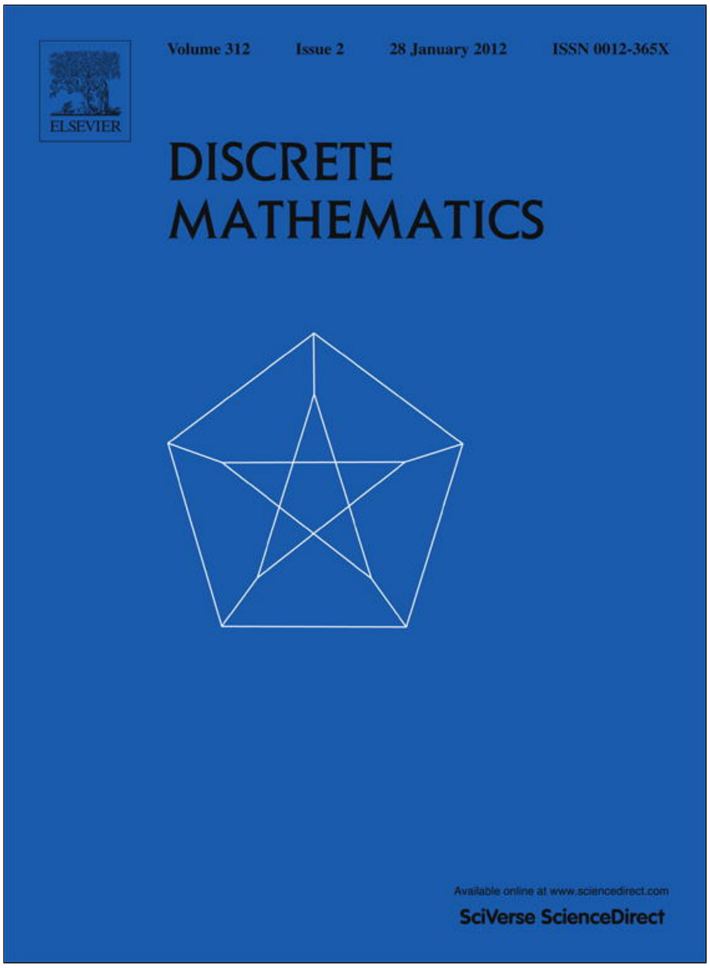

(This is a sample cover image for this issue. The actual cover is not yet available at this time.)

Discrete Mathematics 312 (2012) 2027–2039

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

On coloring problems with local constraints

Flavia Bonomo [a,b,*], Yuri Faenza [c], Gianpaolo Oriolo [a]

[a] Dipartimento di Informatica, Sistemi e Produzione, Università di Roma "Tor Vergata", Rome, Italy

[b] IMAS–CONICET and Departamento de Computación, FCEyN, Universidad de Buenos Aires, Buenos Aires, Argentina

[c] Dipartimento di Matematica Pura e Applicata, Università di Padova, Padua, Italy

ARTICLE INFO

Article history:
Received 9 January 2010
Received in revised form 10 March 2012
Accepted 13 March 2012

Keywords:
Graph coloring
Clique-trees
Unit interval graphs
Computational complexity

ABSTRACT

We deal with some generalizations of the graph coloring problem on classes of perfect graphs. Namely we consider the μ-coloring problem (upper bounds for the color on each vertex), the precoloring extension problem (a subset of vertices colored beforehand), and a problem generalizing both of them, the (γ, μ)-coloring problem (lower and upper bounds for the color on each vertex). We characterize the complexity of all those problems on clique-trees of different heights, providing polynomial-time algorithms for the cases that are easy. These results have interesting corollaries. First, one can observe on clique-trees of different heights the increasing complexity of the chain k-coloring, μ-coloring, (γ, μ)-coloring, and list-coloring. Second, clique-trees of height 2 are the first known example of a class of graphs where μ-coloring is polynomial-time solvable and precoloring extension is NP-complete, thus being at the same time the first example where μ-coloring is polynomially solvable and (γ, μ)-coloring is NP-complete. Last, we show that the μ-coloring problem on unit interval graphs is NP-complete. These results answer three questions from Bonomo et al. [F. Bonomo, G. Durán, J. Marenco, Exploring the complexity boundary between coloring and list-coloring, Annals of Operations Research 169 (1) (2009) 3–16].

1. Introduction

A *coloring* of a graph G is a labeling of its vertices. A coloring satisfying the additional requirement that adjacent vertices have distinct colors is a *proper coloring*. A *k-coloring* is a labeling that maps the vertex set into a set of size k. In this paper, we regard a k-coloring of a graph G as a function $f : V(G) \to \mathbb{N}$ such that $f(v) \leq k$ for every $v \in V(G)$. A graph is *k-colorable* if it admits a proper k-coloring.

The *vertex coloring problem* (also known as *k-colorability problem* or *k-coloring problem*) takes as input a graph G and consists of deciding whether G is k-colorable. This well-known problem is a basic model for scheduling, frequency assignment, and resource allocation problems.

In order to take into account particular constraints arising in practical settings, more elaborate models of vertex coloring have been defined in the literature. A hierarchy of such models was studied in [4]. Two generalizations of the k-coloring problem are precoloring extension [2] and μ-coloring [3].

The *precoloring extension* problem takes as input a graph G, a subset $W \subseteq V(G)$, a natural number k, a proper k-coloring f' of the subgraph of G induced by W, and consists of deciding whether G admits a proper k-coloring f such that $f(v) = f'(v)$ for every $v \in W$. In other words, a prespecified vertex subset is colored beforehand, and the goal is to extend this partial proper coloring to a proper k-coloring of the whole graph.

* Corresponding author at: Dipartimento di Informatica, Sistemi e Produzione, Università di Roma "Tor Vergata", Rome, Italy.
E-mail addresses: fbonomo@dc.uba.ar (F. Bonomo), faenza@math.unipd.it (Y. Faenza), oriolo@disp.uniroma2.it (G. Oriolo).

0012-365X/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2012.03.019

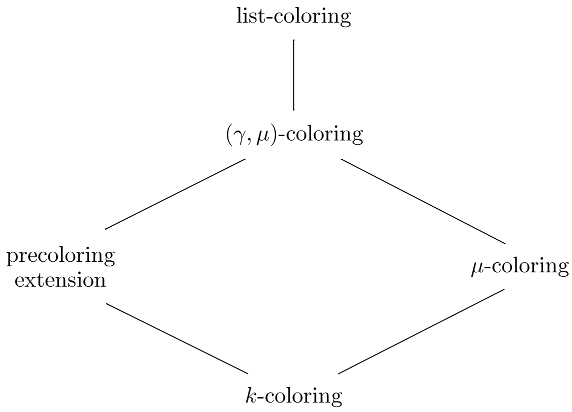

Fig. 1. Scheme of generalizations among these coloring problems.

Given a graph G and a function $\mu : V(G) \to \mathbb{N}$, the *$\mu$-coloring* problem consists of deciding whether G is *μ-colorable*, that is, whether there exists a proper coloring $f : V(G) \to \mathbb{N}$ such that $f(v) \leq \mu(v)$ for every $v \in V(G)$. This model arises in resource allocation problems with conflict between users [3], as well as in train scheduling [6].

A problem generalizing the latter two is the *(γ, μ)-coloring problem* [4], where also lower bounds for the color of each vertex are specified: given a graph G and functions $\gamma, \mu : V(G) \to \mathbb{N}$ such that $\gamma(v) \leq \mu(v)$ for every $v \in V(G)$, the *(γ, μ)-coloring* problem consists of deciding whether there exists a μ-coloring f where additionally $\gamma(v) \leq f(v)$ for every $v \in V(G)$.

Finally, a model generalizing all of the previous problems is the *list-coloring problem* [21], which considers a prespecified set of available colors for each vertex. Given a graph G and a finite list $L(v) \subseteq \mathbb{N}$ for each vertex $v \in V(G)$, the list-coloring problem asks for a *L-coloring* of G, that is, a proper coloring f such that $f(v) \in L(v)$ for every $v \in V(G)$.

The scheme of generalizations, summarized in Fig. 1, implies that all the problems in this hierarchy are polynomially solvable in those graph classes where list-coloring is polynomial and, on the other hand, all the problems are NP-complete in those graph classes where vertex coloring is NP-complete.

The complexity of this family of problems over different classes of graphs has been studied, and there are several examples of classes where k-coloring is polynomial-time solvable but precoloring extension and μ-coloring are NP-complete, like bipartite graphs [3,10], interval graphs [2,4,7] and distance hereditary graphs [4,9]. For the class of split graphs, in contrast precoloring extension is polynomial-time solvable but (γ, μ)-coloring is NP-complete [11,4]. Finally, there are also examples of classes where (γ, μ)-coloring is polynomial-time solvable but list-coloring is NP-complete, like complete bipartite graphs and complete split graphs [4,13]. Note that, however, to the best of our knowledge, so far no class of graphs where (γ, μ)-coloring is NP-complete while μ-coloring is polynomially solvable was known, and finding such a class is mentioned as an open problem in [4].

The problems of precoloring extension and μ-coloring are not directly related; neither is a generalization of the other. Nevertheless, for almost all the graph classes where the complexity of these two problems is known, they are on the same side of the dichotomy "polynomial-time solvable vs. NP-complete". The class of split graphs is the only known class where precoloring extension is polynomial-time solvable [11] while μ-coloring is NP-complete [4]. Again, to the best of our knowledge, so far no class of graphs where μ-coloring is polynomially solvable and precoloring extension is NP-complete was known; finding such a class is mentioned as another open problem in [4].

In this work, we study the complexity of these coloring problems on a class of perfect graphs called clique-trees. A graph G is a clique-tree if the graph $\mathcal{T}(G)$ one obtains from G after identifying true twins is a tree, and its height is the radius of $\mathcal{T}(G)$ (we refer to Section 2 for formal definitions).

We characterize the complexity of each problem on clique-trees of different heights, providing polynomial-time algorithms for the cases that are easy. These results have interesting corollaries. First, one can observe on clique-trees of different heights the increasing complexity of the chain k-coloring, μ-coloring, (γ, μ)-coloring, and list-coloring. In fact, the size of a maximum clique of a clique-tree, and thus its chromatic number, is the maximum sum of the multiplicities of two adjacent vertices that are not twins. Thus, the k-coloring problem can be solved in strongly polynomial time for this class of graphs. On the other side, clique-trees of height 0 are complete graphs, and the list-coloring problem on a complete graph can be modeled as a maximum matching problem on a bipartite graph and thus solved in polynomial time, while it is known that the list-coloring problem is NP-complete for clique-trees of height 1, even when the reduced tree consists of a root and two children [12] and when the multiplicity of all the vertices but the root is 1 [13]. However, as we show in Section 2, (γ, μ)-coloring, and therefore μ-coloring and precoloring extension, are still easy for clique-trees of height 1.

For clique-trees of height 2, we show that only μ-coloring (and, of course, k-coloring) is easy. This class of graphs gives, to the best of our knowledge, the first known example where μ-coloring is polynomially solvable while precoloring extension (and thus (γ, μ)-coloring) is NP-complete: this answers the questions above from [4].

For height 3 or more, also μ-coloring becomes hard, even when the height is fixed. Table 1 summarizes these results.

Table 1
A summary of the complexity results for clique-trees. New results from this paper are bold-faced.

Problem	Height			Fixed $p \geq 3$
	0	1	2	
k-coloring	P	P	P	P
μ-coloring	P	**P**	**P**	**NP-c**
Precoloring	P	**P**	**NP-c**	**NP-c**
(γ, μ)-coloring	P	**P**	**NP-c**	**NP-c**
List-coloring	P	NP-c	NP-c	NP-c

We also study the complexity of the μ-coloring problem on another class of perfect graphs, the class of unit interval graphs. Both the precoloring extension problem and the μ-coloring problem were motivated by and arise in the context of scheduling problems, like job scheduling [2], classroom allocation [3] and train scheduling [6]. Therefore, one of the classes of interest for these problems is the class of interval graphs and, in particular, the class of unit interval graphs.

It is well known that the chromatic number of an interval graph G can be determined in time $O(|V(G)| + |E(G)|)$. While the NP-completeness of precoloring extension on interval graphs was proved in 1992 [2], it took more than ten years, and a quite involved reduction, in order to prove that the problem is hard also for unit interval graphs [14]. Analogously, in [4] it was proved that the μ-coloring problem is NP-complete on interval graphs, and the question of the complexity of μ-coloring on unit interval graphs was raised. We also settle this question and show, via a nontrivial reduction, that this problem is NP-complete.

The paper is organized as follows: we close this section with some notation. In Section 2 we introduce the class of clique-trees and the related notion of height, characterizing the complexity of the aforementioned coloring problems on clique-trees of different heights. In Section 3, we present a (quite involved) proof of the NP-completeness of the μ-coloring problem on unit interval graphs.

We shall consider finite, simple, loopless, undirected graphs. Let G be a graph. Denote by $V(G)$ its vertex set and by $E(G)$ its edge set. Given a vertex v of G, denote by $N_G(v)$ the set of neighbors of v in G and by $N_G[v]$ the set $N_G(v) \cup \{v\}$, and generalize it to a set of vertices $W \subseteq V(G)$ as follows: $N_G(W) = \bigcup_{w \in W} N_G(w) \setminus W$, and $N_G[W] = N_G(W) \cup W$. A graph G is a *tree* if it is connected and $|E(G)| = |V(G)| - 1$. A *rooted tree* is a pair (G, r) consisting of a tree G together with a designated vertex $r \in V(G)$. Vertices v and w of G are *true twins* if either $v = w$ or $N_G[v] = N_G[w]$, that is, they are adjacent and they have the same neighbors.

For each $W \subseteq V(G)$, denote by $G[W]$ the subgraph of G induced by W. When H is an induced subgraph of G, denote by $G \setminus H$ the graph $G[V(G) \setminus V(H)]$.

Given graphs G_1 and G_2 with $V(G_1) \cap V(G_2) = \emptyset$, denote by $G_1 \cup G_2$ the graph such that $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. Given two vertices v and w in a graph G, *contracting* v and w means replacing them with a new vertex whose neighborhood is $N_G(v) \cup N_G(w) - \{v, w\}$. Note that v and w need not be adjacent in G.

For a positive integer k, let $[k] = \{1, 2, \ldots, k\}$.

Given two vertices $a, b \in V(G)$, an *(a, b)-path of length k* in G is a list $(v_0, v_1, \ldots, v_k)$ of distinct vertices such that $v_0 = a$, $v_k = b$, and for $1 \leq i \leq k$, $(v_{i-1}, v_i) \in E(G)$. The *distance* $d(a, b)$ between a and b in G is the minimum length among all (a, b)-paths in G.

A *clique* is a set of pairwise adjacent vertices. A *stable set* is a set of pairwise nonadjacent vertices. Let $A, B \subseteq V(G)$. We say that *A is complete to B* if every vertex of A is adjacent to every vertex of B, and that *A is anticomplete to B* if no vertex of A is adjacent to a vertex of B.

The *chromatic number* of a graph G is the minimum k such that G is k-colorable. A graph is *perfect* [1] if for every induced subgraph H, the chromatic number of H equals the size of a maximum clique in H. The k-coloring problem is known to be polynomial-time solvable on perfect graphs [8].

Throughout the paper, an instance of the μ-coloring problem is a pair (G, μ), where $\mu : V(G) \to \mathbb{N}$. An instance of the (γ, μ)-coloring problem is a triple (G, γ, μ), where $\mu, \gamma : V(G) \to \mathbb{N}$ with $\gamma(v) \leq \mu(v)$ for each $v \in V(G)$. An instance of the precoloring extension problem is a 4-tuple (G, W, f', k) where $W \subseteq V(G)$, $k \in \mathbb{N}$, and $f' : W \to [k]$ is a proper coloring of $G[W]$. Finally, an instance of the list-coloring problem is a pair (G, L), where $L : V(G) \to 2^{(\mathbb{N})}$, that is, $L(v)$ is finite for each $v \in V(G)$.

2. Clique-trees

A graph G is a *clique-tree* if the graph G' obtained by iteratively contracting its true twins is a tree. It is immediate to check that G' does not depend on the order of the contractions, so it is well defined.

The *radius* of a graph G is the value $\min_{v \in V(G)}(\max_{u \in V(G)} d(u, v))$, while the *center* of G is the set of vertices $v \in V(G)$ such that $\max_{u \in V(G)} d(u, v)$ equals the radius. The *height* of a clique-tree G is the radius of the tree obtained from G by identifying true twins. We denote this tree by $\mathcal{T}(G)$ and call *root* any specific vertex from the center of $\mathcal{T}(G)$. (We remark that Brandstädt and Bang Le [5] showed that the class of clique-trees is equivalent to *3-leaf power of trees*, a class of graphs introduced for phylogenetic problems [16].) Given a vertex v of G, the *multiplicity* of v in G is the number of true twins of

v in G (by definition, v is also counted). In this section we characterize the complexity of each problem on clique-trees of different heights, providing polynomial-time algorithms for the cases that are easy. We refer to Table 1 for a summary of those results and the already-known ones from the literature, and to the Introduction for a thoughtful discussion on them.

We start with some useful lemmas.

Lemma 1. *For a positive integer n, let G be a complete graph with vertex set $V \cup V'$, where $V = \{v_1, \ldots, v_n\}$ and $V' = \{v'_1, \ldots, v'_n\}$, $V \cap V' = \emptyset$. Let $\mu : V(G) \to \mathbb{N}$ such that $\mu(v_i) = \mu(v'_i) = 2i$ for i in $[n]$. Let $\gamma : V \to \mathbb{N}$ such that $\gamma(v_i) = 2i - 1$ for i in $[n]$. If f is a solution to (G, μ), then its restriction to $G[V]$ is a solution to $(G[V], (\gamma, \mu))$. Conversely, each solution to $(G[V], (\gamma, \mu))$ can be extended in polynomial time to a solution of (G, μ).*

Proof. Consider a feasible solution to (G, μ), and let $i \in [n]$. Since G is complete and $\mu(v_j) = \mu(v'_j) = 2j$ for $j \in [n]$, vertices $v_1, \ldots, v_i, v'_1, \ldots, v'_i$ use all the colors in $[2i]$. Thus, vertex v_{i+1} uses either color $2i + 1$ or $2(i + 1)$, and the restriction to V gives a solution to $(G[V], (\gamma, \mu))$. Conversely, a solution to $(G[V], (\gamma, \mu))$ leaves unused either $2i$ or $2i - 1$ for each $i \in [n]$, so we can extend it to a solution of (G, μ) by assigning that free color to v'_i. □

The following lemma is implicitly proved in [4]; see Theorem 4.

Lemma 2. *Let G be a complete graph and let $\gamma, \mu : V(G) \to \mathbb{N}$ with $\gamma(w) \leq \mu(w)$ for $w \in V(G)$. Let $\mu_{\max} = \max_{w \in V(G)} \mu(w)$. Let C be a subset of natural numbers within the interval $[1, \mu_{\max}]$. The graph G admits a (γ, μ)-coloring using only colors in C if and only if, for every $1 \leq i \leq j \leq \mu_{\max}$, $|\{w \in V(G) : i \leq \gamma(w) \leq \mu(w) \leq j\}| \leq |C \cap [i, j]|$.*

2.1. Polynomial cases

Theorem 3. *The (γ, μ)-coloring problem can be solved in polynomial time for clique-trees of height at most 1.*

Proof. The statement is true for clique-trees of height 0, that is, complete graphs, since there the more general list-coloring problem can be modeled as a maximum matching problem on a bipartite graph.

Suppose therefore that we are given a clique-tree G of height 1 and $\gamma, \mu : V(G) \to \mathbb{N}$ such that for every $v \in V(G)$, $\gamma(v) \leq \mu(v)$. Let $\mu_{\max}$ be the maximum value of μ over G. We may assume that every color in $[\mu_{\max}]$ belongs to the interval $[\gamma(v), \mu(v)]$, for some $v \in V(G)$. Furthermore, we may assume $\mu(v) - \gamma(v) \leq |N_G(v)|$ for every $v \in V(G)$. It follows from these assumptions that $\mu_{\max} \leq n \cdot \max_{v \in V(G)} |N_G(v)|$.

Let A be the clique corresponding to the root of $\mathcal{T}(G)$ and $B_1, \ldots, B_r$ the cliques corresponding to the leaves of $\mathcal{T}(G)$. For $0 < j \leq i \leq \mu_{\max}$, let $L_A(i, j) = |\{v \in A : j \leq \gamma(v) \leq \mu(v) \leq i\}|$ and $L_k(i, j) = |\{v \in B_k : j \leq \gamma(v) \leq \mu(v) \leq i\}|$, for k in $[r]$. We reduce the problem of finding a (γ, μ)-coloring of G to a linear programming feasibility problem. For j in $[\mu_{\max}]$, we define the integer variable x_j such that $x_j - x_0$ is the number of colors from the set $[j]$ assigned to vertices of A and, based on this definition, we consider the following linear program:

$$x_i - x_{j-1} \geq L_A(i, j) \quad \forall\, i, j : 0 < j \leq i \leq \mu_{\max} \tag{1}$$

$$x_i - x_{j-1} \leq i - j + 1 - \max_{k \in [r]} L_k(i, j) \quad \forall\, i, j : 0 < j \leq i \leq \mu_{\max}. \tag{2}$$

Since $\mu_{\max} \leq n \cdot \max_{v \in V(G)} |N_G(v)|$, the number of variables and constraints is polynomial in the size of G. All the constraints take the form $x_j - x_k \geq \alpha_{jk}$ or $x_j = \alpha_j$. Hence, the constraint matrix is totally unimodular, implying that the associated polytope is integral (see for example [15]). To complete the proof, we verify that G is (γ, μ)-colorable if and only if the linear program (1)–(2) is feasible.

Suppose first that G is (γ, μ)-colorable. Let $x_0 = 0$ and, for j in $[\mu_{\max}]$, let x_j be the number of colors from $[j]$ assigned to vertices of A. Since A is a clique, at least $L_A(i, j)$ colors from $[j, i]$ are assigned to the vertices of A, hence constraints (1) hold. Analogously, for each $k \in [r]$, at least $L_k(i, j)$ colors from $[j, i]$ are assigned to the vertices of B_k; since each B_k is complete to A, constraints (2) hold. Thus, if G is (γ, μ)-colorable, then the linear program (1)–(2) has a feasible solution.

Conversely, suppose that the linear program (1)–(2) is feasible and let x be an integer solution, which exists since the associated polytope is integral. We shall verify that G admits a (γ, μ)-coloring. Let $C = \{j : 1 \leq j \leq \mu_{\max}$ and $x_j - x_{j-1} = 1\}$. By (1), C and A satisfy the hypothesis of Lemma 2, so there is a (γ, μ)-coloring of A using the colors in C. By (2), for each k in $[r]$, the clique B_k and the set of colors $[\mu_{\max}] \setminus C$ satisfy the hypothesis of Lemma 2, so there is a (γ, μ)-coloring of B_k using the colors in $[\mu_{\max}] \setminus C$. Putting things together, there exists a (γ, μ)-coloring of G; note that this coloring may be found by solving $r + 1$ bipartite matching problems. □

Even though the algorithm from the proof of the previous theorem relies on linear programming, one can solve the (γ, μ)-coloring in a clique-tree of height at most 1 using only combinatorial routines. Indeed, it is well-known (see e.g. [20]) that the feasibility of a linear program whose only constraints are upper bounds on the difference of variables (like the one defined by (1)–(2)) is equivalent to testing for the existence of a negative cycle in an appropriate directed graph. In our case, this digraph is complete, has vertices $v_0, \ldots, v_{\mu_{\max}}$, and for $i, j : 0 < j \leq i \leq \mu_{\max}$ the cost of the edge (v_i, v_{j-1}) is $-L_A(i, j)$ and the cost of the edge $v_{j-1}v_i$ is $i - j + 1 - \max_{k \in [r]} L_k(i, j)$. Given an instance of (γ, μ)-coloring on a clique-tree G of height 1, finding a negative cycle in that auxiliary directed graph would lead to a (maybe proper) subgraph H of G such that the (γ, μ)-coloring problem is already infeasible restricted to H.

Since the (γ, μ)-coloring problem is a generalization of the precoloring extension and the μ-coloring problem, we have the following corollary.

Corollary 4. *The precoloring extension and the μ-coloring problem can be solved in polynomial time for clique-trees of height at most* 1.

Theorem 5. *The μ-coloring problem can be solved in polynomial time for clique-trees of height at most* 2.

Proof. Let (G, μ) be an instance of μ-coloring, where G is a clique-tree of height 2. We shall show that (G, μ) can be polynomially reduced to an instance $(G', (\gamma, \mu))$ of the (γ, μ)-coloring problem, where G' is a clique-tree of height 1. Thus, we can invoke Theorem 3.

Let A be the clique of G corresponding to the root of $\mathcal{T}(G)$, $\{B_k\}_{k\in[r]}$ be the cliques of G corresponding to the vertices in level 1 of $\mathcal{T}(G)$, and let $\{C_k^j\}_{k\in[r],j\in[s_k]}$ the cliques of G corresponding to the leaves of $\mathcal{T}(G)$, where for each k in $[r]$ and each j in $[s_k]$, C_k^j is complete to B_k. The graph G' is obtained from G by deleting the cliques corresponding to vertices of level 2 in $\mathcal{T}(G)$, that is, $G' = G[A \bigcup_{k\in[r]} B_k]$. Thus, G' is a clique-tree of height 1.

As for the vector γ, it requires some more definitions. Let $\mu_{\max}$ be the maximum value of μ over G and, for every $i \in [\mu_{\max}]$ and $k \in [r]$, let $l_k^i = \max_{1\le j\le s_k} |\{v \in C_k^j : \mu(v) \le i\}|$. Also we assume that $B_k = \{w_k^1, \ldots, w_k^{|B_k|}\}$, with $\mu(w_k^1) \le \cdots \le \mu(w_k^{|B_k|})$, for $k \in [r]$. For each $k \in [r]$ and $j \in [|B_k|]$, we let $\gamma(w_k^j) = h$, where $(h-1)$ is the largest index $i \in [\mu_{\max}] : l_k^i + j > i$. Meanwhile, we let $\gamma(v) = 1$ for every $v \in A$.

We claim that the resulting γ-vector is such that $|\{v \in B_k : \gamma(v) \le i\}| \le i - l_k^i$ for $i \in [\mu_{\max}]$. Indeed, fix i and suppose first $i - l_k^i < |B_k|$. For $w = w_k^{i-l_i^k+1}, \ldots, w_k^{|B_k|}$, it follows that $\gamma(w) \ge i+1$, so the statement holds. Conversely, if $i - l_k^i \ge |B_k|$, then the statement holds since $|\{v \in B_k : \gamma(v) \le i\}|$ is a subset of B_k.

We finally show that every solution of $(G', (\gamma, \mu))$ can be extended to a solution of (G, μ) and, conversely, every solution to (G, μ) is, restricted to $V(G')$, a solution of $(G', (\gamma, \mu))$. Let f be a (γ, μ)-coloring of G'. It is, in particular, a μ-coloring of G'. For each $k \in [r]$, let $f(B_k) = \{f(v), v \in B_k\}$ and let $Q_k = [\mu_{\max}] \setminus f(B_k)$. We claim that, for each $k \in [r]$ and $j \in [s_k]$, there exists a μ-coloring of C_k^j that uses only colors from Q_k. From Lemma 2, it is enough to show that, if $i \le \mu_{\max}$, then $|\{v \in C_k^j : \mu(w) \le i\}| \le |Q_k \cap [i]| = i - |f(B_k) \cap [i]|$. Since $|f(B_k) \cap [i]| \le |\{v \in B_k : \gamma(v) \le i\}| \le i - l_k^i \le i - |\{v \in C_k^j : \mu(v) \le i\}|$, we are done.

Conversely, let f be a μ-coloring of G. Since $\mu(w_k^i) \le \mu(w_k^{i+1})$ and vertices in B_k are twins when $k \in [r]$, we can permute colors of vertices in B_k to obtain $f(w_k^1) < \cdots < f(w_k^{|B_k|})$, without affecting the feasibility of the problem. We claim that f now induces a (γ, μ)-coloring of G'. Note that we only need to show that, for each $k \in [r]$ and $j \in [|B_k|]$ we have $\gamma(w_k^j) \le f(w_k^j)$. Suppose the contrary and let $h = \gamma(w_k^j) > f(w_k^j)$ for some j. By definition of $\gamma(w_k^j)$, it follows that $l_k^{h-1} + j > h - 1$, i.e. $j > h - 1 - l_k^{h-1}$, and, since $w_k^1, \ldots, w_k^j$ have a color in $[h-1]$, this is a contradiction with f being a feasible μ-coloring of G. □

2.2. NP-complete cases

Theorem 6. *For each integer $p \ge 3$, the μ-coloring problem is NP-complete on clique-trees of height p and the precoloring extension problem is NP-complete on clique-trees of height $p-1$.*

Proof. We divide the proof in three steps, each corresponding to a claim. We first reduce 3-SAT to an instance of list-coloring on a clique-tree of height 1, having certain special properties. Next, we reduce that instance of list-coloring to an instance of precoloring extension on a clique-tree of height 2, again with some special features. It suffices to show that precoloring extension is NP-complete on clique-trees of height 2, and the extension to heights $p \ge 3$ is trivial, so we omit it. Finally, we reduce this latter instance to an instance of μ-coloring on a clique-tree of height 3. This shows that μ-coloring is NP-complete on clique-trees of height 3, and again the extension to bigger heights is trivial (and thus omitted).

Claim 1. *There is a polynomial-time reduction from a 3-SAT instance with n variables and k clauses to a list-coloring instance (G, L), such that $V(G) = V_0 \cup V_1$, where V_0 is a clique of size n, V_1 is a stable set complete to V_0, $L(v) = [2n]$ for each $v \in V_0$, and $L(v) \subseteq [2n]$ for each $v \in V_1$. Note that G is a clique-tree of height* 1.

Proof of Claim 1. We are given a 3-SAT instance with variables $x_1, \ldots, x_n$ and clauses $c_1, \ldots, c_k$. Without loss of generality we assume that there is no clause c_j where both x_i and $\bar{x}_i$ appear.

Our task is to produce a list-coloring instance (G, L) such that there exists a feasible coloring for (G, L) if and only if there exists a feasible solution to the 3-SAT instance. This goes as follows. Define a bijection φ from the set $\bigcup_i \{x_i, \bar{x}_i\}$ to $[2n]$ such that: $\varphi(x_i) = 2i - 1$ and $\varphi(\bar{x}_i) = 2i$. The list-coloring instance (G, L) is defined as follows: $V(G) = \{A \cup B \cup C\}$ with $A = \{v_1, \ldots, v_n\}$, $B = \{v_1', \ldots, v_n'\}$, $C = \{w_1, \ldots, w_k\}$, where A is a clique, $B \cup C$ is a stable set and A is complete to $B \cup C$. Let $L(v_i) = [2n]$ and $L(v_i') = \{2i-1, 2i\} = \{\varphi(x_i), \varphi(\bar{x}_i)\}$ for $i \in [n]$, and $L(w_j) = \{\varphi(x_i) \mid x_i \in c_j\} \cup \{\varphi(\bar{x}_i) \mid \bar{x}_i \in c_j\}$ for $j \in [k]$. It is clear that (G, L) satisfies the conditions in the statement of Claim 1.

Consider now a solution S to the 3-SAT instance: if x_i is true (resp. $\bar{x}_i$ is true) in S, then color v_i with $\varphi(\bar{x}_i)$ (resp. $\varphi(x_i)$) and v_i' with $\varphi(x_i)$ (resp. $\varphi(\bar{x}_i)$). As for the vertices of C, let c_j be any clause and suppose x_i (resp. $\bar{x}_i$) makes c_j true in S. The color $\varphi(x_i)$ (resp. $\varphi(\bar{x}_i)$) is not used in A, so it can be used to color w_j: thus, we obtain a feasible coloring for (G, L). Conversely, every feasible coloring f to (G, L) induces a true/false assignment to the variables $x_1, \ldots, x_n$ such that the Boolean formula is satisfied: namely, make x_i true (resp. $\bar{x}_i$ true) if and only if the color of v_i' is $\varphi(x_i)$ (resp. $\varphi(\bar{x}_i)$). □

Claim 2. *There is a polynomial-time reduction from a list-coloring instance (G, L) satisfying the conditions of Claim 1 to a precoloring extension instance $(G', W, f', 2n)$, where G' is a clique-tree of height 2 of at least n vertices and each vertex from W has no true twins.*

Proof of Claim 2. Let (G, L) be a list-coloring instance satisfying the conditions of Claim 1, i.e., $V(G) = V_0 \cup V_1$, where V_0 is a clique of size n, V_1 is a stable set complete to V_0, $L(v) = [2n]$ for each $v \in V_0$ and $L(v) \subseteq [2n]$ for each $v \in V_1$. $V(G')$ is made of vertices from $V(G)$ plus some sets of additional vertices, which we define below, that are disjoint with each other and with the set $V(G)$. For $v \in V_1$, define the set D_v of $2n - |L(v)|$ vertices. Also, let $V_2 = \bigcup_{v \in V_1} D_v$. The graph G' is the unique graph with the following properties: $V(G') = V_0 \cup V_1 \cup V_2$, $V_0 \cup V_1$ induces G, V_2 is a stable set, and $N_{G'}(D_v) = \{v\}$ for each v such that D_v is nonempty. Note that G' is a clique-tree of height 2, where V_2 are the vertices in the last level. Let $W = V_2$ and let f' be a precoloring such that, for each v in V_1, all vertices of D_v are given distinct colors from $[2n] \setminus L(v)$. Note that $(G', W, f', 2n)$ satisfies the "structural" requirements from the claim.

It is easy to see that every precoloring extension of G', restricted to $V_0 \cup V_1$, is a solution for the list-coloring instance (G, L) and, conversely, every solution of (G, L) can be combined with the precolored vertices in V_2, obtaining a valid $2n$-coloring of G'. Moreover, the size of G' is a polynomial in the size of G. □

From Claims 1 and 2, the precoloring extension problem is NP-complete on clique-trees of height 2.

Claim 3. *There is a polynomial-time reduction from a precoloring extension instance $(G', W, f', 2n)$, where G' is a clique-tree of height p of at least n vertices and each vertex from W has no true twins, to a μ-coloring instance on a clique-tree of height at most $p + 1$.*

Proof of Claim 3. Let $(G', W, f', 2n)$ be a precoloring extension instance where G' is a clique-tree of height p of at least n vertices, and each vertex from W has no true twins.

We now define an instance (G'', μ) of the μ-coloring problem such that G'' is a clique-tree of height at most $p + 1$, G' is an induced subgraph of G'', a solution of (G'', μ) restricted to $V(G')$ is a solution of $(G', W, f', 2n)$, and every solution of $(G', W, f', 2n)$ can be extended to a μ-coloring of G''.

Again, all sets of vertices we add are pairwise disjoint and disjoint from $V(G')$. For each vertex $v \in W$, let E_v be a clique of size $f'(v) - 1$. The edge set of G'' is the union of the following two sets: the set of edges of G' and, for each $v \in W$ such that E_v is non-empty, (i) the set of edges joining v and E_v and (ii) all edges within vertices of E_v. Since each $v \in W$ has no true twins, G' is a clique-tree of height at most $p + 1$.

The function μ is defined by $\mu(v) = 2n$ for each $v \in V(G') \setminus W$, $\mu(v) = f'(v)$ for each $v \in W$, and $\mu(v) = f'(v) - 1$ for each $v \in E_v$. In every μ-coloring of G'', for each v such that $f'(v) > 1$, the vertices of E_v are assigned all the colors in $[f'(v) - 1]$, since E_v is a clique of size $f'(v) - 1$. Hence, each vertex $v \in W$ is colored $f'(v)$. Therefore, every μ-coloring of G'', restricted to $V(G')$, is a solution for the precoloring extension instance $(G', W, f', 2n)$. It is also easy to see that, conversely, every solution of $(G', W, f', 2n)$ can be extended to a μ-coloring of G''. Moreover, the size of G'' is a polynomial in the size of G'. □

From Claims 1–3, we conclude that the μ-coloring problem is NP-complete on clique-trees of height 3. □

Since the (γ, μ)-coloring problem is a generalization of the precoloring extension problem, we have the following corollary.

Corollary 7. *For each integer $p \geq 2$, the (γ, μ)-coloring problem is NP-complete on clique-trees of height p.*

3. Unit interval graphs

A graph G is an *interval graph* if it is the intersection graph of a set of intervals over the real line. A *unit interval graph* is the intersection graph of a set of intervals of length 1, while a *proper interval graph* is the intersection graph of a set of intervals where no interval is properly contained in another. A *claw* is the complete bipartite graph $K_{1,3}$.

Theorem 8 ([19]). *The classes of unit interval graphs, proper interval graphs, and claw-free interval graphs coincide.*

Let $v_1, \ldots, v_n$ be an ordering of the vertices of a graph G. The ordering is *consistent* if there is no triple $i < j < k$ such that $v_k v_i \in E(G)$ and $v_k v_j \notin E(G)$. If, in addition, there is no triple $i < j < k$ such that $v_i v_k \in E(G)$ and $v_i v_j \notin E(G)$ (equivalently, the reverse ordering is also consistent), then the ordering is called *proper consistent*.

Theorem 9 ([17,18]). *A graph is an interval graph if and only if its vertices admit a consistent order, while it is a unit interval graph if and only if its vertices admit a proper consistent order.*

The main result of this section is the following.

Theorem 10. *The μ-coloring problem is NP-complete on unit interval graphs.*

Our proof is based on a reduction from 3-SAT. Recall that in such a problem, we are given a t-variable Boolean formula ϕ in conjunctive normal form with k clauses such that each variable $x_1, \ldots, x_t$ appears in at least two clauses but not twice in the same clause, each clause contains two or three literals, and the goal is to find a satisfying truth assignment or show that none exists.

Given a 3-SAT instance ϕ, we denote by size_ϕ the number of bits required for a binary encoding of ϕ. The definition is quite standard, and we refer the reader to textbooks for details (see for example [15]). For the sake of clarity, we start with reducing 3-SAT to a list-coloring problem with some parity constraints, which we call *Parity list coloring* (*PLC* in short). We then show how to complete the proof, linking PLC to the μ-coloring problem on unit interval graphs.

PARITY LIST COLORING (PLC)

Input: A graph G, a finite list $L(v) \subseteq \mathbb{N}$ for each vertex $v \in V(G)$, and a partition $\mathcal{F}$ of $V(G)$ into classes.

Goal: Find an L-coloring of G such that all the vertices in a same class are assigned colors with the same parity.

We say that a set $F \in \mathcal{F}$ is *trivial* if $|F| = 1$, *nontrivial* otherwise.

3.1. From satisfiability to parity list coloring

As we now show, PLC is NP-complete on complete graphs (recall that list-coloring is easy on such graphs). We first show how to associate to some instance ϕ of 3-SAT an instance $(G, L, \mathcal{F})$ of PLC. We assume that we are given some *ordering* $c_1, \ldots, c_k$ on the clauses of ϕ, and associate therefore to ϕ a string $\ell_1 \ldots \ell_y$ of characters from the alphabet of literals. For instance, if $\phi = (x_1 \vee \bar{x}_2) \wedge (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3)$, we associate to ϕ the string $\ell_1 \ldots \ell_y \equiv x_1\bar{x}_2x_1x_2\bar{x}_3\bar{x}_1x_3$. It follows from the definition of 3-SAT that $y \leq 3k$.

We consider a clause as a set of characters. Associate to each clause c_j for $j \in [k]$ the set $\alpha(c_j) = \{1 \leq i \leq y : \ell_i \in c_j\}$ and to each variable x_i for $i \in [t]$ the set $\beta(x_i) = \{1 \leq i \leq y : \ell_i \equiv x_i \text{ or } \ell_i \equiv \bar{x}_i\}$. Referring again to the previous example, we have that $\alpha(c_1) = \{1, 2\}$, $\alpha(c_2) = \{3, 4, 5\}$, $\alpha(c_3) = \{6, 7\}$, $\beta(x_1) = \{1, 3, 6\}$, $\beta(x_2) = \{2, 4\}$ and $\beta(x_3) = \{5, 7\}$.

We exploit some ideas from Jansen [12] and associate to ϕ an instance $(G_\phi, L_\phi, \mathcal{F}_\phi)$ of parity list coloring, that is defined as follows. The graph G_ϕ is a complete graph of $y + k$ vertices. The vertices of G_ϕ are partitioned into two sets: $T = \{v_1, \ldots, v_y\}$ and $U = \{v_{y+1}, v_{y+2}, \ldots, v_{y+k}\}$. The lists L_ϕ of feasible colors of $V(G_\phi)$ are defined as follows:

for $p \in [y]$, $L_\phi(v_p) = \{2p, 2p + 1\}$;
for $j \in [k]$, $L_\phi(v_{y+j}) = \{2h : h \in \alpha(c_j) \text{ and } \ell_h \equiv x_i, \text{ for some } i \in [t]\} \cup \{2h + 1 : h \in \alpha(c_j) \text{ and } \ell_h \equiv \bar{x}_i, \text{ for some } i \in [t]\}$.

(We remark that, for each $j \in [k - 1]$, each color from $L_\phi(v_{y+j})$ is strictly smaller than each color in $L_\phi(v_{y+j+1})$.)

Finally, let $\mathcal{F}_\phi = \{\{v_i, i \in \beta(x_1)\}, \ldots, \{v_i, i \in \beta(x_t)\}, \{v_{y+1}\}, \ldots, \{v_{y+k}\}\}$.

Lemma 11. *There exists a feasible PLC for $(G_\phi, L_\phi, \mathcal{F}_\phi)$ if and only if there exists a feasible truth assignment for ϕ, and one can construct one from the other in a time polynomial in* size_ϕ.

Proof. *Sufficiency*: let ν be a feasible assignment for ϕ. Consider a coloring f defined as follows: for $p \in [y]$, set $f(v_p) = 2p+1$ if the variable corresponding to ℓ_p is true in ν, and $f(v_p) = 2p$ otherwise; for $j \in [k]$, we choose a literal ℓ_m of c_j that is true (there is at least one since ν is a feasible truth assignment), and set $f(v_{y+j}) = 2m + 1$ if ℓ_m corresponds to a negated variable, $f(v_{y+j}) = 2m$ otherwise. It is straightforward to check that f is feasible for $(G_\phi, L_\phi, \mathcal{F}_\phi)$.

Necessity: from a feasible PLC, we define a truth assignment ν as follows: a variable $\nu(x_i)$ is true if each vertex v_h with $h \in \beta(x_i)$ has odd color, it is false otherwise. It is straightforward to check that ν is a feasible truth assignment. We conclude by pointing out that both producing a feasible truth assignment from a feasible coloring and the converse operation can be performed in time polynomial in size_ϕ. $\square$

As an immediate corollary of the previous lemma, we obtain that PLC is NP-complete on cliques.

3.2. From PLC to μ-coloring on unit interval graphs

Consider an instance of PLC $(\widetilde{G}, \widetilde{L}, \widetilde{\mathcal{F}})$ defined as follows: we are given natural numbers t, y, k and a partition of $[y]$ into sets $\{\beta_1, \ldots, \beta_t\}$, each of which has size at least 2. Let $\widetilde{G}$ be a complete graph such that the set $V(\widetilde{G})$ is the disjoint union of sets T and U, with $T = \{v_1, \ldots, v_y\}$ and $U = \{v_{y+1}, \ldots, v_{y+k}\}$. The ground set of colors is $[2y + 1]$. The list associated to each vertex $v_p \in T$ is $\{2p, 2p + 1\}$, while the lists $L(v_{y+1}), \ldots, L(v_{y+p})$ are such that, for $i \in [p - 1]$, each color from $L(v_{y+i})$ is smaller than each color from $L(v_{y+i+1})$. Finally, we let $\widetilde{\mathcal{F}} = \{\{v_i : i \in \beta_1\}, \ldots, \{v_i : i \in \beta_t\}, \{v_{y+1}\}, \ldots, \{v_{y+k}\}\}$.

Note that each instance of PLC $(G_\phi, L_\phi, \mathcal{F}_\phi)$ associated to a 3-SAT instance ϕ as in Section 3.1 fits the framework $(\widetilde{G}, \widetilde{L}, \widetilde{\mathcal{F}})$ defined above. We now associate with each such instance of PLC $(\widetilde{G}, \widetilde{L}, \widetilde{\mathcal{F}})$ an instance (G, μ) of μ-coloring, with G a unit interval graph. We postpone the complete definition of (G, μ), that is rather technical, to Section 3.3, but give here some crucial properties. Namely, (G, μ) is such that:

(P1) $V(G) = U \cup U' \cup T \cup T' \cup \{v^\star\}$.

(P2) $G[U \cup T]$ is isomorphic to $\widetilde{G}$.
(P3) $G[T \cup T' \cup \{v^\star\}]$ (resp. $G[U \cup U']$) is a unit interval graph admitting a proper consistent order where the vertices of $T \cup \{v^\star\}$ (resp. U) are last (resp. first).
(P4) $N(v^\star) = U \cup T$; U' (resp. T') is anti-complete to $T \cup T'$ (resp. $U \cup U'$).
(P5) For each $v \in U \cup T$, $\mu(v) = \max_{h \in \widetilde{L}(v)} h$; $\mu(v^\star) = 1$.
(P6) (G, μ) can be built in polynomial time.

Above and throughout this last section, when we write that a list of operations can be performed in polynomial time without specifying with respect to which function, we mean polynomial in the size of $(\widetilde{G}, \widetilde{L}, \widetilde{\mathcal{F}})$. Note that each function that is polynomial in y, t, k is also polynomial in the size of $(\widetilde{G}, \widetilde{L}, \widetilde{\mathcal{F}})$.

Lemma 12. *G is a unit interval graph.*

Proof. Observe that U is complete to $T \cup \{v^\star\}$ and that there are no more edges from $U \cup U'$ to $T \cup T' \cup \{v^\star\}$. The statement then follows from (P3). $\square$

It follows from Lemma 11 that, in order to prove Theorem 10, it is enough to prove the next lemma. Note that, with a slight abuse, we often look at $\widetilde{G}$ as the subgraph of G induced by $U \cup T$. Moreover, if f is a coloring for a graph G and $X \subseteq V(G)$, we denote by $f[X]$ the restriction of f to X.

Lemma 13. *Let $(\widetilde{G}, \widetilde{L}, \widetilde{\mathcal{F}})$ and (G, μ) be defined as above. If f is a μ-coloring for (G, μ), then $f[V(G_{\widetilde{G}})]$ is a feasible PLC for $(\widetilde{G}, \widetilde{L}, \widetilde{\mathcal{F}})$; conversely, every feasible PLC for $(\widetilde{G}, \widetilde{L}, \widetilde{\mathcal{F}})$ can be extended in polynomial time to a feasible μ-coloring for (G, μ).*

We divide the proof of Lemma 13 into the proof of two lemmas, whose statements need a few definitions.

Our first aim is to define a "restriction" of $(\widetilde{G}, \widetilde{L}, \widetilde{\mathcal{F}})$ to $G[U]$ and $G[T]$. First, we let $\widetilde{L}[U]$ (resp. $\widetilde{L}[T]$) be the restriction of $\widetilde{L}$ to U (resp. T). We also consider the restriction of $\widetilde{\mathcal{F}}$ to T and U: this deserves a few words. First note that the definition of $\widetilde{\mathcal{F}}$ is such that it does indeed induce a partition of U *and* a partition of T. Moreover, the partition of U is only composed of singletons. Thus, the parity constraints associated to vertices of U can be neglected. Thus, $\widetilde{\mathcal{F}}$ is essentially a partition of T and, in the following, with a slight abuse, we also refer to $\widetilde{\mathcal{F}}$ as a partition of T. Therefore, the restriction of $(\widetilde{G}, \widetilde{L}, \widetilde{\mathcal{F}})$ to $G[U]$ is simply an instance $(G[U], \widetilde{L}[U])$ of list-coloring, while the restriction of $(\widetilde{G}, \widetilde{L}, \widetilde{\mathcal{F}})$ to $G[T]$ is an instance $(G[T], \widetilde{L}[T], \widetilde{\mathcal{F}})$ of PLC.

Finally, we associate to an instance (G, μ) of the μ-coloring on G two "sub-instances" $(G[U \cup U'], \mu)$ and $(G[T \cup T' \cup \{v^\star\}], \mu)$, where we are slightly abusing notations since we are identifying μ respectively with its restriction to $U \cup U'$ and to $T \cup T' \cup \{v^\star\}$.

Lemma 14. *If f is a μ-coloring for $(G[U \cup U'], \mu)$, then $f[U]$ is an $\widetilde{L}[U]$-coloring for $(\widetilde{G}[U], \widetilde{L}[U])$; conversely, every $\widetilde{L}[U]$-coloring for $(\widetilde{G}[U], \widetilde{L}[U])$ can be extended in polynomial time to a μ-coloring for $(G[U \cup U'], \mu)$.*

Lemma 15. *If f is a μ-coloring for $(G[T \cup T' \cup \{v^\star\}], \mu)$, then $f[T]$ is a feasible PLC for $(\widetilde{G}[T], \widetilde{L}[T], \widetilde{\mathcal{F}})$; conversely, every feasible PLC for $(\widetilde{G}[T], \widetilde{L}[T], \widetilde{\mathcal{F}})$ can be extended in polynomial time to a μ-coloring for $(G[T \cup T' \cup \{v^\star\}], \mu)$.*

Assume now that Lemmas 14 and 15 hold. As we show in the following, Lemma 13 holds too.

In order to prove the first statement of Lemma 13, we consider a μ-coloring f for (G, μ). The restrictions $f[U \cup U']$ and $f[T \cup T' \cup \{v^\star\}]$ are trivially a μ-coloring for $(G[U \cup U'], \mu)$ and a μ-coloring for $(G[T \cup T' \cup \{v^\star\}], \mu)$, respectively. It follows from Lemmas 14 and 15 that $f[U]$ is an $\widetilde{L}[U]$-coloring for $(\widetilde{G}[U], \widetilde{L}[U])$ and $f[T]$ is a feasible $\widetilde{L}[U]$-coloring for $(\widetilde{G}[T], \widetilde{L}[T], \widetilde{\mathcal{F}})$. We claim that $f[U \cup T]$ determines a feasible PLC for $(\widetilde{G}, \widetilde{L}, \widetilde{\mathcal{F}})$. That is easy to check, as soon as we observe that, $f(u) \neq f(t)$ for $u \in U$ and $t \in T$, since $T \cup U$ is a clique of G.

We now prove the second statement of Lemma 13. Consider a feasible PLC for $(\widetilde{G}, \widetilde{L}, \widetilde{\mathcal{F}})$. Trivially, $f[U]$ is an $\widetilde{L}[U]$-coloring for $(\widetilde{G}[U], \widetilde{L}[U])$, and $f[T]$ is a feasible coloring for $(\widetilde{G}[T], \widetilde{L}[T], \widetilde{\mathcal{F}})$. It follows from Lemmas 14 and 15 that $f[U]$ can be extended to a μ-coloring f' for $(G[U \cup U'], \mu)$ and $f[T]$ can be extended to a μ-coloring f'' for $(G[T \cup T' \cup \{v^\star\}], \mu)$. Observe also that, for each vertex $u \in U$ and $t \in T$, $f(u) \neq f(t)$, since $T \cup U$ is a clique of $\widetilde{G}$. Moreover, $f''(v^\star) = 1$ (since $\mu(v^\star) = 1$) and $f(u) \neq 1$, for $u \in U$ (color 1 does not belong to $\widetilde{L}(u)$). Thus, it is easy to check that the union of f' and f'' determines a μ-coloring for G, and that trivially it can be obtained in polynomial time given f', f''.

In the rest of the paper, we therefore build the graph G so as to satisfy properties (P1)–(P6), and prove Lemmas 14 and 15.

3.3. Building up the graph G

In this section we construct the graph G, describing explicitly the graphs $G[U \cup U']$ and $G[T \cup T' \cup \{v^\star\}]$ and a proper consistent ordering of their vertices where vertices of U (resp. $T \cup \{v^\star\}$) are first (resp. last).

3.3.1. The graph $G[U \cup U']$ and the proof of Lemma 14

Let $D = \bigcup_{v \in U} \widetilde{L}(v)$. Let $U' = \{w_1, \ldots, w_{2y+1}\}$. $G[U \cup U']$ is the graph whose edges are all and only the following: U, U' are cliques, w_i is complete to U for $i \in [2y+1] \setminus D$; for $i \in D$, $v \in U$ is adjacent to w_i if and only if $i < \min \widetilde{L}(v)$.

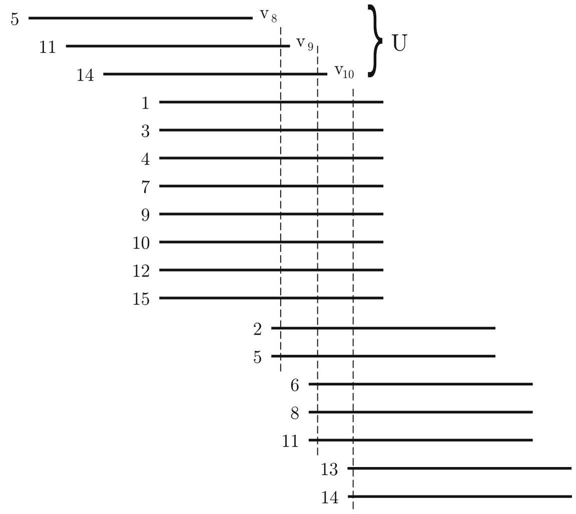

Fig. 2. A unit interval representation of the instance $(G[U \cup U'], \mu)$ arising from an instance $(\widetilde{G}, \widetilde{L})$ with $t = 3, y = 7, k = 3, \widetilde{L}(v_8) = \{2, 5\}, \widetilde{L}(v_9) = \{6, 8, 11\}, \widetilde{L}(v_{10}) = \{11, 14\}$ (the values of μ for each vertex are shown on the left of the corresponding interval). Note that $(\widetilde{G}, \widetilde{L})$ is the PLC instance associated to the 3-Sat instance $\phi = (x_1 \vee \bar{x}_2) \wedge (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3)$.

The proposed order of the vertices starts with the vertices of U, $v_{y+1}, \ldots, v_{y+k}$, followed by the vertices w_i in U' with $i \in [2y+1] \setminus D$, and finally by the vertices w_j in U' with $i \in D$, ordered by their index.

Recall that in the list-coloring instance $(\widetilde{G}[U], \widetilde{L})$, each color in $\widetilde{L}(v_{y+j+1})$ is strictly greater than all colors in $\widetilde{L}(v_{y+j})$ for $j \in [k-1]$. Thus, $N[v_{y+1}] = U \cup \{w_i : i \in [2y+1] \setminus D\}$, and $N[v_{y+j+1}] = N[v_{y+j}] \cup \{w_i : i \in \widetilde{L}(v_{y+j})\}$, for each $j \in [k-1]$. Taking this into account, it is not hard to see that the order proposed above is a proper consistent order. Thus, $G[U \cup U']$ is a unit interval graph. An example can be seen in Fig. 2.

As we already mentioned, $\mu(v) = \max_{h \in \widetilde{L}(v)} h$ for each $v \in U$. Finally, we define μ over U' as $\mu(w_i) = i$ for each $i \in [2y+1]$.

Proof of Lemma 14. Consider a μ-coloring f of $G[U \cup U']$. First note that the vertices in U' form a clique, and $\mu(w_i) = i$ for each $i \in [2y+1]$. This implies that $f(w_i) = i$ for each $i \in [2y+1]$. Now, let $v \in U$. Since $\mu(v) \leq 2y+1$, $f(v)$ is a color given to some vertex in U' nonadjacent to v. By definition of $(G[U \cup U'], \mu)$, a vertex $w_i \in U'$ is nonadjacent to v if and only if either $i > \mu(v)$ or $i \in \widetilde{L}(v)$. Since f is a μ-coloring, $f(v) \in \widetilde{L}(v)$. Thus, $f[U]$ is a feasible solution for $(\widetilde{G}[U], \widetilde{L})$. Conversely, given a feasible $\widetilde{L}[U]$-coloring of $(\widetilde{G}[U], \widetilde{L}[U])$, we can extend it to a μ-coloring of $(G[U \cup U'], \mu)$, by giving to each vertex w of U' color $\mu(w)$. $\square$

3.3.2. *The graph $G[T \cup T' \cup \{v^\star\}]$ and the proof of Lemma 15*

$G[T \cup T' \cup \{v^\star\}]$ is constructed through the following two intermediate steps: we start by building a gadget given by the unit interval graph H_n, where n is any positive integer, and a list $L_{a,b}$ over its vertices, with even integers a, b such that $a \leq b \leq n$. Then, for each $i \in [y-t]$, we build a graph H_y^i and suitably connect those so as to form the unit interval graph $\overline{G}$. This will be $G[T \cup T' \cup \{v^*\}]$. We conclude the paragraph by showing that the latter graph satisfies Lemma 15.

Roughly speaking, the gadget H_n and the list $L_{a,b}$ ensure that, in each feasible $L_{a,b}$-coloring to $(H_n, L_{a,b})$, two given vertices are either given the (even) colors a and b, or the (odd) colors $a+1, b+1$. We shall use this to conclude that two vertices from the same set of $\widetilde{\mathcal{F}}$ (i.e. v, w from the same β_i) are given colors with the same parity. Multiple copies of H_n ensure that *each* pair within *each* set $F \in \widetilde{\mathcal{F}}$ is given colors of the same parity.

We first deal with H_n. For each $n \in \mathbb{N}$ such that $n \geq 3$, H_n is the graph on $4n+8$ vertices that is defined from Fig. 3, where $V(H_n) = \{z_1, \ldots, z_{4n+8}\}$. A_j^i, B_j^i for $i \in \{0, 1\}$ and $j \in \{1, 2\}$ are cliques with the following vertices $A_1^0 = \{z_2, z_3\}, A_2^0 = \{z_4, \ldots, z_{n+1}\}, B_1^0 = \{z_{n+2}, z_{n+3}\}, B_2^0 = \{z_{n+4}, \ldots, z_{2n+1}\}, B_2^1 = \{z_{2n+8}, \ldots, z_{3n+5}\}, B_1^1 = \{z_{3n+6}, z_{3n+7}\}, A_2^1 = \{z_{3n+8}, \ldots, z_{4n+5}\}, A_1^1 = \{z_{4n+6}, z_{4n+7}\}$. The other intervals represent single nodes v_j whose subscript j is denoted on the left of the corresponding interval.

Remark 16. Graph H_n is a symmetric proper interval graph, that is, the one-to-one correspondence $z_j \mapsto z_{4n+9-j}$ is an automorphism of H_n.

Let now $n_1, n_2 \in \mathbb{N}$, such that $n_1 < n_2 \leq n/2$. Define the integers $a = 2n_1$ and $b = 2n_2$ and consider the following list-coloring constraints $L_{a,b}$:

- $L_{a,b}(z_1) = 1$;
- $L_{a,b}(z_2) = L_{a,b}(z_{n+2}) = \{a, a+1\}$;
- $L_{a,b}(z_3) = L_{a,b}(z_{n+3}) = \{b, b+1\}$;

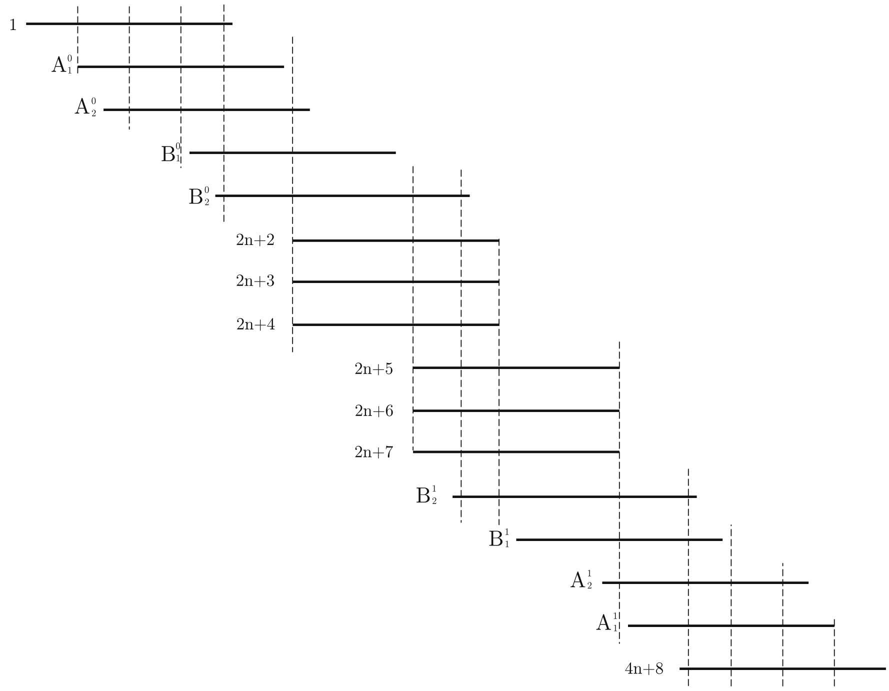

Fig. 3. A unit interval representation of the graph H_n.

- for $z_j \in A_2^0$, $L_{a,b}(z_j) = L_{a,b}(z_{j+n}) = \begin{cases} \{2(j-3), 2(j-3)+1\} & \text{if } j < n_1+3 \\ \{2(j-2), 2(j-2)+1\} & \text{if } n_1+3 \leq j < n_2+2 \\ \{2(j-1), 2j-1\} & \text{if } n_2+2 \leq j; \end{cases}$
- $L_{a,b}(z_{2n+2}) = \{a+1, b\}$;
- $L_{a,b}(z_{2n+3}) = \{b+1, 2n+2\}$;
- $L_{a,b}(z_{2n+4}) = \{1, a\}$;
- $L_{a,b}(z_j) = L_{a,b}(z_{4n+9-j})$ for $j \geq 2n+5$.

Remark 17. Let $n_1, n_2 \in \mathbb{N}$, such that $n_1 < n_2 \leq n/2$, and $(H_n, L_{a,b})$ defined as above. For each $j \in [n] \setminus \{n_1, n_2\}$, there exists a unique vertex $v \in A_2^0$ (resp. B_2^0, A_2^1, B_2^1) such that $L_{a,b}(v) = \{2j, 2j+1\}$. Conversely, for each vertex $v \in A_2^0$ (resp. B_2^0, A_2^1, B_2^1) there exists a value $j \in [n] \setminus \{n_1, n_2\}$ such that $L_{a,b}(v) = \{2j, 2j+1\}$.

Let $\mu : V(H_n) \to \mathbb{N}$ be defined as $\mu(v) = \max_{h \in L_{a,b}(v)} h$.

Lemma 18. *For a, b, μ defined as above, the following properties hold.*

(1) *Every solution of (H_n, μ) is a solution of $(H_n, L_{a,b})$ and conversely every solution of $(H_n, L_{a,b})$ is a solution of (H_n, μ).*
(2) *Let $A^0 = A_1^0 \cup A_2^0$ and $A^1 = A_1^1 \cup A_2^1$. There is no feasible $L_{a,b}$-coloring f of $(H_n, L_{a,b})$, such that there exist $w, w' \in A^0$ (resp. A^1) such that $f(w) = a, f(w') = b+1$, or $f(w) = a+1, f(w') = b$.*
(3) *No pair of vertices of A^0 and A^1 share a color in each feasible $L_{a,b}$-coloring of $(H_n, L_{a,b})$.*
(4) *Each proper coloring of A^0 (resp. A^1) that does not violate constraints from $L_{a,b}$ and from point (2) can be extended in time polynomial in n to a $L_{a,b}$-coloring of $(H_n, L_{a,b})$.*

Proof. (1) One direction is trivial. Thus, we only need to show that for each proper coloring f that is feasible for $(H_n, \mu), f(v) \in L_{a,b}(v)$ holds true for each $v \in V(H_n)$. Let f be a feasible coloring of (H_n, μ). Since $\mu(z_1) = 1$, we have that $f(z_1) = 1$. Recall that $A_1^0 \cup A_2^0 \cup B_1^0 \cup B_2^0$ is a clique, and for each $i \in [n]$ there are exactly two vertices of this clique with $\mu = 2i + 1$ (see Remark 17). As an immediate corollary of Lemma 1 (having excluded color 1, that none of vertices from $A_1^0 \cup A_2^0 \cup B_1^0 \cup B_2^0$ is colored with, being they adjacent to z_1), $f(w) \in L_{a,b}(w)$ for $w \in A^0 \cup B_1^0 \cup B_2^0$. Since H_n is symmetric (cfr. Remark 17), the same holds for $w \in A^1 \cup B_1^1 \cup B_2^1$. Vertex z_{2n+4} is adjacent to the clique $A_2^0 \cup B_2^0$ whose $a-2$ vertices can be colored with colors from 2 to $a-1$, so $f(z_{2n+4}) \in \{1, a\} = L_{a,b}(z_{2n+4})$. Again by symmetry we conclude that $f(z_{2n+5}) \in L_{a,b}(z_{2n+5})$. Vertex z_{2n+2} is adjacent to the clique $A_2^0 \cup B_2^0$ that has $a-2$ vertices colored with colors from 2 to $a-1$ and $b-a-2$ colors in the interval $[a+2, b-1]$. Hence z_{2n+2} can only be given colors $1, a, a+1$ and b. Furthermore, it is adjacent to z_{2n+4} and z_{2n+5}, so it cannot be colored with 1 and a as well. Thus, the only colors left are those from its list. Again, a symmetric argument works for z_{2n+7}. In order to conclude the proof, one can easily show $f(w) \in L_{a,b}(w)$ for $w = z_{2n+3}, z_{2n+6}$ by using similar arguments as those used above.
(2) From Remark 17, z_2 and z_3 are the only vertices from A^0 that can be colored with $a, a+1, b, b+1$. In particular $L_{a,b}(z_2) = \{a, a+1\}$ and $L_{a,b}(z_3) = \{b, b+1\}$. Suppose first $f(z_2) = a$ and $f(z_3) = b+1$. Thus, $f(z_{n+2}) = a+1$ and

$f(z_{n+3}) = b$, which implies that vertex z_{2n+2} cannot be colored. Now suppose $f(z_2) = a+1$ and $f(z_3) = b$. The following assignments of colors are implied: $f(z_{n+2}) = a, f(z_{n+3}) = b+1, f(z_{2n+3}) = 2n+2, f(z_{2n+6}) = b+1, f(z_{2n+4}) = 1$, and $f(z_{2n+5}) = a$. Since $L_{a,b}(z_{2n+7}) = \{a+1, b\}$, either the pair of colors $a, a+1$ or $b, b+1$ are used by vertices adjacent to z_{3n+6} and z_{3n+7}, which then cannot be colored. The symmetric argument works for A^1.

(3) Let f be a feasible $L_{a,b}$-coloring for $(H_n, L_{a,b})$. Recall that $A^0 = A^0_1 \cup A^0_2$; pick any color c such that $f(w) = c$ for some $c \in A^0_2$, and recall that $c \neq a, a+1, b, b+1$. Suppose first that c is even. By Remark 17, $c \neq a, a+1, b, b+1$ and there is a vertex $w' \in B^0_2$ such that $L_{a,b}(w') = \{c, c+1\}$, so $f(w') = c+1$. Repeating the same argument, $f(w'') = c$ for some $w'' \in B^1_2$, and $f(w''') = c+1$ for some $w'' \in A^1_2$. Since by Remark 17 no other vertex of A^1 can be given color c, we conclude the proof for this case. Being the graph symmetric (cfr. Remark 16), we can reverse the argument and settle the case when c is odd. Now pick any color c such that $f(w) = c$ for some $w \in A^0_1 = \{z_2, z_3\}$ and recall that $c \in \{a, a+1, b, b+1\}$. Since f is a feasible $L_{a,b}$-coloring for $(H_n, L_{a,b})$, by part (2) of the Lemma, either $f(z_2) = a$ and $f(z_3) = b$, or $f(z_2) = a+1$ and $f(z_3) = b+1$. Suppose the first holds. Since f is feasible for $(H_n, L_{a,b})$, the following colors are implied: $f(z_{n+2}) = a+1, f(z_{n+3}) = b+1, f(z_{2n+2}) = b, f(z_{2n+3}) = 2n+2, f(z_{2n+6}) = b+1$ and $f(z_{2n+7}) = a+1$. Moreover, note now that $f(z_{2n+5}) = 1$, otherwise z_{3n+7} cannot be colored, so $f(z_{2n+4}) = a$. Repeating the argument, we obtain that $f(z_{4n+6}) = b+1$ and $f(z_{4n+7}) = a+1$. Thus, we showed that if $f(z_2) = a$ and $f(z_3) = b$, then $f(z_{4n+6}) = b+1$ and $f(z_{4n+7}) = a+1$. Since z_{4n+6} and z_{4n+7} are the only vertices from A^1 that can be colored with $a, a+1, b, b+1$, this concludes the proof for this case. We are left to settle the statement for the case $f(z_2) = a+1$ and $f(z_3) = b+1$: note that this is implied by the previous one, since H_n is symmetric.

(4) Let us settle the case for A^0, since the case for A_1 follows by symmetry. Analogously, let $B^0 = B^0_1 \cup B^0_2$ and $B^1 = B^1_1 \cup B^1_2$. Repeating the arguments from the proof of part (3), we observe that in any extension of a proper coloring f of A^0 the following holds: B^1 must use the same colors of A^0, while B^0 and A^1 must use colors $c+1$ for each color c used by A^0 with c even, and colors $c-1$ for each color c used by A^0 with c odd. Moreover, following again the proof of part (3), vertices $z_{2n+2}, \ldots, z_{2n+7}$ can be assigned a color to make the proper coloring feasible. □

Consider $y-t$ vertex disjoint copies $H^1_y, \ldots, H^{y-t}_y$ of H_y and for $i \in [y-t]$ denote their vertex sets by $V(H^i_y) = \{z^i_1, \ldots, z^i_{4y+8}\}$. Let $\overline{G}$ be defined as follows: starting from the graph $\bigcup_{i=1}^{y-t} H^i_y$, for each $i \in [y-t]$, add all edges joining $A^1 \cup \{z^i_{4y+8}\}$ from H^i and $A^0 \cup \{z^{i+1}_1\}$ from H^{i+1}. Then, contract vertex z^i_{4y+8} of H^i with vertex z^{i+1}_1 of H^{i+1}.

It is immediate to check that $\overline{G}$ has $(4y+7)(y-t)+1$ vertices, and

$$z^1_1, \ldots, z^1_{4y+8} = z^2_1, z^2_2, z^2_3, \ldots, z^{y-t}_{4y+8}$$

is a proper consistent order of $V(\overline{G})$. Thus, $\overline{G}$ is a unit interval graph (see Fig. 4 for a unit interval representation of $\overline{G}$ with $y = 3, t = 1$). Moreover, for each $i \in [y-t]$, the subgraph of $\overline{G}$ induced by $\{z^i_1, \ldots, z^i_{4y+8}\}$ is precisely H^i_y. We denote by E^i and F^i respectively the sets A^0 and A^1 from graph H^i_y. Note that $F^i = \{z^i_{3y+8}, \ldots, z^i_{4y+7}\}$ and, with the order defined above, the last vertices of the graphs are those from $F^{y-t} \cup \{z^{y-t}_{4y+8}\}$, i.e. $\{z^{y-t}_{3y+8}, z^{y-t}_{3y+9}, \ldots, z^{y-t}_{4y+8}\}$.

We now define a list $\overline{L}$ for the vertices of $\overline{G}$. For a fixed $j \in [t]$, we let $y_j = |\beta_j| \geq 2$, and $\Delta_j = (\sum_{\ell=1}^{j-1} y_\ell) - (j-1)$; also, denote by $p^1_j, \ldots, p^{y_j}_j$ the elements of β_j, with $p^1_j < \cdots < p^{y_j}_j$. Note that each $i \in [y-t]$ can be written in a unique way as $\Delta_j + \ell$, for some $j \in [t]$ and $\ell \in [y_j - 1]$. For $j \in [t]$ and $\ell \in [y_j - 1]$, define $\overline{L}$ over $\{z^{\Delta_j+\ell}_1, \ldots, z^{\Delta_j+\ell}_{4y+8}\}$ as $\overline{L}_{2p^\ell_j, 2p^{\ell+1}_j}$. Recall that, for $i \in [y-t-1]$, we have $z^i_{4y+8} = z^{i+1}_1$, and those are the only vertices on which $\overline{L}$ is defined twice. Moreover, we have $\overline{L}(z^i_{4y+8}) = \overline{L}(z^{i+1}_1) = 1$. Thus, $\overline{L}$ is well defined. Since $z^i_{4y+8} = z^{i+1}_1$ for $i \in [y-t-1]$, and those vertices can always be colored with 1, $\overline{L}$ is well defined. For each $i \in [y-t]$, for each vertex u of H^i_y set $\mu(u)$ as in Lemma 18. Note that this implies $\mu(u) = \max_{h \in \overline{L}(u)} h$ for each $u \in V(\overline{G})$.

Lemma 19. *For $\overline{G}, \overline{L}, \mu$ defined as above, the following holds.*

(1) *For $i \in [y-t]$, E^i and F^i are cliques. For $i \in [y-t-1]$, z^i_{4y+8} is complete to E^{i+1}. For $i \in \{2, \ldots, y-t\}$, z^i_1 is complete to F^{i-1}.*

(2) *For each $i \in [y-t]$, $p \in [y]$, there exists a unique vertex $v \in E^i$ (resp. F^i) such that $L(v) = \{2p, 2p+1\}$. Conversely, for each vertex $v \in E^i$ (resp. F^i) there exists a value $p \in [y]$ such that $L(v) = \{2p, 2p+1\}$.*

(3) *For each $i, j \in [y-t]$, E^i and F^j share no color, and F^i and F^j share all colors in each feasible $\overline{L}$-coloring of $(\overline{G}, \overline{L})$.*

(4) *Every μ-coloring of $\overline{G}$ is a feasible $\overline{L}$-coloring for $(\overline{G}, \overline{L})$. Conversely, every feasible $\overline{L}$-coloring for $(\overline{G}, \overline{L})$ is a feasible μ-coloring of G.*

(5) *Every feasible $\overline{L}$-coloring of $(\overline{G}, \overline{L})$ is such that, for each $j \in [t]$, either set F^{y-t} uses all colors from the set $\{2p^1_j, 2p^2_j, 2p^3_j, \ldots, 2p^{y_j}_j\}$, and none from the set $\{2p^1_j+1, 2p^2_j+1, 2p^3_j+1, \ldots, 2p^{y_j}_j+1\}$, or it uses all colors from the latter set and none from the former.*

(6) *Each feasible solution to $(\overline{G}[F^{y-t}], \overline{L})$ that satisfies conditions from (5) can be extended in polynomial time to a feasible $\overline{L}$-coloring for $(\overline{G}, \overline{L})$.*

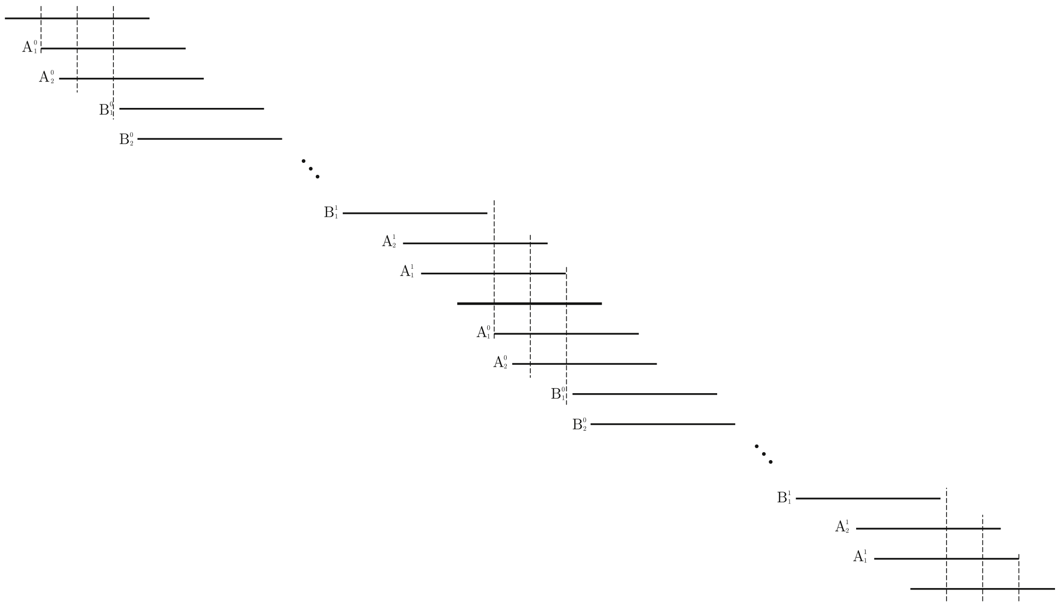

Fig. 4. An example of $\overline{G}$ with $y = 3$, $t = 1$.

Proof. (1) and (2) are immediately shown true, by the very definition of $\overline{G}$, $\overline{L}$ and Remark 17.

(3) From Lemma 18(3) applied to H_y^1, in each feasible $\overline{L}$-coloring f of $(\overline{G}, \overline{L})$, vertices of E^1 share no color with vertices of F^1. By Remark 17, there is exactly one vertex from F^1 and one from E^2 whose list of feasible colors is $\{2p, 2p+1\}$, for each $p \in [y]$. Since both sets are cliques of size y, a color is used in F^1 if and only if it is not used in E^2. We conclude that a color *is* used in E^1 if and only if it *is not* used in F^1, and if and only if it *is* used in E^2. By repeatedly applying this argument, we obtain the thesis.

(4) Trivially, every feasible $\overline{L}$-coloring for $(\overline{G}, \overline{L})$ is a μ-coloring for $(\overline{G}, \mu)$. Now let f be a μ-coloring for $(\overline{G}, \mu)$. We only have to check that $f(v) \in \overline{L}(v)$ for all $v \in V(\overline{G})$. Note that f is a μ-coloring of $(\overline{G}[H_y^i], \mu)$, for each $i \in [y-t]$. Recall that each $i \in [y-t]$ can be written in a unique way as $\Delta_j + \ell$, for some $j \in [t]$ and $\ell \in [y_j - 1]$. Hence, from Lemma 18(1) and the fact that $\overline{L}$ is defined over $\{z_1^{\Delta_j+\ell}, \ldots, z_{4y+8}^{\Delta_j+\ell}\}$ as $\overline{L}_{2p_j^\ell, 2p_j^{\ell+1}}$, f is an $\overline{L}[H_y^i]$-coloring for $(\overline{G}[H_y^i], \overline{L}[H_y^i])$, for each i in $[y-t]$. Thus, it is also an $\overline{L}$-coloring for $(\overline{G}, \overline{L})$.

(5) Let f be a feasible $\overline{L}$-coloring for $(\overline{G}, \overline{L})$ and pick any $i \in [y-t]$. Recall that there exist unique j and ℓ such that $i = \Delta_j + \ell$, and $\ell \in [y_j - 1]$. We first show that F^i either uses both colors $2^\ell p_j, 2^{\ell+1} p_j$ and neither $2^\ell p_j + 1$ nor $2^{\ell+1} p_j + 1$, or uses both colors $2^\ell p_j + 1$ and $2^{\ell+1} p_j + 1$, and neither $2^\ell p_j$ nor $2^{\ell+1} p_j$. By part (2), F^i never uses simultaneously $2^\ell p_j$ and $2^\ell p_j + 1$ in f. From Lemma 18(2), F^i uses either both $2^\ell p_j$ and $2^{\ell+1} p_j$ or both $2^\ell p_j + 1$ and $2^{\ell+1} p_j + 1$. By part (3), it follows that, for each $j \in [t]$, F^{Δ_j+1} either uses all colors from the set $\{2p_j^1, 2p_j^2, 2p_j^3, \ldots, 2p_j^{y_j}\}$, and none from the set $\{2p_j^1 + 1, 2p_j^2 + 1, 2p_j^3 + 1, \ldots, 2p_j^{y_j} + 1\}$, or it uses all colors from the latter set and none from the former. By using again part (3), the thesis follows.

(6) Consider now a feasible $\overline{L}[F^{y-t}]$-coloring f for $(\overline{G}[F^{y-t}], \overline{L}[F^{y-t}])$. From part (5), we know it satisfies the hypothesis from Lemma 18(4) (with respect to the graph H_y^{y-t} and the list $L_{a,b}$, with $a = 2p_t^{y_t-1}$, $b = 2p_t^{y_t}$). Thus, f can be extended to a feasible $\overline{L}[H_y^{y-t}]$-coloring for all vertices of H_y^{y-t}. Repeating the argument used in the proof of claim (3), we can extend that proper coloring to F^{y-t-1} by assigning in the only possible way the set of colors used by F^{y-t} in f. This partial proper coloring of H^{y-t-1} respects the constraints of Lemma 18(4). Hence, we can repeat the argument used above. Iterating over all $i \in [y-t]$, we conclude that f can be extended to a feasible coloring for $(\overline{G}, \overline{L})$. It is immediate to check that it can be done in polynomial time. $\square$

Define $G[T \cup T' \cup \{v^*\}] = \overline{G}$, where we identify T with F^{y-t} and $v^\star$ with z_{8y+15}^{y-t}. Lemma 19(2) and the definition of $\overline{G}$ allow us to identify T with F^{y-t} in such a way that, for each $p \in [y]$, the vertex v_p of T corresponds to the only vertex v in F^{y-t} satisfying $\overline{L}(v) = \widetilde{L}(v_p) = \{2p, 2p+1\}$. Also, $\mu(v^\star) = 1$ and, in the proper consistent order of $\overline{G}$ given, $T \cup \{v^\star\}$ are the last vertices.

Proof of Lemma 15. Let f be a μ-coloring for $(G[T \cup T' \cup \{v^\star\}], \mu)$. By Lemma 19(4), it is also a feasible $\overline{L}$-coloring for $(\overline{G}, \overline{L})$. By Lemma 19(5), for each $F \in \widetilde{\mathcal{F}}$, vertices from F are given colors with the same parity. Thus, f is a feasible PLC for $(\widetilde{G}[T], \widetilde{L}[T], \widetilde{F})$.

Conversely, let f be a feasible PLC coloring for $(\widetilde{G}[T], \widetilde{L}[T], \widetilde{\mathcal{F}})$. Since $\widetilde{L}$ coincides with the restriction of $\overline{L}$ to F^{y-t}, f is also a feasible $\overline{L}$-coloring for $(\overline{G}[F^{y-t}], \overline{L})$ and it satisfies the conditions from Lemma 19(5). Thus, by Lemma 19(6), f can be extended in polynomial time to a feasible $\bar{L}$-coloring for $(\bar{G}, \bar{L})$. Finally, we use Lemma 19(4) to conclude that f is a μ-coloring for $\bar{G}$. □

In order to conclude the proof of Theorem 10, we are left to note that, by definition of (G, μ), properties (P1)–(P6) clearly hold.

Acknowledgments

We thank the two anonymous referees for providing remarks that guided us when revising this manuscript. In particular, we are in debt with one of them for many insights and suggestions, which e.g. led to shortening the proof of Theorem 10. We also thank the editor for his many suggestions that highly improved the readability of the paper. Flavia Bonomo's research was partially supported by ANPCyT PICT-2007-00518 and PICT-2007-00533, CONICET PIP 112-200901-00178, and UBACyT Grants 20020090300094 and 20020100100980 (Argentina). Yuri Faenza's research was supported by the Progetto di Eccellenza 2008–2009 of the Fondazione Cassa di Risparmio di Padova e Rovigo. Gianpaolo Oriolo's research was supported by the Italian Research Project Prin 2009 "Programmazione semidefinita e altre tecniche non lineari per problemi di ottimizzazione discreta".

References

[1] C. Berge, Les problèmes de colorations en théorie des graphes, in: Publications de l'Institut de Statistique de l'Université de Paris, vol. 9, 1960, pp. 123–160.
[2] M. Biró, M. Hujter, Zs. Tuza, Precoloring extension. I. Interval graphs, Discrete Mathematics 100 (1–3) (1992) 267–279.
[3] F. Bonomo, M. Cecowski, Between coloring and list-coloring: μ-coloring, Electronic Notes in Discrete Mathematics 19 (2005) 117–123.
[4] F. Bonomo, G. Durán, J. Marenco, Exploring the complexity boundary between coloring and list-coloring, Annals of Operations Research 169 (1) (2009) 3–16.
[5] A. Brandstädt, V. Bang Le, Structure and linear time recognition of 3-leaf powers, Information Processing Letters 98 (4) (2006) 113–138.
[6] V. Cacchiani, A. Caprara, P. Toth, Solving a real-world train unit assignment problem, in: C. Liebchen, R.K. Ahuja, and J.A. Mesa (Eds.), ATMOS 2007—7th Workshop on Algorithmic Approaches for Transportation Modeling, Optimization, and Systems, Dagstuhl, Germany, 2007.
[7] F. Gavril, Algorithms for minimum coloring, maximum clique, minimum covering by cliques and maximum independent set of a chordal graph, SIAM Journal on Computing 1 (2) (1972) 180–187.
[8] M. Grötschel, L. Lovász, A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization, Combinatorica 1 (1981) 169–197.
[9] P.L. Hammer, F. Maffray, Complete separable graphs, Discrete Applied Mathematics 27 (1) (1990) 85–99.
[10] M. Hujter, Zs. Tuza, Precoloring extension. II. Graph classes related to bipartite graphs, Acta Mathematica Universitatis Comenianae 62 (1) (1993) 1–11.
[11] M. Hujter, Zs. Tuza, Precoloring extension. III. Classes of perfect graphs, Combinatorics, Probability and Computing 5 (1996) 35–56.
[12] K. Jansen, Complexity results for the optimum cost chromatic partition problem, Manuscript, 1997.
[13] K. Jansen, P. Scheffler, Generalized coloring for tree-like graphs, Discrete Applied Mathematics 75 (1997) 135–155.
[14] D. Marx, Precoloring extension on unit interval graphs, Discrete Applied Mathematics 154 (6) (2006) 995–1002.
[15] G. Nemhauser, L. Wolsey, Integer and Combinatorial Optimization, in: Wiley Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, New York, 1988.
[16] N. Nishimura, N. Radge, D.M. Thilikos, On graph powers for leaf-labeled trees, Journal of Algorithms 42 (1) (2002) 69–108.
[17] S. Olariu, An optimal greedy heuristic to color interval graphs, Information Processing Letters 37 (1991) 21–25.
[18] J. Ramalingam, C. Pandu Rangan, A unified approach to domination problems on interval graphs, Information Processing Letters 27 (1988) 271–274.
[19] F.S. Roberts, Indifference graphs, in: F. Harary (Ed.), Proof Techniques in Graph Theory, Academic Press, 1969, pp. 139–146.
[20] A. Schrijver, Theory of Linear and Integer Programming, John Wiley, 1986.
[21] V. Vizing, Coloring the vertices of a graph in prescribed colors, Metody Diskretnogo Analiza 29 (1976) 3–10.