Acta Appl Math (2012) 118:237–249
DOI 10.1007/s10440-012-9687-1

Solutions to Integro-differential Problems Arising on Pricing Options in a Lévy Market

Indranil SenGupta · Maria C. Mariani · Pablo Amster

Received: 1 March 2011 / Accepted: 14 October 2011 / Published online: 8 February 2012

Abstract We study an integro-differential parabolic problem arising in Financial Mathematics. Under suitable conditions, we prove the existence of solutions for a multi-asset case in a general domain using the method of upper and lower solutions and a diagonal argument. We also model the jump in the related integro differential equation and give a solution procedure for that model assuming that the brownian motions are not correlated. For a bounded domain, this model for the jump gives an elegant expression of the solution in terms of hyper-spherical harmonics.

Keywords Integro-differential operator · Financial market · Levy model · Upper and lower solutions · Spherical harmonics

1 Introduction

In recent years there has been an increasing interest on problems arising in Financial Mathematics and in particular on option pricing. The standard approach to this problem leads to the study of equations of parabolic type.

In financial mathematics, usually the Black-Scholes model [7, 9, 15–17, 20] is used for pricing derivatives, by means of a reversed-time parabolic partial differential equation. In this model, an important quantity is the volatility, that is a measure of the fluctuation (risk) in the asset prices and corresponds to the diffusion coefficient in the Black-Scholes equation.

In the standard Black-Scholes model, a basic assumption is that the volatility is constant. Several models that have been proposed in recent years, however, allowed the volatility to be non constant or an stochastic variable. For instance, in [14] a model with stochastic volatility

I. SenGupta (✉) · M.C. Mariani
Department of Mathematical Sciences, University of Texas at El Paso, El Paso, TX, USA
e-mail: isengupta@utep.edu

P. Amster
Departamento de Matemática, Universidad de Buenos Aires, Buenos Aires, Argentina

is proposed. In this model the underlying security S follows, as in the Black-Scholes model, a stochastic process

$$dS_t = \mu S_t\, dt + \sigma_t S_t\, dZ_t,$$

where Z is a standard Brownian motion. Unlike the classical model, the variance $v(t) = \sigma^2(t)$ also follows a stochastic process given by

$$dv_t = \kappa\big(\theta - v(t)\big)\, dt + \gamma\sqrt{v_t}\, dW_t,$$

where W is another standard Brownian motion. The correlation coefficient between W and Z is denoted by ρ:

$$\mathbb{E}(dZ_t, dW_t) = \rho\, dt.$$

This leads to a generalized Black-Scholes equation:

$$\frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\gamma vS\frac{\partial^2 U}{\partial v\partial S} + \frac{1}{2}v\gamma^2\frac{\partial^2 U}{\partial v^2} + rS\frac{\partial U}{\partial S} + \big[\kappa(\theta - v) - \lambda v\big]\frac{\partial U}{\partial v} - rU + \frac{\partial U}{\partial t} = 0.$$

A similar model has been considered in [4], for which the stationary equation has been studied in [2].

More general models with stochastic volatility have been considered for example in [6], where the following problem is derived from the Feynman-Kac relation:

$$\begin{cases} u_t = \frac{1}{2}\operatorname{Tr}(M(x,\tau)D^2u) + q(x,\tau)\cdot Du, \\ u(x,0) = u_0(x) \end{cases}$$

for some diffusion matrix M and a payoff function u_0.

The Black-Scholes models with jumps arise in the fact that the Brownian Random Walk doesn't fit the financial data presenting large fluctuations; the necessity of taking account of large market movements, and a great amount of information arriving suddenly (i.e. a jump) has led to the study of partial integro-differential equations (PIDE) in which the integral term is modeling the jump.

In [3, 20] the following PIDE on the variables t and S is obtained:

$$\frac{1}{2}\sigma^2 S^2 F_{SS} + (r - \lambda p)SF_S - F_t - rF + \lambda\mathbb{E}\big\{F(SY,t) - F(S,t)\big\} = 0. \tag{1.1}$$

Here r denotes the riskless rate, λ the jump intensity, and $p = \mathbb{E}(P-1)$, where $\mathbb{E}$ is the expectation operator and the random variable $P-1$ measures the percentage change in the stock price if the jump-modeled by a Poisson process-occurs (for details see [3, 20]).

The following PIDE is a generalization of (1.1) for d assets with prices $S_1, \dots, S_d$:

$$\begin{aligned} &\sum_{i=1}^{d}\frac{1}{2}\sigma_i^2 S_i^2\frac{\partial^2 F}{\partial S_i^2} + \sum_{i\neq j}\frac{1}{2}\rho_{ij}\sigma_i\sigma_j S_i S_j\frac{\partial^2 F}{\partial S_i\partial S_j} + \sum_{i=1}^{d}(r-\lambda p_i)S_i\frac{\partial F}{\partial S_i} + \frac{\partial F}{\partial t} - rF \\ &\quad + \lambda\int\big[F(S_1Y_1,\dots,S_dY_d,t) - F(S_1,\dots,S_d,t)\big]g(Y_1,\dots,Y_d)\, dY_1\dots dY_d = 0 \end{aligned} \tag{1.2}$$

where

$$\rho_{ij}\, dt = \mathbb{E}\{dz_i, dz_j\}$$

are the correlation coefficients.

We recall that the case in which F is increasing and all jumps are negative corresponds to the evolution of a call option near a crash. In the last section of this paper we shall model the integral term in such a way that the problem admits an elegant solution.

When the volatility is stochastic we may consider the following processes

$$dS = S\sigma\, dZ + S\mu\, dt$$
$$d\sigma = \beta\sigma\, dW + \alpha\sigma\, dt$$

where Z and W are two standard Brownian motion with correlation coefficient ρ. If $F(S,\sigma,t)$ is the price of an option depending on the price of the asset S then, by Ito's lemma [16]:

$$dF(S,\sigma,t) = F_S\, dS + F_\sigma\, d\sigma + \mathcal{L}F\, dt$$

where $\mathcal{L}$ is given by

$$\mathcal{L} = \partial_t + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \frac{1}{2}\beta^2\sigma^2 \frac{\partial^2}{\partial\sigma^2} + \rho\sigma^2 S\beta \frac{\partial^2}{\partial S\partial\sigma}.$$

Under an appropriate choice of the portfolio the stochastic term of the equation vanishes (for details, see [4]).

In some of these models the high-frequency data has been described by a Lévy—like stochastic process, and again the partial integro-differential equations have been widely used for modeling the presence of jumps and critical stochastic processes.

Integro-differential models in exponential Lévy models, where the market price of an asset is represented as the exponential of a Lévy stochastic process have been also discussed by several authors (see for example [8, 12]). The integro-differential equation for an European option reads:

$$\frac{\partial C}{\partial \tau} + L^S C = rC \tag{1.3}$$

with final condition

$$C(T,S) = \tilde{H}(S),$$

where L^S is the integro-differential operator defined by:

$$L^S f(S) = rS\frac{\partial f}{\partial S}(S) + \frac{\sigma^2 S^2}{2}\frac{\partial^2 f}{\partial S^2}(S) + \int \left[f\big(Se^y\big) - f(S) - S\big(e^y - 1\big)\frac{\partial f}{\partial S}(S)\right]\nu(dy)$$

with $\nu(dy)$ a positive Radon measure.

If we introduce the change of variables given by $x = \ln(S)$, $t = T - \tau$, $u(x,t) = C(S,\tau)$ the following problem for $u(x,t)$ is obtained:

$$\frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial u}{\partial x} - ru - \frac{\partial u}{\partial t} + I(u) = Lu - \frac{\partial u}{\partial t} + I(u) = 0 \tag{1.4}$$

where $I(u)$ is the term modeling the jumps,

$$I(u)(x,t) = \int \left[u(x+y,t) - u(x,t) - \big(e^y - 1\big)\frac{\partial u}{\partial x}(x,t)\right]\nu(dy)$$

with the initial condition

$$u(x,0)=u_0(x) \quad x\in\mathbb{R}.$$

We recall that, in this case, the evolution of a call option near a crash is given by a spectrally negative Lévy process.

In a more general context, the previous discussion motivates to consider more general integro-differential parabolic problems. This work is devoted to the study of the solutions to the following general partial integro-differential equation in an unbounded smooth domain $\Omega\times(0,T)$ for some $\Omega\subset\mathbb{R}^d$:

$$\begin{cases} Lu-u_t=\mathcal{G}(\cdot,u) & \text{in } \Omega\times(0,T)\\ u(x,0)=u_0(x) & \text{on } \Omega\times\{0\}\\ u(x,t)=h(x,t) & \text{on } \partial\Omega\times(0,T).\end{cases} \tag{1.5}$$

Here, $L=L(x,t)$ is a second order elliptic operator in non-divergence form, namely

$$Lu:=\sum_{i,j=1}^{d}a^{ij}(x,t)u_{x_ix_j}+\sum_{i=1}^{d}b^i(x,t)u_{x_i}+c(x,t)u,$$

where the coefficients of L belong to the Hölder Space $C^{\delta,\delta/2}(\overline{\Omega}\times[0,T])$ and satisfy the following conditions

$$\Lambda|v|^2\geq\sum_{i,j=1}^{d}a^{ij}(x,t)v_iv_j\geq\lambda|v|^2 \quad (\Lambda\geq\lambda>0)$$

$$\left|b^i(x,t)\right|\leq C, \qquad c(x,t)\leq 0.$$

Moreover, $\mathcal{G}:[0,T]\times C(\overline{\Omega}\times[0,T])\to C(\overline{\Omega}\times[0,T])$ is a completely continuous operator as the ones defined in (1.1), (1.2) and (1.3), modeling the jump. Specifically, we shall assume that $\mathcal{G}$ is continuous, and sends bounded sets into pre-compact sets. Note that for $u\in C(\overline{\Omega}\times[0,T])$ fixed, $\mathcal{G}(\cdot,u)$ defines a continuous function given by $\mathcal{G}(\cdot,u)(x,t)=\mathcal{G}(t,u(x,t))$.

In this model, the case in which all jumps are positive and $\mathcal{G}$ is monotone nonincreasing, in the sense that

$$\text{If } u(x,t)\leq v(x,t) \quad \forall(x,t)\ \Rightarrow\ \mathcal{G}(\cdot,u)\geq\mathcal{G}(\cdot,v)\ \forall t, \tag{1.6}$$

corresponds to the evolution of a call option near a crash.

We shall assume that $u_0\in C^{2+\delta}(\overline{\Omega})$ and h is the restriction to $\partial\Omega\times[0,T]$ of some $C^{2+\delta,1+\delta/2}$ function defined on $\overline{\Omega}\times[0,T]$. Furthermore, we shall assume the following *consistency conditions*:

$$h(x,0)=u_0(x) \quad \forall x\in\partial\Omega \tag{1.7}$$

$$L(x,0)u_0(x)-h_t(x,0)=0 \quad \forall x\in\partial\Omega. \tag{1.8}$$

We shall prove the existence of solutions of (1.5), using the method of upper and lower solutions. We recall that a smooth function u is called an upper (lower) solution of problem

(1.5) if

$$\begin{cases} Lu - u_t \leq (\geq)\, \mathcal{G}(\cdot, u) & \text{in } \Omega \times (0, T) \\ u(x, 0) \geq (\leq)\, u_0(x) & \text{on } \Omega \times \{0\} \\ u(x, t) \geq (\leq)\, h(x, t) & \text{on } \partial\Omega \times (0, T). \end{cases}$$

Our main results read as follows:

Theorem 1.1 *Let L and $\mathcal{G}$ as before, such that (1.6), (1.7) and (1.8) hold, and assume there exist α and β a lower and an upper solution of the problem. Furthermore, assume that $\alpha \leq \beta$ in $\Omega \times (0, T)$. Then problem (1.5) admits a solution u such that $\alpha \leq u \leq \beta$ in $\Omega \times (0, T)$.*

Corollary 1.2 *Let L and $\mathcal{G}$ as before, such that (1.7) and (1.8) hold. Furthermore, assume that $\mathcal{G}(t, u) \geq 0$, $\mathcal{G}(t, 0) = 0$, and that the operator $\mathcal{G}_\lambda(t, u) := \mathcal{G}(t, u) - \lambda u$ satisfies (1.6) for some $\lambda > 0$.*

Let $\beta = k(T - t)^{-\frac{d}{2}} e^{\frac{\theta}{T-t}|x|^2}$. Then there exists $\theta_0 = \theta_0(T, d, \lambda, \Lambda, \|b\|_\infty)$ such that for any $\theta \leq \theta_0$ and u_0, h as before, satisfying $0 \leq u_0(x) \leq \beta(x, 0)$ and $0 \leq h(x, t) \leq \beta(x, t)$ for $x \in \partial\Omega$, $t \in [0, T]$ the problem admits a solution u with $0 \leq u \leq \beta$.

Remark 1.1 Although $\mathcal{G}$ does not depend on ∇u, case (1.3) is still contained in problem (1.5). Indeed, it suffices to observe that the last term of $I(u)$ in (1.4) can be added to the operator L. It is not clear, however, whether or not the previous results can be extended for a general operator $\mathcal{G} = \mathcal{G}(t, u, \nabla u)$. At first sight, it might be expected that existence of solutions can be obtained if $\mathcal{G}$ has some sort of sub-quadratic growth on ∇u, as it happens in the case of a differential equation

$$Lu - u_t = g(\cdot, u, \nabla u)$$

with $g : [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ continuous (see e.g. [5]). However, the standard argument uses local properties of second order differential parabolic operators, such as the maximum principle, and cannot be easily generalized for a functional operator, which is nonlocal.

2 The Method of Upper and Lower Solutions

In this section, we give a proof of Theorem 1.1 and Corollary 1.2. Firstly, we solve an analogous problem in a bounded domain; with this aim, we extend the boundary data to the interior of $\Omega \times (0, T)$ in a convenient way:

Lemma 2.1 *There exists a unique function $\varphi \in C^{2+\delta, 1+\delta/2}(\overline{\Omega} \times [0, T])$ such that*

$$\begin{cases} L\varphi - (\varphi)_t = 0, & \\ \varphi(x, 0) = u_0(x) & x \in \Omega \\ \varphi(x, t) = h(x, t) & (x, t) \in \partial\Omega \times [0, T]. \end{cases}$$

Moreover, if α and β are a lower and an upper solution of the problem with $\alpha \leq \beta$ in $\Omega \times (0, T)$, then

$$\alpha(x, t) \leq \varphi(x, t) \leq \beta(x, t)$$

for $(x, t) \in \overline{\Omega} \times [0, T]$.

Proof Existence and uniqueness follow immediately from [18, Thm. 10.4.1], and the compatibility condition (1.7). By the maximum principle, it is clear that if $\alpha \le \beta$ are a lower and an upper solution, then

$$\alpha(x,t) \le \varphi(x,t) \le \beta(x,t).$$

□

Remark 2.1 As we shall deal with a bounded domain $U \subset \Omega$, the operator $\mathcal{G}$ cannot be applied directly to elements of $[0,T] \times C(\overline{U} \times [0,T])$. However, by the well known Dugundji extension theorem [11, Thm. 5.1], there exists a linear operator $P_{\overline{U}} : C(\overline{U} \times [0,T]) \to C(\overline{\Omega} \times [0,T])$ satisfying

1. $P_{\overline{U}}(u)$ extends the function u to $\overline{\Omega} \times [0,T]$.
2. $\|P_{\overline{U}}(u)\|_\infty = \|u\|_\infty$ for every $u \in C(\overline{U} \times [0,T])$.
3. The range of $P_{\overline{U}}(u)$ is contained in the convex hull of the range of u.

In particular, the last property says that if $u \le v$ then, as the range of $v - u$ is contained in $\mathbb{R}_{\ge 0}$,

$$Range\big(P_{\overline{U}}(v) - P_{\overline{U}}(u)\big) = Range\big(P_{\overline{U}}(v-u)\big) \subset \mathbb{R}_{\ge 0}.$$

In other words, $P_{\overline{U}}$ is monotone nondecreasing, and consequently condition (1.6) implies:

$$u \le v \quad \Rightarrow \quad \mathcal{G}\big(\cdot, P_{\overline{U}}(u)\big) \ge \mathcal{G}\big(\cdot, P_{\overline{U}}(v)\big).$$

For notation convenience, we shall denote again $\mathcal{G}(\cdot, u) := \mathcal{G}(\cdot, P_{\overline{U}}(u))$.

Lemma 2.2 *Let $U \subset \Omega$ a bounded smooth domain, let $\tilde{T} < T$ and let φ be defined as in Lemma 2.1. Then the problem*

$$\begin{cases} Lu - u_t = \mathcal{G}(\cdot, u) & \text{in } U \times (0, \tilde{T}) \\ u(x,0) = u_0(x) & \text{in } U \times \{0\} \\ u(x,t) = \varphi(x,t) & \text{in } \partial U \times (0,\tilde{T}) \end{cases} \tag{2.1}$$

admits at least one solution u with $\alpha \le u(x,t) \le \beta$ for $x \in U$, $0 \le t \le \tilde{T}$.

Proof Set $u^0 = \alpha$ and $V = U \times (0,\tilde{T})$. By standard results, we may define $u^{n+1} \in W_p^{2,1}(V)$ as the unique solution of the problem

$$\begin{cases} Lu^{n+1} - u_t^{n+1} = \mathcal{G}(\cdot, u^n) & \text{in } U \times (0, \tilde{T}) \\ u^{n+1}(x,0) = u_0(x) & \text{in } U \times \{0\} \\ u^{n+1}(x,t) = \varphi_U(x,t) & \text{in } \partial U \times (0,\tilde{T}). \end{cases} \tag{2.2}$$

We claim that

$$\alpha \le u^n(x,t) \le u^{n+1}(x,t) \le \beta \quad \forall (x,t) \in \overline{U} \times [0,\tilde{T}],\ \forall n \in \mathbb{N}_0.$$

Indeed, by the maximum principle it follows that $u^1 \ge \alpha$; moreover,

$$Lu^1 - u_t^1 = \mathcal{G}(\cdot, \alpha) \ge \mathcal{G}(\cdot, \beta) \ge L\beta - \beta_t$$

and hence $u^1 \le \beta$. Inductively,

$$Lu^{n+1} - u_t^{n+1} = \mathcal{G}\big(\cdot, u^n\big) \le \mathcal{G}\big(\cdot, u^{n-1}\big) = Lu^n - u_t^n$$

and thus $u^{n+1} \geq u^n$. In the same way as before, it follows that $u^{n+1} \leq \beta$.

Next, define

$$u(x,t) = \lim_{n\to\infty} u^n(x,t).$$

By the standard L^p-estimates (see e.g. [19, Chap. 7]), the $W_p^{2,1}$-norm of $u^n - u^m$ can be controlled by its L^p-norm and the L^p-norm of its image by the operator $L - \partial_t$, namely:

$$\begin{aligned}
&\left\| D^2\left(u^n - u^m\right)\right\|_{L^p(V)} + \left\|\left(u^n - u^m\right)_t\right\|_{L^p(V)} \\
&\quad \leq c\left(\left\| L\left(u^n - u^m\right) - \left(u^n - u^m\right)_t\right\|_{L^p(V)} + \left\| u^n - u^m\right\|_{L^p(V)}\right).
\end{aligned}$$

By construction,

$$L\left(u^n - u^m\right) - \left(u^n - u^m\right)_t = \mathcal{G}\left(\cdot, u^{n-1}\right) - \mathcal{G}\left(\cdot, u^{m-1}\right).$$

As $\mathcal{G}$ is a completely continuous operator, using the fact that $\alpha \leq u^n \leq \beta$ and Lebesgue's dominated convergence theorem it follows that $\{u^n\}$ is a Cauchy sequence in $W_p^{2,1}(V)$. Hence $u^n \to u$ in the $W_p^{2,1}$-norm, and then u is a strong solution of the problem. □

2.1 Proof of Theorem 1.1

Let us approximate the domain Ω by an non-decreasing sequence $(\Omega_N)_{N\in\mathbb{N}}$ of bounded smooth sub-domains of Ω, which can be chosen in such a way that $\partial\Omega$ is also the union of the non-decreasing sequence $\partial\Omega_N \cap \partial\Omega$.

Then, define u^N as a solution of the problem

$$\begin{cases}
Lu - u_t = \mathcal{G}(\cdot, u) & \text{in } \Omega_N \times (0, T - \frac{1}{N}) \\
u(x,0) = u_0(x) & \text{in } \Omega_N \times \{0\} \\
u(x,t) = \varphi(x,t) & \text{in } \partial\Omega_N \times (0, T - \frac{1}{N})
\end{cases} \tag{2.3}$$

such that $0 \leq u^N \leq \beta$ in $\Omega_N \times (0, T - \frac{1}{N})$. Define $V_N = \Omega_N \times (0, T - \frac{1}{N})$ and choose $p > d$. For $M > N$, we have that

$$\begin{aligned}
&\left\| D^2\left(u^M\right)\right\|_{L^p(V_N)} + \left\|\left(u^M\right)_t\right\|_{L^p(V_N)} \\
&\quad \leq c\left(\left\| Lu^M - \left(u^M\right)_t\right\|_{L^p(V_N)} + \left\| u^M\right\|_{L^p(V_N)}\right) \\
&\quad \leq c\left(\left\|\mathcal{G}\left(\cdot, u^M\right)\right\|_{L^p(V_N)} + \|\beta\|_{L^p(V_N)}\right) \leq C
\end{aligned}$$

for some constant C depending only on N. By the well known Morrey imbedding $W_p^{2,1}(V_N) \hookrightarrow C(\overline{V}_N)$ (see e.g. [1]), there exists a subsequence that converges uniformly on $\overline{V}_N$.

Now, we apply the well known Cantor diagonal argument: for $N = 1$, we extract a subsequence of $u^M|_{\overline{\Omega}_1\times[0,T-1]}$ (still denoted $\{u^M\}$) that converges uniformly to some function u_1 over $\overline{\Omega}_1 \times [0, T-1]$. Next, we extract a subsequence of $u^M|_{\overline{\Omega}_2\times[0,T-\frac{1}{2}]}$ for $M \geq 2$ (still denoted $\{u^M\}$) that converges uniformly to some function u_2 over $\overline{\Omega}_2 \times [0, T - \frac{1}{2}]$, and so on. As the families $\{\Omega_N\}$ and $\{\partial\Omega_N \cap \partial\Omega\}$ are non-decreasing, it is clear that $u_N(x,0) = u_N(x)$ for $x \in \Omega_N$, and that $u_N(x,t) = h(x,t)$ for $x \in \partial\Omega \cap \partial\Omega_N$ and $t \in (0, T - \frac{1}{N})$. Moreover, as u_{N+1} is constructed as the limit of a subsequence of $u^M|_{\overline{\Omega}_{N+1}\times[0,T-\frac{1}{N+1}]}$, which converges uniformly to some function u_N over $\overline{\Omega}_N \times [0, T-\frac{1}{N}]$, it follows that $u_{N+1}|_{\overline{\Omega}_N\times[0,T-\frac{1}{N}]} = u_N$ for every N.

Thus, the diagonal subsequence (still denoted $\{u^M\}$) converges uniformly over compact subsets of $\Omega \times (0,T)$ to the function u defined as $u = u_N$ over $\overline{\Omega}_N \times [0, T-\frac{1}{N}]$. For $V = U \times (0,\tilde{T})$, $U \subset\subset \Omega$ and $\tilde{T} < T$, taking $M, N \geq N_V$ for some N_V large enough we have that

$$\begin{aligned}&\left\|D^2\left(u^N - u^M\right)\right\|_{L^p(V)} + \left\|\left(u^N - u^M\right)_t\right\|_{L^p(V)}\\ &\quad\leq c\left(\left\|L\left(u^N - u^M\right) - \left(u^N - u^M\right)_t\right\|_{L^p(V)} + \left\|u^N - u^M\right\|_{L^p(V)}\right).\end{aligned}$$

By construction,

$$L\left(u^N - u^M\right) - \left(u^N - u^M\right)_t = \mathcal{G}\left(\cdot, u^{N-1}\right) - \mathcal{G}\left(\cdot, u^{M-1}\right).$$

As before, using that $\mathcal{G}$ is continuous, and that $\alpha \leq u^N \leq \beta$, by dominated convergence it follows that $\{u^N\}_{N\geq N_V}$ is a Cauchy sequence in $W_p^{2,1}(V)$. Hence $u^N \to u$ over V for the $W_p^{2,1}$-norm, and then u is a strong solution in V. It follows that u satisfies the equation on $\Omega \times (0,T)$. Furthermore, it is clear that $u(x,0) = u_0(x)$. For $M > N$ we have that $u_M(x,t) = u_N(x,t) = h(x,t)$ for $x \in \partial\Omega \cap \partial\Omega_N$ and $t \in (0, T-\frac{1}{N})$. Thus, u satisfies the boundary condition $u(x,t) = h(x,t)$ on $\partial\Omega \times [0,T)$.

2.2 Proof of Corollary 1.2

A straightforward computation shows that β satisfies:

$$\begin{aligned}L\beta - \beta_t = \beta\Bigg\{&\left(\frac{2\theta}{T-t}\right)^2 \sum_{i,j=1}^{d} a^{ij}x_i x_j + \frac{2\theta}{T-t}\sum_{i=1}^{d} a^{ii}\\ &+ \frac{2\theta}{T-t}\sum_{i=1}^{d} b^i x_i + c - \left[\frac{d}{2(T-t)} + \frac{\theta}{(T-t)^2}|x|^2\right]\Bigg\}.\end{aligned}$$

Using the fact that $\sum_{i=1}^{d} a^{ii} \leq \Lambda$, and that $2\sum_{i=1}^{d} b^i x_i \leq \varepsilon|x|^2 + \frac{1}{\varepsilon}\|b\|_\infty^2$, we deduce that

$$\begin{aligned}\frac{1}{\beta}(L\beta - \beta_t) \leq{}& \left(4\theta\Lambda - 1 + \varepsilon(T-t)\right)\frac{\theta|x|^2}{(T-t)^2}\\ &+ \frac{1}{T-t}\left[2\theta\Lambda - \frac{d}{2} + \frac{1}{\varepsilon}\theta\|b\|_\infty^2 + c\right].\end{aligned}$$

We may take $\varepsilon < \frac{1}{T}$ in such a way that

$$\frac{1 - T\varepsilon}{4\Lambda} = \frac{d\varepsilon}{2\|b\|_\infty^2 + 4\Lambda} := \theta_0,$$

and it follows that

$$L\beta - \beta_t \leq 0 \leq \mathcal{G}(t,\beta)$$

for any $\theta \leq \theta_0$. (Observe that $0 \leq \mathcal{G}(t,\beta)$, follows from the hypothesis of this corollary.)

Thus, $\alpha \equiv 0$ and β are respectively a lower and an upper solution of the problem, and it suffices to repeat the proof of Theorem 1.1 for the equivalent problem $L_\lambda u - u_t = \mathcal{G}_\lambda(t,u)$, with $L_\lambda u := Lu - \lambda u$.

3 Another Approach to Integro-differential Equations in a Lévy Market with Bounded Stocks

In this section we model the *jump* given in (1.2) in a special way so that it admits an elegant solution when the domain is bounded (i.e. the stocks values cannot be arbitrarily large). We shall assume that the brownian motions have no correlation, i.e., $\rho_{ij} = 0$, for all $i \neq j$, so the system (1.2) becomes weakly coupled. As before, we assume d assets $S = (S_1, \dots, S_d)$ and we assume the *boundedness* of the stocks, given by $\sum_{i=1}^{d}(\ln \frac{S_i}{|E|})^2 \leq R'^2$, for some constant R'. Let us define this region by $\mathbb{U}$. Define $\alpha_i = -\frac{1}{2}(\frac{r-\lambda p_i}{\sigma^2/2} - 1)$, for $i = 1, \dots, d$ and $\omega = \sum_{i=1}^{d}\alpha_i - 1$. We consider the equation

$$\begin{aligned}\frac{\partial C}{\partial t} &+ \sum_{i=1}^{d}\frac{1}{2}\sigma_i^2 S_i^2\frac{\partial^2 C}{\partial S_i^2} + \sum_{i=1}^{d}(r-\lambda p_i)S_i\frac{\partial C}{\partial S_i} - rC \\ &+ \lambda|E|^{\omega}\int_{\mathbb{U}} G(S,P)C(P,t)\left(\prod_{i=1}^{d}P_i^{\alpha_i+1}\right)^{-1} dP = 0, \end{aligned} \tag{3.1}$$

for some random variable $P = (P_1, \dots, P_d) \in \mathbb{U}$, where λ is the jump intensity. We take $G(S, P) = g(\ln \frac{S_1}{P_1}, \dots, \ln \frac{S_d}{P_d})$, where g is a probability density function of its variables, $p_i = \mathbb{E}(P_i - 1)$, where $\mathbb{E}$ is the expectation operator and the random variable $P_i - 1$ measures the percentage change in the stock price for S_i if jump occurs. Further, we shall assume that volatility is the same for all the assets, that is $\sigma_i = \sigma$, $i = 1, \dots, d$. So (3.1) becomes

$$\begin{aligned}\frac{\partial C}{\partial t} &+ \frac{1}{2}\sigma^2\sum_{i=1}^{d}S_i^2\frac{\partial^2 C}{\partial S_i^2} + \sum_{i=1}^{d}(r-\lambda p_i)S_i\frac{\partial C}{\partial S_i} - rC \\ &+ \lambda|E|^{\omega}\int_{\mathbb{U}} G(S,P)C(P,t)\left(\prod_{i=1}^{d}P_i^{\alpha_i+1}\right)^{-1} dP = 0. \end{aligned} \tag{3.2}$$

We set

$$S_i = |E|e^{x_i}, \qquad P_i = |E|e^{y_i}, \quad t = T - \frac{\tau}{\sigma^2/2},$$

and

$$C(S_1, \dots, S_d, t) = |E|\exp\left(\sum_{i=1}^{d}\alpha_i\right)u(x_1, \dots, x_d, \tau).$$

Then we get

$$-\frac{\partial u}{\partial \tau} + \gamma u + \Delta u + \lambda\int_{\Omega} g(x-Y)u(Y)\,dY = 0 \tag{3.3}$$

where $\Omega := \mathbb{B}(R') = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d \,|\, \sum_{i=1}^{d}x_i^2 \leq R'^2\}$ and

$$\gamma = \left[\sum_{i=1}^{d}\left(\alpha_i^2 + (k_i - 1)\alpha_i\right) - k'\right], \quad k_i = \frac{r-\lambda p_i}{\sigma^2/2},\ k' = \frac{r}{\sigma^2/2}.$$

We choose a specific form of g, namely

$$g(X)=\frac{1}{N_{R'}}\frac{J_\nu(c|X|)}{(c|X|)^\nu}, \tag{3.4}$$

where J_ν is the Bessel function for order ν, with $\nu=\frac{d-2}{2}$ and $N_{R'}$ is a normalizing constant such that $\int_{\mathbb{B}(R')} g(X)\,dX=1$. To solve the problem (3.3) with g given by (3.4) we need the following two theorems. Proofs of them may be found in [21].

Theorem 3.1 *Suppose* $\mathbf{x}=(r,\eta)$ *and* $\mathbf{y}=(r',\xi)$ *are in* $\mathbb{R}^2$ *where* η *and* ξ *are angular parts of* $\mathbf{x}$ *and* $\mathbf{y}$ *respectively. Then*

$$\int_{S^1} J_0\big(c|\mathbf{x}-\mathbf{y}|\big)e^{ik\xi}\,d\xi=2\pi J_k(cr)J_k\big(cr'\big)e^{ik\eta}.$$

Theorem 3.2 *Suppose* $\mathbf{x}=(r,\eta)$ *and* $\mathbf{y}=(r',\xi)$ *are in* $\mathbb{R}^d$ *where* η *and* ξ *are angular parts of* $\mathbf{x}$ *and* $\mathbf{y}$ *respectively and* $\nu=\frac{d-2}{2}$. *Then*

$$\int_{S^{d-1}}\frac{J_\nu(c|\mathbf{x}-\mathbf{y}|)}{(c|\mathbf{x}-\mathbf{y}|)^\nu}S_k^s(\xi)\,d\xi=\frac{2^{3\nu+1}}{\pi^{\nu-1}}\Delta_n(\nu,cr)\Delta_d\big(\nu,cr'\big)S_k^s(\eta),$$

where $\Delta_m(\nu,r)=(\frac{\pi}{2r})^\nu J_{\nu+m}(r)$.

We consider here the case $d\geq 3$. The case $d=2$ will be similar and simpler. We denote $\mathbf{H}_l$ as the space of degree l spherical harmonics on the d-sphere.

Theorem 3.3 *If* g *is given by* (3.4), *then there exists a solution of* (3.3) *of the form*

$$u(x,\tau)=\sum_{N=0}^{\infty}\sum_{l=1}^{h(N,p)}T_{Nl}(\tau)R_{Nl}(r)S_N^l(\eta), \tag{3.5}$$

where $x=(x_1,\dots,x_d)=(r,\eta)$, *and*

$$T'_{Nl}(\tau)=\Lambda T_{Nl}(\tau),$$

and

$$\gamma R_{Nl}(r)+r^{1-d}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial R_{Nl}(r)}{\partial r}\right)$$
$$-R_{Nl}(r)\frac{N(N+d-2)}{r^2}+\zeta\Delta_N(\nu,cr)I=\Lambda R_{Nl}(r),$$

where Λ *is a constant,* $\zeta=\frac{2^{3\nu+1}\lambda}{\pi^{\nu-1}N_{R'}}$ *and*

$$I=\int_0^{R'}\Delta_N\big(\nu,cr'\big)R_{Nl}\big(r'\big)r'^{p+1}\,dr'.$$

Initial values for $T_{Nl}(\tau)$ *and the boundary values of* $R(r)$ *are obtained from the initial-boundary conditions of the original given problem.*

Proof With the substitution of (3.5) integral term of (3.3) becomes (with $Y=(Y_1,\dots,Y_d)=(r',\xi)$) with the use of Theorem 3.2

$$\begin{aligned}
&\lambda\int_{\mathbb{B}(R')} g(x-Y)u(Y)\,dY\\
&\quad=\frac{\lambda}{N_{R'}}\int_{\mathbb{B}(R')}\frac{J_\nu(c|x-Y|)}{(c|x-Y|)^\nu}u(Y)\,dY\\
&\quad=\frac{\lambda}{N_{R'}}\sum_{N=0}^{\infty}\sum_{l=1}^{h(N,p)}\int_0^{R'} r'^{p+1}\,dr'\int_{S^{d-1}}\frac{J_\nu(c|x-Y|)}{(c|x-Y|)^\nu}T_{Nl}(\tau)R_{Nl}(r')S_N^l(\xi)\,d\xi\\
&\quad=\frac{\lambda}{N_{R'}}\sum_{N=0}^{\infty}\sum_{l=1}^{h(N,p)}\int_0^{R'} r'^{p+1}dr'\frac{2^{3\nu+1}}{\pi^{\nu-1}}\Delta_N(\nu,cr)\Delta_N(\nu,cr')T_{Nl}(\tau)R_{Nl}(r')S_N^l(\eta)\\
&\quad=\sum_{N=0}^{\infty}\sum_{l=1}^{h(N,p)}\frac{2^{3\nu+1}\lambda}{\pi^{\nu-1}N_{R'}}\Delta_N(\nu,cr)T_{Nl}(\tau)\left(\int_0^{R'}\Delta_N(\nu,cr')R_{Nl}(r')r'^{p+1}\,dr'\right)S_N^l(\eta).
\end{aligned}$$

Therefore (3.3) becomes

$$\begin{aligned}
&-\sum_{N=0}^{\infty}\sum_{l=1}^{h(N,p)}T'_{Nl}(\tau)R_{Nl}(r)S_N^l(\eta)+\gamma\sum_{N=0}^{\infty}\sum_{l=1}^{h(N,p)}T_{Nl}(\tau)R_{Nl}(r)S_N^l(\eta)\\
&\quad+\sum_{N=0}^{\infty}\sum_{l=1}^{h(N,p)}T_{Nl}(\tau)r^{1-d}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial R_{Nl}(r)}{\partial r}\right)S_N^l(\eta)\\
&\quad-\sum_{N=0}^{\infty}\sum_{l=1}^{h(N,p)}T_{Nl}(\tau)R_{Nl}(r)\frac{N(N+d-2)}{r^2}S_N^l(\eta)\\
&\quad+\sum_{N=0}^{\infty}\sum_{l=1}^{h(N,p)}\frac{2^{3\nu+1}\lambda}{\pi^{\nu-1}N_{R'}}\Delta_N(\nu,cr)T_{Nl}(\tau)\left(\int_0^{R'}\Delta_N(\nu,cr')R_{Nl}(r')r'^{p+1}dr'\right)S_N^l(\eta)=0.
\end{aligned}$$

Since $S_N^l(\eta)$ are linearly independent comparing the coefficients we have the following equations for $N=0,1,\dots$ and $l=1,\dots,h(N,p)$.

$$\begin{aligned}
T'_{Nl}(\tau)R_{Nl}(r)&=\gamma T_{Nl}(\tau)R_{Nl}(r)+T_{Nl}(\tau)r^{1-d}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial R_{Nl}(r)}{\partial r}\right)\\
&\quad-T_{Nl}(\tau)R_{Nl}(r)\frac{N(N+d-2)}{r^2}\\
&\quad+\frac{2^{3\nu+1}\lambda}{\pi^{\nu-1}N_{R'}}\Delta_N(\nu,cr)T_{Nl}(\tau)\left(\int_0^{R'}\Delta_N(\nu,cr')R_{Nl}(r')r'^{p+1}\,dr'\right).
\end{aligned}$$

Therefore we have the following equations

$$T'_{Nl}(\tau)=\Lambda T_{Nl}(\tau), \tag{3.6}$$

and

$$\gamma R_{Nl}(r) + r^{1-d}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial R_{Nl}(r)}{\partial r}\right)$$
$$- R_{Nl}(r)\frac{N(N+d-2)}{r^2} + \zeta \Delta_N(\nu, cr) I = \Lambda R_{Nl}(r), \tag{3.7}$$

where Λ is a constant, $\zeta = \frac{2^{3\nu+1}\lambda}{\pi^{\nu-1}N_{R'}}$ and

$$I = \int_0^{R'} \Delta_N\big(\nu, cr'\big) R_{Nl}\big(r'\big) r'^{p+1}\, dr'.$$

Initial values for $T_{Nl}(\tau)$ and the boundary values of $R(r)$ are obtained from the given problem. □

Solution of (3.6) is given by

$$T_{Nl}(\tau) = T_{Nl}(0)e^{\Lambda\tau}.$$

Solution of (3.7) can be obtained by standard techniques such as *homotopy perturbation method* (see [10, 13]). Here we give an outline of that. Observe that (3.7) can be rewritten as

$$\frac{\partial^2 R_{Nl}(r)}{\partial r^2} + \frac{d-1}{r}\frac{\partial R_{Nl}(r)}{\partial r} + \left(\gamma - \frac{N(N+d-2)}{r^2} - \Lambda\right) R_{Nl}(r)$$
$$+ \zeta \Delta_N(\nu, cr) \int_0^{R'} \Delta_N\big(\nu, cr'\big) R_{Nl}\big(r'\big) r'^{p+1} dr' = 0. \tag{3.8}$$

By homotopy perturbation technique, we construct a homotopy

$$H(v, p) = \frac{\partial^2 v(r)}{\partial r^2} - \frac{\partial^2 y_0(r)}{\partial r^2} + p\frac{\partial^2 y_0(r)}{\partial r^2} - p\left[\left(\frac{N(N+d-2)}{r^2} + \Lambda - \gamma\right) v(r)\right.$$
$$\left. - \frac{d-1}{r}\frac{\partial v(r)}{\partial r} - \zeta \Delta_N(\nu, cr) \int_0^{R'} \Delta_N\big(\nu, cr'\big) v\big(r'\big) r'^{p+1}\, dr'\right] = 0, \tag{3.9}$$

where $y_0(r)$ is the initial approximation. According to homotopy perturbation theory, we can first use the embedding parameter p as a *small parameter*, and assume that the solution of (3.9) can be written as a power series in p. That is

$$v(r) = v_0(r) + pv_1(r) + p^2 v_2(r) + \cdots. \tag{3.10}$$

Setting $p = 1$, we can get the solution for (3.8) as

$$R_{Nl}(r) = v_0(r) + v_1(r) + v_2(r) + \cdots. \tag{3.11}$$

Substituting (3.10) in (3.9) and equating the coefficients of like powers of p, we obtain

$$p^0: \quad \frac{\partial^2 v_0(r)}{\partial r^2} - \frac{\partial^2 y_0(r)}{\partial r^2} = 0, \tag{3.12}$$

$$p^1: \quad \frac{\partial^2 v_1(r)}{\partial r^2} + \frac{\partial^2 y_0(r)}{\partial r^2} - \left[\left(\frac{N(N+d-2)}{r^2} + \Lambda - \gamma\right) v_0(r) - \frac{d-1}{r}\frac{\partial v_0(r)}{\partial r}\right.$$
$$\left. - \zeta \Delta_N(\nu, cr) \int_0^{R'} \Delta_N\left(\nu, cr'\right) v_0\left(r'\right) r'^{p+1} dr'\right] = 0, \tag{3.13}$$

$$p^k: \quad \frac{\partial^2 v_k(r)}{\partial r^2} - \left[\left(\frac{N(N+d-2)}{r^2} + \Lambda - \gamma\right) v_{k-1}(r) - \frac{d-1}{r}\frac{\partial v_{k-1}(r)}{\partial r}\right.$$
$$\left. - \zeta \Delta_N(\nu, cr) \int_0^{R'} \Delta_N\left(\nu, cr'\right) v_{k-1}\left(r'\right) r'^{p+1} dr'\right] = 0, \quad k \geq 2. \tag{3.14}$$

Then starting with an initial approximation $y_0(r)$ and solving successively the above equations we can find $v_k(r)$ for $k = 0, 1, 2, \ldots$. Therefore we can get the k-th approximation of the exact solution (3.11) as $R_{Nl}^k(r) = v_0(r) + v_1(r) + \cdots + v_{k-1}(r)$. Observe that according to homotopy perturbation theory $\lim_{k\to\infty} R_{Nl}^k(r) = R_{Nl}(r)$.

References

1. Adams, R.: Sobolev Spaces. Academic Press, San Diego (1975)
2. Amster, P., Averbuj, C., Mariani, M.C.: Solutions to a stationary nonlinear Black-Scholes type equation. J. Math. Anal. Appl. **276**, 231–238 (2002)
3. Andersen, L., Andreasen, J.: Jumps diffusion models: volatility smile fitting and numerical methods for pricing. Rev. Deriv. Res. **4**, 231–262 (2000)
4. Avellaneda, M., Zhu, Y.: Risk neutral stochastic volatility model. Int. J. Theor. Appl. Finance **1**(2), 289–310 (1998)
5. Bebernes, J.W., Schmitt, K.: On the existence of maximal and minimal solutions for parabolic partial differential equations. Proc. Am. Math. Soc. **73**(2), 211–218 (1979)
6. Berestycki, H., Busca, J., Florent, I.: Computing the implied volatility in stochastic volatility models. Commun. Pure Appl. Math. (2004). doi:10.1002/cpa.20039
7. Black, F., Scholes, M.: The pricing of options and corporate liabilities. J. Polit. Econ. **81**, 637–659 (1973)
8. Cont, R., Tankov, P.: Financial Modeling with Jumps Processes. Chapman & Hall/CRC Press, London/Boca Raton (2003)
9. Duffie, D.: Dynamic Asset Pricing Theory. Princeton University Press, Princeton (1996)
10. Dehghan, M., Shakeri, F.: Solution of an integro-differential equation arising in oscillating magnetic fields using He's homotopy perturbation method. Prog. Electromagn. Res. **78**, 361–376 (2008)
11. Dugundji, J.: An extension of Tietze's theorem. Pac. J. Math. **1**, 353–367 (1951)
12. Geman, H.: Pure jump Lévy processes for asset price modeling. J. Bank. Finance **26**, 1297–1316 (2002)
13. He, J.H.: Homotopy perturbation technique. Comput. Methods Appl. Mech. Eng. **178**, 257–262 (1999)
14. Heston, S.: A closed-form solution for options with stochastic volatility with applications to bond and currency options. Rev. Financ. Stud. **6**, 327 (1993)
15. Hull, J.C.: Options, Futures, and Other Derivatives. Prentice Hall, New York (1997)
16. Ikeda, N., Watanabe, S.: Stochastic Differential Equations and Diffusion Processes. North-Holland, Amsterdam (1989)
17. Jarrow, R.: Modelling Fixed Income Securities and Interest Rate Options. McGraw-Hill, New York (1997)
18. Krylov, N.: Lectures on Elliptic and Parabolic Equations in Hölder Spaces. Graduate Studies in Math., vol. 12. AMS, Providence (1996)
19. Lieberman, G.: Second Order Parabolic Differential Equations. World Scientific, Singapore (1996)
20. Merton, R.: Continuous-Time Finance. Blackwell, Cambridge (2000)
21. SenGupta, I.: Differential operator related to the generalized superradiance integral equation. J. Math. Anal. Appl. **369**, 101–111 (2010)