non-pyramidal configuration if and only if there exists a relation $(p_1, \dots, p_n) \in \mathcal{R}_A$ with $\prod_i p_i \neq 0$, that is, if $\mathcal{R}_A$ is not contained in a coordinate hyperplane.

DEFINITION 2.12. Let V_1, V_2 be two $\Bbbk$-vector spaces of respective dimensions $h_1 + 1, h_2 + 1$ and $X \subset \mathbb{P}(V_1), Y \subset \mathbb{P}(V_2)$ be two projective varieties. Recall that the *join* of X and Y is the projective variety

$$\mathrm{J}_{h_1,h_2}(X, Y) = \overline{\{[x : y] : \ [x] \in X, [y] \in Y\}} \subset \mathbb{P}(V_1 \times V_2),$$

that is, the cone over the join $\mathrm{J}_{h_1,h_2}(X, Y)$ is the product of the cones over X and Y. We set

$$\mathrm{J}_{h_1,h_2}(\emptyset, Y) = \{[\underbrace{0 : \ldots : 0}_{h_1+1} : y] \in \mathbb{P}(V_1 \times V_2), \ [y] \in Y\} \subset \mathbb{P}(V_1 \times V_2).$$

We define analogously $\mathrm{J}_{h_1,h_2}(X, \emptyset)$.

We define $\mathbb{P}^h = \mathbb{P}(\Bbbk^{h+1})$. Observe that, for any $Y \subset \mathbb{P}^{h_2}$, $Y \cong \mathrm{J}_{h_1,h_2}(\emptyset, Y) \subset \mathbb{P}^{h_1+h_2+1}$ for any $h_1 \in \mathbb{N}$. If X and Y are non-empty, then $\dim \mathrm{J}_{h_1,h_2}(X, Y) = \dim X + \dim Y + 1$.

Remark that given $X_i \subset \mathbb{P}(V_i)$, $\dim V_i = h_i + 1$, $i = 1, 2, 3$, we have

$$\mathrm{J}_{h_1+h_2+1,h_3}(\mathrm{J}_{h_1,h_2}(X_1, X_2), X_3) = \mathrm{J}_{h_1,h_2+h_3+1}(X_1, \mathrm{J}_{h_2,h_3}(X_2, X_3)) \subset \mathbb{P}(V_1 \times V_2 \times V_3).$$

We denote this variety by $\mathrm{J}_{h_1,h_2,h_3}(X_1, X_2, X_3)$.

Given two projective toric varieties X_{A_1} and X_{A_2}, their join is also a toric variety.

REMARK 2.13. (1) Let $T = S_1 \times S_2$ be a splitting of T as a product of tori, and $A_1 = \{\lambda_1, \dots, \lambda_k\} \subset \mathcal{X}(S_1)$, $A_2 = \{\lambda_{k+1}, \dots, \lambda_n\} \subset \mathcal{X}(S_2)$ be two regular configurations.

Let $V_1 = \bigoplus_{i=1}^{k} \Bbbk v_i$, $s_1 \cdot v_i = \lambda_i(s_1)v_i$ for all $s_1 \in S_1$, and $V_2 = \bigoplus_{i=k+1}^{n} \Bbbk v_i$, $s_2 \cdot v_i = \lambda_i(s_2)v_i$ for all $s_2 \in S_2$. Then $V = V_1 \times V_2$ is a T-module for the product action $(s_1, s_2) \cdot (w_1, w_2) = (s_1 \cdot w_1, s_2 \cdot w_2)$. Moreover, V decomposes into simple submodules as $V = \bigoplus_{i=1}^{k} \Bbbk(v_i, 0) \oplus \bigoplus_{i=k+1}^{n} \Bbbk(0, v_i)$.

Consider the S_i-toric varieties $X_{A_i} \subset \mathbb{P}(V_i)$ $(i = 1, 2)$ and let $A = A_1 \times \{0\} \cup \{0\} \times A_2 \subset \mathcal{X}(S_1) \times \mathcal{X}(S_2) = \mathcal{X}(T)$. The projective toric variety associated to A is then the join $X_A = \mathrm{J}_{k-1,n-k-1}(X_{A_1}, X_{A_2})$.

(2) In the particular case when $A \subset M = M_1 \oplus M_2$ is a k-pyramidal configuration with $A_1 \subset M_1, A_2 \subset M_2$ as in Definition 2.10, let $S_1 = \mathrm{Hom}_{\mathbb{Z}}(M_1, \Bbbk^*)$, $S_2 = \mathrm{Hom}_{\mathbb{Z}}(M_2, \Bbbk^*)$, $T = S_1 \times S_2$ and V as above. We then have that $X_A = \mathrm{J}_{k-1,n-k-1}(\mathbb{P}(V_1), X_{A_2})$; that is, X_A is the *cone over* X_{A_2} *with vertex* $\mathbb{P}(V_1)$.

Next, we describe the toric varieties associated to configurations with repeated weights. Recall that a projective variety is called *non-degenerate* if it is not contained in a proper linear subspace.

LEMMA 2.14. *Let $A = \{\lambda_1, \dots, \lambda_1, \dots, \lambda_h, \dots, \lambda_h\} \subset \mathcal{X}(T)$ be a configuration of n weights, with λ_i appearing $k_i + 1$ times and $\lambda_i \neq \lambda_j$ if $i \neq j$. If we set $k = \sum_i k_i = n - h$, then the smallest linear subspace that contains X_A has codimension k.*

In particular, X_A is a non-degenerate variety if and only if the configuration A has no repeated elements.

Proof. Let $\mathcal{B} = \{v_{1,1}, \dots, v_{1,k_1+1}, \dots, v_{h,1}, \dots, v_{h,k_h+1}\}$ be a basis of associated eigenvectors of V, with $t \cdot v_{i,j_i} = \lambda_i(t)v_{i,j_i}$ for all $i = 1, \dots, h$, $j_i = 1, \dots, k_i + 1$. Consider a hyperplane $\Pi \subset \mathbb{P}(\bigoplus_{i=1}^{h}(\bigoplus_{j_i=1}^{k_i+1} \Bbbk v_{i,j_i}))$ of equation

$$\sum_{i,j_1,\dots,j_h} c_{i,j_i} x_{i,j_i} = 0,$$

where x_{i,j_i} are the coordinates in the basis $\mathcal{B}$. Then $X_A \subset \Pi$ if and only if $[t \cdot \sum v_{i,j_i}] \subset \Pi$ for all $t \in T$. As $[t \cdot \sum v_{i,j_i}] = [\sum \lambda_i(t)v_{i,j_i}] \in \Pi$, this is equivalent to the equalities

$$\sum_{i=1}^{h} \sum_{j_i=1}^{k_i+1} c_{i,j_i} \lambda_i(t) = 0, \quad t \in T.$$

Since $\{\lambda_1, \dots, \lambda_h\}$ are different weights, we deduce that $\sum_{j_i=1}^{k_i+1} c_{i,j_i} = 0$ for all $i = 1, \dots, h$. It follows that the maximum codimension of a subspace that contains X_A is $\sum_{i=1}^{h} k_i = k$.

On the other hand, clearly

$$X_A \subset H =, \left\{ \sum_{i=1}^{h} x_i \sum_{j_i=1}^{k_i+1} v_{i,j_i} : x_i \in \Bbbk \right\},$$

where the subspace $H \subset \mathbb{P}(V)$ has codimension k. □

LEMMA 2.15. *Let $A = \{\lambda_1, \dots, \lambda_1, \dots, \lambda_h, \dots, \lambda_h\} \subset \mathcal{X}(T)$ be a configuration of n weights, with λ_i appearing $k_i + 1$ times and $\lambda_i \neq \lambda_j$ if $i \neq j$. Set $k = \sum_i k_i = n - h$ and let*

$$V = \bigoplus_{i=1}^{h} \left(\bigoplus_{j_i=1}^{k_i+1} \Bbbk v_{i,j_i} \right) = \left(\bigoplus_{i=1}^{h} \left(\bigoplus_{j_i=1}^{k_i} \Bbbk v_{i,j_i} \right) \right) \oplus \left(\bigoplus_{i=1}^{h} \Bbbk v_{i,j_{k_i+1}} \right),$$

with $t \cdot v_{i,j_i} = \lambda_i(t) v_{i,j_i}$ for all $t \in T$, $i = 1, \dots, h$, $j_i = 1, \dots, k_i + 1$.

Let $C = \{\lambda_1, \dots, \lambda_h\}$ and consider $X_C \subset \mathbb{P}(\bigoplus_{i=1}^{h} \Bbbk v_{i,j_{k_i+1}})$. Then X_A is isomorphic to the cone $\mathrm{J}_{k-1,h-1}(\emptyset, X_C)$ over the non-degenerate projective toric variety X_C.

Proof. Let $f : V \to V$ be the linear isomorphism defined by

$$f((x_{i,j_i})_{i=1,\dots,h, j_i=1,\dots,k_i}, (x_{i,j_{k_i+1}})_{i=1,\dots,h}) = ((x_{i,j_i} - x_{i,j_{k_i+1}})_{i=1,\dots,h,j_i=1,\dots,k_i}, (x_{i,j_{k_i+1}})_{i=1,\dots,h}).$$

The associated projective map clearly sends X_A to the join $\mathrm{J}_{k-1,h-1}(\emptyset, X_C)$. □

In Proposition 2.17, we combine Remark 2.13 and Lemmas 2.14 and 2.15, in order to describe a projective toric variety as a cone over a non-degenerate projective *toric* variety that is not a cone (that is, the associated configuration is non-pyramidal).

REMARK 2.16. Let $X \subset \mathbb{P}^{n-1}$ be a *non-linear* irreducible projective variety. Let $H \subset \mathbb{P}^{n-1}$ be the minimal linear subspace containing X, and let k be the codimension of H. Then $H \cong \mathbb{P}^{n-k-1}$ and if X' denotes the variety X as a subvariety of H, then $X = \mathrm{J}_{k-1,n-k-1}(\emptyset, X')$. Since X' is non-degenerate, it follows that there exists $Y \subset \mathbb{P}^{m-1}$ such that $X' = \mathrm{J}_{h-1,m-1}(\mathbb{P}^{h-1}, Y)$, where $n - k - 1 = h + m - 1$. Hence, we have an identification

$$X = \mathrm{J}_{k-1,h-1,m-1}(\emptyset, \mathbb{P}^{h-1}, Y).$$

In particular, $\dim X = h + \dim Y$.

Observe that $Y \subset \mathbb{P}^{m-1}$ is a non-degenerate subvariety. Moreover, we can assume that Y is not a cone. In this case, we define $X_{\mathrm{nd}} = Y$. Moreover, if X is an equivariantly embedded toric variety, then we can choose X_{nd} as X_{C_2} in the following proposition.

When X is linear, $X = H$, $m = 1$ and Y is empty.

PROPOSITION 2.17. *Let $A = \{\lambda_1, \dots, \lambda_1, \dots, \lambda_h, \dots, \lambda_h\} \subset \mathcal{X}(T)$ be a configuration of n weights, with λ_i appearing $k_i + 1$ times and $\lambda_i \neq \lambda_j$ if $i \neq j$. Set $k = \sum_i k_i = n - h$ and assume that $C = \{\lambda_1, \dots, \lambda_h\}$ is r-pyramidal. Then there exists a splitting $T = S_1 \times S_2$ such that, after reordering of the elements in C, it holds that $C = C_1 \cup C_2$, where $C_1 = \{\lambda_1, \dots, \lambda_r\}$ is a basis of $\mathcal{X}(S_1)$ and $C_2 = \{\lambda_{r+1}, \dots, \lambda_h\} \subset \mathcal{X}(S_2)$ is a non-pyramidal configuration, as in Definition 2.10. Moreover, we have that*

$$X_A = \mathrm{J}_{k-1,r-1,h-r-1}(\emptyset, \mathbb{P}^{r-1}, X_{C_2}).$$

In the special case when X_A is linear, C_2 is empty.

Proof. We set $V = \bigoplus_{i=1}^{h} (\bigoplus_{j_i=1}^{k_i+1} \Bbbk v_{i,j_i})$, with $t \cdot v_{i,j_i} = \lambda_i(t) v_{i,j_i}$ for all $t \in T$, $i = 1, \dots, h$, $j_i = 1, \dots, k_i + 1$, and $w_i = v_{i,k_i+1}$.

Assume that C is an r-pyramidal configuration, and let $X_C \subset \mathbb{P}(\bigoplus_{i=1}^h \Bbbk w_i)$. Then there exists a splitting $T = S_1 \times S_2$ such that, after the reordering of C, $C_1 = \{\lambda_1, \dots, \lambda_r\}$ is a basis of $\mathcal{X}(S_1)$ and $C_2 = \{\lambda_{r+1}, \dots, \lambda_h\} \subset \mathcal{X}(S_2)$ is a non-pyramidal configuration. Hence,

$$X_C = \mathrm{J}_{r-1,h-r-1}(X_{\mathrm{id}_r}, X_{C_2}) = \mathrm{J}_{r-1,h-r-1}\left(\mathbb{P}\left(\bigoplus_{i=1}^r \Bbbk v_{i,k_i+1}\right), X_{C_2}\right).$$

By Lemma 2.15, we can assume that $X_A = \mathrm{J}_{k-1,h-1}(\emptyset, X_C)$, and so

$$\begin{aligned} X_A &= \mathrm{J}_{k-1,h-1}(\emptyset, \mathrm{J}_{r-1,h-r-1}(X_{\mathrm{id}_r}, X_{C_2})) \\ &= \mathrm{J}_{k-1,h-1}\left(\emptyset, \mathrm{J}_{r-1,h-r-1}\left(\mathbb{P}\left(\bigoplus_{i=1}^r \Bbbk v_{i,k_i+1}\right), X_{C_2}\right)\right) \\ &= \mathrm{J}_{k-1,r-1,h-r-1}\left(\emptyset, \mathbb{P}\left(\bigoplus_{i=1}^r \Bbbk v_{i,k_i+1}\right), X_{C_2}\right), \end{aligned}$$

as claimed. □

2.3. *Dual of a projective toric variety*

We recall the classical notion of the dual variety of a projective variety.

DEFINITION 2.18. Let V be a $\Bbbk$-vector space of finite dimension and denote by $V^\vee$ its dual $\Bbbk$-vector space. Let $X \subset \mathbb{P}(V)$ be an irreducible projective variety. The *dual variety* of X is defined as the closure of the hyperplanes intersecting the regular part X_{reg} of X non-transversally:

$$X^* = \overline{\{[f] \in \mathbb{P}(V^\vee) : \exists x \in X_{\mathrm{reg}},\ f|_{T_x X} \equiv 0\}} \subset \mathbb{P}(V^\vee).$$

As usual, $T_x X$ denotes the embedded tangent space of X at $x \in X_{\mathrm{reg}}$.

Note that $\mathbb{P}(V)^* = \emptyset$. We set by convention $\emptyset^* = \mathbb{P}(V^\vee)$.

Self-duality is not an intrinsic property, it depends on the projective embedding. It can be proved that X^* is an irreducible projective variety and that $(X^*)^* = X$ (see, for example, [**12**]).

For a generic variety $X \subset \mathbb{P}(V)$, $\operatorname{codim} X^* = 1$. If $\operatorname{codim} X^* \neq 1$, then it is said that X has *defect* $\operatorname{codim} X^* - 1$.

DEFINITION 2.19. An irreducible projective variety $X \subset \mathbb{P}(V)$ is called *self-dual* if X is isomorphic to X^* as embedded projective varieties, that is, if there exists a (necessarily linear) isomorphism $\varphi : \mathbb{P}(V) \to \mathbb{P}(V^\vee)$ such that $\varphi(X) = X^*$.

A self-dual projective variety $X \subset \mathbb{P}^{n-1}$ of dimension $d - 1 < n - 1$ (that is, which is not a hypersurface) has positive defect $n - d - 1$. The defect of the whole projective space $\mathbb{P}^{n-1}$ is $n - 1$.

REMARK 2.20. Recall that given a basis $\mathcal{B} = \{v_1, \dots, v_n\}$ of V, we can identify $\mathbb{P}(V)$ with $\mathbb{P}(V^\vee)$ by means of $v_i \leftrightsquigarrow v_i^\vee$, where $\{v_1^\vee, \dots, v_n^\vee\}$ is the dual basis of $\mathcal{B}$. Then, via the choice of a basis of V, we can look at the dual variety inside the same projective space. Self-duality can be reformulated as follows: $X \subset \mathbb{P}(V)$ is self-dual if there exists $\varphi \in \operatorname{Aut}(\mathbb{P}(V))$ such that $\varphi(X) = X^*$.

Let V be a T-module of finite dimension n over a d-dimensional torus T and let A be the associated configuration of weights. In view of the considerations of the preceding subsections, we assume from now on and without loss of generality that $A = \{\lambda_1, \ldots, \lambda_n\} \subset \mathcal{X}(T)$ is a regular configuration, possibly with repeated elements, such that $\langle A \rangle_{\mathbb{Z}} = \mathcal{X}(T)$.

The regularity of A implies in particular the existence of a splitting $T = \Bbbk^* \times S$ as in Remark 2.7. Then X_A is a $(d-1)$-dimensional subvariety of the $(n-1)$-dimensional projective space $\mathbb{P}(V)$ and the lattice $\mathcal{R}_A$ has rank $n-d$.

The dual variety X_A^* has the following interpretation. For $[\xi] \in \mathbb{P}(V^\vee)$, let $f_\xi \in \Bbbk[T]$, $f_\xi(t) = \xi(t \cdot \sum v_i) \in \Bbbk[T]$. Then X_A^* is obtained as the closure of the set of those $[\xi] \in \mathbb{P}(V^\vee)$ such that there exists $t \in T$ with $f_\xi(t) = (\partial f_\xi / \partial t_i(t)) = 0$ for all $i = 1, \ldots, n$.

$$X_A^* = \overline{\left\{ \xi \in \mathbb{P}(V^\vee) : \exists t \in T, f_A(t) = \frac{\partial f_\xi}{\partial t_1}(t) = \frac{\partial f_\xi}{\partial t_2}(t) = \ldots = \frac{\partial f_\xi}{\partial t_d}(t) = 0 \right\}}.$$

In [6], a rational parametrization of the dual variety X_A^* was obtained. We adapt this result to our notation. As before, $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis of eigenvectors, $t \cdot v_i = \lambda_i(t) v_i$, and $\mathcal{B}_A = \{u_1, \ldots, u_{n-d}\}$ is a basis of $\mathcal{R}_A$. We denote by $\mathcal{R}_{A,\Bbbk}$ the $(n-d)$-dimensional $\Bbbk$-vector space $\mathcal{R}_A \otimes_{\mathbb{Z}} \Bbbk$ and we identify $\mathbb{P}(V)$ with $\mathbb{P}(V^\vee)$ by means of the chosen basis $\mathcal{B}$ of eigenvectors (and its dual basis) as in Remark 2.20.

Proposition 2.21 [6, Proposition 4.1]. *Let $T = \Bbbk^* \times S, V, A, \mathcal{B}, \mathcal{B}_A$ as before. Then the mapping $\mathbb{P}(\mathcal{R}_{A,\Bbbk}) \times S \to \mathbb{P}(V)$ defined by*

$$([a_1 : \ldots : a_n], s) \mapsto s \cdot \left[\sum a_i v_i \right]$$

has image dense in X_A^*; that is, the morphism

$$(\Bbbk^*)^{n-d} \times T \to \mathbb{P}(V), \quad (c, t) \mapsto t \cdot \left[\sum c_i u_i \right]$$

is a rational parametrization of X_A^*, and

$$X_A^* = \overline{\bigcup_{p \in \mathbb{P}(\mathcal{R}_{A,\Bbbk})} \mathcal{O}(p)} = \overline{T \cdot \mathbb{P}(\mathcal{R}_{A,\Bbbk})}.$$

This last equality, which expresses the dual variety as the closure of the union of the torus orbits of all the classes in the vector space of relations of the configuration A, is the starting point of our classification of self-dual projective toric varieties, which we describe in what follows.

3. *Characterization of self-duality in terms of orbits*

Let T be a torus of dimension d and V be a rational T-module of dimension n with associated configuration of weights $A = \{\lambda_1, \ldots, \lambda_n\}$. We assume that $\langle A \rangle_{\mathbb{Z}} = \mathcal{X}(T)$ and keep the notation of the preceding section. Given $p = [\sum p_i v_i] \in \mathbb{T}^{n-1}$, we denote by $m_p([\sum x_i v_i]) = [\sum p_i x_i v_i]$ the diagonal linear isomorphism defined by p.

3.1. *Non-pyramidal configurations*

In this subsection, we characterize self-dual projective toric varieties associated to a configuration of weights A which define a *non-pyramidal* configuration, in terms of the orbits of the torus action.

Note that the whole projective space $\mathbb{P}(V)$ can be seen as a toric projective variety associated to a $\dim V$-pyramidal configuration and its dual variety is empty. But we now show that, for non-pyramidal configurations, the dimension of the dual variety X_A^* cannot be smaller than the dimension of the toric variety X_A. This result has been proved by Zak [**17**] for any non-degenerate smooth projective variety.

Lemma 3.1. *If A is a non-pyramidal configuration, then* $\dim X_A^* \geqslant \dim X_A$.

Proof. Indeed, if A is not a pyramidal configuration, then, by Remark 2.11, we know that there exists $p = \sum p_i v_i \in \mathcal{R}_{A,\Bbbk}$ such that $p_i \neq 0$ for all $i = 1, \dots, n$. Hence, if we identify $\mathbb{P}(V)$ with $\mathbb{P}(V^\vee)$ by means of the dual basis, then

$$X_A^* = \overline{T \cdot \mathbb{P}(\mathcal{R}_{A,\Bbbk})} \supset \overline{\mathcal{O}([p])} = m_p\left(\overline{\mathcal{O}\left(\left[\sum_i v_i\right]\right)}\right) = m_p(X_A).$$

Since $p \in \mathbb{T}^{n-1}$, we have that $\dim m_p(X_A) = \dim X_A$ and the result follows. □

We identify $\mathbb{P}(V)$ with $\mathbb{P}(V^\vee)$ by means of the chosen basis $\mathcal{B}$ of eigenvectors (and its dual basis) as in Remark 2.20. The following is the main result of this subsection.

Theorem 3.2. *Let $A \subset \mathcal{X}(T)$ be a non-pyramidal configuration. The following assertions are equivalent.*

1. *The variety X_A is a self-dual projective variety.*
2. *There exists $p_0 \in \mathbb{P}(\mathcal{R}_{A,\Bbbk}) \cap \mathbb{T}^{n-1}$ such that $\mathbb{P}(\mathcal{R}_{A,\Bbbk}) \subset \overline{\mathcal{O}(p_0)}$.*
3. *There exists $p_0 \in \mathbb{P}(\mathcal{R}_{A,\Bbbk}) \cap \mathbb{T}^{n-1}$ such that $X_A^* = m_{p_0}(X_A)$.*
4. *For all $q \in \mathbb{P}(\mathcal{R}_{A,\Bbbk}) \cap \mathbb{T}^{n-1}$, $\mathbb{P}(\mathcal{R}_{A,\Bbbk}) \subset \overline{\mathcal{O}(q)}$.*
5. *For all $q \in \mathbb{P}(\mathcal{R}_{A,k}) \cap \mathbb{T}^{n-1}$, $X_A^* = m_q(X_A)$.*

Proof. We prove (1) ⇒ (5) and (2) ⇒ (4), the rest of the implications being trivial.

(1) ⇒ (5): By Proposition 2.21,

$$X_A^* = \overline{\bigcup_{p \in \mathbb{P}(\mathcal{R}_{A,\Bbbk})} \mathcal{O}(p)} \supset \overline{\bigcup_{p \in \mathbb{P}(\mathcal{R}_{A,\Bbbk}) \cap \mathbb{T}^{n-1}} \mathcal{O}(p)} \supset \overline{\mathcal{O}(q)} = m_q(X_A),$$

for all $q \in \mathbb{P}(\mathcal{R}_{A,\Bbbk}) \cap \mathbb{T}^{n-1}$. Since $\dim X_A = \dim X_A^*$, equality holds in the last equation.

(2) ⇒ (4): Let $p_0 \in \mathbb{P}(\mathcal{R}_{A,\Bbbk}) \cap \mathbb{T}^{n-1}$ be such that $\mathbb{P}(\mathcal{R}_{A,\Bbbk}) \subset \overline{\mathcal{O}(p_0)}$. If $q \in \mathbb{P}(\mathcal{R}_{A,\Bbbk}) \cap \mathbb{T}^{n-1}$, then $q \in \overline{\mathcal{O}(p_0)} \cap \mathbb{T}^{n-1} = \mathcal{O}(p_0)$. Then $\mathcal{O}(q) = \mathcal{O}(p_0)$ and the result follows. □

The equivalence between (1) and (5) in Theorem 3.2 implies that as soon as the dual of an equivariantly embedded projective toric variety of the form X_A has the same dimension of the variety, there exists a linear isomorphism between them.

Theorem 3.3. *Let $A \subset \mathcal{X}(T)$ be a configuration of weights that is non-pyramidal. Then X_A is self-dual if and only if* $\dim X_A = \dim X_A^*$.

This result is not true in general for projective toric varieties not equivariantly embedded, even for rational planar curves (for which the dual is again a curve, but not necessarily isomorphic).

3.2. *The general case*

We now address the complete characterization of self-dual projective toric varieties associated to an arbitrary configuration of weights $A \subset \mathcal{X}(T)$. We keep the notation of the preceding section.

We begin by recalling a well-known result about duality of projective varieties.

LEMMA 3.4 [**16**, Theorem 1.23]. *Let $X \subset \mathbb{P}^n$ be a non-linear irreducible subvariety.*

(1) *Assume that X is contained in a hyperplane $H = \mathbb{P}^{n-1}$. If ${X'}^*$ is the dual variety of X, when we consider X as a subvariety of $\mathbb{P}^{n-1}$, then X^* is the cone over ${X'}^*$ with vertex p corresponding to H.*

(2) *Conversely, if X^* is a cone with vertex p, then X is contained in the corresponding hyperplane H.*

When X is linear, $(X')^*$ is empty.

As an immediate application of Lemma 3.4, we have the following characterization of self-dual equivariantly embedded projective toric hypersurfaces. Note that the only linear varieties that are self-dual are the subspaces of dimension $k-1$ in $\mathbb{P}^{2k-1}$. In particular, the only hyperplanes that are self-dual are points in $\mathbb{P}^1$.

COROLLARY 3.5. *Let T be an algebraic torus and $A \subset \mathcal{X}(T)$ be a configuration such that X_A is a non-linear hypersurface. Then X_A is self-dual if and only if X_A is not a cone.*

Proof. Assume that X_A is a cone. Then, by Lemma 3.4, it follows that X_A^* is contained in a hyperplane, hence X_A is not self-dual.

If X_A is not a cone, then A is non-pyramidal (see Remark 2.13), and it follows from Lemma 3.1 that $\dim X_A^* \geqslant \dim X_A$. If $\dim X_A^* > \dim X_A$, then $X_A^* = \mathbb{P}(V)$ and hence $X_A = (X_A^*)^* = \emptyset$, which is a contradiction. It follows that $\dim X_A^* = \dim X_A$ and hence Theorem 3.3 implies that X_A is self-dual. □

Applying Lemma 3.4, we can reduce the study of duality of projective varieties to the study of non-degenerate projective varieties that are not a cone.

PROPOSITION 3.6. *Let $X \subset \mathbb{P}^{n-1}$ be an irreducible projective variety. Let $k-1$ be the codimension of the minimal subspace of $\mathbb{P}^{n-1}$ containing X. Then, with the notation of* Remark 2.16, *the following assertions hold*:

(1) *If $X = \mathrm{J}_{k-1,k-1,m-1}(\emptyset, \mathbb{P}^{k-1}, X_{\mathrm{nd}})$, with $X_{\mathrm{nd}} \subset \mathbb{P}^{m-1}$ self-dual, then X is self-dual.*

(2) *If X is self-dual, then $\dim X_{\mathrm{nd}} = \dim(X_{\mathrm{nd}})^*$, and $h = k$, that is*

$$X = \mathrm{J}_{k-1,k-1,m-1}(\emptyset, \mathbb{P}^{k-1}, X_{\mathrm{nd}}), \tag{3.1}$$

Proof. Let $X = \mathrm{J}_{k-1,h-1,m-1}(\emptyset, \mathbb{P}^{h-1}, X_{\mathrm{nd}})$. Applying recursively Lemma 3.4 (see Remark 2.16), we obtain that

$$\begin{aligned} X^* &= \mathrm{J}_{k-1,h-1,m-1}(\emptyset, \mathbb{P}^{h-1}, X_{\mathrm{nd}})^* = \mathrm{J}_{k-1,h+m-1}(\emptyset, \mathrm{J}_{h-1,m-1}(\mathbb{P}^{h-1}, X_{\mathrm{nd}}))^* \\ &= \mathrm{J}_{k-1,h+m-1}(\mathbb{P}^{k-1}, \mathrm{J}_{h-1,m-1}(\mathbb{P}^{h-1}, X_{\mathrm{nd}})^*) \\ &= \mathrm{J}_{k-1,h+m-1}(\mathbb{P}^{k-1}, \mathrm{J}_{h-1,m-1}(\emptyset, X_{\mathrm{nd}}^*)) \\ &= \mathrm{J}_{k-1,h-1,m-1}(\mathbb{P}^{k-1}, \emptyset, X_{\mathrm{nd}}^*) = \mathrm{J}_{h-1,k-1,m-1}(\emptyset, \mathbb{P}^{k-1}, X_{\mathrm{nd}}^*) \\ &= \mathrm{J}_{h-1,k+m-1}(\emptyset, \mathrm{J}_{k-1,m-1}(\mathbb{P}^{k-1}, X_{\mathrm{nd}}^*)). \end{aligned}$$

In particular, $\dim X^* = k + \dim X_{\mathrm{nd}}^*$, and the maximal subspace that contains X^* has codimension h.

To prove (1), assume that $h = k$ and X_{nd} is self-dual. Then $X^* = \mathrm{J}_{k-1,k-1,m-1}(\emptyset, \mathbb{P}^{k-1}, X_{\mathrm{nd}}^*)$. Since X_{nd} is self-dual, there exists an isomorphism $\varphi : \mathbb{P}^{m-1} \to \mathbb{P}^{m-1}$ such that $\varphi(X_{\mathrm{nd}}) = X_{\mathrm{nd}}^*$. It is clear that φ extends to an isomorphism $\tilde{\varphi} : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ such that $\tilde{\varphi}(X) = X^*$.

In order to prove (2), assuming X is self-dual and writing X as in Remark 2.16, it follows that $h = k$, and hence $h + \dim X_{\mathrm{nd}} = \dim X = \dim X^* = k + \dim X_{\mathrm{nd}}$. □

In our toric setting, Proposition 3.6 can be improved, so that we obtain a geometric characterization of self-dual projective toric varieties.

THEOREM 3.7. *Let A be an arbitrary lattice configuration. Then X_A is self-dual if and only if $\dim X_A = \dim X_A^*$ and the smallest linear subspaces containing X_A and X_A^* have the same (co)dimension.*

Proof. By Proposition 2.17,

$$X_A = \mathrm{J}_{k-1,h-1,m-1}(\emptyset, \mathbb{P}^{h-1}, X_{C_2}),$$

where $C_2 \subset A$ is a non-pyramidal configuration without repeated weights. By Theorem 3.3, $X_{C_2} \subset \mathbb{P}^{m-1}$ is self-dual if and only if $\dim X_{C_2} = \dim X_{C_2}^*$. The result follows now from Proposition 3.6. □

Combining Proposition 2.17 and Theorem 3.7, we obtain the following explicit combinatorial description of self-dual toric varieties.

THEOREM 3.8. *Let $A = \{\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2, \ldots, \lambda_h, \ldots, \lambda_h\} \subset \mathcal{X}(T)$ be a configuration of n weights with each λ_i appearing $k_i + 1$ times, $\lambda_i \neq \lambda_j$ if $i \neq j$. Let $C = \{\lambda_1, \ldots, \lambda_h\}$ be the associated configuration without repeated weights. Then X_A is self-dual if and only if the following assertions hold.*

(1) *The configuration C is k-pyramidal, where $k = n - h = \sum k_i$.*

(2) *There exists a splitting $T = S_1 \times S_2$ such that, after reordering of the elements in C, it holds that $C = C_1 \cup C_2$, where $C_1 = \{\lambda_1, \ldots, \lambda_k\}$ is a basis of $\mathcal{X}(S_1)$ and $C_2 = \{\lambda_{k+1}, \ldots, \lambda_h\} \subset \mathcal{X}(S_2)$ is a non-pyramidal configuration, as in Definition 2.10. Moreover, the S_2-toric projective variety $X_{C_2} \subset \mathbb{P}(\bigoplus_{i=k+1}^{h} \Bbbk w_i)$, $t \cdot w_i = \lambda_i(t) w_i$, is self-dual.*

It follows from Theorem 3.8 that if X_A is a self-dual toric variety with A pyramidal, then there are repeated weights in A. The converse of this statement does not hold. In the next example, we show a family of *non*-pyramidal configurations A with repetitions such that X_A is self-dual.

EXAMPLE 3.9. Let $C = \{c_1, \dots, c_s\} \subset \mathbb{Z}^{n-1}$ be any non-pyramidal configuration, such that X_C is self-dual. Then the configuration $A = \{e_1, e_1, (0, c_1), \dots, (0, c_s)\} \subset \mathbb{Z}^n$ has repeated weights, and X_A is self-dual by Theorem 3.8. It is straightfoward to check that A is non-pyramidal. Note that these configurations become pyramidal when we avoid repetitions.

4. *Characterizations of self-duality in combinatorial terms*

In this section, we characterize self-duality of projective toric varieties of type X_A in combinatorial terms. We make explicit calculations for the algebraic torus $(\mathbb{k}^*)^d$ acting on $\mathbb{k}^n$, in order to give an interpretation of the conditions of Theorem 3.2 in terms of the configuration A and in terms if its *Gale dual configuration*, whose definition we recall below.

We refer the reader to [**18**, Chapter 6] for an account of the basic combinatorial notions we use in what follows.

4.1. *Explicit calculations for* $(\mathbb{k}^*)^d$ *acting on* $\mathbb{k}^n$

Let $T = (\mathbb{k}^*)^d$. We identify the lattice of characters $\mathcal{X}(T)$ with $\mathbb{Z}^d$. Thus, any character $\lambda \in \mathcal{X}(T)$ is of the form $\lambda(t) = t^m$, where $m \in \mathbb{Z}^d$ and $t^m = t_1^{m_1} \dots t_d^{m_d}$. We take the canonical basis of $\mathbb{k}^n$ as the basis of eigenvectors of the action of T; that is, if $A = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{Z}^d$, T acts on $\mathbb{k}^n$ by $t \cdot (z_1, \dots, z_n) = (t^{\lambda_1} z_1, \dots, t^{\lambda_n} z_n)$ for all $t = (t_1, \dots, t_d) \in T$; then

$$X_A = \overline{\mathcal{O}([1 : \dots : 1])} = \overline{\{[t^{\lambda_1} : \dots : t^{\lambda_n}] : t \in (\mathbb{k}^*)^d\}} \subset \mathbb{P}^{n-1}.$$

By abuse of notation, we also set $A \in \mathcal{M}_{d \times n}(\mathbb{Z})$ the matrix with columns the weights λ_i. In view of the reductions made in Section 2, we assume without loss of generality that the first row of A is $(1, \dots, 1)$ and that the columns of A span $\mathbb{Z}^d$.

The homogeneous ideal I_A in $\mathbb{k}[x_1, \dots, x_n]$ of the associated projective toric variety X_A is the binomial ideal [**15**]

$$I_A = \left\langle x^a - x^b : \ a, b \in \mathbb{N}^n, \sum_{i=1}^n a_i \lambda_i = \sum_{i=1}^n b_i \lambda_i \right\rangle.$$

Thus, $X_A = \{[x] \in \mathbb{P}^{n-1} : \ x^a = x^b, \ \forall \, a, b \in \mathbb{N}^n \text{ such that } Aa = Ab\}$, and it is easy to see that

$$X_A = \{[x] \in \mathbb{P}^{n-1} : \ x^{v^+} - x^{v^-} = 0, \ \forall \, v \in \mathcal{R}_A\},$$

where $v_i^+ = \max\{v_i, 0\}$, $v_i^- = -\min\{v_i, 0\}$ (and so $v = v^+ - v^-$).

For $p \in \mathbb{T}^{n-1}$ we then have

$$m_p(X_A) = \overline{\mathcal{O}(p)} = \{[x] \in \mathbb{P}^{n-1} : p^{v^-} x^{v^+} - p^{v^+} x^{v^-} = 0, \ \forall \, v \in \mathcal{R}_A\}.$$

4.2. *Characterization of self-duality in terms of the Gale dual configuration*

If A is a non-pyramidal configuration, then Theorem 3.2 can be rephrased in terms of a geometric condition on the Gale dual of A.

DEFINITION 4.1. Let $A \in \mathcal{M}_{d \times n}(\mathbb{Z})$ with rank d. Let $\mathcal{B}_A = \{u_1, \dots, u_{n-d}\} \subset \mathbb{Z}^n$ be a basis of $\mathcal{R}_A$.

We say that the matrix $B_A \in \mathcal{M}_{n\times(n-d)}(\mathbb{Z})$ with columns the vectors u_i is a *Gale dual matrix* of A. Let $\mathcal{G}_A = \{b_1, \dots, b_n\} \subset \mathbb{Z}^{n-d}$ be the configuration of rows of B_A, that is, $B_A = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ (observe that we allow repeated elements). The configuration $\mathcal{G}_A$ is called a *Gale dual configuration* of A. Remark that $\sum_{i=1}^n b_i = 0$.

Remark 4.2. (1) Since $\mathcal{R}_A$ is an affine invariant of the configuration A, it follows that two affinely equivalent configurations share their Gale dual configurations.

(2) The configuration A is non-pyramidal if and only if $b_i \neq 0$ for all $i = 1, \dots, n$.

(3) When A is regular, $\mathcal{R}_A$ is the integer kernel $\mathrm{Ker}_{\mathbb{Z}}(A)$ of the matrix A.

(4) For any Gale dual matrix of A, the morphism $\gamma : \mathbb{k}^{n-d} \to \mathbb{k}^n$, $\gamma(s) = (\langle s, b_1\rangle, \dots, \langle s, b_n\rangle)$ gives a parametrization of $\mathcal{R}_{A,\mathbb{k}}$, where we denote $\langle s, b_i\rangle = \sum_{j=1}^{n-d} s_j b_{ij}$.

Remark 4.3. By Theorem 3.2, X_A is self-dual if and only if there exists $p_0 \in \mathbb{P}(\mathcal{R}_{A,\mathbb{k}}) \cap \mathbb{T}^{n-1}$ such that $\mathbb{P}(\mathcal{R}_{A,\mathbb{k}}) \subset \overline{\mathcal{O}(p_0)}$. By the remarks in Section 4.1, it follows that X_A is self-dual if and only if, for some such p_0, we have that $p_0^{v^-} w^{v^+} - p_0^{v^+} w^{v^-} = 0$ for all $w \in \mathcal{R}_{A,\mathbb{k}}$ and $v \in \mathcal{R}_A$.

Assume that X_A is self-dual. Then, given any choice of Gale dual configuration, we deduce that, for all $s \in \mathbb{k}^{n-d} \setminus \{0\}$ and $j = 1, \dots, n-d$, we have that

$$p_0^{u_j^-} (\langle s, b_1\rangle, \langle s, b_2\rangle, \dots, \langle s, b_n\rangle)^{u_j^+} = p_0^{u_j^+} (\langle s, b_1\rangle, \langle s, b_2\rangle, \dots, \langle s, b_n\rangle)^{u_j^-},$$

for (some, or in fact all) $p_0 \in \mathbb{P}(\mathcal{R}_{A,\mathbb{k}}) \cap \mathbb{T}^{n-1}$.

Since this gives an equality in the polynomial ring $\mathbb{k}[s_1, \dots, s_{n-d}]$, both sides must have the same irreducible factors. But $\langle s, b_i\rangle$ and $\langle s, b_k\rangle$ are associated irreducible factors if and only if b_i and b_k are collinear vectors. We deduce that, for any line L in B-space $\mathbb{Z}^{n-d}$ and for all j,

$$\sum_{b_i \in L, b_{ij} > 0} b_{ij} = - \sum_{b_i \in L, b_{ij} < 0} b_{ij}.$$

Hence, $\sum_{b_i \in L} b_{ij} = 0$ for all $j = 1, \dots, n-d$ or, equivalently, $\sum_{b_i \in L} b_i = 0$.

In fact, this last condition is not only necessary but also sufficient. We give a proof of both implications using results about the tropicalization of the dual variety X_A as described in [**6**].

First, we recall that given a dual Gale configuration $\mathcal{G}_A = \{b_1, \dots, b_n\}$, and a subset $J \subset \{1, \dots, n\}$, the *flat* S_J of $\mathcal{G}_A$ associated to J is the subset of all the indices $i \in \{1, \dots, n\}$ such that b_i belongs to the subspace generated by $\{b_j : j \in J\}$.

Theorem 4.4. *Let $A \in \mathcal{M}_{d\times n}(\mathbb{Z})$ be a non-pyramidal configuration, and B_A be a Gale dual for A as in* (4.1). *Then X_A is self-dual if and only if, for any line L through the origin in $\mathbb{Z}^{n-d}$, we have that $\sum_{b_i \in L} b_i = 0$.*

Proof. Since we are dealing with affine invariants, we can assume that A is a regular configuration. By Theorem 3.3, we know that X_A is self-dual if and only if $\dim X_A$ equals $\dim X_A^*$. Given a vector $v \in \mathbb{Z}^n$, we define a new vector $\sigma(v) \in \{0,1\}^n$ by $\sigma(v)_i = 0$ if $v_i \neq 0$ and $\sigma(v)_i = 1$ if $v_i = 0$.

If follows from [**6**, Corollary 4.5] that $\dim X_A = \dim X_A^*$ if and only if, for any vector $v \in \mathcal{R}_A$, the vector $(1, \dots, 1) - \sigma(v)$ lies in the row span F of the matrix A. But since we are assuming that $(1,\dots,1) \in F$, this is equivalent to the condition that $\sigma(v) \in F$. By duality, this is in turn equivalent to the fact that, for any $j = 1,\dots,n-d$, the inner product

$$\langle \sigma(v), u_j \rangle = \sum_{v_i=0} b_{ij} = 0;$$

that is to say, X_A is self-dual if and only if, for any $v \in \mathcal{R}_A$, the sum $\sum_{v_i=0} b_i = 0$. But the sets S of non-zero coordinates of the vectors in the space of linear relations $\mathcal{R}_A$ coincide with the flats of the Gale configuration $\mathcal{G}_A$. So, X_A is self-dual if and only if, for any flat $S \subset \{1,\dots,n\}$, the sum $\sum_{i\in S} b_i = 0$. It is clear that this happens if and only if the same condition holds for all the one-dimensional flats, that is, if, for any line L through the origin, the sum $\sum_{b_i \in L} b_i = 0$. □

The assumption that A is a non-pyramidal configuration in Theorem 4.4 is crucial, as the following example shows.

EXAMPLE 4.5. Let A be a configuration such that $\mathcal{R}_A$ has rank 1. Then $\mathcal{R}_A$ is spanned by a single vector, whose coordinates add up to 0. So, the condition in Theorem 4.4 that the sum of the b_i in this line equals 0 is satisfied. But by Corollary 3.5 if A is a pyramid, then X_A is not self-dual.

4.3. *Geometric characterization of self-dual configurations*

In this paragraph, we characterize the non-pyramidal configurations $A \subset \mathbb{Z}^d$ whose Gale dual configurations are as in Theorem 4.4. We keep the assumptions that $\langle A\rangle_{\mathbb{Z}} = \mathbb{Z}^d$ and that A is non-pyramidal. We begin with some basic definitions about configurations.

DEFINITION 4.6. Given $a = (a_1,\dots,a_n) \in \mathcal{R}_{A,\Bbbk}$, we call $\{i : a_i \neq 0\}$ the *support* of the relation and define $\operatorname{supp}(a) = \{i : a_i \neq 0\}$. We say that λ_i belongs to the relation if $i \in \operatorname{supp}(a)$.
Recall that any affine relation $a \in \mathcal{R}_{A,\Bbbk}$ satisfies $\sum_i a_i = 0$. It is said that a is a *circuit* if there is no non-trivial affine dependency relation with support strictly contained in $\operatorname{supp}(a)$. In other words, a circuit is a minimal affine dependency relation.

REMARK 4.7. Let C be a circuit of a configuration A and let F be the minimal face of Conv(A) containing C. If d' denotes the dimension of the affine span of F, then C has at most $d'+2$ elements.

DEFINITION 4.8. Two elements b, b' of a configuration B are *parallel* if they generate the same straight line through the origin. In particular, $b \neq 0$ and $b' \neq 0$. The elements b, b' are *antiparallel* if they are parallel and point into opposite directions.
Two elements λ, λ' of a configuration A are *coparallel* if they belong exactly to the same circuits.

REMARK 4.9. (1) Coparallelism is an equivalence relation. We denote by $\operatorname{cc}(\lambda)$ the coparallelism class of the element $\lambda \in A$.
(2) It is easy to see that λ and λ' are coparallel if and only if they belong to the same affine dependency relations.
(3) The definition of coparallelism can be extended to pyramidal configurations as follows. If $\lambda \in A$ is such that it does not belong to any dependency relation, then $\operatorname{cc}(\lambda) = \{\lambda\}$. Otherwise,

$\mathrm{cc}(\lambda)$ consists, as above, of all elements of A belonging to the same circuits as λ. The condition that A is not a pyramid is then equivalent to the condition that $|\,\mathrm{cc}(\lambda)| \geqslant 2$ for all $\lambda \in A$.

LEMMA 4.10. *Let $\mathcal{G}_A = \{b_1, \dots, b_n\}$ be a Gale dual of A. Then λ_i is coparallel to λ_j if and only if b_i and b_j are parallel elements of $\mathcal{G}_A$.*

Proof. Let B_A be the $(n \times (n-d))$-matrix with rows given by $\mathcal{G}_A$ as in Definition 4.1. As A is not a pyramid, no row b_i of B_A is zero. Any element $a \in \mathcal{R}_{A,\Bbbk}$ is of the form $B_A \cdot m$ for some $m \in \Bbbk^{n-d}$. Then λ_i is coparallel to λ_j if and only if, for any non-zero $m \in \Bbbk^{n-d}$, it holds that $\langle b_i, m\rangle \neq 0$ precisely when $\langle b_j, m\rangle \neq 0$. It is clear that this happens if and only if $b_i = \alpha b_j$ for a non-zero constant $\alpha \in \Bbbk$, that is, if and only if b_i, b_j are parallel. □

DEFINITION 4.11. Let $A = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{Z}^d$ be a configuration. A subconfiguration $C' \subset A$ is called *facial* if there exists a face F of the convex hull $\mathrm{Conv}(A) \subset \mathbb{R}^d$ of A such that $C' = A \cap F$.

A subconfiguration $C \subset A$ is a *face complement* if $A \setminus C$ is a facial subconfiguration of A.

REMARK 4.12. Let $A = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{Z}^d$ be a configuration. A subconfiguration $C = \{\lambda_{i_1}, \dots, \lambda_{i_h}\} \subset A$ is a face complement if and only if there exists a dependency relation such that

$$\sum_{j=1}^{h} r_{i_j} b_{i_j} = 0, \quad r_{i_j} > 0.$$

Indeed, a dependency relation $\sum_{j=1}^{h} r_{i_j} b_{i_j} = 0$ with all $r_{i_j} > 0$ can be extended with zero coordinates to a relation $r = (r_1, \dots, r_n)$ among $b_1, \dots, b_n$. Thus, r lies in the row space of A and so there exists $\ell = (\ell_1, \dots, \ell_d)$ such that $r_i = \langle \ell, \lambda_i\rangle$. It follows that the linear form associated to ℓ vanishes on the complement of C, and all the points of C lie in the same open half-space delimited by the kernel of ℓ.

LEMMA 4.13. *Let $A = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{Z}^d$ be a configuration. A coparallelism class $C = \{\lambda_{i_1}, \dots, \lambda_{i_h}\} \subset A$ is a face complement if and only if there exist $j, k \in \{1, \dots, h\}$ such that b_{i_j} and b_{i_k} are antiparallel.*

Proof. If C is a coparallelism class, then we know by Lemma 4.10 that all $b_{i_1}, \dots, b_{i_h}$ are parallel. It is then clear that a dependency relation r as in Remark 4.12 exists if and only if two of the vectors b_{i_j}, b_{i_k} are antiparallel. □

DEFINITION 4.14. Let $A \subset \mathbb{Z}^d$ be a configuration and $C \subset A$ be a face complement. We say that C is a *parallel face complement* if C and $A \setminus C$ lie in parallel hyperplanes.

Note that in this case both C and $A \setminus C$ are facial.

EXAMPLE 4.15. In Figure 1, there are three configurations of six lattice points in three-dimensional space (the six vertices in each polytope). The two vertices marked with big dots in each of the configurations define a coparallelism class C. In the first polytope (1), C is not a face complement; in the second polytope (2), C is a face complement but not a parallel face complement; in the third polytope (3), C is a parallel face complement. The characterization

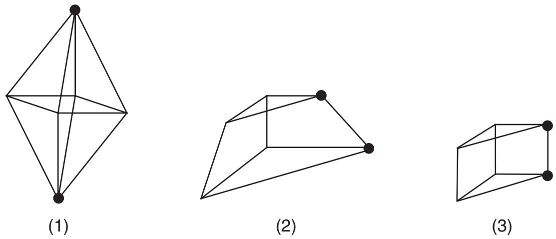

FIGURE 1. *Only configuration* (3) *is self-dual.*

in our next theorem proves that only the toric variety corresponding to this last configuration is self-dual.

It is straightforward to check that if A_1, A_2 are affinely equivalent configurations and φ is an affine linear map sending bijectively A_1 to A_2, then φ preserves coparallelism classes, faces and parallelism relations. Indeed, all these notions can be read in a common Gale dual configuration. Moreover, we can translate Theorem 4.4 as follows.

THEOREM 4.16. *Let $A \subset \mathbb{Z}^d$ be a non-pyramidal configuration. The projective toric variety X_A is self-dual if and only if any coparallelism class of A is a parallel face complement.*

Proof. Let $\mathcal{G}_A$ be a Gale dual of A as in Definition 4.1. By Lemma 4.10, coparallelism classes $C = \{\lambda_{i_1}, \ldots, \lambda_{i_h}\}$ in A are in correspondence with parallel vectors $b_{i_1}, \ldots, b_{i_h}$ in the dual space (that is, lines containing vectors of $\mathcal{G}_A$). But now, C is a parallel face complement if and only if there exists $\ell = (\ell_1, \ldots, \ell_d)$ such that $\langle \ell, \lambda_i \rangle = 0$ for all $\lambda_i \notin C$ and $\langle \ell, \lambda_{i_j} \rangle = 1$ for all $j = 1, \ldots, h$. Reciprocally, the sum of the vectors $\sum_{j=1}^{h} b_{i_j} = 0$ implies the existence of such an ℓ as in Remark 4.12. The result now follows from Theorem 4.4. □

We have the following easy lemma.

LEMMA 4.17. *Assume that A is a non-pyramidal self-dual lattice configuration. Then, for any $\mu \in A$, the coparallelism class $\operatorname{cc}(\mu)$ has at least two elements and it is a facial subconfiguration of A.*

Proof. It follows from Definition 4.14 that there exists a linear function f taking value 0 on $A \setminus \operatorname{cc}(\mu)$ and value 1 on $\operatorname{cc}(\mu)$. Then $\operatorname{cc}(\mu)$ is the facial subconfiguration of A supported by the hyperplane $(f-1) = 0$. If $\operatorname{cc}(\mu) = \{\mu\}$, then, by Theorem 4.16, $\{\mu\}$ is a vertex, and hence A would be a pyramid. It follows that so $|\operatorname{cc}(\mu)| \geqslant 2$, for any $\mu \in A$. □

We give in Lemma 5.4(2) an example of a self-dual lattice configuration A that contains an interior point of $\operatorname{Conv}(A)$. However, this cannot happen if X_A is not a hypersurface, as the following proposition shows.

PROPOSITION 4.18. *Let $A \subset \mathcal{X}(T)$ be a configuration without repetitions such that X_A is self-dual, with* $\operatorname{codim} X_A > 1$. *Then, the interior of the convex hull* $\operatorname{Conv}(A)$ *does not contain* *elements of A and, for any facial subconfiguration C' of A, at most one point of C' lies in the relative interior of* $\mathrm{Conv}(C')$.

Proof. Since $A = \{\lambda_1, \dots, \lambda_n\} \subset \mathcal{X}(T)$ has no repeated elements, it follows from Theorem 3.8 that A is non-pyramidal. Then, as X_A is not a hypersurface, if follows from Remark 2.4 that $n \geqslant d+3$, where d is the dimension of the affine span of A.

Assume that there exists $\mu \in A$ belonging to the relative interior of an s-dimensional face F of $\mathrm{Conv}(A)$. Therefore, μ is a convex combination of the vertices of F, and thus $\mathrm{cc}(\mu) \subset F$. But, by Lemma 4.17, $\mathrm{cc}(\mu)$ is a facial subconfiguration of A, and thus a facial subconfiguration of $F \cap A$, which intersects the relative interior of F. Then $\mathrm{cc}(\mu) = F \cap A$. Let $\mathrm{cc}(\mu) = \{\mu, \lambda_1, \dots, \lambda_r\}$. We claim that $\{\lambda_1, \dots, \lambda_r\}$ are affinely independent and thus $\mathrm{cc}(\mu)$ is a circuit. Indeed, for any $i = 1, \dots, r$, $\mathrm{cc}(\lambda_i) = \mathrm{cc}(\mu) = F \cap A$, and so there cannot be any non-trivial affine dependence relation involving only $\{\lambda_1, \dots, \lambda_r\}$. In particular, $r = s+1$, $\{\lambda_1, \dots, \lambda_{s+1}\}$ are the vertices of F and μ is the only point in $F \cap A$ belonging to the relative interior of $\mathrm{Conv}(F)$.

Therefore, if the relative interior of $\mathrm{Conv}(A)$ contains one element $\mu \in A$, it follows that A is a circuit, and hence $n = d+2$ (see Remark 4.7). That is, X_A is a hypersurface. □

Example 4.19. Consider the self-dual configuration A given by the columns of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

The associated toric variety has dimension 3 in $\mathbb{P}^5$, so it is not a hypersurface. No point of $A = \mathrm{Conv}(A) \cap \mathbb{Z}^4$ lies in the interior, but there are two facial subconfigurations of A (namely, the segments with vertices $\{(1,0,0,0),(1,0,2,0)\}$ and $\{(0,1,0,0),(0,1,0,2)\}$, respectively) which do have a point of A in their relative interior. Note that $X_A = \{(x_1, \dots, x_6) \in \mathbb{P}^5 / x_2^2 - x_1x_3 = x_4^2 - x_5x_6\}$ is not smooth. It is a complete intersection, but the four fixed points $(0,0,1,0,0,0), (1,0,0,0,0,0), (0,0,0,1,0,0), (0,0,0,0,0,1)$ are not regular, as can be checked by the drop in rank of the Jacobian matrix. This could be seen directly in the geometry of the configuration. The convex hull of A is a simple polytope (in fact, it is a simplex) of dimension 3 lying in the hyperplane $H = \{(y_1, \dots, y_4) \in \mathbb{R}^4 / y_1 + y_2 = 1\}$, but fixing the origin at any of the four vertices, the first lattice points in the three rays from that vertex do not form a basis of the lattice $H \cap \mathbb{Z}^4$. Note that there is a splitting of the 4-torus T as a product of tori of dimension 2 corresponding to the first three and last three weights in A.

We end this paragraph by showing another interesting combinatorial property of configurations associated to self-dual toric varieties.

Proposition 4.20. *Let $A = \{\lambda_1, \dots, \lambda_n\} \subset \mathcal{X}(T)$ be a non-pyramidal configuration such that X_A is self-dual and let D be an arbitrary non-empty subset of A. Then, either D is a pyramidal configuration or X_D is self-dual and, moreover, D is a facial subconfiguration of A.*

Proof. Assume that $D = \{\lambda_1, \dots, \lambda_s\} \subset A$ is non-pyramidal, and consider $\mathcal{R}_D \subset \mathbb{Z}^s$. It is clear that $\mathcal{R}_D \times \{0\} \subset \mathcal{R}_A$. Hence, if $\mathcal{B}_D$ is a basis of $\mathcal{R}_D$, then there exists a $\mathbb{Q}$-basis of $\mathcal{R}_A \otimes \mathbb{Q}$ of the form $\mathcal{G}_D \times \{0\} \cup \mathcal{C}$. Let $\mathcal{B}_A$ be a $\mathbb{Z}$-basis of $\mathcal{R}_A$, and $\mathcal{G}_A = \{b_1, \dots, b_n\}$ be its associated

Gale dual configuration. Then there exists an invertible $\mathbb{Q}$-matrix M such that

$$B' = \left(\begin{array}{c|c} B_D & C_1 \\ \hline 0 & C_2 \end{array}\right) = \begin{pmatrix} b_1 \\ \vdots \\ b_s \\ b_{s-1} \\ \vdots \\ b_n \end{pmatrix} M.$$

Since A is self-dual, it follows from Theorem 4.4 that the rows b_i are such that the sum of vectors b_i in the same line through the origin is zero. Hence, the matrix B' satisfies the same property. As D is non-pyramidal, no row of B_D is zero. Therefore, (B_D, C_1), and hence B_D, also satisfy the property that the sum of all its row vectors in a line through the origin is equal to zero. Hence, X_D is self-dual. Moreover, the sum of the row vectors of C_2 is zero, and it follows from Remark 4.12 that D is facial. □

5. *Families of self-dual projective varieties*

In this section, we use our previous results in order to obtain new families of projective toric varieties that are self-dual. In particular, we obtain many new examples of non-smooth self-dual projective varieties. We also identify all the smooth self-dual projective varieties of the form X_A. We retrieve in this (toric) case Ein's result, without needing to rely on Hartshorne's conjecture.

5.1. *Projective toric varieties associated to Lawrence configurations*

DEFINITION 5.1. We say that a configuration A of $2n$ lattice points is *Lawrence* if it is affinely equivalent to a configuration whose associated matrix has the form

$$\begin{pmatrix} \mathrm{Id}_n & \mathrm{Id}_n \\ 0 & M \end{pmatrix}, \tag{5.1}$$

where Id_n denotes the $n \times n$ identity matrix. Equivalently, A is a Lawrence configuration if it is affinely equivalent to a Cayley sum of n subsets, each one containing the vector 0 and one of the column vectors of M.

Lawrence configurations are a special case of *Cayley configurations* (see [**3**]). The Lawrence configuration associated to the matrix (5.1) is the Cayley configuration of the two-point configurations consisting of the origin and one column vector of M. In the smooth case, Cayley configuration of strictly equivalent polytopes correspond to toric fibrations (see [**8**]).

It is straightforward to verify that if A is Lawrence, then the following conditions hold:

(i) $\mathcal{R}_A = \{\left(\begin{smallmatrix} -v \\ v \end{smallmatrix}\right) : v \in \mathrm{Ker}_{\mathbb{Z}}(M)\}$; and
(ii) A is pyramidal if and only if M is pyramidal.

We immediately deduce from Theorem 4.4 the following result.

COROLLARY 5.2. *If A is a non-pyramidal Lawrence matrix, then X_A is self-dual.*

EXAMPLE 5.3. The well-known fact that the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{m-1}$ in $\mathbb{P}^{2m-1}$ is self-dual follows directly from Corollary 5.2, the image of the Segre morphism

$$\varphi(x, y) = [y_0x_0 : y_1x_0 : \ldots : y_mx_0 : y_0x_1 : y_1x_1 : \ldots : y_mx_1],$$

where $x = [x_0, x_1], y = [y_0 : y_1 : \ldots : y_m]$, is a projective toric variety with associated matrix

$$A = \begin{pmatrix} 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 \\ 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

The sum of the first two rows equals the sum of the last m rows. It is easy to see that A is affinely equivalent to the configuration A' with associated matrix

$$A' = \begin{pmatrix} \mathrm{Id}_n & \mathrm{Id}_n \\ 0 \cdots 0 & 1 \cdots 1 \end{pmatrix}. \tag{5.2}$$

The matrix A' is a non-pyramidal Lawrence matrix, hence $X_{A'} = X_A$ is self-dual.

We finish this paragraph by proving that Segre embeddings of $\mathbb{P}^1 \times \mathbb{P}^{m-1}$, $m \geqslant 2$ are the unique smooth self-dual projective toric varieties that are not a hypersurface. We begin with an easy lemma that classifies all smooth hypersurfaces of the form X_A.

Lemma 5.4. *Let A be a lattice configuration such that X_A is a smooth hypersurface. Then, A is of one of the following forms*:

(1) *A consists of two equal points, and so $X_A = \{(1:1)\} = \{(x_0 : x_1) \in \mathbb{P}^1 / x_0 - x_1 = 0\}$;*
(2) *A consists of three collinear points with one of them the mid point of the others, and so $X_A = \{(x_0 : x_1 : x_2) \in \mathbb{P}^2 / x_1^2 - x_0 x_2 = 0\}$; and*
(3) *A consists of four points a, b, c, d with $a + c = b + d$, and so $X_A = \{(x_0, x_1, x_2, x_3) \in \mathbb{P}^3 / x_0 x_3 - x_1 x_2 = 0\}$ is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$.*

Proof. When X_A is a hypersurface, an equation for X_A is given by $b_A(x) = \prod_{b_i > 0} x_i^{b_i} - \prod_{b_i < 0} x_i^{-b_i}$, where the transpose of the row vector $(b_1, \ldots, b_n)$ is a choice of Gale dual of A. The cases (1)–(3) in the statement correspond to the row vectors $(1, 1), (1, -2, 1)$ and $(1, -1, -1, 1)$ (or any permutation of the coordinates), and it is straightforward to check that X_A is smooth. It is easy to verify that in any other case, there exists a point $x \in X_A$ where b_A and all its partial derivatives vanish at x. □

We saw in Example 4.19 that a non-pyramidal self-dual lattice configuration A with $\operatorname{codim}(X_A) > 1$ can have a point in the interior of a proper face. Moreover, more complicated situations can happen.

Example 5.5. Consider the following dimension 3 configuration $A \subset \mathbb{Z}^4$, $A = \{(1,0,0,2), (1,0,0,0), (0,1,0,0), (0,1,0,2), (0,0,1,0), (0,0,1,1)\}$. Then, $\mathbb{Z}A = \mathbb{Z}^4$ and X_A is self-dual because the following is a choice of Gale dual $B \in \mathbb{Z}^{6\times 2}$:

$$B = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

All the points in A are vertices of the polytope $P := \mathrm{Conv(A)}$, but $A \neq P \cap \mathbb{Z}^4$. Indeed, there is a lattice point in the middle of each of the segments $[(1,0,0,2), (1,0,0,0)]$, $[0,1,0,0), (0,1,0,2)]$, which are faces of P. It is clear that X_A is not smooth (for instance, looking at the first lattice points in all the edges emanating from $(1,0,0,0)$), nor embedded by a complete linear system.

However, the following result shows that when X_A is smooth and self-dual, the situation is nicer.

LEMMA 5.6. *Let A be a lattice configuration without repeated points such that X_A is self-dual and* smooth. *Then, unless X_A is the quadratic rational normal curve in* (2) *of Lemma* 5.4, *no facial subconfiguration $C \subseteq A$ contains a point of A in the relative interior of* $\mathrm{Conv}(C)$.

Proof. Assume $A = \{\lambda_1, \ldots, \lambda_n\}$ has no repeated points and there exists $\mu \in A$ and a proper face F of $\mathrm{Conv}(A)$ containing μ in its relative interior. Then $F \cap A$ is not a pyramid, and it follows from Proposition 4.20 that $X_{F\cap A}$ is self-dual. Since $X_{F\cap A}$ is also smooth, Proposition 4.18 implies that $X_{F\cap A}$ is a hypersurface. We deduce from Lemma 5.4 that $F \cap A$ has dimension 1 and consists (up to reordering) of three points $\{\lambda_1, \lambda_2, \lambda_3\}$ with $\lambda_1 + \lambda_3 = 2\lambda_2$. We can choose a Gale dual B of A of the form

$$B = \left(\begin{array}{c|c} B_1 & C_1 \\ \hline 0 & C_2 \end{array}\right),$$

with B_1 the 3×1 column vector with transpose $(1, -2, 1)$. We see that the coparallelism class of each λ_i is contained in $F \cap A$ and no class can consist of a single element because A is not a pyramid. Therefore, $\mathrm{cc}(\lambda_i) = F \cap A, i = 1, 2, 3$; that is, any two of the first three rows of B are linearly dependent. We can thus find another choice of Gale dual B' of A of the form

$$B = \left(\begin{array}{c|c} B_1 & 0 \\ \hline 0 & C_2 \end{array}\right).$$

Then there is a splitting of the torus and X_A cannot be smooth, with arguments similar to those in Example 4.19, because A has no repeated points and so there is no linear equation in the ideal I_A. □

We now characterize the Segre embeddings $\mathbb{P}^1 \times \mathbb{P}^{m-1}$ in $\mathbb{P}^{2m-1}$ from Example 5.3 in terms of the Gale dual configuration.

LEMMA 5.7. *A toric variety $X_A \subset \mathbb{P}^{2m-1}$ is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{m-1}$ if and only if any Gale dual $B \in \mathbb{Z}^{2m\times r}$ of A has the following form*: $r = m - 1$ *and, up to reordering, the rows of $b_1, \ldots, b_{2m}$ of B satisfy* $\det(b_1, \ldots, b_{m-1}) = 1, b_1 + \ldots + b_m = 0$ *and* $b_{m+j} + b_j = 0$, *for all* $j = 1, \ldots, m$.

Proof. It is clear that any Gale dual to the matrix A' in (5.2) is of this form. And it is also straightforward to check that any matrix B as in the statement is a Gale dual of this A'. □

We can now prove the complete characterization of smooth self-dual varieties X_A.

THEOREM 5.8. *The only self-dual smooth non-linear projective toric varieties equivariantly embedded are the toric hypersurfaces described in* (2) *and* (3) *of Lemma* 5.4 *and the Segre embeddings* $\mathbb{P}^1 \times \mathbb{P}^{m-1}$ *in* $\mathbb{P}^{2m-1}$ *for* $m \geqslant 3$.

Proof. We proceed by induction in the codimension of A. By Lemma 5.4, the result is true when X_A is a hypersurface. Assume then that $\operatorname{codim}(X_A) > 1$. Now, by Lemma 5.6, we know that all the points in A are vertices of $\operatorname{Conv}(A)$. Let C be a coparallelism class and let $D := A \setminus C$. Then X_D is smooth and it is non-pyramidal. Indeed, we can choose a Gale dual B of A of the form

$$B = \left(\begin{array}{c|c} b_{11} & \\ \vdots & 0 \\ b_{r1} & \\ \hline b_{r+1,1} & \\ \vdots & D_2 \\ b_{n1} & \end{array}\right),$$

where $(b_{11}, 0), \dots, (b_{r1}, 0)$ correspond to the elements of C. If D is a pyramid, then it is easy to show that at least one row of D_2 must be zero, and it follows that the corresponding point of the configuration belongs also to C, and thus is a contradiction.

Hence, it follows from Proposition 4.20 that X_D is self-dual with $\operatorname{codim}(X_D) = \operatorname{codim}(X_A) - 1 < \operatorname{codim}(X_A)$ and no point of D belongs to the relative interior of $\operatorname{Conv}(D)$. Therefore, by induction, X_D is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{m'-1}$ in $\mathbb{P}^{2m'-1}$ for $m' \geqslant 2$ (including the hypersurface case $\mathbb{P}^1 \times \mathbb{P}^1$). In particular, $|D| = 2m'$ is even.

Assume $C = \{\mu_1, \dots, \mu_r\}$. Let $B_D \in \mathbb{Z}^{2m' \times (m'-1)}$ be a choice of Gale dual of D as in Lemma 5.7, with rows $e'_1, \dots, e'_{m'}, -e'_1, \dots, -e'_{m'}$ with $\{e'_1, \dots, e'_{m'-1}\}$ a basis of $\mathbb{Z}^{m'-1}$ and $e'_1 + \dots + e'_m = 0$. Add another integer affine relation with coprime entries as the first column, to form a matrix B' whose columns are a $\mathbb{Q}$-basis of relations of A of the form

$$B' = \left(\begin{array}{c|c} B_1 & 0 \\ \hline B_2 & B_D \end{array}\right).$$

Now, each coparallelism class of any $\mu \in D$ (with respect to D) has two elements when $m' > 2$, and so it cannot be 'broken' when considering coparallelism classes in A, since it is not a pyramid. Then, via column operations, we can assume that B_2 is of the form $B_2^t = (0, \dots, 0, a, 0, \dots, 0, -a)$, $(a \in \mathbb{Z}_{\geqslant 0})$. In case $m' = 2$, then $B_D^t = (1, -1, -1, 1)$ and the unique coparallelism class could be broken, but at most in two pieces with two elements each, and again we have the same formulation for B_2. In both cases, if $a = 0$, then we have a splitting, which implies that either there is a repeated point (if $B_1^t = (1, -1)$) or X_A is not smooth. Then $a \geqslant 1$. Consider the subconfiguration E of A obtained by forgetting the two columns corresponding to the rows m' and $2m'$ of B_D. Since the vectors b_i with complementary indices add up to zero, it follows that E is facial and again, X_E is smooth. We deduce that $a = 1$ and $B_1^t = \pm(1, -1)$, which implies that X_A is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{m'+1}$ in $\mathbb{P}^{2m'+1}$. □

5.2. *Non-Lawrence families of examples*

We have the following obvious corollaries of Theorem 5.8.

Corollary 5.9. *Let $A \in \mathcal{M}_{d\times n}(\mathbb{Z})$ with maximal rank d associated to a regular configuration of weights and let $X_A \subset \mathbb{P}^{n-1}$ be the projective toric variety associated to A. Assume that X_A is not a hypersurface, non-linear, smooth and self-dual. Then, n is even.*

As the defect of the Segre embedding $X_m = \mathbb{P}^1 \times \mathbb{P}^{m-1}$ in $\mathbb{P}^{2m-1}$ for any $m \geqslant 2$ equals $2m-2-m = m-2 = \dim X_m - 2$, we recover for smooth varieties X_A the following result (see for instance [**10**]) for any projective smooth variety.

Corollary 5.10. *If $X_A \subset \mathbb{P}^{n-1}$ is a non-linear smooth projective variety such that $\dim X < n-2$ with defect $k > 0$, then $\dim X \equiv k(2)$.*

We use the previous corollaries together with Theorem 4.4 to construct families of non-regular self-dual varieties.

Example 5.11. Consider the families of matrices $\{A_\alpha\}, \{B_\alpha\}$ for $\alpha \in \mathbb{Z}$, $\alpha \neq 0$, defined by

$$A_\alpha = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \alpha & 0 & -\alpha & 0 \\ 0 & 0 & 1 & 0 & -\alpha & 0 & \alpha \end{pmatrix}, \quad B_\alpha = \begin{pmatrix} 2\alpha & 0 \\ -\alpha & 0 \\ -\alpha & 0 \\ 1 & 1 \\ -1 & -1 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}.$$

Clearly, B_α is a choice of a Gale dual matrix of A_α.

Observe that as $\alpha \neq 0$, the configuration A_α is not a pyramid and $\dim(X_{A_\alpha}) = 4$. Moreover, it is easy to show that if $\alpha \neq \alpha'$, then X_{A_α} and $X_{A_{\alpha'}}$ are not isomorphic as embedded varieties because they have different degrees. The degree of X_{A_α} is the normalized volume of the convex hull of the points in the configuration A_α (see [**15**]) and it can be computed easily in terms of the Gale dual configuration.

Since the conditions of Theorem 4.4 hold, it follows that X_{A_α} is self-dual for all $\alpha \in \mathbb{Z}, \alpha \neq 0$. Moreover, $n = 7$ is odd and so we deduce from Corollary 5.9 that X_{A_α} is a singular variety. The difference between its dimension and its defect is $4 - 1 = 3 \not\equiv 0$ (2).

We can generalize Example 5.11 in order to construct families of non-degenerate projective toric self-dual varieties of arbitrary dimension greater than or equal to 3 and of arbitrary codimension greater than or equal to 2.

Example 5.12. *Families of self-dual varieties of any dimension at least* 3 . Let any $r \geqslant 2$ and $\alpha_1, \dots, \alpha_r$ be non-zero integer numbers satisfying $\sum_{i=1}^r \alpha_i = 0$. Consider the planar lattice configuration

$$\mathcal{G}_\alpha = \{(\alpha_1, 0), \dots, (\alpha_r, 0), (0,1), (0,-1), (1,1), (-1,-1)\}.$$

Let A be any lattice configuration with Gale dual $\mathcal{G}_\alpha$. Then A is not a pyramid and the associated projective toric variety $X_A \subset \mathbb{P}^{r+3}$ is self-dual by Theorem 4.4, with dimension $\dim X_A = (r+4) - 2 - 1 = r + 1$.

When $r = 2$, the dimension of X_{A_α} is 3. The case $\alpha_1, \alpha_2 = \pm 1$ corresponds to the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$. Already, for $\alpha_1, \alpha_2 = \pm 2$, the configuration A_α does not contain all the lattice points in its convex hull. If we add those 'remaining' points to the configuration, then the associated toric variety is no longer self-dual.

EXAMPLE 5.13. *Families of self-dual varieties of any codimension at least* 2 . Using the same ideas of the previous example, we can construct pairs (A, B) with A a non-pyramidal configuration, and B its Gale dual satisfying the hypothesis of Theorem 4.4, so that X_A is self-dual, with arbitrary codimension $m \geqslant 2$.

For any $r \geqslant 2$ set $n = 2m + r$. As usual, $e_1, \ldots, e_m$ denotes the canonical basis in $\mathbb{Z}^m$. For any choice of non-zero integers $\alpha_1, \ldots, \alpha_r$ with $\sum_{i=1}^{r} \alpha_i = 0$, consider the following lattice configuration in $\mathbb{Z}^m$:

$$\mathcal{G}_\alpha := \{\alpha_1 e_1, \ldots, \alpha_r e_1, e_2, -e_2, \ldots, e_m, -e_m, e_1 + \ldots + e_m, -(e_1 + \ldots + e_m)\}.$$

For any lattice configuration $A_\alpha \subset \mathbb{Z}^n$ with this Gale dual, A_α is not a pyramid and its associated self-dual toric variety $X_{A_\alpha} \subset \mathbb{P}^n$ has dimension $m + r - 1$ and codimension m.

6. *Strongly self-dual varieties*

We are interested now in characterizing a particular interesting case of self-dual projective toric varieties.

DEFINITION 6.1. Let A be a regular lattice configuration without repetitions. We say that the projective variety $X_A \subset \mathbb{P}^{n-1}$ is *strongly self-dual* if X_A coincides with X_A^* under the canonical identification between $\mathbb{P}^{n-1}$ and its dual projective space as in Remark 2.20.

We deduce from Theorem 3.2 the following characterization of strongly self-dual projective toric varieties of the form X_A.

PROPOSITION 6.2. *Let A be a regular lattice configuration without repetitions. Then X_A is strongly self-dual if and only if $\mathbb{P}(\mathcal{R}_{A,\Bbbk}) \subset X_A$.*

Proof. If X_A is strongly self-dual, then the containment $\mathbb{P}(\mathcal{R}_{A,\Bbbk}) \subset X_A^*$ implies that the condition $\mathbb{P}(\mathcal{R}_{A,\Bbbk}) \subset X_A$ is necessary.

Assume that this condition holds and A has no repetitions. As we already observed, Theorem 3.8 implies that A is not pyramidal. Then it follows from Theorem 3.2 that, for any $q \in \mathbb{P}(\mathcal{R}_{A,\Bbbk}) \cap \mathbb{T}^{n-1} \subset X_A \cap \mathbb{T}^{n-1}$, $m_q(X_A) = X_A^*$. But since $q \in \mathcal{O}([1 : \ldots : 1])$, we deduce that $m_q(X_A) = \overline{\mathcal{O}(q)} = \overline{\mathcal{O}([1 : \ldots : 1])} = X_A$, that is, $X_A^* = X_A$. □

Using the same notation of Theorem 4.4, we have the following theorem.

THEOREM 6.3. *Let A be a non-pyramidal regular lattice configuration A of n weights spanning $\mathbb{Z}^d$ and let B_A be a Gale dual of A. Then :*

$$X_A \text{ is strongly self-dual } \Leftrightarrow \begin{cases} \text{(a) For any line } L \text{ through the origin we have } \displaystyle\sum_{b_i \in L} b_i = 0. \\ \text{(b) } \displaystyle\prod_{\substack{j=1 \\ b_{ji}>0}}^{n} b_{ji}^{b_{ji}} = \prod_{\substack{j=1 \\ b_{ji}<0}}^{n} b_{ji}^{-b_{ji}}, \quad i = 1, \ldots, n-d. \end{cases}$$

In the above statement, we use the convention that $0^0 = 1$.

Proof. Assume that X_A is strongly self-dual. Then (a) holds by Theorem 4.4. By Proposition 6.2, we know that $\mathbb{P}(\mathcal{R}_{A,\Bbbk}) \cap \mathbb{T}^{n-1} \subset X_A \cap \mathbb{T}^{n-1}$, and this last variety is cut out by the $(n-d)$ binomials

$$\prod_{\substack{j=1\\ b_{ji}>0}}^{n} x_j^{b_{ji}} = \prod_{\substack{j=1\\ b_{ji}<0}}^{n} x_j^{-b_{ji}}, \quad \forall\, i = 1, \ldots, n-d.$$

Then we have the following equalities, for all $s \in \Bbbk^{n-d}$:

$$\prod_{\substack{j=1\\ b_{ji}>0}}^{n} \langle s, b_j\rangle^{b_{ji}} = \prod_{\substack{j=1\\ b_{ji}<0}}^{n} \langle s, b_j\rangle^{-b_{ji}}, \quad \forall\, i = 1, \ldots, n-d. \tag{6.1}$$

We get the condition (b) by evaluating, respectively, at $s = e_1, \ldots, e_{n-d}$.

Conversely, condition (a) implies the equalities (6.1) of the polynomials in s on both sides up to constant, as in Remark 4.3. Then condition (b) ensures that this constant is 1. Therefore, $\mathbb{P}(\mathcal{R}_{A,\Bbbk}) \cap \mathbb{T}^{n-1} \subset X_A \cap \mathbb{T}^{n-1}$, and so X_A is strongly self-dual by Proposition 6.2. □

Example 6.4. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & -2\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

Observe that A is non-pyramidal. A Gale dual matrix B_A for A is given by the transpose of the matrix $\left(\begin{smallmatrix} -2 & -2 & -2 & -2 & 4 & 1 & 1 & 1 & 1\\ 1 & 1 & 2 & 0 & -2 & -1 & 0 & -1 & 0 \end{smallmatrix}\right)$.

Clearly, B_A satisfies the conditions of Theorem 6.3 and hence X_A is strongly self-dual. But note that A is not a Lawrence configuration.

We conclude this section with the complete characterization of strongly self-dual varieties of type X_A, with A a non-pyramidal Lawrence matrix.

Theorem 6.5. *Let A be a non-pyramidal Lawrence configuration consisting of $2n$ points in $\mathbb{Z}^{n+d}$, as in (5.1). Then X_A is strongly self-dual if and only if there exists a subset I of rows of the lower matrix $M = (m_{jk})$ such that $\sum_{j\in I} m_{jk}$ is an odd number for all $k = 1, \ldots, n$.*

Proof. By Corollary 5.2, X_A is self-dual for any non-pyramidal Lawrence configuration A. Thus, X_A is strongly self-dual if and only if condition (b) in Theorem 6.3 is satisfied. If $\mathcal{G}_M = \{c_1, \ldots, c_n\} \subset \mathbb{Z}^{n-d}$ is a Gale dual configuration for M, then $\{-c_1, \ldots, -c_n, c_1, \ldots, c_n\}$ defines a Gale dual configuration for A. Condition (b) is then equivalent in this case to the equalities

$$(-1)^{\sum_{j=1}^{n} c_{ji}} = 0, \quad i = 1, \ldots, n-d.$$

This is in turn equivalent to the condition that, for all $v \in \mathcal{R}_M$, the sum $\sum_{j=1}^{n} v_j \equiv 0\ (2)$. But this is equivalent to the fact that the vector $(1,\ldots,1)$ lies in the row span of M when we reduce all its entries modulo 2. Denoting classes in $\mathbb{Z}_2$ with an overline, this condition means that there exist $\alpha_1,\ldots,\alpha_d \in \mathbb{Z}_2 = \{0,1\}$ such that

$$(1,\ldots,1) = \sum_{i=1}^{d} \alpha_i(\overline{m_{i1}},\ldots,\overline{m_{in}}) = \sum_{\alpha_i=1} (\overline{m_{i1}},\ldots,\overline{m_{in}}).$$

It suffices to call $I = \{i \in \{1,\ldots,d\} : \alpha_i = 1\}$. □

Example 6.6. The Segre embeddings in Example 5.3 have associated Lawrence matrices as in (5.2), where M is a matrix with a single row with all entries equal to 1. They clearly satisfy the hypotheses of Theorem 6.5. Then, for any $m > 1$, the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{m-1}$ is a strongly-self-dual projective toric variety.

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