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Corresponding Author	Family name	Amster
	Particle	
	Given Name	Pablo
	Suffix	
	Division	Departamento de Matemática, FCEyN
	Organization	UBA Ciudad Universitaria
	Address	Pabellón I, 1428, Buenos Aires, Argentina
	E-mail	pamster@dm.uba.ar
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ORIGINAL RESEARCH

Non-asymptotic and potential Landesman-Lazer conditions for a nonlinear beam equation

Pablo Amster

Abstract Existence of solutions for a nonlinear fourth order ordinary differential equation arising in beam theory is considered.

We obtain solutions by a degree argument under a non-asymptotic condition on the nonlinear terms of the problem. Moreover, assuming a potential Landesman-Lazer condition, we prove the existence of at least one solution by variational methods.

Keywords Nonlinear beam equation · Symmetric solutions · Landesman-Lazer conditions · Degree theory · Variational methods

Mathematics Subject Classification 34B15 · 47H11 · 47J30

1 Introduction

We study the problem

$$\begin{cases} u^{(4)}(t) + g\big(t, u(t)\big) = 0 \\ u''(0) = u''(T) = 0 \\ u'''(0) = -f\big(u(0)\big) \\ u'''(T) = f\big(u(T)\big), \end{cases} \tag{1}$$

where $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous. In addition, we shall assume that g is symmetric in t, namely

$$g(t, u) = g(T - t, u). \tag{2}$$

P. Amster (✉)
Departamento de Matemática, FCEyN, UBA Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina
e-mail: pamster@dm.uba.ar

Problem (1) arises on a model for the deflection of a beam resting on elastic bearings. The existence of symmetric solutions when (2) holds has been considered by Grossinho and Ma by variational methods. It has been proved (see [2], Theorem 2) that, if f and $g(t,\cdot)$ are nondecreasing, then (1) has a symmetric solution if and only if

$$2f(a)+\int_0^T g(t,a)dt=0 \quad \text{for some } a\in\mathbb{R}.$$

Moreover, if no monotonicity condition is assumed, the authors have proved (see [2], Theorem 5) the existence of a symmetric solution of (1) for the *sublinear* case, namely

$$\frac{g(t,s)}{s}\to 0 \quad \text{as } |s|\to\infty$$

uniformly in t, and

$$\frac{f(s)}{s}\to 0 \quad \text{as } |s|\to\infty,$$

assuming a growth condition for f and g, and that one of the following hypotheses holds:

(i) $g(t,s)\to\pm\infty$ as $s\to\pm\infty$ uniformly in t, and f is bounded from below.
(ii) $f(s)\to\pm\infty$ as $s\to\pm\infty$, and g is bounded from below.

In [1], solutions have been obtained by the use of Mawhin's coincidence degree theory [5], for bounded g, assuming an asymptotic Landesman-Lazer type conditions (see e.g. [3, 6]). In more precise terms, if we denote respectively by $g_{sup}^{\pm}, g_{inf}^{\pm}, f_{sup}^{\pm}, f_{inf}^{\pm}$ the upper and lower limits of g and f as $u\to\pm\infty$, existence of solutions is guaranteed by one of the following conditions:

$$2f_{sup}^{-}+\int_0^T g_{sup}^{-}(t)\,dt<0<2f_{inf}^{+}+\int_0^T g_{inf}^{+}(t)\,dt \tag{3}$$

or

$$2f_{sup}^{+}+\int_0^T g_{sup}^{+}(t)\,dt<0<2f_{inf}^{-}+\int_0^T g_{inf}^{-}(t)\,dt. \tag{4}$$

In this setting, it may be argued that the boundedness of g is, in some sense, essential: indeed, if we consider for example

$$g(t,u)=-u+\sin t, \qquad f\equiv 0,$$

then there are no solutions of (1) for $T=\pi$ although (4) is satisfied. However, this counterexample is obviously due to the interaction of g with the spectrum of the linear operator $Lu:=u^{(4)}$ over the set of symmetric functions such that $u''(0)=u'''(0)=0$; thus, different ways of avoiding the problem may be suggested, without restricting ourselves to the case in which g is bounded.

In first place, observe that L is nonnegative: in consequence, under appropriate assumptions no interference with the spectrum should be expected when (3) holds instead of (4).

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In second place, we might consider also that g is sublinear (although not necessarily f).

Finally, the boundedness restriction can be dropped if we assume that g and f are both bounded from below in (4), or from above in (3).

These three situations are reflected in the following cases that shall be considered in our result, together with an appropriate generalization of (3) and (4):

- *Case 1*: $ug(t,u) \geq k(t)$ for $|u| \geq u_0$ and some $k \in L^1(0,T)$, and $sf(s) \geq l$ for $|s| \geq s_0$ and some $l \in \mathbb{R}$.
- *Case 2*:
$$\left|g(t,u)\right| \leq \left(\frac{2}{T}\right)^4 \frac{r}{T}$$
for $a - r \leq u \leq b + r$ and $t \in [0,T]$, for some constants $a < b$ and $r > 0$.

  For example, if g grows at most linearly, i.e. $|g(u)| \leq \alpha|u| + \beta$, with $2\alpha T < (\frac{2}{T})^4$, then it suffices to take r large enough, $a = -cr$, and $b = cr$ for some constant $c \in (1,2)$.
- *Case 3*: There exists $k \in L^1(0,T)$ and $l \in \mathbb{R}$ such that either
$$g(t,u) \geq k(t) \quad \text{and} \quad f \geq l$$
or
$$g(t,u) \leq k(t) \quad \text{and} \quad f \leq l.$$

In order to establish non-asymptotic conditions for g and f let us define, for $a < b$ and $r > 0$, the functions

$$g_{sup}^a(t) = \sup_{|u-a|<r} g(t,u), \qquad g_{sup}^b(t) = \sup_{|u-b|<r} g(t,u),$$

and

$$g_{inf}^a(t) = \inf_{|u-a|<r} g(t,u), \qquad g_{inf}^b(t) = \inf_{|u-b|<r} g(t,u).$$

In Case 2, the constants a, b and r are given; in Cases 1 and 3, a and b are arbitrary, but the value of r depends on the data, and it shall be explicitly computed (see (11) and (12) below). We are now able to state our existence result as follows.

Theorem 1.1 *Assume that one of the situations of the previous Cases* 1, 2 *or* 3 *holds, and that*

$$\int_0^T g_{sup}^a(t)\,dt + 2f(a) < 0 < \int_0^T g_{inf}^b(t)\,dt + 2f(b) \tag{5}$$

or

$$\int_0^T g_{sup}^b(t)\,dt + 2f(b) < 0 < \int_0^T g_{inf}^a(t)\,dt + 2f(a). \tag{6}$$

Then (1) *admits at least one classical solution.*

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Remark 1.2 It is easy to verify that all the above-mentioned existence results from [1] and [2] can be deduced from Theorem 1.1.

A proof of Theorem 1.1 will be given in Sect. 2, by topological degree methods.

On the other hand, we might expect existence results of a different type, due to the variational structure of problem (1). In particular, we shall prove that existence of solutions can be proved under a *potential* Landesman-Lazer condition; namely, in terms of the primitives

$$G(t,u) := \int_0^u g(t,s)\,ds, \qquad F(u) := \int_0^u f(s)\,ds.$$

Although a more general situation could be considered, we shall assume for simplicity that g is bounded. Let us define

$$\limsup_{s\to\pm\infty} \frac{G(t,s)}{s} := G^{\pm}_{sup}(t), \qquad \liminf_{s\to\pm\infty} \frac{G(t,s)}{s} := G^{\pm}_{inf}(t),$$

and the (possibly infinite) limits:

$$\limsup_{s\to\pm\infty} \frac{F(s)}{s} := F^{\pm}_{sup}, \qquad \liminf_{s\to\pm\infty} \frac{F(s)}{s} := F^{\pm}_{inf}.$$

Then we obtain:

Theorem 1.3 *Assume that* (2) *holds, and that* g *is bounded, with*

$$2F^{-}_{sup} + \int_0^T G^{-}_{sup}(t)\,dt < 0 < 2F^{+}_{inf} + \int_0^T G^{+}_{inf}(t)\,dt \tag{7}$$

or

$$2F^{+}_{sup} + \int_0^T G^{+}_{sup}(t)\,dt < 0 < 2F^{-}_{inf} + \int_0^T G^{-}_{inf}(t)\,dt. \tag{8}$$

Then (1) *admits at least one classical solution.*

Remark 1.4 It is worth to observe that the potential conditions in Theorem 1.3 cannot be deduced from the assumptions of Theorem 1.1. Indeed, (5) and (6) require some specific behavior on f and g over an interval of length $2r$, where r is determined by the data and cannot be arbitrarily small. But it might happen, for instance, that g and f oscillate rapidly, so that (5) or (6) do not hold, but one of the conditions in Theorem 1.3 is fulfilled.

Remark 1.5 As before, the boundedness condition can be dropped when the first of the assumption holds. However, in this case an uniformness condition on the limits is needed, in order to avoid some sort of "$\infty - \infty$" indeterminacy. For example, a sufficient condition when g is unbounded is the following:

$$G(t,u) \ge \xi(t)|u| + C, \qquad F(s) \ge \theta|s| + C$$

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for $|u| \geq u_0$, $|s| \geq s_0$ and some $\xi \in L^1(0,T)$, $\theta \in \mathbb{R}$ such that $\int_0^T \xi(t)\,dt + 2\theta > 0$. Note that ξ and θ are not necessarily nonnegative, although in this case the inequalities in (7) still make sense. It might happen, however, that for example $F^+_{inf} = -\infty$ or $F^-_{sup} = +\infty$, as far as $G(t,u)$ goes to $+\infty$ fast enough, uniformly in t.

2 Non-asymptotic conditions: a degree argument

A proof of Theorem 1.1 could be given in the context of coincidence degree theory. For completeness, we shall present a direct argument, using the homotopy invariance of the Leray-Schauder degree.

In order to put the problem in an appropriate setting, let us firstly observe that the linear associated problem for a symmetric function $\varphi \in L^2(0,T)$ and a constant c given by

$$u^{(4)}(t) = \varphi(t), \qquad u''(0) = 0, \qquad u'''(0) = c \tag{9}$$

has a symmetric solution $u \in H^4(0,T)$ if and only if

$$\overline{\varphi} + \frac{2c}{T} = 0,$$

where $\overline{\varphi}$ denotes the average of φ over the interval $(0,T)$. Thus, the linear operator $\mathcal{L} : H^4(0,T) \to L^2(0,T) \times \mathbb{R}$ given by $\mathcal{L}u := (u^{(4)}, u'''(0))$ admits a continuous right inverse $\mathcal{K} : \{(\varphi,c) : \overline{\varphi} + \frac{2c}{T} = 0\} \to H^4(0,T)$, given by $\mathcal{K}(\varphi,c) = u$, the unique solution of (9) such that $u(0) = 0$. We shall apply a degree argument to the Fredholm operator $F_\lambda : E \to E$, where E denotes the Banach space $C_{sym}([0,T])$ of symmetric continuous functions, given by

$$\begin{aligned} F_\lambda(u) &= u - u(0) - \overline{g(\cdot,u)} - \frac{2f(u(0))}{T} \\ &\quad - \lambda\mathcal{K}\Big(g(\cdot,u) - \overline{g(\cdot,u)} - \frac{2f(u(0))}{T}, f\big(u(0)\big)\Big). \end{aligned}$$

A straightforward computation shows that, for $\lambda \in (0,1]$, then $F_\lambda(u) = 0$ if and only if u is a symmetric solution of

$$\begin{cases} u^{(4)}(t) + \lambda g\big(t,u(t)\big) = 0 \\ u''(0) = u''(T) = 0 \\ u'''(0) = -\lambda f\big(u(0)\big) \\ u'''(T) = \lambda f\big(u(T)\big). \end{cases} \tag{10}$$

We shall prove the existence of at least one solution of the problem in $\overline{\Omega} \subset E$, where

$$\Omega := \big\{u \in E : \big\|u - u(0)\big\| < r, u(0) \in (a,b)\big\}.$$

Observe that if the problem has a solution $u \in \partial\Omega$, then there is nothing to prove. Thus, we may suppose that the problem has no solutions in $\partial\Omega$. Next, we shall prove

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that F_λ does not vanish on $\partial\Omega$ for $\lambda \in (0,1)$. With this aim, let us prove in first place that if $F_\lambda u = 0$ for $\lambda \in (0,1)$ and $u \in \overline{\Omega}$, then $\|u - u(0)\| < r$. Indeed, if $F_\lambda u = 0$, then u solves (10). We shall proceed as follows, according to the different cases:

- *Case 1*: Multiplying the equation by $u - u(0)$, and using the fact that

$$\int_0^T g\big(t, u(t)\big)\,dt + 2f\big(u(0)\big) = 0,$$

we obtain:

$$\begin{aligned}
&\int_0^T u^{(4)}(t)\big(u(t) - u(0)\big)\,dt \\
&\quad = -\lambda\left(\int_0^T g\big(t,u(t)\big)u(t)\,dt + 2f\big(u(0)\big)u(0)\right) \\
&\quad \le -\lambda\left(\int_{|u|\le u_0} g\big(t,u(t)\big)u(t)\,dt + \int_{|u|>u_0} k(t)\,dt + 2\min\Big\{\inf_{|s|<s_0} sf(s), l\Big\}\right)
\end{aligned}$$

Integrating by parts the left-hand side, it is seen that $\|u''\|_{L^2}^2 < C$ for some appropriate constant C, and hence

$$\big\|u - u(0)\big\|_\infty < \left(\frac{T}{2}\right)^{3/2}\frac{C}{2} := r. \tag{11}$$

- *Case 2*: Let us define $I_r = [a - r, b + r]$, and

$$\gamma_r(I) = T \sup_{t\in[0,T],u\in I_r} \big|g(t,u)\big|.$$

From the equation, it follows easily that

$$\big\|u - u(0)\big\|_\infty \le \lambda\left(\frac{T}{2}\right)^4 \gamma_r(I).$$

From the hypothesis, we obtain that $\|u - u(0)\|_\infty < r$.

- *Case 3*: Assume that $g(t,u) \ge k(t)$ and $f \ge l$, and write: $-u^{(4)}(t) = \lambda[g(t,u(t)) - k(t)] + \lambda k(t)$. We deduce that

$$\big\|u^{(4)}\big\|_{L^1} \le \lambda \int_0^T g\big(t,u(t)\big)\,dt + \lambda\big(\|k\|_{L^1} - T\overline{k}\big).$$

Moreover, as $\int_0^T g(t,u(t))\,dt + 2f(u(0)) = 0$, we conclude:

$$\big\|u^{(4)}\big\|_{L^1} < \|k\|_{L^1} - T\overline{k} - 2l := C.$$

From the symmetry of u, this implies that

$$\big\|u'''\big\|_\infty < \frac{C}{2}, \qquad \big\|u''\big\|_\infty < \frac{TC}{4}, \qquad \big\|u'\big\|_\infty < \frac{T^2C}{8},$$

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and finally

$$\left\|u-u(0)\right\|_\infty < \frac{T^3C}{16} := r. \tag{12}$$

The proof is analogous if $g(t,u) \le k(t)$ and $f \le l$, with $C = \|k\|_{L^1} + T\overline{k} + 2l$.

It remains to prove that $u(0) \ne a, b$. Suppose for example that $u(0) = b$, and that (5) holds. As $|u(t) - b| < r$ for every $t \in [0,T]$, we deduce that

$$0 = \int_0^T g\big(t, u(t)\big)\,dt + 2f(b) \ge \int_0^T g^b_{inf}(t)\,dt + 2f(b) > 0,$$

a contradiction. The proof is analogous if $u(0) = a$, and also if (6) holds instead of (5).

To conclude, let us verify that F_0 does not vanish on $\partial\Omega$, and that the degree of F_0 at 0 over the domain Ω is different from zero.

Indeed, if $F_0(u) = 0$ for some $u \in \overline{\Omega}$, then $u \in \mathbb{R}$. Hence, $u - u(0) = 0$ and $\int_0^T g(t,u)\,dt + 2f(u(0)) = 0$. Moreover, if $u = a$ or $u = b$, a contradiction is obtained as before. Thus, $u \notin \partial\Omega$. From the definition of the Leray-Schauder degree,

$$deg(F_0, \Omega, 0) = deg_B(F_0|_{\mathbb{R}}, \Omega \cap \mathbb{R}, 0).$$

For $u \in \mathbb{R}$, it is clear that $F_0(u) = -\frac{1}{T}(\int_0^T g(t,u)\,dt + f(u))$. From (5) or (6), it follows that $F_0(a)F_0(b) < 0$; thus, the degree is ± 1, and the proof is complete.

3 Proof of Theorem 1.3

Let $\mathcal{H}$ be the Hilbert space defined by

$$\mathcal{H} := \big\{u \in H^2(0,T) : u(T-t) = u(t)\big\},$$

and consider the functional $I : \mathcal{H} \to \mathbb{R}$ given by:

$$I(u) = \int_0^T \frac{u''(t)^2}{2} + G\big(t, u(t)\big)\,dt + 2F\big(u(0)\big).$$

A simple computation shows that $I \in C^1(\mathcal{H}, \mathbb{R})$, with

$$DI(u)(\varphi) = \int_0^T u''(t)\varphi''(t) + g\big(t,u(t)\big)\varphi(t)\,dt + 2f\big(u(0)\big)\varphi(0).$$

Thus, if u is a critical point of I, then taking all the symmetric test functions $\varphi \in C_0^\infty(0,T)$ it follows that $u^{(4)}(t) + g(t,u(t)) = 0$ in the weak sense. As u is continuous, it follows that u is a classical solution of the equation. Furthermore, as u and φ are symmetric, for any $\varphi \in \mathcal{H}$ we obtain

$$\int_0^T u''(t)\varphi''(t)\,dt = -2u''(0)\varphi'(0) + 2u'''(0)\varphi(0) - \int_0^T g\big(t,u(t)\big)\varphi(t)\,dt.$$

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Taking φ such that $\varphi'(0) = 0 \neq \varphi(0)$ we deduce that

$$u'''(0) + f\big(u(0)\big) = 0.$$

Finally, taking φ such that $\varphi(0) = 0 \neq \varphi'(0)$ we conclude that also $u''(0) = 0$.

For the sake of completeness, let us recall the following well known facts from critical point theory:

Definition 3.1 Let E be a Banach space and $I \in C^1(E, \mathbb{R})$. It is said that I satisfies (PS) if any sequence $\{u_n\} \subset E$ such that $|I(u_n)| \le c$ for some constant c and $DI(u_n) \to 0$, has a convergent subsequence in E.

Theorem 3.2 *Let E be a Banach space and let $I \in C^1(E, \mathbb{R})$ satisfy (PS). Furthermore, assume that I is coercive. Then I achieves a minimum.*

Theorem 3.3 (*Rabinowitz*, [7]) *Let E be a Banach space and let $I \in C^1(E, \mathbb{R})$ satisfy (PS). Furthermore, assume that $E = E_1 \oplus E_2$, with $\dim(E_1) < \infty$, and*

$$\max_{u \in E_1 : \|u\| = R} I(x) < \inf_{u \in E_2} I(u)$$

for some $R > 0$. Then I has at least one critical point.

3.1 Palais-Smale condition

In this section, we prove that I satisfies the Palais-Smale condition.

Let $u_n \in \mathcal{H}$ satisfy $|I(u_n)| \le c$, $DI(u_n) \to 0$. If $\{u_n\}$ is bounded, then taking a subsequence we may assume that u_n converges to some function u, both for the weak topology and the C^1-norm. As

$$DI(u_n)(u) = \int_0^T u_n''(t)u''(t) + g\big(t, u_n(t)\big)u(t)\, dt + 2f\big(u_n(0)\big)u(0) \to 0$$

and $\int_0^T u_n''(t)u''(t)\, dt \to \int_0^T u''(t)^2\, dt$, we deduce that

$$\int_0^T u''(t)^2 + g\big(t, u(t)\big)\, dt + 2f\big(u(0)\big)u(0) = 0.$$

Moreover, as $\{u_n\}$ is bounded then also

$$DI(u_n)(u_n) = \int_0^T u_n''(t)^2 + g\big(t, u_n(t)\big)u_n(t)\, dt + 2f\big(u_n(0)\big)u_n(0) \to 0,$$

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and hence $\int_0^T u_n''(t)^2\,dt \to \int_0^T u''(t)^2\,dt$. Thus,

$$\int_0^T \left(u_n''(t) - u''(t)\right)^2 dt$$
$$= \int_0^T u_n''(t)^2\,dt + \int_0^T u''(t)^2\,dt - 2\int_0^T u_n''(t)u''(t)\,dt \to 0,$$

which implies that $u_n \to u$ in $\mathcal{H}$.

Next, we shall prove that if $\{u_n\}$ is a sequence such that $|I(u_n)| \le c$ and $DI(u_n) \to 0$, then $\{u_n\}$ is bounded.

If (7) holds, the claim follows immediately from the fact that I is coercive, which will be proved in Sect. 3.2 below. We shall give an argument when (8) holds, which in fact is also valid under assumption (7).

Suppose that $\{u_n\}$ is unbounded, such that $|I(u_n)| \le c$ and $DI(u_n) \to 0$. Writing

$$u_n(t) = u_n(0) + \int_0^t u_n'(s)\,ds$$

and

$$u_n'(t) = \int_{T/2}^t u_n''(s)\,ds,$$

we deduce that

$$\left\| u_n - u_n(0) \right\|_\infty \le \frac{T}{2}\|u_n'\|_\infty \le \frac{T^{3/2}}{4}\left\|u_n''\right\|_{L^2}.$$

Define $v_n := \frac{\tilde{u}_n}{\|u_n\|}$. As $DI(u_n)(v_n) \to 0$, from the boundedness of g we deduce that $\frac{\|u_n''\|_{L^2}^2}{\|u_n\|}$ is bounded, and hence $\frac{\|u_n''\|_{L^2}}{\|u_n\|} \to 0$. From the previous inequalities, this implies that $v_n \to 0$ in the sense of H^2; in particular, $|u_n(0)| \to \infty$ and $\|\tilde{u}_n\|_\infty = o(|u_n(0)|)$. Passing to a subsequence, we may suppose for example that $u_n(0) \to +\infty$.

Using again that $DI(u_n)(v_n) \to 0$ and that g is bounded, we conclude that $\frac{\|u_n''\|_{L^2}^2}{\|u_n\|} \to 0$. Thus, if we divide $I(u_n)$ by $u_n(0)$, and use the fact that $\frac{\tilde{u}_n(t)}{u_n(0)} \to 0$ uniformly, we obtain:

$$0 = \lim_{n\to\infty} \int_0^T \frac{G(t, u_n(t))}{u_n(0)}\,dt + 2\frac{F(u_n(0))}{u_n(0)}$$
$$\le \int_0^T G_{sup}^+(t)\,dt + 2F_{sup}^+ < 0,$$

a contradiction. The proof is similar if $u_n(0) \to -\infty$.

3.2 End of the proof

We shall prove that if (7) holds, then I is coercive, and if condition (8) holds, then Rabinowitz Theorem applies.

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Indeed, assume (7), and suppose that $I(u_n)$ is bounded from above for some sequence $\{u_n\}$ such that $\|u_n\| \to \infty$. As before, we deduce that $\frac{\|u_n''\|^2_{L^2}}{\|u_n\|}$ is bounded, and $\|\tilde{u}_n\|_\infty = o(|u_n(0)|)$. Then

$$\liminf_{n\to\infty} \int_0^T \frac{G(t, u_n(t))}{|u_n(0)|}\, dt + 2\frac{F(u_n(0))}{|u_n(0)|} > 0.$$

This implies that $I(u_n) \to \infty$, a contradiction.

Finally, let us assume that (8) holds, and consider

$$E_1 := \mathbb{R}, \qquad E_2 := \{u \in \mathcal{H} : u(0) = 0\}.$$

Taking $R \in \mathbb{R}$, a simple computation shows that

$$I(\pm R) \to -\infty \quad \text{as } R \to +\infty.$$

Moreover, it is immediate to see that $I|_{E_2}$ is bounded from below, and Theorem 3.3 applies for R large enough.

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