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A characterization of the left exact categories whose exact completions are toposes

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Dedicated to S. Mac Lane on the occasion of his 90th birthday

Abstract

We characterize the categories with finite limits whose exact completions are toposes and discuss some examples and counter-examples.

MSC: 18A35; 18B25; 03F50

1. Introduction

Many categories of interest arise as exact completions of a left exact category. For example, realizability toposes are exact completions [24] as well as the presheaf topos $\mathbf{Sets}^{\mathbf{C}^{op}}$ for every small left exact category $\mathbf{C}$ (see [5]). More recently, in computer science there has been a lot of interest in the exact completion of the category of topological spaces [3]. Although a simple construction of the exact completion of a left exact category was given in [6], the resulting category will be usually more difficult to work with directly than the category giving rise to it. So it is interesting to be able to deduce important properties of the former in terms of easily checkable properties of the latter. For example, in [8] it was shown that the exact completion $\mathbf{C}_{ex}$ of a

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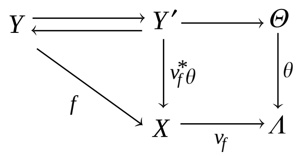
category with finite limits $\mathbf{C}$ is locally cartesian closed if and only if $\mathbf{C}$ has *weak dependent products*. This was used to prove that the exact completion of the category of topological spaces is locally cartesian closed. In this paper, we provide necessary and sufficient conditions on a category with finite limits for its exact completion to be a topos. This characterization rests on the following definition.

Definition 1.1. A *generic proof* is a map $\theta : \Theta \to \Lambda$ such that for every map $f : Y \to X$ there exists a $v_f : X \to \Lambda$ such that f factors through $v_f^*\theta$ and $v_f^*\theta$ factors through f.



The characterization can now be stated as follows.

Theorem 1.2. $\mathbf{C}_{\mathrm{ex}}$ *is a topos if and only if* $\mathbf{C}$ *has weak dependent products and a generic proof.*

The proof will be given in Section 4 after reviewing the basic technical facts about regular categories in Section 2 and about completions in Section 3. In the final section we discuss some examples and counter-examples.

2. Regular and exact categories

In this section we quickly review regular and exact categories [2,10,4]. A category with finite limits is *regular* if every kernel pair has a coequalizer and regular epis are stable under pullback. It follows that a regular category has stable regular-epi/mono factorizations. A diagram as below is called an *exact sequence* if it is both a pullback and a coequalizer. That is, if it is a coequalizer diagram and e_0, e_1 is the kernel pair of e.

$$X' \underset{e_1}{\overset{e_0}{\rightrightarrows}} X \overset{e}{\twoheadrightarrow} X''$$

We now present as a lemma, two well-known facts about regular categories whose proofs can be found in the references mentioned above. As usual, we denote by α^* the operation of pulling back along the map α.

Lemma 2.1. *In a regular category*

1. *exact sequences are stable under pullback*;
2. *let the following diagram be such that both rows are exact and the two left hand squares are pullbacks* (*so that* $e_0^* f \cong e_1^* f$). *Then the right hand square*

is a pullback.

$$\begin{array}{ccccc} Y' & \underset{d_1}{\overset{d_0}{\rightrightarrows}} & Y & \overset{d}{\twoheadrightarrow} & Y'' \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f''}\downarrow \\ X' & \underset{e_1}{\overset{e_0}{\rightrightarrows}} & X & \overset{e}{\twoheadrightarrow} & X'' \end{array}$$

Definition 2.2 (Equivalence relations and exact categories).

1. A (*binary*) *relation* on an object A is an object R together with a jointly monic pair of arrows $r_1, r_2 : R \to A$. For every object X we write $R_X = \{(r_1.f, r_2.f) | f : X \to R\}$ for the corresponding relation (in the usual sense) on the set $Hom(X, A)$.
2. An *equivalence relation* on an object A is a relation on A in the sense above such that for every X, R_X is an equivalence relation in the usual sense. An equivalence relation is *effective* if it is the kernel pair of some arrow.
3. A category is *exact* if it is regular and every equivalence relation is effective.

When considering categories with finite limits (as we are doing), there is an equivalent elementary definition of equivalence relation (see for example [4]). We use the one above for convenience. Finally, recall that a functor between regular categories is *exact* if it preserves finite limits and exact sequences.

3. Regular and exact completions

In this section we review some of the material in [6,5] concerning the regular and exact completions of a left exact category. For any category with finite limits $\mathbf{C}$, it is possible to build a category $\mathbf{C}_{\text{reg}}$ and an embedding $\mathbf{C} \to \mathbf{C}_{\text{reg}}$ that induces an equivalence of categories between the category of functors from $\mathbf{C}$ to $\mathbf{D}$ that preserve finite limits and the category of exact functors from $\mathbf{C}_{\text{reg}}$ to $\mathbf{D}$. The category $\mathbf{C}_{\text{reg}}$ has an easy description and it can be found in [5]. We will only need its characterization which we recall below.

Definition 3.1. An object X is (*regular*) *projective* if for every regular epi $e : A \twoheadrightarrow B$ and map $g : X \to B$ there exists a map $f : X \to A$ such that $e.f = g$.

We say that a category *has enough projectives* if for every object A there exists a projective X and a regular epi $q : X \twoheadrightarrow A$. We say that q is a *projective cover of* A.

Proposition 3.2. *A regular category* $\mathbf{D}$ *is a regular completion if and only if it has enough projectives, projectives are closed under finite limits and every object is a subobject of a projective. Moreover, in this case,* $\mathbf{D}$ *is the regular completion of its full subcategory of projectives.*

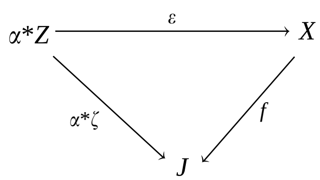
As we mentioned, it is also possible to obtain the exact completion of a category with finite limits. For any category $\mathbf{C}$ with finite limits, there exists an exact category $\mathbf{C}_{\text{ex}}$ and an embedding $\mathbf{C} \to \mathbf{C}_{\text{ex}}$ that induces for every exact category $\mathbf{E}$, an equivalence between the category of functors from **C** to **E** that preserve finite limits and the category of exact functors from $\mathbf{C}_{\text{ex}}$ to **E**. There also exists a characterization of the categories that arise as exact completions.

Proposition 3.3. *An exact category* **E** *is an exact completion if and only if it has enough projectives and projectives are closed under finite limits in* **E**. *Moreover, in this case* **E** *is the exact completion of its full subcategory of projectives.*

The explicit description of $\mathbf{C}_{\text{ex}}$ can be found [5]. It should also be mentioned that finite limits are not really needed to build regular and exact completions. As explained in [9], it is enough to assume *weak* finite limits. Moreover, there are interesting toposes that arise as exact completions of categories with only weak finite limits (see [9,23]). In this paper we prove our results assuming finite limits. It is to be expected that these results generalize to the case of weak finite limits.

A key ingredient in our characterization is the use of the locally cartesian closed structure of toposes. The technical development that allowed this argument is the characterization of the categories with weak finite limits whose exact completions are locally cartesian closed [8]. This result is also reviewed in [3] where strong finite limits are assumed.

Definition 3.4. A *weak dependent product* of a map $f : X \to J$ along a map $\alpha : J \to I$ consists of maps $\zeta : Z \to I$ and $\varepsilon : \alpha^* Z \to X$ such that $f.\varepsilon = \alpha^* \zeta$. Moreover, the pair ε, ζ is weakly universal in the sense that for any other pair of maps $\zeta' : Z' \to I$ and $\varepsilon' : \alpha^* Z' \to X$ such that $f.\varepsilon' = \alpha^* \zeta'$ there exists a not necessarily unique $f' : \alpha^* \zeta' \to \alpha^* \zeta$ such that $\varepsilon.f' = \varepsilon'$.



As explained in [3], a weak dependent product of f along α can also be expressed as a natural epi $\mathbf{C}/I(\_, \zeta) \twoheadrightarrow \mathbf{C}/J(\alpha^*(\_), f)$. See also Remark 3.2 in [8]. The result in [8], specialized as in [3] to the setting where strong finite limits are assumed, is the following.

Proposition 3.5. $\mathbf{C}_{\text{ex}}$ *is locally cartesian closed iff* **C** *has weak dependent products.*

We now restate some of the results in [24] where it is proved that in an exact completion, it is enough to be able to classify subobjects of projectives in order to be able to deduce the existence of an honest subobject classifier.

Definition 3.6. A *classifier of subobjects of projectives* is a mono $\tau : \Upsilon \rightarrowtail \Lambda$ such that for every projective X and subobject $m : U \rightarrowtail X$ there exists a unique $v_m : X \to \Lambda$

such that the square below is a pullback.

$$\begin{array}{ccc} U & \longrightarrow & \Upsilon \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \tau} \\ X & \underset{v_m}{\longrightarrow} & \Lambda \end{array}$$

We say that τ is a *weak* classifier of subobjects of projectives if the v_m is not necessarily unique. Moreover, we say that the classifier (weak or not) is *projective* if Λ is.

Lemma 3.7. $\mathbf{C}_{\mathrm{ex}}$ *has a subobject classifier if and only if it has a classifier of subobjects of projectives.*

Proof. The "only if" direction is trivial as the subobject classifier classifies subobjects of projectives in particular. For the converse, let $\mathbf{y}:\mathbf{C} \to \mathbf{C}_{\mathrm{ex}}$ denote the left exact embedding of the projectives in an exact completion. The existence of a classifier of subobjects of projectives $\tau:\Upsilon \rightarrowtail \Lambda$ in $\mathbf{C}_{\mathrm{ex}}$ induces an isomorphism $\mathbf{C}_{\mathrm{ex}}(\mathbf{y}X, \Lambda) \to Sub(\mathbf{y}X)$ natural in X. That is, the functor Sub is representable "over projectives". In [24] (Lemma 5.1) it is proved that for certain *suitable* functors (of which Sub is an example), representability is implied by representability over projectives. □

Finally we recall the definition of a *generic mono* which is used in [16,21] to study some examples and counter-examples, see also Section 5.1 below.

Definition 3.8. A *generic mono* in a category $\mathbf{D}$ is a mono $\tau:\Upsilon \rightarrowtail \Lambda$ such that every mono $u:U \rightarrowtail A$ in $\mathbf{D}$ arises as a pullback of τ along a not necessarily unique map.

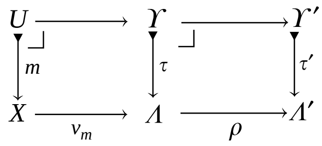
4. The characterization

In this section we prove Theorem 1.2.

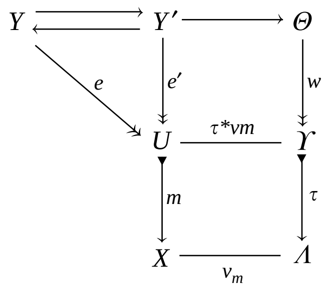
Lemma 4.1. *The following are equivalent.*

1. $\mathbf{C}$ *has a generic proof.*
2. $\mathbf{C}_{\mathrm{ex}}$ *has a projective weak classifier of subobjects of projectives.*
3. $\mathbf{C}_{\mathrm{ex}}$ *has a weak classifier of subobjects of projectives.*
4. $\mathbf{C}_{\mathrm{reg}}$ *has a generic mono.*

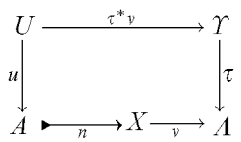
Proof. It is easy to show that 4 implies 3. To prove that 3 implies 2 let $\tau' : \Upsilon' \to \Lambda'$ be a weak classifier of subobjects of projectives in $\mathbf{C}_{\mathrm{ex}}$. Let $\rho:\Lambda \twoheadrightarrow \Lambda'$ be a projective cover. Then define $\tau:\Upsilon \rightarrowtail \Lambda$ to be the pullback of τ' along ρ. To prove that this τ is a weak classifier, let X be projective and let $m:U \rightarrowtail X$. By hypothesis, there exists a $\chi_m:X \to \Lambda'$ such that m is the pullback of τ' along χ_m. Now, as X is projective and ρ is a regular epi, there exists a $v_m:X \to \Lambda$ such that $\rho.v_m = \chi_m$. It follows that m is the pullback of τ along v_m as in the following diagram.



We now prove that 2 implies 1. So assume that $\mathbf{C}_{ex}$ has a projective weak classifier of subobjects of projectives $\tau : \Upsilon \to \Lambda$. Let $w : \Theta \to \Upsilon$ be a projective cover. Now, consider any $f : Y \to X$ between projectives and consider its regular-epi/mono factorization $f = m.e$ in $\mathbf{C}_{ex}$. By assumption, there exists a $v_m : X \to \Lambda$ such that $v_m^* \tau \cong m$. Now let $e' : Y' \twoheadrightarrow U$ be the pullback of w along $\tau^* v_m$ as in the diagram below. As v_m and $\tau.w$ are maps between projectives and projectives are closed under finite limits in $\mathbf{C}_{ex}$, Y' is projective. Then, e and e' factor through each other and hence, so do f and $m.e'$ which is the pullback of $\tau.w$ along v_m. So $\tau.w$ is a generic proof in $\mathbf{C}$.



To prove 1 implies 4, let $\theta : \Theta \to \Lambda$ be a generic proof in $\mathbf{C}$. Let $\tau.w = \theta$ be the regular-epi/mono factorization in $\mathbf{C}_{reg}$ with $\tau : \Upsilon \to \Lambda$. We will prove that τ is our desired generic mono in $\mathbf{C}_{reg}$. Now let $u : U \rightarrowtail A$ be an arbitrary mono in $\mathbf{C}_{reg}$. By Proposition 3.2, there is a mono $n : A \rightarrowtail X$ into a projective X. Let $m : U \rightarrowtail X$ be the subobject of the projective X given by $n.u$. Let $e : Y \twoheadrightarrow U$ a projective cover and define $f = m.e$. As θ is a generic proof, there exists a map $v : X \to \Lambda$ such that f factors through $v^* \theta = (v^* \tau).((\tau^* v)^* w)$ and vice versa. Then, by Lemma 2.1, m and $v_f^* \tau$ determine the same subobject. That is, m is a pullback of τ. But then the following diagram is also a pullback and this shows that τ is a generic mono.

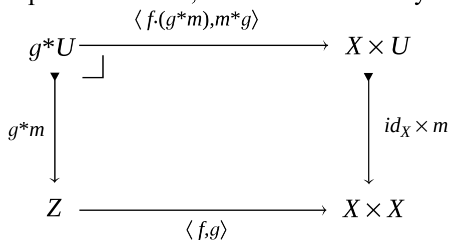


□

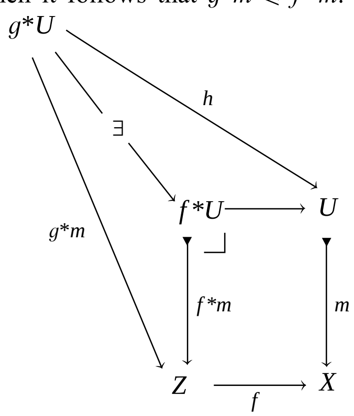
For any category $\mathbf{C}$ with finite limits let us denote by Prf the contravariant functor that for every object X in $\mathbf{C}$, $Prf(X) = \widetilde{\mathbf{C}/X}$ the poset reflection of the slice. It operates on arrows by pullback. This functor is called the *proof theoretic power set* functor in [15]. An application of Yoneda shows that $\mathbf{C}$ has a generic proof if and only if there exists an object Λ and a natural epi $\mathbf{C}(X, \Lambda) \twoheadrightarrow Prf(X)$. That is, if Prf is *weakly representable*. By Lemma 4.1, this is equivalent to the fact that $Sub:(\mathbf{C}_{\mathrm{ex}})^{\mathrm{op}} \to \mathbf{Sets}$ is weakly representable over projectives. That is, that there exists a natural epi $\mathbf{C}_{\mathrm{ex}}(\mathbf{y}X, \mathbf{y}\Lambda) \twoheadrightarrow Sub(\mathbf{y}X)$. The strategy for the proof of Theorem 1.2 is to build an honest classifier of subobjects of projectives out of a weak one. In order to do this, we are going to use the locally cartesian closed structure of the exact completion to build an equivalence relation on Λ. The quotient of this equivalence relation will classify subobjects of projectives. Then, using Lemma 3.7 we will be able to conclude that there exists a subobject classifier. The following lemma explains how to build the equivalence relation.

Lemma 4.2. *Let* $\mathbf{E}$ *be locally cartesian closed and let* $m: U \rightarrowtail X$. *Then there exists an arrow* $m': U' \rightarrowtail X \times X$ *such that* $\langle f, g\rangle: Z \to X \times X$ *factors through* m' *if and only if* f *and* g *pullback* m *to the same subobject* (*i.e.* $f^*m \cong g^*m$).

Proof. Consider $\gamma = m \times id_X : U \times X \rightarrowtail X \times X$ and $\delta = id_X \times m : X \times U \rightarrowtail X \times X$ as objects in the slice $\mathbf{E}/(X \times X)$. We can then build the mono $m' = \gamma^\delta \times \delta^\gamma : U' \rightarrowtail X \times X$ using the product and exponentiation in the slice. Let $\langle f, g\rangle : Z \to X \times X$ factor through m'. That is, we have an arrow $\langle f, g\rangle \to m'$ in the slice $\mathbf{E}/(X \times X)$. This is uniquely determined by arrows $\langle f, g\rangle \to \gamma^\delta$ and $\langle f, g\rangle \to \delta^\gamma$. Let us concentrate on the arrow on the left. It is uniquely determined by an arrow $\langle f, g\rangle \times \delta \to \gamma$. Products in the slice are just pullbacks in $\mathbf{E}$, so we have an easy description of the domain of this arrow



So, our arrow $\langle f, g\rangle \times \delta \to \gamma$ is just an arrow $\langle h, h'\rangle : g^*U \to U \times X$ such that $(m \times id_X).\langle h, h'\rangle = \langle f, g\rangle.g^*m = (id_X \times m).\langle f.(g^*m), m^*g\rangle$. This implies $m.h = f.g^*m$ and then it follows that $g^*m \leqslant f^*m$.



Similarly, the arrow $\langle f,g\rangle \to \delta^{\gamma}$ implies that $f^*m \leqslant g^*m$. On the other hand, if we start assuming that $g^*m \leqslant f^*m$, by following the proof above from bottom to top, it is easy to prove that there is an arrow $\langle f,g\rangle \to \gamma^{\delta}$. Using the same idea, starting from $f^*m \leqslant g^*m$ it is easy to prove the existence of $\langle f,g\rangle \to \delta^{\gamma}$. So, if f and g pullback m to the same subobject, then $\langle f,g\rangle$ factors through m'. □

Clearly, "pulling back an arrow with codomain X to the same thing" determines an equivalence relation on the hom-sets $\mathbf{E}(\_,X)$. It is not difficult to see then that the $m' = \langle m_0, m_1\rangle$ built above determines an equivalence relation $m_0, m_1 : U' \to X$ on X. Notice that U' can be defined by $U' := x,x' : X \vdash (Ux \to Ux') \wedge (Ux' \to Ux)$ using the internal logic.

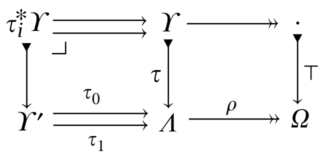
Proposition 4.3. *If* $\mathbf{C}_{\text{ex}}$ *is locally cartesian closed then the following are equivalent*:

1. $\mathbf{C}_{\text{ex}}$ *is a topos*,
2. $\mathbf{C}$ *has a generic proof.*

Proof. By Lemma 4.1 it is enough to show that $\mathbf{C}_{\text{ex}}$ is a topos if and only if $\mathbf{C}_{\text{ex}}$ has a weak classifier of subobjects of projectives. To see that 1 implies 2, notice that the subobject classifier is trivially a weak classifier of subobjects of projectives. To prove that 2 implies 1, let $\tau : \Upsilon \rightarrowtail \Lambda$ be a weak classifier of subobjects of projectives. By hypothesis, the slice $\mathbf{C}_{\text{ex}}/(\Lambda \times \Lambda)$ is cartesian closed. So we can apply Lemma 4.2 to obtain a mono $\tau' : \Upsilon' \rightarrowtail \Lambda \times \Lambda$ with the properties specified. We can then take the quotient:

$$\Upsilon' \underset{\tau_1}{\overset{\tau_0}{\rightrightarrows}} \Lambda \overset{\rho}{\twoheadrightarrow} \Omega$$

Trivially, $\langle \tau_0, \tau_1\rangle = \tau'$ factors through τ'. Then $\tau_0^*\tau \cong \tau_1^*\tau$ by Lemma 4.2. Also, τ pulls the equivalence relation τ_0, τ_1 back to another equivalence relation. As $\mathbf{C}_{\text{ex}}$ is exact, we can take its effective quotient and obtain the top exact sequence in the diagram below. Using the universal property of coequalizers we obtain the map $\top$ making the right hand square commute. It follows by Lemma 2.1 that the right hand square is a pullback. That is, $\rho^*\top = \tau$.



We now prove that Ω classifies subobjects of projectives. It will then follow by Lemma 3.7 that $\mathbf{C}_{\text{ex}}$ is a topos. So let X be projective and let $m : U \rightarrowtail X$ be an arbitrary subobject. Then, as Λ is a weak classifier, there exists a v_m such that $m = v_m^*\tau = v_m^*(\rho^*\top) = (\rho.v_m)^*\top$. This means that $\top$ is also a weak classifier of sub-objects of projectives. We need to prove that there is only one arrow classifying each subobject. So let $f', g' : X \to \Omega$ pull $\top$ back to the same subobject. As X is projective, it follows that f' and g' factor through ρ, say via f and g. Then f and g pullback τ to the same subobject. So there exists an h such that $\langle f,g\rangle = \tau'.h$ by Lemma 4.2. But then $f = \tau_0.h$ and $g = \tau_1.h$. So $\rho.f = \rho.\tau_0.h = \rho.\tau_1.h = \rho.g$. That is, $f' = g'$. □

Together with Proposition 3.5 we obtain our main result.

Corollary 4.4 (Theorem 1.2). $\mathbf{C}_{\mathrm{ex}}$ *is a topos if and only if* $\mathbf{C}$ *has weak dependent products and a generic proof.*

It is worth mentioning that every epi splits in the presence of a proof classifier (i.e. a generic proof for which the v_f is required to be unique). The details of this can be found in [20].

5. Examples

5.1. Realizability toposes

For any partial combinatory algebra $\mathbf{A}$ we define $\mathbf{P_A}$ to be the category whose objects are pairs $X = (|X|, \|\_\|_X)$ with $|X|$ a set and $\|\_\|_X : |X| \to \mathbf{A}$. We usually omit subscripts. An arrow $f : Y \to X$ in this category is a function $f : |Y| \to |X|$ such that there exists an $a \in \mathbf{A}$ such that for every $y \in |Y|, a\|y\|$ is defined and $a\|y\| = \|fy\|$. This is the well known category of *partitioned assemblies*. It is not difficult to prove that $\mathbf{P_A}$ has finite limits and in [8] it is shown that $\mathbf{P_A}$ has weak dependent products. It is also well-known that realizability toposes [11] are the exact completions of the categories of partitioned assemblies and it follows by our characterization that the latter categories must have generic proofs. Let us present them explicitly. Let $\wp\mathbf{A}$ be the object in $\mathbf{P_A}$ with underlying set the set of subsets of $\mathbf{A}$ and associated function, some chosen constant function. Let Θ have underlying set $\{(U,a) | U \subseteq \mathbf{A}$ and $a \in U\}$ and associated function the second projection. We have an obvious map $\theta : \Theta \to \wp\mathbf{A}$ with first projection as underlying function. We now prove that θ is a generic proof. Let $f : Y \to X$ be realized by a_f. Then define $v : X \to \wp\mathbf{A}$ by $vx = \{\|y\| \ | fy = x\}$. Let P be the pullback of v and θ. It has underlying set $|P| = \{(x, vx, a) | a \in vx\}$ and $\|(x,U,a)\| = \langle \|x\|, a\rangle$. It is easy to see that f factors through $\pi : P \to X$ via the function $y \mapsto (fy, vfy, \|y\|)$ which is realized by $a \mapsto \langle a_f a, a\rangle$. Now for each $(x,U,a) \in |P|$ there exists a $gx \in |Y|$ such that $fgx = x$ and $\|gx\| = a$. Using *choice* we obtain a function $g : |P| \to |Y|$ that is realized by the projection $(x,U,a) \mapsto a$. It clearly holds that $f.g = \pi$. Notice that this first example of a generic proof also provides a very simple presentation of realizability toposes.

Consider now the category **Ass** of assemblies as described for example in [17]. This category is equivalent to $(\mathbf{P_A})_{\mathrm{reg}}$ (see [5]). In [23], van Oosten introduces a topos using tripos theory and then shows that its subcategory of projectives is the category **Ass**. Using the same idea as in the case of partitioned assemblies it is easy to build directly a generic proof in **Ass** which again provides an alternative presentation of the topos. Moreover, in [21] it is shown that under certain hypotheses (valid in these realizability cases), the existence of a generic proof is equivalent to the existence of a generic mono. Using this fact, a sequence of new examples can be obtained as follows. Let $\mathbf{C}_{\mathrm{reg}(0)} = \mathbf{C}$ and $\mathbf{C}_{\mathrm{reg}(n+1)} = (\mathbf{C}_{\mathrm{reg}(n)})_{\mathrm{reg}}$. Under the hypotheses mentioned above it follows that if $\mathbf{C}$ has weak dependent products and a generic mono then, for each n the exact completion of $\mathbf{C}_{\mathrm{reg}(n)}$ is a topos.

5.2. Presheaf toposes

It is well known that many presheaf toposes arise as exact completions. In this section we review this fact (which can be proved without our characterization) and give explicit constructions of the generic proofs involved. In order to present presheaf toposes as exact completions it is useful to introduce another free construction, the *coproduct completion*. For any category $\mathbf{C}$ there exists a unique (up to equivalence) category $\mathbf{C}_+$ with small coproducts and such that for every category $\mathbf{D}$ with coproducts there exists an equivalence of categories between the category of functors from $\mathbf{C}$ to $\mathbf{D}$ and the category of coproduct-preserving functors from $\mathbf{C}_+$ to $\mathbf{D}$. The objects of $\mathbf{C}_+$ are families of objects $\{X_i\}_{i\in I}$ in $\mathbf{C}$ indexed by a set I and maps between $\{X_i\}_{i\in I}$ and $\{Y_j\}_{j\in J}$ are families $f = \{f_i : X_i \to Y_{\phi i}\}_{i\in I}$ with f_i in $\mathbf{C}$ and ϕ a function from I to J. In coproduct completions, pullback along injections exist and coproducts are stable and disjoint (see [7]).

Definition 5.1. An object X is *indecomposable* if the corresponding covariant hom-functor $Hom(X, \_)$ preserves coproducts.

It is worth mentioning that in the presence of stable and disjoint coproducts, an object X is indecomposable if and only if X is not initial and cannot be decomposed as a coproduct of non-initial objects. The following proposition appears as Lemma 4.2 in [9].

Proposition 5.2. *A category* $\mathbf{E}$ *is the coproduct completion of a small category* $\mathbf{C}$ *if and only if* $\mathbf{E}$ *is locally small with small coproducts and there exists a small subcategory* $\mathbf{C}$ *of* $\mathbf{E}$ *consisting of indecomposable objects and such that every object in* $\mathbf{E}$ *is isomorphic to a coproduct of objects in* $\mathbf{C}$.

The relation between coproduct completions and presheaf toposes is the following.

Proposition 5.3. *Let* $\mathbf{C}$ *be a small category. If* $\mathbf{C}_+$ *has finite limits then* $(\mathbf{C}_+)_{\mathrm{ex}}$ *is equivalent to* $\mathbf{Sets}^{\mathbf{C}^{op}}$.

Proof. This is the argument used in the Corollary in p. 130 of [5]. See also Corollary 4.3 in [9]. □

(Notice that $\mathbf{C}$ need not have finite limits, as in the example of G-sets below.) It follows from the previous proposition that such a $\mathbf{C}_+$ must have a generic proof. We now give an explicit description of them. For any $f : Y \to X$ we write $\hat{f}$ for the corresponding element in $Prf(X)$.

Lemma 5.4. *If* **C** *is a small category then the functor Prf on the category* $\mathbf{C}_+$ *takes values in* **Sets**.

Proof. Let $f : Y \to X$ in $\mathbf{C}_+$. We can assume that Y is a small coproduct $\coprod_{i \in I} C_i$ of objects in **C**. It follows that f is determined by a family of maps $\{f_i : C_i \to X\}_{i \in I}$. Reordering things a little bit it is easy to see that $\hat{f}$ is determined by a family $\{U_C\}_{C \in \mathbf{C}}$ where each U_C is a subset (maybe empty) of $\mathbf{C}_+(C,X)$. That is $\hat{f}$ is determined by a subset of the small coproduct $\coprod_{C \in \mathbf{C}} \mathbf{C}_+(C,X)$. Hence, $Prf(X)$ is bounded by the set $Sub(\coprod_{C \in \mathbf{C}} \mathbf{C}_+(C,X))$. □

Notice that for any category with stable coproducts the functor Prf carries coproducts to products. That is, there exists a natural isomorphism

$$\prod_{i \in I} Prf(X_i) \cong Prf\left(\coprod_{i \in I} X_i\right).$$

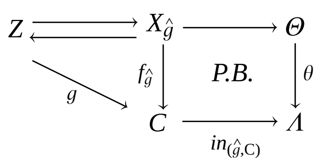
We can now describe the generic proofs.

Lemma 5.5. *If* **C** *is a small category then* $\mathbf{C}_+$ *has a generic proof.*

Proof. Let $\mathbf{P} = \{(p, C) \mid C \in \mathbf{C},\ \ p \in Prf(C)\}$. It is a set because **C** is small and by Lemma 5.4 so is each $Prf(C)$. For each $(p, C) \in \mathbf{P}$ *choose* a map $f_p : X_p \to C$ such that $\hat{f}_p = p$. Now consider the following small coproduct of maps:

$$\coprod_{(p,C) \in \mathbf{P}} f_p : \coprod_{(p,C) \in \mathbf{P}} X_p \to \coprod_{(p,C) \in \mathbf{P}} C.$$

Denote this map by $\theta : \Theta \to \Lambda$. We now prove that θ is a generic proof. To do this, consider first an indecomposable object C. We can assume it is in **C**. Let $g : Z \to C$ be any map and consider the following diagram.



So Λ weakly classifies proofs of indecomposable objects. Now for an arbitrary X. Again, without loss of generality we can assume that $X = \coprod_{i \in I} C_i$ with I a set and for each $i \in I$, C_i in **C**. The following calculation shows that Λ weakly represents Prf.

$$\begin{aligned} Prf(X) = Prf\left(\coprod_{i \in I} C_i\right) &\cong \prod_{i \in I} Prf(C_i) \twoheadleftarrow \prod_{i \in I} \mathbf{G}(C_i, \Lambda) \\ &\cong \mathbf{G}\left(\coprod_{i \in I} C_i, \Lambda\right) = \mathbf{G}(X, \Lambda). \qquad \square \end{aligned}$$

5.2.1. G-sets

For any group G it is possible to consider the presheaf topos $\mathbf{Sets}^G$ of G-sets [12,18,1]. It is well known that the indecomposable G-sets are the non-empty ones with only one orbit and that every G-set is a small coproduct of these. Moreover, every indecomposable is isomorphic to a G-set given by a *coset space* in G (see Proposition 4 in Section 3 of Chapter 1 in [1]). By Proposition 5.2 we can conclude that the presheaf topos of G-sets is the coproduct completion of its small full subcategory Q of coset spaces of G. Notice that Q does not have finite limits in general. In any case, we can conclude, using Proposition 5.3, that the exact completion of the category of G-sets is a presheaf topos. Indeed, $(\mathbf{Sets}^G)_{\text{ex}}$ is equivalent to $\mathbf{Sets}^{Q^{\text{op}}}$.

Due to its connection with Läuchli's abstract notion of realizability and completeness result [14], it may be of interest to pay special attention to the exact completion of the topos of $\mathbb{Z}$-sets. In [19] (see also [15]) the hyperdoctrine that assigns to each object X of $\mathbf{Sets}^{\mathbb{Z}}$ the small Heyting algebra $Prf(X) = (\widetilde{\mathbf{Sets}^{\mathbb{Z}}}/X)$ is used to give an abstract account of Läuchli's completeness result. For every X in $\mathbf{Sets}^{\mathbb{Z}}$, the lattice of subobjects of X in $(\mathbf{Sets}^{\mathbb{Z}})_{\text{ex}}$ is isomorphic to the $Prf(X)$ above. It may be interesting to look at Läuchli's result from this perspective.

5.2.2. Presheaves on a frame

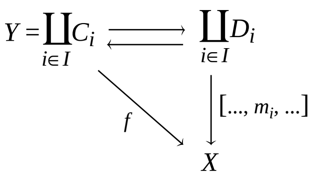
Let H be a frame in the sense of the theory of locales [13]. The coproduct completion H_+ has the following description. Its objects are pairs $X = (|X|, \|\_\|_X)$ with X a set and $\|\_\|_X : |X| \to H$ a function valued on the frame. An arrow $f : X \to Y$ in H_+ is a function $f : |X| \to |Y|$ such that for every x in $|X|$, $\|x\| \leqslant \|fx\|$. As H has finite limits, it follows by results in [5] that H_+ also has finite limits. In [22] it is proved that this category is regular and cartesian closed. It is actually a quasi-topos, but in any case, finite limits are enough to conclude that $(H_+)_{\text{ex}}$ is the presheaf topos on H. We introduce the explicit description of H here in order to present another example of a generic proof. Let $|\Lambda|$ be the set of subsets of H and for every $U \in |\Lambda|$, $\|U\| = \top$. This is the chaotic H-valued set of subsets of H. Let $|\Theta| = \{(U, a) \mid a \in U \in |\Lambda|\}$ and $\|(U, a)\| = a$. Using the same idea as for $\mathbf{P_A}$, it is not difficult to prove that the first projection $\Theta \to \Lambda$ is a generic proof in H_+.

5.3. Smallness and weak representability

Let $\rightarrow$ be the category with only two objects $\bot$ and $\top$ and a unique arrow $! : \bot \to \top$. It is small and has finite limits. It follows by Proposition 5.3 that $(\rightarrow_+)_{\text{ex}}$ is equivalent to $\mathbf{Sets}^{\rightarrow^{\text{op}}}$ which is equivalent to $\mathbf{Sets}^{\rightarrow}$. In this section we prove that the exact completion of $\mathbf{Sets}^{\rightarrow}$ is not a topos. This shows that $\mathbf{C}_{\text{ex}}$ a topos does not imply that $(\mathbf{C}_{\text{ex}})_{\text{ex}}$ also is. Notice first that the indecomposable objects in $\mathbf{Sets}^{\rightarrow}$ are those functors whose value at $\top$ is a singleton and that every non-initial object is a small coproduct of indecomposable objects. In fact, the indecomposable objects in this topos behave as non-empty sets. In particular, every epi between indecomposable objects splits. In contrast with the case of G-sets, there is a proper class of non-isomorphic indecomposable objects.

Lemma 5.6. *For the topos* **Sets**$^{\rightarrow}$, *Prf is a functor to* **Sets**.

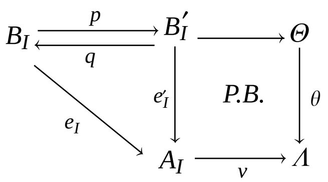
Proof. This result seems to be folklore but we give a proof for completeness. We are going to prove that for every X, $Prf(X)$ is bounded by the set of sets of subobjects of X, that is, by $Sub(Hom(X,\Omega))$. Let $f: Y \to X$. We can assume that $Y = \coprod_{i\in I} C_i$ is a small coproduct of indecomposable objects. Then f is the unique map given by universality of the coproduct and a family $\{f_i : C_i \to X\}_{i\in I}$. Now, each f_i factors as an epi $e_i : C_i \twoheadrightarrow D_i$ followed by a mono $m_i : D_i \rightarrowtail X$. Quotients of indecomposable objects are indecomposable so every e_i splits.



This diagram shows that f is denotes the same element in $Prf(X)$ (via the split epis) as the universal map given by a coproduct and a family of subobjects of X. In turn, such a map denotes the same element as a map given in the same fashion but where there are no repetitions of subobjects. It follows that $Prf(X)$ is bounded by $Sub(Sub(X))$. As **Sets**$^{\rightarrow}$ is a topos, it is well powered so $Sub(X)$ is a set and hence so is $Prf(X)$. □

Proposition 5.7. **Sets**$^{\rightarrow}$ *does not have a generic proof.*

Proof. Assume that there is a generic proof $\theta : \Theta \to \Lambda$. Let I be an arbitrary set which we think of as "size". Let A_I be the functor determined $A_I \perp = I$ and $A_I \top = \{*\}$ and B_I the one determined by $B_I \perp = I$, $B_I \perp = I$ and $B_I ! = id$. There is an obvious epi map $e_I : B_I \twoheadrightarrow A_I$ whose underlying function $B_I \perp \twoheadrightarrow A_I \perp$ is the identity. By assumption, there exists a map $v : A_I \to \Lambda$ such that the following happens.



But $A_I \top$ has only one element. Then, the construction of pullbacks in presheaf categories gives that $B'_I \top = \{t \mid \theta_\top t = v_\top *\} \rightarrowtail \Theta\top$. In order to achieve a contradiction, we will show that $I \rightarrowtail B'_I \top$ and hence that $I \rightarrowtail \Theta\top$ for each I. As $e_I = e'_I . p$ is epi, e'_I is also epi. We now show that $e'_I = e_I . q$ epi implies that q is epi. Let $q_\perp$ and $q_\top$ be the components of the natural transformation q. As the $\perp$ component of e_I is an isomorphism (between the sets $B_I \perp$ and $A_I \perp$) it follows that for $e_I . q_\perp$ to be epi, $q_\perp$ must be epi. Now, $B_I ! . q_\perp = q_\top . B'_I !$ by naturality and hence as the left hand side of the equation is epi, $q_\top : B'_I \top \to B_I \top$ is also epi. Among sets, the surjection

$B'_I\top \twoheadrightarrow B_I\top = I$ has a section. Hence, among sets there is a monic function $I \rightarrowtail \Theta\top$. But this is for every I! So $\Theta\top$ is not a set, a contradiction. □

Notice that this also shows that even for the restricted class of presheaf toposes, the condition that *Prf* takes values in **Sets** does not imply the existence of a generic proof. Actually, in [20] it is shown that for any small category **C**, the presheaf topos $\mathbf{Sets}^{\mathbf{C}^{op}}$ has a generic proof if and only if **C** is a groupoid. This extends the well-known characterization of boolean presheaf toposes [12,10,18]. We still do not know of a non-boolean topos with a generic proof.

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