

Invariance of a Shift-Invariant Space in Several Variables

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Abstract. In this article we study invariance properties of shift-invariant spaces in higher dimensions. We state and prove several necessary and sufficient conditions for a shift-invariant space to be invariant under a given closed subgroup of \mathbb{R}^d , and prove the existence of shift-invariant spaces that are exactly invariant for each given subgroup. As an application we relate the extra invariance to the size of support of the Fourier transform of the generators of the shift-invariant space.

This work extends recent results obtained for the case of one variable to several variables.

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1. Introduction

A *shift-invariant space* (SIS) of $L^2(\mathbb{R})$ is a closed subspace that is invariant under translations by integers. These spaces are important in approximation theory, wavelets, sampling and frames. They also serve as models in many applications in signal and image processing.

An important and interesting question regarding these spaces is whether they have the property to be invariant under translations other than integers. A limit case is when the space is invariant under translations by all real numbers. In this case the space is called *translation invariant*. However there exist shift-invariant spaces with some *extra* invariance that are not necessarily translation invariant. That is, there are some intermediate cases between shift-invariance and translation invariance. The question is then, how can we identify them?

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Recently, Hogan and Lakey defined the *discrepancy* of a shift-invariant space as a way to quantify the *non-translation invariance* of the subspace, (see [9]). The discrepancy measures how far a unitary norm function of the subspace, can move away from it, when translated by non integers. A translation invariant space has discrepancy zero.

In another direction, Aldroubi et al, see [1] studied shift-invariant spaces of $L^2(\mathbb{R})$ that have some extra invariance. They show that if S is a shift-invariant space, then its *invariance set*, is a closed additive subgroup of \mathbb{R} containing \mathbb{Z} . The invariance set associated to a shift-invariant space is the set M of real numbers satisfying that for each $p \in M$ the translations by p of every function in S , belongs to S . As a consequence, since every additive subgroup of \mathbb{R} is either discrete or dense, there are only two possibilities left for the extra invariance. That is, either S is invariant under translations by the group $(1/n)\mathbb{Z}$, for some positive integer n (and not invariant under any bigger subgroup) or it is translation invariant. They found different characterizations, in terms of the Fourier transform, of when a shift invariant space is $(1/n)\mathbb{Z}$ -invariant.

A natural question arises in this context. Are the characterizations of extra invariance that hold on the line, still valid in several variables?

A shift-invariant space in $L^2(\mathbb{R}^d)$ is a closed subspace that is invariant under translations by the group \mathbb{Z}^d . The invariance set $M \subseteq \mathbb{R}^d$ associated to a shift-invariant space S , that is, the set of vectors that leave S invariant when translated by its elements, is again, as in the 1-dimensional case, a closed subgroup of \mathbb{R}^d (see Proposition 2.1). The problem of the extra invariance can then be reformulated as finding necessary and sufficient conditions for a shift-invariant space to be invariant under a closed additive subgroup $M \subseteq L^2(\mathbb{R}^d)$ containing \mathbb{Z}^d .

The main difference here with the one dimensional case, is that the structure of the subgroups of \mathbb{R}^d when d is bigger than one, is not as simple.

The results obtained for the 1-dimensional case translate very well in the case in which the invariance set M is a lattice, (i.e. a discrete group) or when M is dense, that is $M = \mathbb{R}^d$. However, there are subgroups of \mathbb{R}^d that are neither discrete nor dense. So, can there exist shift-invariant spaces which are M -invariant for such a subgroup M and are not translation invariant?

In this paper we studied the extra invariance of shift-invariant spaces in higher dimensions. We were able to obtain several characterizations paralleling the 1-dimensional results. In addition our results show the existence of shift-invariant spaces that are *exactly* M -invariant for every closed subgroup $M \subseteq L^2(\mathbb{R}^d)$ containing \mathbb{Z}^d . By ‘exactly M -invariant’ we mean that they are not invariant under any other subgroup containing M . We apply our results to obtain estimates on the size of the support of the Fourier transform of the generators of the space.

The paper is organized in the following way. Section 2 contains some notations, definitions and preliminary results that will be needed throughout. We review the structure of closed additive subgroups of \mathbb{R}^d in Section 3. In Section 4 we extend some results, known for shift-invariant spaces in \mathbb{R}^d , to M -invariant spaces when M is a closed subgroup of \mathbb{R}^d containing \mathbb{Z}^d . The necessary and

sufficient conditions for the M -invariance of shift-invariant spaces are stated and proved in Section 5. Finally, Section 6 contains some applications of our results.

2. Preliminaries

2.1. Notation and Definitions

Given a subspace W of a Hilbert space H , we denote by \overline{W} its closure and by W^\perp its orthogonal complement. The inner product in H will be denoted by $\langle \cdot, \cdot \rangle$.

We normalize the Fourier transform of $f \in L^1(\mathbb{R}^d)$ as

$$\widehat{f}(\omega) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle \omega, x \rangle} dx.$$

The Fourier transform extends to a unitary operator on $L^2(\mathbb{R}^d)$. Given $\Phi \subseteq L^2(\mathbb{R}^d)$, we set $\widehat{\Phi} = \{\widehat{f} : f \in \Phi\}$.

For $y \in \mathbb{R}^d$, we write $e^{-2\pi i \langle \omega, y \rangle}$ as $e_y(\omega)$ and the translation operator t_y is $t_y f(x) = f(x - y)$. Note that $\widehat{(t_y f)}(\omega) = e_y(\omega) \widehat{f}(\omega)$.

Let G be a subset of \mathbb{R}^d , we will say that a function f defined in \mathbb{R}^d is G -periodic if $t_x f = f$ for all $x \in G$. A subset $A \subseteq \mathbb{R}^d$ is G -periodic if its indicator function (denoted by χ_A) is G -periodic.

A *shift-invariant space* (SIS) is a closed subspace S of $L^2(\mathbb{R}^d)$ such that $t_k f \in S$ for every $k \in \mathbb{Z}^d$ and $f \in S$.

Given $\Phi \subseteq L^2(\mathbb{R}^d)$, we define

$$E(\Phi) = \{t_k \varphi : \varphi \in \Phi, k \in \mathbb{Z}^d\}.$$

The SIS generated by Φ is

$$S(\Phi) = \overline{\text{span}} E(\Phi) = \overline{\text{span}} \{t_k \varphi : \varphi \in \Phi, k \in \mathbb{Z}^d\}.$$

We call Φ a *set of generators* for $S(\Phi)$. When $\Phi = \{\varphi\}$, we simply write $S(\varphi)$.

The *length* of a shift-invariant space S is the minimum cardinality of the sets Φ such that $S = S(\Phi)$. A SIS of length one is called a *principal* SIS. A SIS of finite length is a *finitely generated* SIS (FSIS).

We will write $W = U \oplus V$ to denote the *orthogonal* direct sum of closed subspaces of $L^2(\mathbb{R}^d)$, i.e., the subspaces U, V must be closed and orthogonal, and W is their direct sum.

The Lebesgue measure of a set $E \subseteq \mathbb{R}^d$ is denoted by $|E|$.

The cardinality of a finite set F is denoted by $\#F$.

2.2. The invariance set

Let $S \subseteq L^2(\mathbb{R}^d)$ be a SIS. We define the set

$$M := \{x \in \mathbb{R}^d : t_x f \in S, \forall f \in S\}. \quad (1)$$

If Φ is a set of generators for S , it is easy to check that, $x \in M$ if and only if for all $\varphi \in \Phi$, $t_x \varphi \in S$.

In case that $M = \mathbb{R}^d$, Wiener's theorem (see [8], [13]) states that there exists a measurable set $E \subseteq \mathbb{R}^d$ satisfying

$$S = \{f \in L^2(\mathbb{R}^d) : \text{supp}(\widehat{f}) \subseteq E\}.$$

We want to characterize S when M is not all \mathbb{R}^d . We will first study the structure of the set M .

Proposition 2.1. *Let S be a SIS of $L^2(\mathbb{R}^d)$ and let M be defined as in (1). Then M is an additive closed subgroup of \mathbb{R}^d containing \mathbb{Z}^d .*

For the proof of this proposition we will need the following lemma. Recall that an additive semigroup is a non-empty set with an associative additive operation.

Lemma 2.2. *Let H be a closed semigroup of \mathbb{R}^d containing \mathbb{Z}^d , then H is a group.*

Proof. Let π be the quotient map from \mathbb{R}^d onto $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Since H is a semigroup containing \mathbb{Z}^d , we have that $H + \mathbb{Z}^d = H$, thus

$$\pi^{-1}(\pi(H)) = \bigcup_{h \in H} h + \mathbb{Z}^d = H + \mathbb{Z}^d = H.$$

This shows that $\pi(H)$ is closed in \mathbb{T}^d and therefore compact.

By [10, Theorem 9.16], we have that a compact semigroup of \mathbb{T}^d is necessarily a group, thus $\pi(H)$ is a group and consequently H is a group. \square

Proof of Proposition 2.1. Since S is a SIS, $\mathbb{Z}^d \subseteq M$.

We now show that M is closed. Let $x_0 \in \mathbb{R}^d$ and $\{x_n\}_{n \in \mathbb{N}} \subseteq M$, such that $\lim_{n \rightarrow \infty} x_n = x_0$.

Then

$$\lim_{n \rightarrow \infty} \|t_{x_n} f - t_{x_0} f\| = 0.$$

Therefore, $t_{x_0} f \in \overline{S}$. But S is closed, so $t_{x_0} f \in S$.

It is easy to check that M is a semigroup of \mathbb{R}^d , hence we conclude from Lemma 2.2 that M is a group. \square

In what follows, we will give some characterizations concerning closed subgroups of \mathbb{R}^d .

3. Closed subgroups of \mathbb{R}^d

Throughout this section we describe the additive closed subgroups of \mathbb{R}^d containing \mathbb{Z}^d . We first study closed subgroups of \mathbb{R}^d in general.

When two groups G_1 and G_2 are isomorphic we will write $G_1 \approx G_2$. Here and subsequently all the vector subspaces will be real.

3.1. General case

We will state in this section, some basic definitions and properties of closed subgroups of \mathbb{R}^d , for a detailed treatment and proofs we refer the reader to [5].

Definition 3.1. Given M a subgroup of \mathbb{R}^d , the *range* of M , denoted by $\mathbf{r}(M)$, is the dimension of the subspace generated by M as a real vector space.

It is known that every closed subgroup of \mathbb{R}^d is either discrete or contains a subspace of at least dimension one (see [5, Proposition 3]).

Definition 3.2. Given M a closed subgroup of \mathbb{R}^d , there exists a subspace V whose dimension is the largest of the dimensions of all the subspaces contained in M . We will denote by $\mathbf{d}(M)$ the dimension of V . Note that $\mathbf{d}(M)$ can be zero.

Note that $0 \leq \mathbf{d}(M) \leq \mathbf{r}(M) \leq d$.

The next theorem establishes that every closed subgroup of \mathbb{R}^d is the direct sum of a subspace and a discrete group.

Theorem 3.3. *Let M be a closed subgroup of \mathbb{R}^d such that $\mathbf{r}(M) = r$ and $\mathbf{d}(M) = p$. Let V be the subspace contained in M as in Definition 3.2. There exists a basis $\{u_1, \dots, u_d\}$ for \mathbb{R}^d such that $\{u_1, \dots, u_r\} \subseteq M$ and $\{u_1, \dots, u_p\}$ is a basis for V . Furthermore,*

$$M = \left\{ \sum_{i=1}^p t_i u_i + \sum_{j=p+1}^r n_j u_j : t_i \in \mathbb{R}, n_j \in \mathbb{Z} \right\}.$$

Corollary 3.4. *If M is a closed subgroup of \mathbb{R}^d such that $\mathbf{r}(M) = r$ and $\mathbf{d}(M) = p$, then*

$$M \approx \mathbb{R}^p \times \mathbb{Z}^{r-p}.$$

3.2. Closed subgroups of \mathbb{R}^d containing \mathbb{Z}^d

We are interested in closed subgroups of \mathbb{R}^d containing \mathbb{Z}^d . For their understanding, the notion of dual group is important.

Definition 3.5. Let M be a subgroup of \mathbb{R}^d . Consider the set

$$M^* := \{x \in \mathbb{R}^d : \langle x, m \rangle \in \mathbb{Z} \quad \forall m \in M\}.$$

Then M^* is a subgroup of \mathbb{R}^d called the *dual group* of M . In particular, $(\mathbb{Z}^d)^* = \mathbb{Z}^d$.

Now we will list some properties of the dual group.

Proposition 3.6. *Let M, N be subgroups of \mathbb{R}^d .*

- i) M^* is a closed subgroup of \mathbb{R}^d .
- ii) If $N \subseteq M$, then $M^* \subseteq N^*$.
- iii) If M is closed, then $\mathbf{r}(M^*) = d - \mathbf{d}(M)$ and $\mathbf{d}(M^*) = d - \mathbf{r}(M)$.
- iv) $(M^*)^* = \overline{M}$.

Let H be a subgroup of \mathbb{Z}^d with $\mathbf{r}(H) = q$, we will say that a set $\{v_1, \dots, v_q\} \subseteq H$ is a *basis* for H if for every $x \in H$ there exist unique $k_1, \dots, k_q \in \mathbb{Z}$ such that

$$x = \sum_{i=1}^q k_i v_i.$$

Note that $\{v_1, \dots, v_d\} \subseteq \mathbb{Z}^d$ is a basis for \mathbb{Z}^d if and only if the determinant of the matrix A which has $\{v_1, \dots, v_d\}$ as columns is 1 or -1 .

Given $B = \{v_1, \dots, v_d\}$ a basis for \mathbb{Z}^d , we will call $\tilde{B} = \{w_1, \dots, w_d\}$ a *dual basis* for B if $\langle v_i, w_j \rangle = \delta_{i,j}$ for all $1 \leq i, j \leq d$.

If we denote by \tilde{A} the matrix with columns $\{w_1, \dots, w_d\}$, the relation between B and \tilde{B} can be expressed in terms of matrices as $\tilde{A} = (A^*)^{-1}$.

The closed subgroups M of \mathbb{R}^d containing \mathbb{Z}^d , can be described with the help of the dual relations. Since $\mathbb{Z}^d \subseteq M$, we have that $M^* \subseteq \mathbb{Z}^d$. So, we need first the characterization of the subgroups of \mathbb{Z}^d . This is stated in the following theorem.

Theorem 3.7. *Let H be a subgroup of \mathbb{Z}^d with $\mathbf{r}(H) = q$, then there exist a basis $\{w_1, \dots, w_d\}$ for \mathbb{Z}^d and unique integers a_1, \dots, a_q satisfying $a_{i+1} \equiv 0 \pmod{a_i}$ for all $1 \leq i \leq q-1$, such that $\{a_1 w_1, \dots, a_q w_q\}$ is a basis for H . The integers a_1, \dots, a_q are called invariant factors.*

The proof of the previous result can be found in [6].

Remark 3.8. Under the assumptions of the above theorem we obtain

$$\mathbb{Z}^d/H \approx \mathbb{Z}_{a_1} \times \dots \times \mathbb{Z}_{a_q} \times \mathbb{Z}^{d-q}.$$

We are now able to characterize the closed subgroups of \mathbb{R}^d containing \mathbb{Z}^d . The proof of the following theorem can be found in [5], but we include it here for the sake of completeness.

Theorem 3.9. *Let $M \subseteq \mathbb{R}^d$. The following conditions are equivalent:*

- i) M is a closed subgroup of \mathbb{R}^d containing \mathbb{Z}^d and $\mathbf{d}(M) = d - q$.
- ii) There exist a basis $\{v_1, \dots, v_d\}$ for \mathbb{Z}^d and integers a_1, \dots, a_q satisfying $a_{i+1} \equiv 0 \pmod{a_i}$ for all $1 \leq i \leq q-1$, such that

$$M = \left\{ \sum_{i=1}^q k_i \frac{1}{a_i} v_i + \sum_{j=q+1}^d t_j v_j : k_i \in \mathbb{Z}, t_j \in \mathbb{R} \right\}.$$

Furthermore, the integers q and a_1, \dots, a_q are uniquely determined by M .

Proof. Suppose i) is true. Since $\mathbb{Z}^d \subseteq M$ and $\mathbf{d}(M) = d - q$, we have that $M^* \subseteq \mathbb{Z}^d$ and $\mathbf{r}(M^*) = q$. By Theorem 3.7, there exist invariant factors a_1, \dots, a_q and $\{w_1, \dots, w_d\}$ a basis for \mathbb{Z}^d such that $\{a_1 w_1, \dots, a_q w_q\}$ is a basis for M^* .

Let $\{v_1, \dots, v_d\}$ be the dual basis for $\{w_1, \dots, w_d\}$.

Since M is closed, it follows from item iv) of Proposition 3.6 that $M = (M^*)^*$. So, $m \in M$ if and only if

$$\langle m, a_j w_j \rangle \in \mathbb{Z} \quad \forall 1 \leq j \leq q. \quad (2)$$

As $\{v_1, \dots, v_d\}$ is a basis, given $u \in \mathbb{R}^d$, there exist $u_i \in \mathbb{R}$ such that $u = \sum_{i=1}^d u_i v_i$. Thus, by (2), $u \in M$ if and only if $u_i a_i \in \mathbb{Z}$ for all $1 \leq i \leq q$.

We finally obtain that $u \in M$ if and only if there exist $k_i \in \mathbb{Z}$ and $u_j \in \mathbb{R}$ such that

$$u = \sum_{i=1}^q k_i \frac{1}{a_i} v_i + \sum_{j=q+1}^d u_j v_j.$$

The proof of the other implication is straightforward.

The integers q and a_1, \dots, a_q are uniquely determined by M since $q = d - \mathbf{d}(M)$ and a_1, \dots, a_q are the invariant factors of M^* . \square

As a consequence of the proof given above we obtain the following corollary.

Corollary 3.10. *Let $\mathbb{Z}^d \subseteq M \subseteq \mathbb{R}^d$ be a closed subgroup with $\mathbf{d}(M) = d - q$. If $\{v_1, \dots, v_d\}$ and a_1, \dots, a_q are as in Theorem 3.9, then*

$$M^* = \left\{ \sum_{i=1}^q n_i a_i w_i : n_i \in \mathbb{Z} \right\},$$

where $\{w_1, \dots, w_d\}$ is the dual basis of $\{v_1, \dots, v_d\}$.

Example 3.11. Assume that $d = 3$. If $M = \frac{1}{2}\mathbb{Z} \times \frac{1}{3}\mathbb{Z} \times \mathbb{R}$, then $v_1 = (1, 1, 0)$, $v_2 = (3, 2, 0)$ and $v_3 = (0, 0, 1)$ verify the conditions of Theorem 3.9 with the invariant factors $a_1 = 1$ and $a_2 = 6$. On the other hand $v'_1 = (1, 1, 0)$, $v'_2 = (3, 2, 1)$ and $v'_3 = (0, 0, 1)$ verify the same conditions. This shows that the basis in Theorem 3.9 is not unique.

Remark 3.12. If $\{v_1, \dots, v_d\}$ and a_1, \dots, a_q are as in Theorem 3.9, let us define the linear transformation T as

$$T : \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad T(e_i) = \begin{cases} \frac{1}{a_i} v_i & \text{if } 1 \leq i \leq q \\ v_i & \text{if } q+1 \leq i \leq d. \end{cases}$$

Then T is an invertible transformation that satisfies

$$M = T(\mathbb{Z}^q \times \mathbb{R}^{d-q}).$$

If $\{w_1, \dots, w_d\}$ is the dual basis for $\{v_1, \dots, v_d\}$, the inverse of the adjoint of T is defined by

$$(T^*)^{-1} : \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad (T^*)^{-1}(e_i) = \begin{cases} a_i w_i & \text{if } 1 \leq i \leq q \\ w_i & \text{if } q+1 \leq i \leq d. \end{cases}$$

By Corollary 3.10, it is true that

$$M^* = (T^*)^{-1}(\mathbb{Z}^q \times \{0\}^{d-q}).$$

4. The structure of principal M -invariant spaces

Throughout this section M will be a closed subgroup of \mathbb{R}^d containing \mathbb{Z}^d and M^* its dual group defined as in the previous section.

Definition 4.1. We will say that a closed subspace S of $L^2(\mathbb{R}^d)$ is M -invariant if $t_m f \in S$ for all $m \in M$ and $f \in S$.

Given $\Phi \subseteq L^2(\mathbb{R}^d)$, the M -invariant space generated by Φ is

$$S_M(\Phi) = \overline{\text{span}}(\{t_m \varphi : m \in M, \varphi \in \Phi\}).$$

If $\Phi = \{\varphi\}$ we write $S_M(\varphi)$ and we say that $S_M(\varphi)$ is a principal M -invariant space. For simplicity of notation, when $M = \mathbb{Z}^d$, we write $S(\varphi)$ instead of $S_{\mathbb{Z}^d}(\varphi)$.

Principal SISs have been completely characterized by [3] as follows.

Theorem 4.2. *Let $f \in L^2(\mathbb{R}^d)$ be given. If $g \in S(f)$, then there exists a \mathbb{Z}^d -periodic function η such that $\widehat{g} = \eta \widehat{f}$.*

Conversely, if η is a \mathbb{Z}^d -periodic function such that $\eta \widehat{f} \in L^2(\mathbb{R}^d)$, then the function g defined by $\widehat{g} = \eta \widehat{f}$ belongs to $S(f)$.

The aim of this section is to generalize the previous theorem to the M -invariant case. In case that M is discrete, Theorem 4.2 follows easily by rescaling. The difficulty arises when M is not discrete.

Theorem 4.3. *Let $f \in L^2(\mathbb{R}^d)$ and M a closed subgroup of \mathbb{R}^d containing \mathbb{Z}^d . If $g \in S_M(f)$, then there exists an M^* -periodic function η such that $\widehat{g} = \eta \widehat{f}$.*

Conversely, if η is an M^ -periodic function such that $\eta \widehat{f} \in L^2(\mathbb{R}^d)$, then the function g defined by $\widehat{g} = \eta \widehat{f}$ belongs to $S_M(f)$.*

Theorem 4.3 was proved in [3] for the lattice case. We adapt their arguments to this more general case.

We will first need some definitions and properties.

By Remark 3.12, there exists a linear transformation $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $M = T(\mathbb{Z}^q \times \mathbb{R}^{d-q})$ and $M^* = (T^*)^{-1}(\mathbb{Z}^q \times \{0\}^{d-q})$, where $q = d - \mathbf{d}(M)$.

We will denote by \mathcal{D} the section of the quotient \mathbb{R}^d/M^* defined as

$$\mathcal{D} = (T^*)^{-1}([0, 1]^q \times \mathbb{R}^{d-q}). \quad (3)$$

Therefore, $\{\mathcal{D} + m^*\}_{m^* \in M^*}$ forms a partition of \mathbb{R}^d .

Given $f, g \in L^2(\mathbb{R}^d)$ we define

$$[f, g](\omega) := \sum_{m^* \in M^*} f(\omega + m^*) \overline{g(\omega + m^*)},$$

where $\omega \in \mathcal{D}$. Note that, as $f, g \in L^2(\mathbb{R}^d)$ we have that $[f, g] \in L^1(\mathcal{D})$, since

$$\begin{aligned} \int_{\mathbb{R}^d} f(\omega) \overline{g(\omega)} d\omega &= \sum_{m^* \in M^*} \int_{\mathcal{D}+m^*} f(\omega) \overline{g(\omega)} d\omega \\ &= \sum_{m^* \in M^*} \int_{\mathcal{D}} f(\omega + m^*) \overline{g(\omega + m^*)} d\omega \\ &= \int_{\mathcal{D}} [f, g](\omega) d\omega. \end{aligned} \quad (4)$$

From this, it follows that if $f \in L^2(\mathbb{R}^d)$, then $\{f(\omega + m^*)\}_{m^* \in M^*} \in \ell^2(M^*)$ a.e. $\omega \in \mathcal{D}$.

The Cauchy-Schwarz inequality in $\ell^2(M^*)$, gives the following a.e. pointwise estimate

$$|[f, g]|^2 \leq [f, f][g, g] \quad (5)$$

for every $f, g \in L^2(\mathbb{R}^d)$.

Given an M^* -periodic function η and $f, g \in L^2(\mathbb{R}^d)$ such that $\eta f \in L^2(\mathbb{R}^d)$, it is easy to check that

$$[\eta f, g] = \eta [f, g]. \quad (6)$$

The following lemma is an extension to general subgroups of \mathbb{R}^d of a result which holds for the discrete case.

Lemma 4.4. *Let $f \in L^2(\mathbb{R}^d)$, M a closed subgroup of \mathbb{R}^d containing \mathbb{Z}^d and \mathcal{D} defined as in (3). Then,*

$$S_M(f)^\perp = \{g \in L^2(\mathbb{R}^d) : [\widehat{f}, \widehat{g}](\omega) = 0 \text{ a.e. } \omega \in \mathcal{D}\}.$$

Proof. Since the span of the set $\{t_m f : m \in M\}$ is dense in $S_M(f)$, we have that $g \in S_M(f)^\perp$ if and only if $\langle \widehat{g}, e_m \widehat{f} \rangle = 0$ for all $m \in M$. As e_m is M^* -periodic, using (4) and (6), we obtain that $g \in S_M(f)^\perp$ if and only if

$$\int_{\mathcal{D}} e_m(\omega) [\widehat{f}, \widehat{g}](\omega) d\omega = 0, \quad (7)$$

for all $m \in M$.

At this point, what is left to show is that if (7) holds then $[\widehat{f}, \widehat{g}](\omega) = 0$ a.e. $\omega \in \mathcal{D}$. For this, taking into account that $[\widehat{f}, \widehat{g}] \in L^1(\mathcal{D})$, it is enough to prove that if $h \in L^1(\mathcal{D})$ and $\int_{\mathcal{D}} h e_m = 0$ for all $m \in M$ then $h = 0$ a.e. $\omega \in \mathcal{D}$.

We will prove the preceding property for the case $M = \mathbb{Z}^q \times \mathbb{R}^{d-q}$. The general case will follow from a change of variables using the description of M and \mathcal{D} given in Remark 3.12 and (3).

Suppose now $M = \mathbb{Z}^q \times \mathbb{R}^{d-q}$, then $\mathcal{D} = [0, 1)^q \times \mathbb{R}^{n-q}$. Take $h \in L^1(\mathcal{D})$, such that

$$\iint_{[0,1)^q \times \mathbb{R}^{n-q}} h(x, y) e^{-2\pi i(kx+ty)} dx dy = 0 \quad \forall k \in \mathbb{Z}^q, t \in \mathbb{R}^{d-q}. \quad (8)$$

Given $k \in \mathbb{Z}^q$, define $\alpha_k(y) := \int_{[0,1]^q} h(x, y) e^{-2\pi i k x} dx$ for a.e. $y \in \mathbb{R}^{d-q}$. It follows from (8) that

$$\int_{\mathbb{R}^{d-q}} \alpha_k(y) e^{-2\pi i t y} dy = 0 \quad \forall t \in \mathbb{R}^{d-q}. \quad (9)$$

Since $h \in L^1(\mathcal{D})$, by Fubini's Theorem, $\alpha_k \in L^1([0, 1]^q)$. Thus, using (9), $\alpha_k(y) = 0$ a.e. $y \in \mathbb{R}^{d-q}$. That is

$$\int_{[0,1]^q} h(x, y) e^{-2\pi i k x} dx = 0 \quad (10)$$

for a.e. $y \in \mathbb{R}^{d-q}$. Define now $\beta_y(x) := h(x, y)$. By (10), for a.e. $y \in \mathbb{R}^{d-q}$ we have that $\beta_y(x) = 0$ for a.e. $x \in [0, 1]^q$. Therefore, $h(x, y) = 0$ a.e. $(x, y) \in [0, 1]^q \times \mathbb{R}^{d-q}$ and this completes the proof. \square

Now we will give a formula for the orthogonal projection onto $S_M(f)$.

Lemma 4.5. *Let P be the orthogonal projection onto $S_M(f)$. Then, for each $g \in L^2(\mathbb{R}^d)$, we have $\widehat{P}g = \eta_g \widehat{f}$, where η_g is the M^* -periodic function defined by*

$$\eta_g := \begin{cases} [\widehat{g}, \widehat{f}] / [\widehat{f}, \widehat{f}] & \text{on } E_f + M^* \\ 0 & \text{otherwise,} \end{cases}$$

and E_f is the set $\{\omega \in \mathcal{D} : [\widehat{f}, \widehat{f}](\omega) \neq 0\}$.

Proof. Let \widehat{P} be the orthogonal projection onto $\widehat{S}_M(\widehat{f})$. Since $\widehat{P}g = \widehat{P}\widehat{g}$, it is enough to show that $\widehat{P}\widehat{g} = \eta_g \widehat{f}$.

We first want to prove that $\eta_g \widehat{f} \in L^2(\mathbb{R}^d)$. Combining (4), (5) and (6)

$$\int_{\mathbb{R}^d} |\eta_g \widehat{f}|^2 = \int_{\mathcal{D}} |\eta_g|^2 [\widehat{f}, \widehat{f}] \leq \int_{\mathcal{D}} [\widehat{g}, \widehat{g}] = \|g\|_{L^2}^2,$$

and so, $\eta_g \widehat{f} \in L^2(\mathbb{R}^d)$. Define the linear map

$$Q : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d), \quad Q\widehat{g} = \eta_g \widehat{f},$$

which is well defined and has norm not greater than one. We will prove that $Q = \widehat{P}$.

Take $\widehat{g} \in \widehat{S}_M(\widehat{f})^\perp = (S_M(f)^\perp)^\wedge$. Then Lemma 4.4 gives that $\eta_g = 0$, hence $Q\widehat{g} = 0$. Therefore, $Q = \widehat{P}$ on $\widehat{S}_M(\widehat{f})^\perp$.

On the other hand, on $E_f + M^*$,

$$\eta_{(t_m f)} = [e_m \widehat{f}, \widehat{f}] / [\widehat{f}, \widehat{f}] = e_m \quad \forall m \in M.$$

Since $\widehat{f} = 0$ outside of $E_f + M^*$, we have that $Q(\widehat{t_m f}) = e_m \widehat{f}$. As Q is linear and bounded, and the set $\text{span}\{t_m f : m \in M\}$ is dense in $S_M(f)$, $Q = \widehat{P}$ on $\widehat{S}_M(\widehat{f})$. \square

Proof of Theorem 4.3. Suppose that $g \in S_M(f)$, then $Pg = g$, where P is the orthogonal projection onto $S_M(f)$. Hence, by Lemma 4.5, $\widehat{g} = \eta_g \widehat{f}$.

Conversely, if $\eta \widehat{f} \in L^2(\mathbb{R}^d)$ and η is an M^* -periodic function, then g , the inverse transform of $\eta \widehat{f}$ is also in $L^2(\mathbb{R}^d)$ and satisfies, by (6), that $\eta_g = [\eta \widehat{f}, \widehat{f}] / [\widehat{f}, \widehat{f}] = \eta$ on $E_f + M^*$.

On the other hand, since $\text{supp}(\widehat{f}) \subseteq E_f + M^*$, we have that $\eta_g \widehat{f} = \eta \widehat{f}$.

So, $\widehat{Pg} = \eta_g \widehat{f} = \eta \widehat{f} = \widehat{g}$. Consequently, $Pg = g$, and hence $g \in S_M(f)$. \square

5. Characterization of M-invariance

Given M a closed subgroup of \mathbb{R}^d containing \mathbb{Z}^d , our goal is to characterize when a SIS S is an M -invariant space. For this, we will construct a partition $\{B_\sigma\}_{\sigma \in \mathcal{N}}$ of \mathbb{R}^d , where each B_σ will be an M^* -periodic set and the index set \mathcal{N} will be properly chosen later (see(14)). Using this partition, for each $\sigma \in \mathcal{N}$, we define the subspaces

$$U_\sigma = \{f \in L^2(\mathbb{R}^d) : \widehat{f} = \chi_{B_\sigma} \widehat{g}, \text{ with } g \in S\}. \quad (11)$$

The main theorem of this section characterizes the M -invariance of S in terms of the subspaces U_σ .

Theorem 5.1. *If $S \subseteq L^2(\mathbb{R}^d)$ is a SIS and M is a closed subgroup of \mathbb{R}^d containing \mathbb{Z}^d , then the following are equivalent.*

- i) S is M -invariant.
- ii) $U_\sigma \subseteq S$ for all $\sigma \in \mathcal{N}$.

Moreover, in case any of the above holds, we have that S is the orthogonal direct sum

$$S = \bigoplus_{\sigma \in \mathcal{N}} U_\sigma.$$

Before proving the theorem let us carefully define the partition $\{B_\sigma\}_{\sigma \in \mathcal{N}}$, in such a way that each B_σ is an M^* -periodic set.

Let Ω be the section of the quotient $\mathbb{R}^d / \mathbb{Z}^d$ given by

$$\Omega = (T^*)^{-1}([0, 1)^d), \quad (12)$$

where T is as in Remark 3.12. Then Ω tiles \mathbb{R}^d by \mathbb{Z}^d translations, that is

$$\mathbb{R}^d = \bigcup_{k \in \mathbb{Z}^d} \Omega + k. \quad (13)$$

Now, for each $k \in \mathbb{Z}^d$, consider $(\Omega + k) + M^*$. Although these sets are M^* -periodic, they are not a partition of \mathbb{R}^d . So, we need to choose a subset \mathcal{N} of \mathbb{Z}^d such that if $\sigma, \sigma' \in \mathcal{N}$ and $\sigma + M^* = \sigma' + M^*$, then $\sigma = \sigma'$. Thus \mathcal{N} should be a section of the quotient \mathbb{Z}^d / M^* . We can choose for example the set \mathcal{N} given by

$$\mathcal{N} = (T^*)^{-1}(\{0, \dots, a_1 - 1\} \times \dots \times \{0, \dots, a_q - 1\} \times \mathbb{Z}^{d-q}), \quad (14)$$

where a_1, \dots, a_q are the invariant factors of M . Hence, given $\sigma \in \mathcal{N}$ we define

$$B_\sigma = \Omega + \sigma + M^* = \bigcup_{m^* \in M^*} (\Omega + \sigma) + m^*. \quad (15)$$

We give three basic examples of this construction.

Example 5.2.

(1) Let $M = \frac{1}{n}\mathbb{Z} \subseteq \mathbb{R}$, then $M^* = n\mathbb{Z}$, $\Omega = [0, 1)$ and $\mathcal{N} = \{0, \dots, n-1\}$. Given $\sigma \in \{0, \dots, n-1\}$, we have

$$B_\sigma = \bigcup_{m^* \in n\mathbb{Z}} ([0, 1) + \sigma) + m^* = \bigcup_{j \in \mathbb{Z}} [\sigma, \sigma + 1) + nj.$$

Figure 1 illustrates the partition for $n = 4$. In the picture, the black dots represent the set \mathcal{N} . The set B_2 is the one which appears in gray.

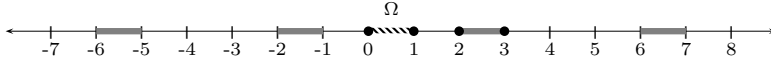


FIGURE 1. Partition of the real line for $M = \frac{1}{4}\mathbb{Z}$.

(2) Let $M = \frac{1}{2}\mathbb{Z} \times \mathbb{R}$, then $\Omega = [0, 1)^2$, $M^* = 2\mathbb{Z} \times \{0\}$ and $\mathcal{N} = \{0, 1\} \times \mathbb{Z}$. So, the sets $B_{(i,j)}$ are

$$B_{(i,j)} = \bigcup_{k \in \mathbb{Z}} ([0, 1)^2 + (i, j)) + (2k, 0)$$

where $(i, j) \in \mathcal{N}$. See Figure 2, where the sets $B_{(0,0)}$, $B_{(1,1)}$ and $B_{(-1,-1)}$ are represented by the squares painted in light gray, gray and dark gray respectively. As in Figure 1, the set \mathcal{N} is represented by the black dots.

(3) Let $M = \{k\frac{1}{3}v_1 + tv_2 : k \in \mathbb{Z} \text{ and } t \in \mathbb{R}\}$, where $v_1 = (1, -2)$ and $v_2 = (0, 1)$. Then, $\{v_1, v_2\}$ satisfy conditions in Theorem 3.9. By Corollary 3.10, $M^* = \{k3w_1 : k \in \mathbb{Z}\}$, where $w_1 = (1, 0)$ and $w_2 = (2, 1)$.

Note that the sets Ω and \mathcal{N} can be expressed in terms of w_1 and w_2 as

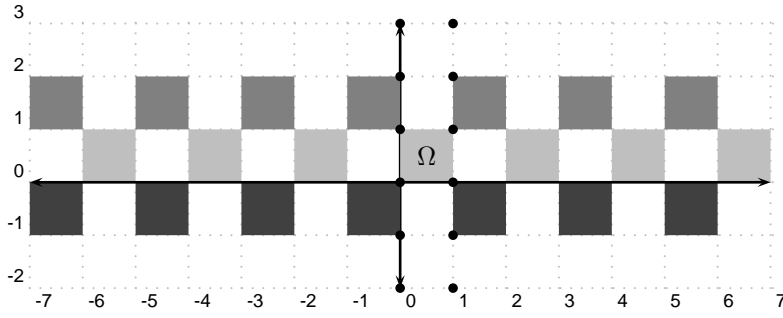
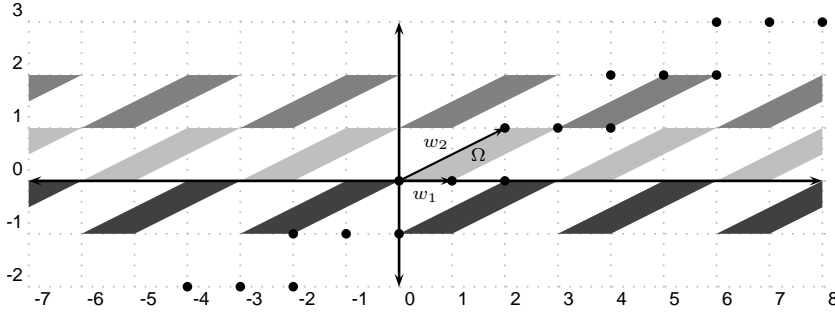
$$\Omega = \{tw_1 + sw_2 : t, s \in [0, 1)\}$$

and

$$\mathcal{N} = \{aw_1 + kw_2 : a \in \{0, 1, 2\}, k \in \mathbb{Z}\}.$$

This is illustrated in Figure 3. In this case the sets $B_{(0,0)}$, $B_{(3,1)}$ and $B_{(0,-1)}$ correspond to the light gray, gray and dark gray regions respectively. And again, the black dots represent the set \mathcal{N} .

Once the partition $\{B_\sigma\}_{\sigma \in \mathcal{N}}$ is defined, we state a lemma which will be necessary to prove Theorem 5.1.


 FIGURE 2. Partition of the plane for $M = \frac{1}{2}\mathbb{Z} \times \mathbb{R}$.

 FIGURE 3. Partition for $M = \{k\frac{1}{3}(1, -2) + t(0, 1) : k \in \mathbb{Z} \text{ and } t \in \mathbb{R}\}$.

Lemma 5.3. *Let S be a SIS and $\sigma \in \mathcal{N}$. Assume that the subspace U_σ defined in (11) satisfies $U_\sigma \subseteq S$. Then, U_σ is an M -invariant space and in particular is a SIS.*

Proof. Let us first prove that U_σ is closed. Suppose that $f_j \in U_\sigma$ and $f_j \rightarrow f$ in $L^2(\mathbb{R}^d)$. Since $U_\sigma \subseteq S$ and S is closed, f must be in S . Further,

$$\|\widehat{f}_j - \widehat{f}\|_2^2 = \|(\widehat{f}_j - \widehat{f})\chi_{B_\sigma}\|_2^2 + \|(\widehat{f}_j - \widehat{f})\chi_{B_\sigma^c}\|_2^2 = \|\widehat{f}_j - \widehat{f}\chi_{B_\sigma}\|_2^2 + \|\widehat{f}\chi_{B_\sigma^c}\|_2^2.$$

Since the left-hand side converges to zero, we must have that $\widehat{f}\chi_{B_\sigma^c} = 0$ a.e. $\omega \in \mathbb{R}^d$, and $\widehat{f}_j \rightarrow \widehat{f}\chi_{B_\sigma}$ in $L^2(\mathbb{R}^d)$. So, as $\widehat{f}_j \rightarrow \widehat{f}$ in $L^2(\mathbb{R}^d)$, we conclude that

$$\widehat{f} = \widehat{f}\chi_{B_\sigma},$$

which proves that $f \in U_\sigma$ and so U_σ is closed.

Note that, since $\mathbb{Z}^d \subseteq M$, the \mathbb{Z}^d -invariance of U_σ is a consequence of the M -invariance.

So, given $m \in M$ and $f \in U_\sigma$, we will prove that $e_m \widehat{f} \in \widehat{U}_\sigma$. Since $f \in U_\sigma$, there exists $g \in S$ such that $\widehat{f} = \chi_{B_\sigma} \widehat{g}$. Hence,

$$e_m \widehat{f} = e_m(\chi_{B_\sigma} \widehat{g}) = \chi_{B_\sigma}(e_m \widehat{g}). \quad (16)$$

If we can find a \mathbb{Z}^d -periodic function ℓ_m verifying

$$e_m(\omega) = \ell_m(\omega) \quad \text{a.e. } \omega \in B_\sigma, \quad (17)$$

then, we can rewrite (16) as

$$e_m \widehat{f} = \chi_{B_\sigma}(\ell_m \widehat{g}).$$

By Theorem 4.2, $\ell_m \widehat{g} \in \widehat{S(g)} \subseteq \widehat{S}$ and so, $e_m \widehat{f} \in \widehat{U}_\sigma$.

Let us now define the function ℓ_m . Note that, since e_m is M^* -periodic,

$$e_m(\omega + \sigma) = e_m(\omega + \sigma + m^*) \quad \text{a.e. } \omega \in \Omega, \forall m^* \in M^*. \quad (18)$$

For each $k \in \mathbb{Z}^d$, set

$$\ell_m(\omega + k) = e_m(\omega + \sigma) \quad \text{a.e. } \omega \in \Omega. \quad (19)$$

It is clear that ℓ_m is \mathbb{Z}^d -periodic and combining (18) with (19), we obtain (17). \square

Proof of Theorem 5.1. i) \Rightarrow ii): Fix $\sigma \in \mathcal{N}$ and $f \in U_\sigma$. Then $\widehat{f} = \chi_{B_\sigma} \widehat{g}$ for some $g \in S$. Since χ_{B_σ} is an M^* -periodic function, by Theorem 4.3, we have that $f \in S_M(g) \subseteq S$, as we wanted to prove.

ii) \Rightarrow i): Suppose that $U_\sigma \subseteq S$ for all $\sigma \in \mathcal{N}$. Note that Lemma 5.3 implies that U_σ is M -invariant, and we also have that the subspaces U_σ are mutually orthogonal since the sets B_σ are disjoint.

Take $f \in S$. Then, since $\{B_\sigma\}_{\sigma \in \mathcal{N}}$ is a partition of \mathbb{R}^d , we can decompose f as $f = \sum_{\sigma \in \mathcal{N}} f^\sigma$ where f^σ is such that $\widehat{f^\sigma} = \widehat{f} \chi_{B_\sigma}$. This implies that $f \in \bigoplus_{\sigma \in \mathcal{N}} U_\sigma$ and consequently, S is the orthogonal direct sum

$$S = \bigoplus_{\sigma \in \mathcal{N}} U_\sigma.$$

As each U_σ is M -invariant, so is S . \square

5.1. Characterization of M -invariance in terms of fibers

A useful tool in the theory of shift-invariant spaces is based on early work of Helson [8]. An $L^2(\mathbb{R}^d)$ function is decomposed into ‘‘fibers.’’ This produces a characterization of SIS in terms of closed subspaces of $\ell^2(\mathbb{Z}^d)$ (the fiber spaces).

For a detailed description of this approach, see [7] and the references therein.

Given $f \in L^2(\mathbb{R}^d)$ and $\omega \in \Omega$, the *fiber* \widehat{f}_ω of f at ω is the sequence

$$\widehat{f}_\omega = \{\widehat{f}(\omega + k)\}_{k \in \mathbb{Z}^d}.$$

If f is in $L^2(\mathbb{R}^d)$, then the fiber \widehat{f}_ω belongs to $\ell^2(\mathbb{Z}^d)$ for almost every $\omega \in \Omega$.

Given a subspace V of $L^2(\mathbb{R}^d)$ and $\omega \in \Omega$, the *fiber space* of V at ω is

$$J_V(\omega) = \overline{\{\widehat{f}_\omega : f \in V\}},$$

where the closure is taken in the norm of $\ell^2(\mathbb{Z}^d)$.

The map assigning to each ω the fiber space $J_V(\omega)$ is known in the literature as the *range function* of V .

The *dimension function* is defined by

$$\dim_V : \Omega \rightarrow \mathbb{N}_0 \cup \{\infty\}, \quad \dim_V(\omega) := \dim(J_V(\omega)).$$

For a proof that, for almost every ω , $J_V(\omega)$ is a well-defined closed subspace of $\ell^2(\mathbb{Z}^d)$ and that shift-invariant spaces can be characterized through range functions, see [7], [8].

Proposition 5.4 ([8]). *If S is a SIS, then*

$$S = \{f \in L^2(\mathbb{R}^d) : \widehat{f}_\omega \in J_S(\omega) \text{ for a.e. } \omega \in \Omega\}.$$

Proposition 5.5. *Let S_1 and S_2 be SISs. Then we have, $S = S_1 \dot{\oplus} S_2$, if and only if*

$$J_S(\omega) = J_{S_1}(\omega) \dot{\oplus} J_{S_2}(\omega), \quad \text{a.e. } \omega \in \Omega.$$

Let $\Phi = \{\varphi_1, \dots, \varphi_\ell\}$ be a finite collection of functions in $L^2(\mathbb{R}^d)$. Then the *Gramian* G_Φ of Φ is the $\ell \times \ell$ matrix of \mathbb{Z}^d -periodic functions

$$[G_\Phi(\omega)]_{ij} = \left\langle (\widehat{\varphi}_i)_\omega, (\widehat{\varphi}_j)_\omega \right\rangle = \sum_{k \in \mathbb{Z}^d} \widehat{\varphi}_i(\omega + k) \overline{\widehat{\varphi}_j(\omega + k)}, \quad \omega \in \Omega. \quad (20)$$

So, we have the following relation whose proof is straightforward.

Proposition 5.6. *Let S be an FSIS generated by Φ . Therefore,*

$$\dim_S(\omega) = \text{rank}[G_\Phi(\omega)] \quad \text{a.e. } \omega \in \Omega.$$

Now, if $f \in L^2(\mathbb{R}^d)$ and $\sigma \in \mathcal{N}$, let f^σ denote the function defined by

$$\widehat{f}^\sigma = \widehat{f} \chi_{B_\sigma}.$$

Let P_σ be the orthogonal projection onto S_σ , where

$$S_\sigma := \{f \in L^2(\mathbb{R}^d) : \text{supp}(\widehat{f}) \subseteq B_\sigma\}.$$

Therefore

$$f^\sigma = P_\sigma f \quad \text{and} \quad U_\sigma = P_\sigma(S) = \{f^\sigma : f \in S\}.$$

Moreover, if $S = S(\Phi)$ with Φ a countable subset of $L^2(\mathbb{R}^d)$, then

$$J_{U_\sigma}(\omega) = \overline{\text{span}\{(\widehat{\varphi}^\sigma)_\omega : \varphi \in \Phi\}}. \quad (21)$$

Remark 5.7. Note that $(\widehat{\varphi}^\sigma)_\omega = \{\chi_{B_\sigma}(\omega + k) \widehat{\varphi}(\omega + k)\}_{k \in \mathbb{Z}^d}$. Then, since $\chi_{B_\sigma}(\omega + k) \neq 0$ if and only if $k \in \sigma + M^*$,

$$\chi_{B_\sigma}(\omega + k) \widehat{\varphi}(\omega + k) = \begin{cases} \widehat{\varphi}(\omega + k) & \text{if } k \in \sigma + M^* \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, if $\sigma \neq \sigma'$, $J_{U_\sigma}(\omega)$ is orthogonal to $J_{U_{\sigma'}}(\omega)$ for a.e. $\omega \in \Omega$.

Combining Theorem 5.1 with Proposition 5.4 and (21) we obtain the following proposition.

Theorem 5.8. *Let S be a SIS generated by Φ . The following statements are equivalent.*

- (i) S is M -invariant.
- (ii) $(\widehat{\varphi^\sigma})_\omega \in J_S(\omega)$ a.e. $\omega \in \Omega$ for all $\varphi \in \Phi$ and $\sigma \in \mathcal{N}$.

Now, using Proposition 5.5, Proposition 5.6, Theorem 5.8 and Remark 5.7, we give a slightly simpler characterization of M -invariance for the finitely generated case.

Theorem 5.9. *If S is an FSIS generated by Φ , then the following statements are equivalent.*

- (a) S is M -invariant.
- (b) For almost every $\omega \in \Omega$, $\dim_S(\omega) = \sum_{\sigma \in \mathcal{N}} \dim_{U_\sigma}(\omega)$.
- (c) For almost every $\omega \in \Omega$, $\text{rank}[G_\Phi(\omega)] = \sum_{\sigma \in \mathcal{N}} \text{rank}[G_{\Phi^\sigma}(\omega)]$,
where $\Phi^\sigma = \{\varphi^\sigma : \varphi \in \Phi\}$.

6. Applications of M -invariance

In this section we present two applications of the results given before. First, we will estimate the size of the supports of the Fourier transforms of the generators of an FSIS which is also M -invariant. Finally, given M a closed subgroup of \mathbb{R}^d containing \mathbb{Z}^d , we will construct a SIS S which is exactly M -invariant. That is, S is not invariant under any other closed subgroup containing M .

Theorem 6.1. *Let S be an FSIS generated by $\{\varphi_1, \dots, \varphi_\ell\}$, and define*

$$E_j = \{\omega \in \Omega : \dim_S(\omega) = j\}, \quad j = 0, \dots, \ell.$$

If S is M -invariant and \mathcal{D}' is any measurable section of \mathbb{R}^d/M^ , then*

$$|\{y \in \mathcal{D}' : \widehat{\varphi}_h(y) \neq 0\}| \leq \sum_{j=0}^{\ell} j |E_j| \leq \ell,$$

for each $h = 1, \dots, \ell$.

Proof. The measurability of the sets E_j follows from the results of Helson [8], e.g., see [4] for an argument of this type.

Fix any $h \in \{0, \dots, \ell\}$. Note that, as a consequence of Remark 5.7, if $J_{U_\sigma}(\omega) = \{0\}$, then $\widehat{\varphi}_h(\omega + \sigma + m^*) = 0$ for all $m^* \in M^*$.

On the other hand, since $\{\Omega + \sigma + m^*\}_{\sigma \in \mathcal{N}, m^* \in M^*}$ is a partition of \mathbb{R}^d , if $\omega \in \Omega$ and $\sigma \in \mathcal{N}$ are fixed, there exists a unique $m^*_{(\omega, \sigma)} \in M^*$ such that $\omega + \sigma + m^*_{(\omega, \sigma)} \in \mathcal{D}'$.

So,

$$\{\sigma \in \mathcal{N} : \widehat{\varphi}_h(\omega + \sigma + m_{(\omega, \sigma)}^*) \neq 0\} \subseteq \{\sigma \in \mathcal{N} : \dim_{U_\sigma}(\omega) \neq 0\}.$$

Therefore

$$\begin{aligned} \#\{\sigma \in \mathcal{N} : \widehat{\varphi}_h(\omega + \sigma + m_{(\omega, \sigma)}^*) \neq 0\} &\leq \#\{\sigma \in \mathcal{N} : \dim_{U_\sigma}(\omega) \neq 0\} \\ &\leq \sum_{\sigma \in \mathcal{N}} \dim_{U_\sigma}(\omega) \\ &= \dim_S(\omega). \end{aligned}$$

Consequently, by Fubini's Theorem,

$$\begin{aligned} |\{y \in \mathcal{D}' : \widehat{\varphi}_h(y) \neq 0\}| &= \sum_{\sigma \in \mathcal{N}} |\{\omega \in \Omega : \widehat{\varphi}_h(\omega + \sigma + m_{(\omega, \sigma)}^*) \neq 0\}| \\ &= |\{(\omega, \sigma) \in \Omega \times \mathcal{N} : \widehat{\varphi}_h(\omega + \sigma + m_{(\omega, \sigma)}^*) \neq 0\}| \\ &= \int_{\Omega} \#\{\sigma \in \mathcal{N} : \widehat{\varphi}_h(\omega + \sigma + m_{(\omega, \sigma)}^*) \neq 0\} dw \\ &\leq \int_{\Omega} \dim_S(\omega) dw = \sum_{j=0}^{\ell} j |E_j| \leq \ell. \end{aligned}$$

□

When M is not discrete, the previous theorem shows that, despite the fact that \mathcal{D}' has infinite measure, the support of $\widehat{\varphi}_h$ in \mathcal{D}' has finite measure.

On the other hand, if M is discrete, the measure of \mathcal{D}' is equal to the measure of the section \mathcal{D} given by (3). That is

$$|\mathcal{D}'| = |\mathcal{D}| = a_1 \dots a_d,$$

where a_1, \dots, a_d are the invariant factors. Thus, if $a_1 \dots a_d - \ell > 0$, it follows that

$$|\{y \in \mathcal{D}' : \widehat{\varphi}_h(y) = 0\}| \geq a_1 \dots a_d - \ell. \quad (22)$$

Corollary 6.2. *Let $\varphi \in L^2(\mathbb{R}^d)$ be given. If the SIS $S(\varphi)$ is M -invariant for some closed subgroup M of \mathbb{R}^d such that $\mathbb{Z}^d \subsetneq M$, then $\widehat{\varphi}$ must vanish on a set of infinite Lebesgue measure.*

Proof. Let \mathcal{D} be the measurable section of \mathbb{R}^d/M^* defined in (3). Then,

$$\mathbb{R}^d = \bigcup_{m^* \in M^*} \mathcal{D} + m^*,$$

thus

$$|\{y \in \mathbb{R}^d : \widehat{\varphi}(y) = 0\}| = \sum_{m^* \in M^*} |\{y \in \mathcal{D} + m^* : \widehat{\varphi}(y) = 0\}|.$$

If M is discrete, by (22), we have

$$|\{y \in \mathbb{R}^d : \widehat{\varphi}(y) = 0\}| \geq \sum_{m^* \in M^*} (|\mathcal{D}| - 1) = +\infty. \quad (23)$$

The last equality is due to the fact that M^* is infinite and $|\mathcal{D}| > 1$ (since $M \neq \mathbb{Z}^d$).

If M is not discrete, by Theorem 6.1, $|\{y \in \mathcal{D} + m^* : \widehat{\varphi}(y) = 0\}| = +\infty$, hence $|\{y \in \mathbb{R}^d : \widehat{\varphi}(y) = 0\}| = +\infty$. \square

It is known that on the real line, the SIS generated by a function φ with compact support can only be invariant under integer translations. That is, $t_x\varphi \notin S(\varphi)$ for all $x \in \mathbb{R} \setminus \mathbb{Z}$. The following proposition extends this result to \mathbb{R}^d .

Proposition 6.3. *If a nonzero function $\varphi \in L^2(\mathbb{R}^d)$ has compact support, then $S(\varphi)$ is not M -invariant for any M closed subgroup of \mathbb{R}^d such that $\mathbb{Z}^d \subsetneq M$. In particular,*

$$t_x\varphi \notin S(\varphi) \quad \forall x \in \mathbb{R}^d \setminus \mathbb{Z}^d. \quad (24)$$

Proof. The first part of the proposition is a straightforward consequence of Corollary 6.2. To show (24), take $x \in \mathbb{R}^d \setminus \mathbb{Z}^d$ and suppose that $t_x\varphi \in S(\varphi)$. If M is the closed subgroup generated by x and \mathbb{Z}^d , then $S(\varphi)$ must be M -invariant, which is a contradiction. \square

As a consequence of Theorem 6.1, in case that $M = \mathbb{R}^d$, we obtain the following corollary.

Corollary 6.4. *If $\varphi \in L^2(\mathbb{R}^d)$ and $S(\varphi)$ is \mathbb{R}^d -invariant, then*

$$|\text{supp}(\widehat{\varphi})| \leq 1.$$

Let M be a closed subgroup of \mathbb{R}^d containing \mathbb{Z}^d . The next theorem, states that there exists an M -invariant space S that is *not* invariant under any vector outside M . We will say in this case that S is *exactly* M -invariant.

Note that because of Proposition 2.1, an M -invariant space is exactly M -invariant if and only if it is not invariant under any closed subgroup M' containing M .

Theorem 6.5. *For each closed subgroup M of \mathbb{R}^d containing \mathbb{Z}^d , there exist a shift-invariant space of $L^2(\mathbb{R}^d)$ which is exactly M -invariant.*

Proof. Let M be a subgroup of \mathbb{R}^d containing \mathbb{Z}^d . We will construct a principal shift-invariant space that is exactly M -invariant.

Suppose that $0 \in \mathcal{N}$ and take $\varphi \in L^2(\mathbb{R}^d)$ satisfying $\text{supp}(\widehat{\varphi}) = B_0$, where B_0 is defined as in (15). Let $S = S(\varphi)$.

Then, $U_0 = S$ and $U_\sigma = \{0\}$ for $\sigma \in \mathcal{N}$, $\sigma \neq 0$. So, as a consequence of Theorem 5.1, it follows that S is M -invariant.

Now, if M' is a closed subgroup such that $M \subsetneq M'$, we will show that S can not be M' -invariant.

Since $M \subseteq M'$, $(M')^* \subseteq M^*$. Consider a section \mathcal{H} of the quotient $M^*/(M')^*$ containing the origin. Then, the set given by

$$\mathcal{N}' := \{\sigma + h : \sigma \in \mathcal{N}, h \in \mathcal{H}\},$$

is a section of $\mathbb{Z}^d/(M')^*$ and $0 \in \mathcal{N}'$.

If $\{B'_\gamma\}_{\gamma \in \mathcal{N}'}$ is the partition defined in (15) associated to M' , for each $\sigma \in \mathcal{N}$ it holds that $\{B'_{\sigma+h}\}_{h \in \mathcal{H}}$ is a partition of B_σ , since

$$B_\sigma = \Omega + \sigma + M^* = \bigcup_{h \in \mathcal{H}} \Omega + \sigma + h + (M')^* = \bigcup_{h \in \mathcal{H}} B'_{\sigma+h}. \quad (25)$$

We will show now that $U'_0 \not\subseteq S$, where U'_0 is the subspace defined in (11) for M' . Let $g \in L^2(\mathbb{R}^d)$ such that $\hat{g} = \hat{\varphi}\chi_{B'_0}$. Then $g \in U'_0$. Moreover, since $\text{supp}(\hat{\varphi}) = B_0$, by (25), $\hat{g} \neq 0$.

Suppose that $g \in S$, then $\hat{g} = \eta\hat{\varphi}$ where η is a \mathbb{Z}^d -periodic function. Since $M \not\subseteq M'$, there exists $h \in \mathcal{H}$ such that $h \neq 0$. By (25), \hat{g} vanishes in B'_h . Then, the \mathbb{Z}^d -periodicity of η implies that $\eta(y) = 0$ a.e. $y \in \mathbb{R}^d$. So $\hat{g} = 0$, which is a contradiction.

This shows that $U'_0 \not\subseteq S$. Therefore, S is not M' -invariant. \square

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