

## Mixed discriminants

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**Abstract** The mixed discriminant of  $n$  Laurent polynomials in  $n$  variables is the irreducible polynomial in the coefficients which vanishes whenever two of the roots coincide. The Cayley trick expresses the mixed discriminant as an  $A$ -discriminant. We show that the degree of the mixed discriminant is a piecewise linear function in the Plücker coordinates of a mixed Grassmannian. An explicit degree formula is given for the case of plane curves.

**Keywords**  $A$ -discriminant · Degree · Multiple root · Cayley polytope · Tropical discriminant · Matroid strata · Mixed Grassmannian

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Dedicated to the memory of our friend Mikael Passare (1959–2011).

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## 1 Introduction

A fundamental topic in mathematics and its applications is the study of systems of  $n$  polynomial equations in  $n$  unknowns  $x = (x_1, x_2, \dots, x_n)$  over an algebraically closed field  $K$ :

$$f_1(x) = f_2(x) = \dots = f_n(x) = 0. \quad (1.1)$$

Here we consider Laurent polynomials with fixed support sets  $A_1, A_2, \dots, A_n \subset \mathbb{Z}^n$ :

$$f_i(x) = \sum_{a \in A_i} c_{i,a} x^a \quad (i = 1, 2, \dots, n). \quad (1.2)$$

If the coefficients  $c_{i,a}$  are generic then, according to *Bernstein's Theorem* [3], the number of solutions of (1.1) in the algebraic torus  $(K^*)^n$  equals the *mixed volume*  $MV(Q_1, Q_2, \dots, Q_n)$  of the Newton polytopes  $Q_i = \text{conv}(A_i)$  in  $\mathbb{R}^n$ . However, for special choices of the coefficients  $c_{i,a}$ , two or more of these solutions come together in  $(K^*)^n$  and create a point of higher multiplicity. The conditions under which this happens are encoded in an irreducible polynomial in the coefficients, whose zero locus is the *variety of ill-posed systems* [18, §I-4]. While finding this polynomial is usually beyond the reach of symbolic computation, it is often possible to describe some of its invariants. Our aim here is to characterize its degree.

An isolated solution  $u \in (K^*)^n$  of (1.1) is a *non-degenerate multiple root* if the  $n$  gradient vectors  $\nabla_x f_i(u)$  are linearly dependent, but any  $n - 1$  of them are linearly independent. This condition means that  $u$  is a regular point on the curve defined by any  $n - 1$  of the equations in (1.1). We define the *discriminantal variety* as the closure of the locus of coefficients  $c_{i,a}$  for which the associated system (1.1) has a non-degenerate multiple root. If the discriminantal variety is a hypersurface, we define the *mixed discriminant* of the system (1.1) to be the unique (up to sign) irreducible polynomial  $\Delta_{A_1, \dots, A_n}$  with integer coefficients in the unknowns  $c_{i,a}$  which defines it. Otherwise we say that the system is *defective* and set  $\Delta_{A_1, \dots, A_n} = 1$ .

In the non-defective case, we may identify  $\Delta_{A_1, \dots, A_n}$  with an *A-discriminant* in the sense of Gel'fand, Kapranov and Zelevinsky [12].  $A$  is the *Cayley matrix* (2.1) of  $A_1, \dots, A_n$ . This matrix has as columns the vectors in the lifted configurations  $e_i \times A_i \in \mathbb{Z}^{2n}$  for  $i = 1, \dots, n$ . The relationship between  $\Delta_{A_1, \dots, A_n}$  and the *A-discriminant* will be made precise in Sect. 2. In Sect. 3 we focus on the case  $n = 2$ . Here, the mixed discriminant  $\Delta_{A_1, A_2}$  expresses the condition for two plane curves  $\{f_1 = 0\}$  and  $\{f_2 = 0\}$  to be tangent at a common smooth point. In Theorem 3.3 we present a general formula for the bidegree  $(\delta_1, \delta_2)$  of  $\Delta_{A_1, A_2}$ . In nice special cases, to be described in Corollary 3.15, that formula simplifies to

$$\begin{aligned} \delta_1 &= \text{area}(Q_1 + Q_2) - \text{area}(Q_1) - \text{perim}(Q_2), \\ \delta_2 &= \text{area}(Q_1 + Q_2) - \text{area}(Q_2) - \text{perim}(Q_1), \end{aligned} \quad (1.3)$$

where  $Q_i$  is the convex hull of  $A_i$  and  $Q_1 + Q_2$  is their Minkowski sum. The *area* is normalized so that a primitive triangle has area 1. The *perimeter* of  $Q_i$  is the cardinality of  $\partial Q_i \cap \mathbb{Z}^2$ .

The formula (1.3) applies in the classical case, where  $f_1$  and  $f_2$  are dense polynomials of degree  $d_1$  and  $d_2$ . Here,  $\Delta_{A_1, A_2}$  is the classical *tact invariant* [17, §96] whose bidegree equals

$$(\delta_1, \delta_2) = (d_2^2 + 2d_1d_2 - 3d_2, d_1^2 + 2d_1d_2 - 3d_1). \quad (1.4)$$

See Benoist [2] and Nie [15] for the analogous formula for  $n$  dense polynomials in  $n$  variables. The right-hand side of (1.3) is always an upper bound for the bidegree of the mixed discriminant, but in general the inequality is strict. Indeed, consider two sparse polynomials

$$f_1(x_1, x_2) = c_{10} + c_{11}x_1^{d_1} + c_{12}x_2^{d_1} \quad \text{and} \quad f_2(x_1, x_2) = c_{20} + c_{21}x_1^{d_2} + c_{22}x_2^{d_2}, \quad (1.5)$$

with  $d_1$  and  $d_2$  positive coprime integers, then the bidegree drops from (1.4) to

$$(d_2^2 + 2d_1d_2 - 3d_2 \cdot \min\{d_1, d_2\}, d_1^2 + 2d_1d_2 - 3d_1 \cdot \min\{d_1, d_2\}). \quad (1.6)$$

In Sect. 4 we prove that the degree of the mixed discriminant, in the natural  $\mathbb{Z}^n$ -grading, is a piecewise polynomial function in the coordinates of the points in  $A_1, A_2, \dots, A_n$ .

**Theorem 1.1** *The degree of the mixed discriminant cycle is a piecewise linear function in the Plücker coordinates on the mixed Grassmannian  $G(2n, \mathcal{I})$ . It is linear on the tropical matroid strata of  $G(2n, \mathcal{I})$  determined by the configurations  $A_1, \dots, A_n$ . The formula on each maximal stratum is unique modulo the linear forms on  $\wedge^{2n} \mathbb{R}^m$  that vanish on  $G(2n, \mathcal{I})$ .*

Here, the *cycle* refers to the mixed discriminant raised to a power that expresses the index in  $\mathbb{Z}^n$  of the sublattice affinely spanned by  $A_1 \cup \dots \cup A_n$ . The mixed Grassmannian  $G(2n, \mathcal{I})$  parameterizes all  $2n$ -dimensional subspaces of  $\mathbb{R}^m$  that arise as row spaces of Cayley matrices (2.1) with  $m = \sum_{i=1}^n |A_i|$  columns, and  $\mathcal{I}$  is the partition of  $\{1, \dots, m\}$  specified by the  $n$  configurations. This Grassmannian is regarded as a subvariety in the exterior power  $\wedge^{2n} \mathbb{R}^m$ , via the Plücker embedding by the maximal minors of the matrix (2.1). See Definition 4.2 for details. The mixed Grassmannian admits a combinatorial stratification into *tropical matroid strata*, and our assertion says that the degree of the mixed discriminant cycle is a polynomial on these strata. The proof of Theorem 1.1 is based on *tropical algebraic geometry*, and specifically on the combinatorial construction of the tropical discriminant in [7].

## 2 Cayley configurations

Let  $A_1, \dots, A_n$  be configurations in  $\mathbb{Z}^n$ , defining Laurent polynomials as in (1.2). We shall relate the mixed discriminant  $\Delta_{A_1, \dots, A_n}$  to the  $A$ -discriminant, where  $A$  is the Cayley matrix

$$A = \text{Cay}(A_1, \dots, A_n) = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{1} \\ A_1 & A_2 & \dots & A_n \end{pmatrix} \quad (2.1)$$

This matrix has  $2n$  rows and  $m = \sum_{i=1}^n |A_i|$  columns, so  $\mathbf{0} = (0, \dots, 0)$  and  $\mathbf{1} = (1, \dots, 1)$  denote row vectors of appropriate lengths. The columns of the “lower” matrix  $[A_1 \ A_2 \ \dots \ A_n]$  consist of the vectors in each  $A_i$  in any fixed arbitrary order.

We introduce  $n$  new variables  $y_1, y_2, \dots, y_n$  and encode our system (1.1) by one auxiliary polynomial with support in  $A$ , via the *Cayley trick*:

$$\phi(x, y) = y_1 f_1(x) + y_2 f_2(x) + \dots + y_n f_n(x).$$

We denote by  $\Delta_A$  the  $A$ -discriminant as defined in [12]. That is,  $\Delta_A$  is the unique (up to sign) irreducible polynomial with integer coefficients in the unknowns  $c_{i,a}$  which vanishes whenever the hypersurface  $\{(x, y) \in (K^*)^{2n} : \phi(x, y) = 0\}$  is not smooth. Equivalently,

$\Delta_A$  is the defining equation of the dual variety  $(X_A)^\vee$  when this variety is a hypersurface. Here,  $X_A$  denotes the projective toric variety in  $\mathbb{P}^{m-1}$  associated with the Cayley matrix  $A$ . If  $(X_A)^\vee$  is not a hypersurface, then no such unique polynomial exists. We then set  $\Delta_A = 1$  and refer to  $A$  as a *defective configuration*. It is useful to keep track of the lattice index

$$i(A) = i(A, \mathbb{Z}^{2n}) = [\mathbb{Z}^{2n} : \mathbb{Z} \cdot A],$$

where  $\mathbb{Z} \cdot A$  is the  $\mathbb{Z}$ -linear span of the columns of  $A$ . The *discriminant cycle* is the polynomial

$$\tilde{\Delta}_A = \Delta_A^{i(A)}.$$

The same construction makes sense for the mixed discriminant and it results in the *mixed discriminant cycle*  $\tilde{\Delta}_{A_1, \dots, A_n}$ . The exponents  $i(A)$  will be compatible in the following theorem.

**Theorem 2.1** *The mixed discriminant equals the  $A$ -discriminant of the Cayley matrix:*

$$\tilde{\Delta}_{A_1, \dots, A_n} = \Delta_A.$$

This result is more subtle than it may seem at first glance. It implies that  $(A_1, \dots, A_n)$  is defective if and only if  $A$  is defective. The two discriminantal varieties can differ in that case.

**Example 2.2** Let  $n = 2$  and consider the Cayley matrix

$$A = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \\ A_1 & A_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

The corresponding system (1.1) consists of two univariate quadrics in different variables:

$$f_1(x_1) = c_{10} + c_{11}x_1 + c_{12}x_1^2 = 0 \quad \text{and} \quad f_2(x_2) = c_{20} + c_{21}x_2 + c_{22}x_2^2 = 0.$$

This system cannot have a non-degenerate multiple root, for any choice of coefficients  $c_{ij}$ , so the  $(A_1, A_2)$ -discriminantal variety is empty. On the other hand, the  $A$ -discriminantal variety is non-empty. It has codimension two and is defined by  $c_{11}^2 - 4c_{10}c_{12} = c_{21}^2 - 4c_{20}c_{22} = 0$ .

**Proof of Theorem 2.1** We may assume  $i(A) = 1$ . Let  $u \in (K^*)^n$  be a non-degenerate multiple root of  $f_1(x) = \dots = f_n(x) = 0$ . Our definition ensures the existence of a unique (up to scaling) vector  $v \in (K^*)^n$  such that  $\sum_{i=1}^n v_i \nabla_x f_i(u)$  is the zero vector. The pair  $(u, v) \in (K^*)^{2n}$  is a singular point of the hypersurface defined by  $\phi(x, y) = 0$ . By projecting into the space of coefficients  $c_{i,a}$ , we see that the  $(A_1, \dots, A_n)$ -discriminantal variety is contained in the  $A$ -discriminantal variety. Example 2.2 shows that this containment can be strict.

We now claim that  $\Delta_A \neq 1$  implies  $\Delta_{A_1, \dots, A_n} \neq 1$ . This will establish the proposition because  $\Delta_{A_1, \dots, A_n}$  is a factor of  $\Delta_A$ , and  $\Delta_A$  is irreducible, so the two discriminants are equal. Each point  $(u, v) \in (K^*)^{2n}$  defines a point on  $X_A$ . If  $\Delta_A \neq 1$ , the dual variety  $(X_A)^\vee$  is a hypersurface in the dual projective space  $(\mathbb{P}^{m-1})^\vee$ . Moreover, see e.g. [13], a generic hyperplane in the dual variety is tangent to the toric variety  $X_A$  at a single point.

Consider a generic point on the conormal variety of  $X_A$  in  $\mathbb{P}^{2n-1} \times (\mathbb{P}^{m-1})^\vee$ . It is represented by a pair  $((u, v), c)$ , where  $(u, v) \in (K^*)^{2n}$  and  $c$  is the coefficient vector of a polynomial  $\phi(x, y)$  such that  $(u, v)$  is the unique singular point on  $\{\phi(x, y) = 0\}$ . The coefficient vector  $c$  defines a point on the  $(A_1, \dots, A_n)$ -discriminantal variety unless we can

relabel the matrices  $A_1, \dots, A_n$  in such a way that the gradients of  $f_1, \dots, f_{n-1}$  are linearly dependent at  $u$ . Assuming that this holds, we let

$$\sum_{i=1}^{n-1} t_i \nabla_x f_i(u) = 0$$

be the dependency relation and set  $t = (t_1, \dots, t_{n-1}, 0) \neq \mathbf{0}$ . The point  $((t + u, v), c)$  lies on the conormal variety of  $X_A$ . This implies that the generic hyperplane defined by  $c$  is tangent to  $X_A$  at two distinct points  $(u, v) \neq (t + u, v)$ , which cannot happen. It follows that  $\Delta_{A_1, \dots, A_n} \neq 1$ , as we wanted to show. This concludes our proof.  $\square$

**Example 2.3** Let  $n = 2$  and  $A_1 = A_2 = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$ , a unit square. Then

$$\begin{aligned} f_1(x_1, x_2) &= a_{00} + a_{10}x_1 + a_{01}x_2 + a_{11}x_1x_2, \\ f_2(x_1, x_2) &= b_{00} + b_{10}x_1 + b_{01}x_2 + b_{11}x_1x_2. \end{aligned}$$

The Cayley configuration  $A$  is the standard 3-dimensional cube. The  $A$ -discriminant is known to be the *hyperdeterminant* of format  $2 \times 2 \times 2$ , by [12, Chapter 14], which equals

$$\begin{aligned} \Delta_{A_1, A_2} &= a_{00}^2 b_{11}^2 - 2a_{00}a_{01}b_{10}b_{11} - 2a_{00}a_{10}b_{01}b_{11} - 2a_{00}a_{11}b_{00}b_{11} \\ &\quad + 4a_{00}a_{11}b_{01}b_{10} + a_{01}^2 b_{10}^2 + 4a_{01}a_{10}b_{00}b_{11} - 2a_{01}a_{10}b_{01}b_{10} \\ &\quad - 2a_{01}a_{11}b_{00}b_{10} + a_{10}^2 b_{01}^2 - 2a_{10}a_{11}b_{00}b_{01} + a_{11}^2 b_{00}^2. \end{aligned}$$

Theorem 2.1 tells us that this hyperdeterminant coincides with the mixed discriminant of  $f_1$  and  $f_2$ . Note that the bidegree equals  $(\delta_1, \delta_2) = (2, 2)$ , and therefore (1.3) holds.

We now shift gears and focus on defective configurations. We know from Theorem 2.1 that  $(A_1, \dots, A_n)$  is defective if and only if the associated Cayley configuration  $A$  is defective. While there has been some recent progress on characterizing defectiveness [7, 10, 14], the problem of classifying defective configurations  $A$  remains open, except in cases when the *codimension* of  $A$  is at most four [6, 8] or when the toric variety  $X_A$  is smooth or  $\mathbb{Q}$ -factorial [4, 9]. Recall that  $X_A$  is *smooth* if and only if, at each every vertex of the polytope  $Q = \text{conv}(A)$ , the first elements of  $A$  that lie on the incident edge directions form a basis for the lattice spanned by  $A$ . The variety  $X_A$  is  $\mathbb{Q}$ -factorial when  $Q$  is a *simple* polytope, that is, when every vertex of  $Q$  lies in exactly  $\dim(Q)$  facets. Note that smooth implies  $\mathbb{Q}$ -factorial.

We set  $\dim(A) = \dim(Q)$ , and we say that  $A$  is *dense* if  $A = Q \cap \mathbb{Z}^d$ . A subset  $F \subset A$  is called a *face* of  $A$ , denoted  $F \prec A$ , if  $F$  is the intersection of  $A$  with a face of the polytope  $Q$ . We denote by  $s_n$  the standard  $n$ -simplex and by  $\sigma_n$  the configuration of its *vertices*.

When  $A$  is the Cayley configuration of  $A_1, \dots, A_n \subset \mathbb{Z}^n$ , the codimension of  $A$  is  $m - 2n$ . This number is usually rather large. For instance, if all  $n$  polytopes  $Q_i = \text{conv}(A_i)$  are full-dimensional in  $\mathbb{R}^n$  then  $\text{codim}(A) \geq n \cdot (n - 1)$ , and thus, for  $n \geq 3$ , we are outside the range where defective configurations have been classified. However, if  $n = 2$  and the configurations  $A_1$  and  $A_2$  are full-dimensional we can classify all defective configurations.

**Proposition 2.4** *Let  $A_1, A_2 \subset \mathbb{Z}^2$  be full-dimensional configurations. Then,  $(A_1, A_2)$  is defective if and only if, up to affine isomorphism,  $A_1$  and  $A_2$  are both translates of  $p \cdot \sigma_2$ , for some positive integer  $p$ .*

*Proof* Let  $A = \text{Cay}(A_1, A_2)$ . Both  $A_1$  and  $A_2$  appear as faces of  $A$ . In order to prove that  $A$  is non-defective, it suffices to exhibit a 3-dimensional non-defective subconfiguration (see [5, Proposition 3.1] or [10, Proposition 3.13]). Let  $u_1, u_2, u_3$  be non-collinear points in  $A_1$  and

$v_1, v_2$  distinct points in  $A_2$ . The subconfiguration  $\{u_1, u_2, u_3, v_1, v_2\}$  of  $A$  is 3-dimensional and non-defective if and only if no four of the points lie in a hyperplane or, equivalently, if the vector  $v_2 - v_1$  is not parallel to any of the vectors  $u_j - u_i$ ,  $j \neq i$ . We can always find such subconfigurations unless  $A_1$  and  $A_2$  are the vertices of triangles with parallel edges. In the latter case, we can apply an affine isomorphism to get  $A_1 = p \cdot \sigma_2$  and  $A_2$  a translate of  $\pm q \cdot \sigma_2$ , where  $p$  and  $q$  are positive integers. The total degree of the mixed discriminant equals

$$\begin{aligned}\deg(\Delta_{p \cdot \sigma_2, q \cdot \sigma_2}) &= (p^2 + q^2 + pq - 3 \min\{p, q\}^2) / \gcd(p, q)^2, \\ \deg(\Delta_{p \cdot \sigma_2, -q \cdot \sigma_2}) &= (p + q)^2 / \gcd(p, q)^2.\end{aligned}$$

The first formula follows from (1.6) and it is positive unless  $p = q$ . The second formula will be derived in Example 3.10. It always gives a positive number. This concludes our proof.

Similar arguments can be used to study the case when one of the configurations is one-dimensional. However, it is more instructive to classify such defective configurations from the bidegree of the mixed discriminant. This will be done in Sect. 3.

**Corollary 2.5** *Let  $A_1$  and  $A_2$  be full-dimensional configurations in  $\mathbb{Z}^2$ . Then the mixed discriminantal variety of  $(A_1, A_2)$  is either a hypersurface or empty.*

**Remark 2.6** The same result holds in  $n$  dimensions when the toric variety  $X_A$  is smooth and  $A_1, \dots, A_n$  are full-dimensional configurations in  $\mathbb{Z}^n$ . Under these hypotheses,  $(A_1, \dots, A_n)$  is defective if and only if each  $A_i$  is affinely equivalent to  $p \cdot \sigma_n$ , with  $p \in \mathbb{N}$ . In particular, the mixed discriminantal variety of  $(A_1, \dots, A_n)$  is either a hypersurface or empty. The “if” direction is straightforward: we may assume  $i(A) = 1$  and  $p = 1$  by replacing  $\mathbb{Z}^n$  with the lattice spanned by  $pe_1, \dots, pe_n$ . Then, the system (1.1) consists of linear equations, and it is clearly defective. The “only-if” direction is derived from results in [9]:  $(A_1, \dots, A_n)$  is defective if and only if the  $(2n - 1)$ -dimensional polytope  $Q = \text{conv}(A)$  is isomorphic to a Cayley polytope of at least  $t + 1 \geq n + 1$  configurations of dimension  $k < t$  that have the same normal fan. As we already have a Cayley structure of  $n$  configurations in dimension  $n$ , we deduce  $t = n$  and  $k = n - 1$ . Then, we should have  $Q \simeq s_{n-1} \times s_n \simeq s_n \times s_{n-1}$ . After an affine transformation, all  $n$  polytopes  $Q_i$  are standard  $n$ -simplices and all  $A_i$  are translates of  $\sigma_n$ . This shows that  $A$  has an “inverted” Cayley structure of  $n + 1$  copies of  $\sigma_{n-1}$ .

We expect Proposition 2.4 to hold in  $n$  dimensions without the smoothness hypothesis in Remark 2.6. Clearly, whenever the mixed volume of  $Q_1, \dots, Q_n$  is 1, then there are no multiple roots and we have  $\Delta_{A_1, \dots, A_n} = 1$ . The following result gives a necessary and sufficient condition for being in this situation: up to affine equivalence, this is just the linear case.

**Proposition 2.7** *If  $A_1, \dots, A_n$  are  $n$ -dimensional configurations in  $\mathbb{Z}^n$  then the mixed volume  $\text{MV}(Q_1, \dots, Q_n)$  is 1 if and only if, up to affine isomorphism, each  $A_i$  is a translated of  $\sigma_n$ .*

*Proof* We shall prove the “only-if” direction by induction on  $n$ . Suppose that  $\text{MV}(Q_1, Q_2, \dots, Q_n) = 1$ . By the Aleksandrov–Fenchel inequality, we have  $\text{vol}(Q_i) = 1$  for all  $i$ , where the volume form is normalized so that the standard  $n$ -simplex has volume 1. Since the mixed volume function is monotone in each coordinate, for any choice of edges  $l_i$  in  $Q_i$  we have

$$0 \leq \text{MV}(l_1, l_2, \dots, l_n) \leq \text{MV}(l_1, Q_2, \dots, Q_n) \leq \text{MV}(Q_1, \dots, Q_n) = 1.$$

Since all polytopes  $Q_i$  are full-dimensional, we can pick  $n$  linearly independent edges  $l_1, \dots, l_n$ . Therefore  $\text{MV}(l_1, \dots, l_n) > 0$  and  $\text{MV}(l_1, l_2, \dots, l_n) = \text{MV}(l_1, Q_2, \dots, Q_n) = 1$ . In particular, the edge  $l_1$  has length one. After a change of coordinates we may assume that  $l_1 = e_n$ , the  $n$ th standard basis vector. Denote by  $\pi$  the projection of  $\mathbb{Z}^n$  onto  $\mathbb{Z}^n / \mathbb{Z}e_n \simeq \mathbb{Z}^{n-1}$  and the corresponding map of  $\mathbb{R}$ -vector spaces. We then have  $\text{MV}(\pi(Q_2), \dots, \pi(Q_n)) = 1$ .

By the induction hypothesis, we can transform the first  $n - 1$  coordinates and translate the configurations  $A_2, \dots, A_n$  so that  $\pi(A_2) = \dots = \pi(A_n) = \sigma_{n-1}$ . This means that  $A_i \subset \sigma_{n-1} \times \mathbb{Z}e_n$ . Now, let  $a_i$  be a point in  $A_i$  not lying in the coordinate hyperplane  $x_n = 0$ . Then  $1 \leq \text{vol}(\text{conv}(\sigma_{n-1}, a_i)) \leq \text{vol}(Q_i) = 1$ , and we conclude that  $Q_i = \text{conv}(\sigma_{n-1}, a_i)$ . But, since  $\text{vol}(Q_i) = 1$ , it follows that  $a_i = b_i \pm e_n$ , for some  $b_i \in \sigma_{n-1}$ . By repeating this process with an edge of  $A_1$  containing the point  $a_1$ , we see that all  $b_i$ 's are equal and that the sign of  $e_n$  in all  $a_i$ 's is the same. This shows that, after an affine isomorphism and a suitable translation, we have  $A_1 = \dots = A_n = \sigma_n$ , yielding the result.

### 3 Two curves in the plane

In this section we study the condition for two plane curves to be tangent. This condition is the mixed discriminant in the case  $n = 2$ . Our primary goal is to prove Theorem 3.3, which gives a formula for the bidegree of the mixed discriminant cycle of two full-dimensional planar configurations  $A_1$  and  $A_2$ . Remark 3.11 addresses the degenerate case when one of the  $A_i$  is one-dimensional. Our main tool is the connection between discriminants and principal determinants. In order to make this connection precise, and to define all the terms appearing in (3.3), we recall some basic notation and facts. We refer to [10, 12] for further details.

Let  $A \subset \mathbb{Z}^d$  and  $Q$  the convex hull of  $A$ . As is customary in toric geometry, we assume that  $A$  lies in a rational hyperplane that does not pass through the origin. This holds for Cayley configurations (2.1). Given any subset  $B \subset A$  we denote by  $\mathbb{Z} \cdot B$ , respectively  $\mathbb{R} \cdot B$ , the linear span of  $B$  over  $\mathbb{Z}$ , respectively over  $\mathbb{R}$ . For any face  $F \prec A$  we define the *lattice index*

$$i(F, A) := [\mathbb{R} \cdot F \cap \mathbb{Z}^d : \mathbb{Z} \cdot F].$$

We set  $i(A) = i(A, A) = [\mathbb{Z}^d : \mathbb{Z} \cdot A]$ . We consider the  $A$ -discriminant  $\Delta_A$  and the *principal  $A$ -determinant*  $E_A$ . They are defined in [12] under the assumption that  $i(A) = 1$ . If  $i(A) > 1$  then we change the ambient lattice from  $\mathbb{Z}^d$  to  $\mathbb{Z} \cdot A$ , and we define the associated *cycles*

$$\tilde{E}_A = E_A^{i(A)} \quad \text{and} \quad \tilde{\Delta}_A = \Delta_A^{i(A)}.$$

The expressions on the right-hand sides are computed relative to the lattice  $\mathbb{Z} \cdot A$ .

**Remark 3.1** The principal  $A$ -determinant of [12, Chapter 10] is a polynomial  $E_A$  in the variables  $c_\alpha$ ,  $\alpha \in A$ . Its Newton polytope is the *secondary polytope* of  $A$ , and its degree is  $(d + 1) \text{vol}(\text{conv}(A))$ , where  $\text{vol} = \text{vol}_{\mathbb{Z}A}$  is the normalized lattice volume for  $\mathbb{Z} \cdot A$ . We always have  $\deg(\tilde{E}_A) = (d + 1) \text{vol}_{\mathbb{Z}^d}(\text{conv}(A))$ , where  $\text{vol}_{\mathbb{Z}^d}$  is the normalized lattice volume for  $\mathbb{Z}^d$ .

We state the factorization formula of Gel'fand, Kapranov and Zelevinsky [12, Theorem 1.2, Chapter 10] for the principal  $A$ -determinant as in Esterov [10, Proposition 3.10]:

$$\tilde{E}_A = \pm \tilde{\Delta}_A \cdot \prod_{F \prec A} \tilde{\Delta}_F^{u(F, A)}. \quad (3.1)$$

The product runs over all proper faces of  $A$ . The exponents  $u(F, A)$  are computed as follows. Let  $\pi$  denote the projection to  $\mathbb{R} \cdot A / \mathbb{R} \cdot F$  and  $\Omega$  the normalized volume form on  $\mathbb{R} \cdot A / \mathbb{R} \cdot F$ .

This form is normalized with respect to the lattice  $\pi(\mathbb{Z}^d)$ , so that the fundamental domain with respect to integer translations has volume  $(\dim(\mathbb{R} \cdot A) - \dim(\mathbb{R} \cdot F))!$ . We set

$$u(F, A) := \Omega(\text{conv}(\pi(A)) \setminus \text{conv}(\pi(A \setminus F))).$$

**Remark 3.2** The positive integers  $u(F, A)$  are denoted  $c^{F,A}$  in [10]. If  $i(A) = 1$  then  $u(F, A)$  is the *subdiagram volume* associated with  $F$ , as in [12, Theorem 3.8, Chapter 5].

We now specialize to the case of Cayley configurations  $A = \text{Cay}(A_1, A_2)$ , where  $A_1, A_2 \subset \mathbb{Z}^2$  are full-dimensional. Here,  $A$  is a 3-dimensional configuration in the hyperplane  $x_1 + x_2 = 1$  in  $\mathbb{R}^4$ . Note that  $i(A, \mathbb{Z}^4) = i(A_1 \cup A_2, \mathbb{Z}^2)$ . The configurations  $A_1$  and  $A_2$  are facets of  $A$ .

We say that  $F$  is a *vertical* face of  $A$  if  $F \prec A$  but  $F \not\prec A_i, i = 1, 2$ . The vertical facets of  $A$  are either triangles or two-dimensional Cayley configurations defined by edges  $e \prec A_1$  and  $f \prec A_2$ . This happens if  $e$  and  $f$  are parallel and have the same orientation, that is, if they have the same inward normal direction when viewed as edges in  $Q_1$  and  $Q_2$ . We call such edges *strongly parallel* and denote the vertical facet they define by  $V(e, f)$ .

Let  $\mathcal{E}_i$  denote the set of edges of  $A_i$  and set

$$\mathcal{P} = \{(e, f) \in \mathcal{E}_1 \times \mathcal{E}_2 : e \text{ is strongly parallel to } f\}.$$

We write  $\ell(e)$  for the *normalized length* of an edge  $e$  with respect to the lattice  $\mathbb{Z}^2$ . For  $v \in A_1$  we define

$$\text{mm}(v) = \text{MV}(Q_1, Q_2) - \text{MV}(\text{conv}(A_1 \setminus v), Q_2), \quad (3.2)$$

and similarly for  $v \in A_2$ . This quantity is the *mixed multiplicity* of  $v$  in  $(A_1, A_2)$ .

**Theorem 3.3** *Let  $A_1$  and  $A_2$  be full-dimensional configurations in  $\mathbb{Z}^2$ . Then*

$$\begin{aligned} \delta_1 := \deg_{A_1}(\tilde{\Delta}_{A_1, A_2}) &= \text{area}(Q_2) + 2 \text{MV}(Q_1, Q_2) \\ &\quad - \sum_{(e, f) \in \mathcal{P}} \min\{u(e, A_1), u(f, A_2)\} \ell(f) - \sum_{v \in \text{Vert } A_1} \text{mmult}(v). \end{aligned} \quad (3.3)$$

$$\begin{aligned} \delta_2 := \deg_{A_2}(\tilde{\Delta}_{A_1, A_2}) &= \text{area}(Q_1) + 2 \text{MV}(Q_1, Q_2) \\ &\quad - \sum_{(e, f) \in \mathcal{P}} \min\{u(e, A_1), u(f, A_2)\} \ell(e) - \sum_{v \in \text{Vert } A_2} \text{mm}(v). \end{aligned}$$

Theorem 3.3 is the main result in this section. We shall derive it from the following formula [which is immediate from (3.1)] for the bidegree of our mixed discriminant:

$$\text{bideg}(\tilde{\Delta}_{A_1, A_2}) = \text{bideg}(\tilde{E}_A) - \sum_{k=1}^2 \sum_{F \prec A_k} u(F, A) \text{bideg}(\tilde{\Delta}_F) - \sum_{\substack{F \prec A \\ \text{vertical}}} u(F, A) \text{bideg}(\tilde{\Delta}_F). \quad (3.4)$$

Note the need for the *cycles*  $\tilde{\Delta}_{A_1, A_2}$ ,  $\tilde{E}_A$  and  $\tilde{\Delta}_F$  in this formula. We shall prove Theorem 3.3 by studying each term on the right-hand side of (3.4), one dimension at a time. A series of lemmas facilitates the exposition.

**Lemma 3.4** *The bidegree  $\text{bideg}(E_A)$  of the principal  $A$ -determinantal cycle  $\tilde{E}_A$  equals*  

$$(3 \text{ area}(Q_1) + \text{area}(Q_2) + 2 \text{ MV}(Q_1, Q_2), \text{area}(Q_1) + 3 \text{ area}(Q_2) + 2 \text{ MV}(Q_1, Q_2)).$$
(3.5)

*Proof* By [12], the total degree of  $\tilde{E}_A$  is  $4 \text{ vol}(Q)$ . From any triangulation of  $A$  we can see

$$\text{vol}(Q) = \text{area}(Q_1) + \text{area}(Q_2) + \text{MV}(Q_1, Q_2).$$

Examining the tetrahedra in a triangulation reveals that the bidegree is given by (3.5).  $\square$

Any pyramid is a defective configuration; hence, the vertical facets of  $Q$  that are triangles do not contribute to the right-hand side of (3.4) and can be safely ignored from now on. In particular, we see that the only non-defective vertical facets are the trapezoids  $V(e, f)$  for  $(e, f) \in \mathcal{P}$ . The following lemma explains their contribution to (3.4).

**Lemma 3.5** *Let  $V(e, f)$  be the vertical facet of  $A$  associated with  $(e, f) \in \mathcal{P}$ . Then*

- (1)  $\text{bideg}(\tilde{\Delta}_{V(e, f)}) = (\ell(f), \ell(e))$ ,
- (2)  $u(V(e, f), A) = \min\{u(e, A_1), u(f, A_2)\}$ .

*Proof* The configuration  $V(e, f)$  is the Cayley lift of two one-dimensional configurations. Its discriminantal cycle is the resultant of two univariate polynomials of degree  $\ell(e)$  and  $\ell(f)$ , so (1) holds. In order to prove (2), we note that  $u(V(e, f), Q)$  equals the normalized length of a segment in  $\mathbb{R}^3/\mathbb{R} \cdot V(e, f)$  starting at the origin and ending at the projection of a point in  $A_1$  or  $A_2$ . This image is the closest point to the origin in the line generated by the projection of  $Q$ . Thus, the multiplicity  $u(V(e, f), A)$  is the minimum of  $u(e, A_1)$  and  $u(f, A_2)$ .  $\square$

We next study the horizontal facets of  $A$  given by  $A_1$  and  $A_2$ .

**Lemma 3.6** *The discriminant cycle of the plane curve defined by  $A_i$  has total degree*

$$\deg(\tilde{\Delta}_{A_i}) = 3 \text{ area}(Q_i) - \sum_{e \in \text{Edges } A_i} u(e, A_i) \deg(\tilde{\Delta}_e) - \sum_{v \in \text{Vert } A_i} u(v, A_i),$$

where  $u(v, A_i) = \text{area}(Q_i) - \text{area}(\text{conv}(A_i \setminus v))$ .

*Proof* This is a special case of (3.4) because  $\deg(\tilde{E}_{A_i}) = 3 \cdot \text{area}(Q_i)$  and  $\deg(\tilde{\Delta}_v) = 1$  for any vertex  $v \in A_i$ . The statement about  $u(v, A_i)$  is just its definition.  $\square$

Next, we consider the edges of  $A$ . The vertical edges are defective since they consist of just two points. Thus we need only examine the edges of  $A_1$  and  $A_2$ .

**Lemma 3.7** *Let  $e$  be an edge of  $A_i$ . Then  $u(e, A) = u(e, A_i)$ .*

*Proof* Consider the projection  $\pi : Q_i \rightarrow \mathbb{R}^2/\mathbb{R} \cdot e$ . The image  $\pi(Q_i)$  is a segment of length  $M_1 = \max\{\ell([0, \pi(m)]) : m \in A_i\}$ , while  $\text{conv}(A_i \setminus e)$  projects to a segment of length  $M_2 = \max\{\ell([0, \pi(m)]) : m \in (A_i \setminus e)\}$ . Thus  $u(e, A_i) = M_1 - M_2$ . Next, consider the projection  $Q \rightarrow \mathbb{R}^3/\mathbb{R} \cdot e$ . The images of  $A$  and  $\text{conv}(A \setminus e)$  under this projection are trapezoids in  $\mathbb{R}^3/\mathbb{R} \cdot e$ . Their set-theoretic difference is a triangle of height 1 and base  $M_2 - M_1$ .  $\square$

**Lemma 3.8** *Let  $v$  be a vertex of  $A_i$ . Then  $u(v, A) = u(v, A_i) + \text{mm}(v)$ .*

*Proof* Suppose  $v \in A_1$ . The volume form  $\Omega$  is normalized with respect to the lattice  $\mathbb{Z}^3$ . The volume of our Cayley polytope  $\text{Cay}(A_1, A_2)$  equals  $\text{area}(Q_1) + \text{area}(Q_2) + \text{MV}(Q_1, Q_2)$ , and the analogous formula holds for  $\text{conv}(A \setminus v) = \text{Cay}(A_1 \setminus v, A_2)$ . We conclude

$$\begin{aligned} u(v, A) &= \text{vol}(\text{Cay}(A_1, A_2)) - \text{vol}(\text{Cay}(A_1 \setminus v, A_2)) \\ &= \text{area}(Q_1) - \text{area}(\text{conv}(A_1 \setminus v)) + \text{MV}(Q_1, Q_2) - \text{MV}(\text{conv}(A_1 \setminus v), Q_2) \\ &= u(v, A_1) + \text{mm}(v). \end{aligned}$$

□

*Proof of Theorem 3.3.* By symmetry, it suffices to prove (3.3). We start with the  $A_1$ -degree of the principal  $A$ -determinantal cycle  $\tilde{E}_A$  given in (3.5). In light of (3.1), we subtract the  $A_1$ -degrees of the various discriminant cycles corresponding to all faces of  $A$ . Besides the contribution from  $A_1$ , having  $u(A_1, A) = 1$  and given by Lemma 3.6, only the vertices and the vertical facets contribute. Using Lemmas 3.7 and 3.8, we derive the desired formula.

At this point, the reader may find it an instructive exercise to derive (1.4) and (1.6) from Theorem 3.3, and ditto for  $(\delta_1, \delta_2) = (2, 2)$  in Example 2.3. Here are two further examples.

*Example 3.9* Let  $A_1$  and  $A_2$  be the dense triangles  $(d_1 s_2) \cap \mathbb{Z}^2$  and  $(-d_2 s_2) \cap \mathbb{Z}^2$ . Here,  $i(A) = 1$  and  $\tilde{\Delta}_{A_1, A_2} = \Delta_{A_1, A_2}$ . We have  $\text{MV}(d_1 s_2, -d_2 s_2) = 2d_1 d_2$  and  $\mathcal{P} = \emptyset$ . Computation of the mixed areas in (3.2) yields  $\text{mm}(v) = d_2$  for vertices  $v \in A_1$  and  $\text{mm}(v) = d_1$  for vertices  $v \in A_2$ . We conclude

$$\text{bideg}(\Delta_{A_1, A_2}) = (d_2^2 + 4d_1 d_2 - 3d_2, d_1^2 + 4d_1 d_2 - 3d_1).$$

*Example 3.10* Let  $A_1 = d_1 \sigma_2$  and  $A_2 = -d_2 \sigma_2$ . This is the sparse version of Example 3.9. Now,  $i(A) = g^2$ , where  $g = \gcd(d_1, d_2)$ , and  $\tilde{\Delta}_{A_1, A_2} = \Delta_{A_1, A_2}^{g^2}$ . We still have  $\text{MV}(d_1 s_2, -d_2 s_2) = 2d_1 d_2$  and  $\mathcal{P} = \emptyset$ , but  $\text{mm}(v) = d_1 d_2$  for all vertices  $v \in A$ . Hence

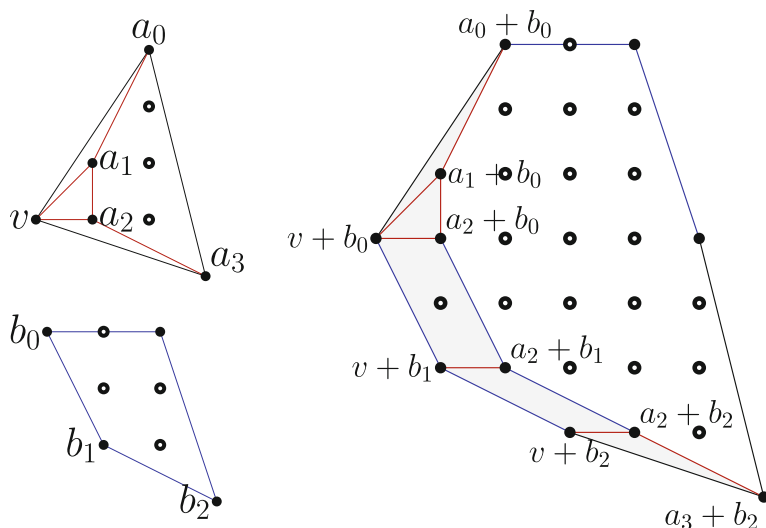
$$\text{bideg}(\Delta_{A_1, A_2}) = \frac{1}{g^2} (d_2^2 + d_1 d_2, d_1^2 + d_1 d_2).$$

*Remark 3.11* From (3.4) we may also derive formulas for the bidegree of the mixed discriminant in the case when one of the configurations, say  $A_2$ , is one-dimensional. The main differences with the proof of Theorem 3.3 is that now we must treat  $A_2$  as an edge, rather than a facet, and that it is enough for an edge  $e$  of  $Q_1$  to be parallel to  $Q_2$  in order to have a non-defective vertical facet of  $Q$ . Clearly, there are at most two possible edges of  $Q_1$  parallel to  $Q_2$ . The  $A_2$ -degree of the mixed discriminant cycle now has a very simple expression:

$$\delta_2 = \text{area}(Q_1) - \sum_{e \parallel Q_2} u(e, A_1) \ell(e), \quad (3.6)$$

where the sum runs over all edges  $e$  of  $Q_1$  which are parallel to  $Q_2$ .

In particular, if no edge of  $Q_1$  is parallel to  $Q_2$ , then  $\delta_2 > 0$  and  $(A_1, A_2)$  is not defective. If only one edge  $e$  of  $Q_1$  is parallel to  $Q_2$  then  $\delta_2 = 0$  if and only if  $\text{area}(Q_1) = u(e, A_1) \ell(e)$  but this happens only if there is a single point of  $A_1$  not lying in the edge  $e$ . This means that  $A = \text{Cay}(A_1, A_2)$  is a pyramid and hence is defective. Finally, if there are two edges  $e_1$  and  $e_2$  of  $Q_1$  parallel to  $Q_2$  then  $\delta_2 = 0$  if and only if  $\text{area}(Q_1) = u(e_1, A_1) \ell(e_1) + u(e_2, A_1) \ell(e_2)$ . This can only happen if all the points of  $A_1$  lie either in  $e_1$  or  $e_2$ . In this case,  $A$  is the Cayley lift of three one-dimensional configurations, and it is defective as well.



**Fig. 1** Geometric computation of the mixed multiplicity  $\text{mm}(v)$  via a suitable mixed subdivision of the two polygons. The region  $D$  is shown in grey.

Our next goal is to provide a sharp geometric bound for the sum of the mixed multiplicities. We start by providing a method to compute such invariants by means of mixed subdivisions.

**Lemma 3.12** *Let  $A_1, A_2$  be full-dimensional in  $\mathbb{Z}^2$  and  $v \in A_1$ . Any mixed subdivision of  $Q^* = \text{conv}(A_1 \setminus v) + Q_2$  extends to a mixed subdivision of  $Q = Q_1 + Q_2$ . The mixed multiplicity  $\text{mm}(v)$  is the sum of the Euclidean areas of the mixed cells in the closure  $D$  of  $Q \setminus Q^*$ .*

*Proof* Let  $\mathcal{E}'_2(v)$  denote the collection of edges in  $A_2$  whose inner normal directions lie in the relative interior of the dual cone to a vertex  $v$  of  $A_1$ . Equivalently,  $\mathcal{E}'_2(v)$  consists of those edges  $[b, b']$  of  $A_2$  such that  $v + b$  and  $v + b'$  are both vertices of  $A_1 + A_2$ . See Fig. 1.

First, assume  $\mathcal{E}'_2(v) = \emptyset$ . Then, there exists a unique  $b \in A_2$  such that  $v + b$  is a vertex of  $Q$ . It follows that there exist  $a_0, \dots, a_r \in A_1$  such that  $D$  is a union of triangles of the form

$$D = \bigcup_{i=1}^r \text{conv}(\{v + b, a_{i-1} + b, a_i + b\}), \quad (3.7)$$

and  $[a_{i-1} + b, a_i + b]$  are edges in the subdivision of  $Q^*$ . Then, we can extend the subdivision of  $Q^*$  by adding the triangles in (3.7). This does not change the mixed areas and  $\text{mm}(v) = 0$ .

Suppose now that  $\mathcal{E}'_2(v) = \{f_1, \dots, f_s\}$ ,  $s \geq 1$ , with indices in counterclockwise order. Let  $b_0, \dots, b_s$  be the vertices of  $A_2$  such that  $f_i$  is the segment  $[b_{i-1}, b_i]$ . The pairs  $v + b_{i-1}, v + b_i$ , for  $i = 1, \dots, s$ , define edges of  $Q$  which lie in the boundary of  $D$ . Let  $a_0, a_r \in A_1$  be the vertices of the edges of  $Q_1$  adjacent to  $v$ . We insert  $r - 1$  points in  $A_1$  to form a counterclockwise oriented sequence  $a_0, a_1, \dots, a_r$  of vertices of  $\text{conv}(A_1 \setminus v)$ . Then  $a_0 + b_0$  and  $a_r + b_s$  are vertices of  $Q^*$ , and the boundary of  $D$  consists of the  $s$  segments  $[v + b_{j-1}, v + b_j]$ , together with segments of the form  $[a_{i-1} + b_j, a_i + b_j]$  or  $[a_i + b_{j-1}, a_i + b_j]$ . Figure 1 depicts the case  $r = 3, s = 2$ . Given this data, we subdivide  $D$  into mixed and unmixed cells. The unmixed cells are triangles  $\{v + b_j, a_{i+1} + b_j, a_i + b_j\}$

coming from the edges  $[a_i + b_j, a_{i+1} + b_j]$  of  $Q^*$ . The mixed cells are parallelograms  $\{v + b_j, v + b_{j+1}, a_i + b_{j+1}, a_i + b_j\}$  built from the edges  $[a_i + b_j, a_i + b_{j+1}]$  of  $Q^*$ . This subdivision is compatible with that of  $Q^*$ .  $\square$

We write  $\mathcal{E}'_1$  for the set of all edges of  $A_1$  that are not strongly parallel to an edge of  $A_2$ . The set  $\mathcal{E}'_2$  is defined analogously.

**Proposition 3.13** *Let  $A_1, A_2 \in \mathbb{Z}^2$  be two-dimensional configurations. Then*

- (1) *The sum of the lengths of all edges in the set  $\mathcal{E}'_j$  is a lower bound for the sum of the mixed multiplicities over all vertices of the other configuration  $A_i$ . In symbols,*

$$\sum_{v \in \text{Vert } A_i} \text{mm}(v) \geq \sum_{e \in \mathcal{E}'_j} \ell(e) \quad \text{for } j \neq i.$$

- (2) *If  $\mathcal{E}'_j = \emptyset$  then  $\sum_{v \in \text{Vert } A_i} \text{mm}(v) = 0$ .*

- (3) *If  $i(A_1) = i(A_2) = 1$  and the three toric surfaces corresponding to  $A_1$ ,  $A_2$  and  $A_1 + A_2$  are smooth then the bound in (1) is sharp.*

*Proof* We keep the notation of the proof of Lemma 3.12. Recall that the set  $\mathcal{E}'_2$  is the union of the sets  $\mathcal{E}'_2(v)$ , where  $v$  runs over all vertices in  $A_1$ . By Lemma 3.12, the mixed multiplicity  $\text{mm}(v)$  is the sum of the Euclidean areas of the mixed cells in  $D$ . Each mixed cell is a parallelogram  $\{v + b_{k-1}, v + b_k, a_i + b_{k-1}, a_i + b_k\}$ , so its area is  $\ell([b_{k-1}, b_k]) \cdot \ell([v, a_i]) \cdot |\det(\tau_{k-1}, \eta_i)|$ , where  $\tau_{k-1}$  and  $\eta_i$  are primitive normal vectors to the edges  $[b_k, b_{k+1}]$  and  $[v, a_i]$ . Thus  $\text{mm}(v) \geq \sum_{e \in \mathcal{E}'_2(v)} \ell(e)$ . Since  $\mathcal{E}'_2$  is the disjoint union of the sets  $\mathcal{E}'_2(v)$ , summing over all vertices  $v$  of  $A_1$  gives the desired lower bound. Part (2) also follows from Lemma 3.12, as  $\mathcal{E}'_2 = \emptyset$  implies that the subdivision of  $D$  has no mixed cells.

It remains to prove (3). In the notation of Lemma 3.12, the assumption that  $X_{A_1}$  is smooth implies that the segment  $[a_0, a_1]$  is an edge in  $\text{conv}(A_1 \setminus v)$ , so  $r = 1$ . Therefore, all mixed cells in  $D$  are parallelograms with vertices  $\{v + b_{k-1}, v + b_k, a_i + b_{k-1}, a_i + b_k\}$  for  $i = 0, 1, k = 1, \dots, s$ . This parallelogram has Euclidean area  $|\det(a_i - v, b_k - b_{k-1})|$ , but, since  $X_{A_1+A_2}$  is smooth, we have:

$$|\det(a_i - v, b_k - b_{k-1})| = \ell([v, a_i]) \cdot \ell([b_{k-1}, b_k]) = 1 \cdot \ell([b_{k-1}, b_k]).$$

Since  $\mathcal{E}'_2(v) = \{[b_0, b_1], \dots, [b_{s-1}, b_s]\}$ , this equality and Lemma 3.12 yield the result.  $\square$

**Remark 3.14** The equality  $\sum_{v \in \text{Vert } A_i} \text{mm}(v) = \sum_{e \in \mathcal{E}'_j} \ell(e)$  in case  $i(A_1) = i(A_2) = 1$  and the toric surfaces of  $A_1$ ,  $A_2$  and  $A_1 + A_2$  are smooth, can be interpreted and proved with tools from toric geometry. Indeed, in this case, let  $X_1, X_2$  and  $X$  be the associated toric varieties. Then, there are birational maps  $\pi_i: X \rightarrow X_i$  defined by the common refinement of the associated normal fans,  $i = 1, 2$ . The map  $\pi_1$  is given by successive toric blow-ups of fixed points of  $X_1$  corresponding to vertices  $v$  of  $A_1$  for which  $\mathcal{E}'_2(v) \neq \emptyset$ . The lengths of the corresponding edges occur as the intersection product of the invariant (exceptional) divisor associated to the edge with the ample line bundle associated to  $A_2$ , pulled back to  $X$ .

If  $A_i$  is dense, then it is immediate to check that  $u(e, A_i) = 1$  for all edges  $e \prec A_i$ . We conclude with a geometric upper bound for the bidegree of the mixed discriminant.

**Corollary 3.15** *Let  $A_1$  and  $A_2$  be full-dimensional configurations in  $\mathbb{Z}^2$ . Then:*

- (1) *The bidegree satisfies  $\deg_{A_i}(\tilde{\Delta}_{A_1, A_2}) \leq \text{area}(Q_j) + 2 \text{MV}(Q_1, Q_2) - \text{perim}(Q_j)$ ,  $j \neq i$ .*

- (2) Equality holds in (1) if  $i(A_1) = i(A_2) = 1$  and the three toric surfaces corresponding to  $A_1$ ,  $A_2$  and  $A_1 + A_2$  are smooth.
- (3) Equality holds in (1) if  $Q_1$ ,  $Q_2$  have the same normal fan and one of  $A_1$  or  $A_2$  is dense.

*Proof* Assume  $i = 1$ . Statement (1) follows from (3.3) and

$$\begin{aligned} \sum_{(e,f) \in \mathcal{P}} \min\{u(e, A_1), u(f, A_2)\} \ell(f) + \sum_{v \in A_1} \text{mm}(v) &\geq \sum_{f \in \mathcal{E}_2 \setminus \mathcal{E}'_2} \ell(f) \\ &+ \sum_{f \in \mathcal{E}'_2} \ell(f) = \text{perim}(A_2). \end{aligned}$$

Statement (2) follows from Proposition 3.13 (3) and the fact that the smoothness condition implies  $u(e, A_1) = u(f, A_2) = 1$  for all edges  $e \prec A_1$ ,  $f \prec A_2$ . Finally, if  $Q_1$  and  $Q_2$  have the same normal fan then  $\mathcal{E}'_1 = \mathcal{E}'_2 = \emptyset$ , and, by Theorem 3.13 (2), all mixed multiplicities vanish. Density of  $A_1$  or  $A_2$  implies  $\min\{u(e, A_1), u(f, A_2)\} = 1$  for every pair  $(e, f) \in \mathcal{P}$ . Hence

$$\sum_{(e,f) \in \mathcal{P}} \min\{u(e, A_1), u(f, A_2)\} \ell(f) = \text{perim}(Q_2).$$

□

Corollary 3.15 establishes the degree formula (1.3). We end this section with an example for which that formula holds, even though conditions (2) and (3) do not. It also shows that, unlike for resultants [7, §6], the bidegree of the mixed discriminant can decrease when removing a single point from  $A$  without altering the lattice or the convex hulls of the configurations.

*Example 3.16* Consider the dense configurations  $A_1 := \{(0, 0), (1, 0), (1, 1), (0, 1)\}$  and  $A_2 := \{(0, 0), (1, 3), (-1, 2), (0, 1), (0, 2)\}$ . The vertex  $v = (0, 0)$  of  $A_2$  is a singular point. However, its mixed multiplicity equals 1, so it agrees with the lattice length of the associated edge  $[(0, 0), (1, 0)]$  in  $A_1$ . Theorem 3.3 implies that the bidegree of the mixed discriminant  $\Delta_{A_1, A_2}$  equals  $(\delta_1, \delta_2) = (12, 8)$ . If we remove the point  $(0, 1)$  from  $A_2$ , the mixed multiplicity of  $v$  is raised to 2 and the bidegree of the mixed discriminant decreases to  $(12, 7)$ .

#### 4 The degree of the mixed discriminant is Piecewise linear

Theorem 3.3 implies that the bidegree of the mixed discriminant of  $A_1, A_2 \subset \mathbb{Z}^2$  is piecewise linear in the maximal minors of the Cayley matrix  $A = \text{Cay}(A_1, A_2)$ . In this section we prove Theorem 1.1 which extends the same statement to arbitrary Cayley configurations, and we describe suitable regions of linearity. Theorem 1.1 allows us to obtain formulas for the multidegree of the mixed discriminant by linear algebraic methods, provided we are able to compute it in sufficiently many examples. This may be done by using the *ray shooting algorithm* of [7, Theorem 2.2], which has become a standard technique in tropical geometry. We start by an example in dimension 3, which was computed using Rincón's software [16].

*Example 4.1* Consider the following three trinomials in three variables:

$$\begin{aligned} f &= a_1 x + a_2 y^p + a_3 z^p, \\ g &= b_1 x^q + b_2 y + b_3 z^q, \\ h &= c_1 x^r + c_2 y^r + c_3 z. \end{aligned}$$

Here  $p, q$  and  $r$  are arbitrary integers different from 1. By the degree we mean the triple of integers that records the degrees of the mixed discriminant cycle  $\Delta_{f,g,h}$  in the unknowns  $(a_1, a_2, a_3)$ ,  $(b_1, b_2, b_3)$ , and  $(c_1, c_2, c_3)$ . It equals the following triple of piecewise polynomials:

$$\begin{aligned} & (2pqr + q^2r + qr^2 - q - r - 1 - p \min\{q, r\}, \\ & 2pqr + p^2r + pr^2 - p - r - 1 - q \min\{r, p\}, \\ & 2pqr + p^2q + pq^2 - p - q - 1 - r \min\{p, q\}). \end{aligned}$$

These three polynomials are linear functions in the  $6 \times 6$ -minors of the  $6 \times 9$  Cayley matrix  $A$  that represents  $(f, g, h)$ . The space of all systems of three trinomials will be defined as a certain *mixed Grassmannian*. The  $6 \times 6$ -minors represent its Plücker coordinates.

Given  $m \in \mathbb{N}$  with  $n \leq m$ , consider a partition  $\mathcal{I} = \{I_1, \dots, I_n\}$  of the set  $[m] = \{1, \dots, m\}$ . Let  $G(d, m)$  denote the *affine cone over the Grassmannian* of  $d$ -dimensional linear subspaces of  $\mathbb{R}^m$ , given by its Plücker embedding in  $\wedge^d \mathbb{R}^m$ . Thus  $G(d, m)$  is the subvariety of  $\wedge^d \mathbb{R}^m$  cut out by the quadratic Plücker relations. For instance, for  $d = 2, m = 4$ , this is the hypersurface  $G(2, 4)$  in  $\wedge^2 \mathbb{R}^4 \simeq \mathbb{R}^6$  defined by the unique Plücker relation  $x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$ .

**Definition 4.2** The *mixed Grassmannian*  $G(d, \mathcal{I})$  associated to the partition  $\mathcal{I}$  is defined as the linear subvariety of  $G(d, m)$  consisting of all subspaces that contain the vectors  $e_{I_j} := \sum_{i \in I_j} e_i$  for  $j = 1, \dots, n$ . Here “linear subvariety” means that  $G(d, \mathcal{I})$  is the intersection of  $G(d, m)$  with a linear subspace of the  $\binom{m}{d}$ -dimensional real vector space  $\wedge^d \mathbb{R}^m$ .

The condition that a subspace  $\xi$  contains  $e_{I_j}$  translates into a system of  $n(m-d)$  linearly independent linear forms in the Plücker coordinates that vanish on  $G(d, \mathcal{I})$ . These linear forms are obtained as the coordinates of the exterior products  $\xi \wedge e_{I_j}$  for  $j = 1, \dots, n$ .

We should stress one crucial point. As an abstract variety, the mixed Grassmannian  $G(d, \mathcal{I})$  is isomorphic to the ordinary Grassmannian  $G(d-n, m-n)$ , where the isomorphism maps  $\xi$  to its image modulo  $\text{span}(e_{I_1}, \dots, e_{I_n})$ . However, we always work with the Plücker coordinates of the ambient Grassmannian  $G(d, m)$  in  $\wedge^d \mathbb{R}^m$ . We do **not** consider the mixed Grassmannian  $G(d-n, m-n)$  in its Plücker embedding in  $\wedge^{d-n} \mathbb{R}^{m-n}$ .

Our mixed Grassmannian has a natural decomposition into finitely many strata whose definition involves *oriented matroids*. On each stratum, the degree of the mixed discriminant cycle is a linear function in the Plücker coordinates. In order to define tropical matroid strata and to prove Theorem 1.1, it will be convenient to regard the mixed discriminant as the  $A$ -discriminant  $\Delta_A$  of the Cayley matrix  $A$ . In fact, we shall consider  $\Delta_A$  for arbitrary matrices  $A \in \mathbb{Z}^{d \times m}$  of rank  $d$  such that  $e_{[m]} = (1, 1, \dots, 1)$  is in the row span of  $A$ . Then,  $A$  represents a point  $\xi$  in the Grassmannian  $G(d, \{[m]\})$ . This is the proper subvariety of  $G(d, m)$  consisting of all points whose subspace contains  $e_{[m]}$ .

In what follows we assume some familiarity with matroid theory and tropical geometry. We refer to [7, 11] for details. Given a  $d \times m$ -matrix  $A$  of rank  $d$  as above, we let  $M^*(A)$  denote the corresponding dual matroid on  $[m]$ . This matroid has rank  $m-d$ . A subset  $I = \{i_1, \dots, i_r\} \subseteq [m]$  is *independent* in  $M^*(A)$  if and only if  $e_{i_1}^*, \dots, e_{i_r}^*$  are linearly independent when restricted to  $\ker(A)$ , where  $e_1^*, \dots, e_m^*$  denotes the standard dual basis. The *flats* of the matroid  $M^*(A)$  are the subsets  $J \subseteq [m]$  such that  $[m] \setminus J$  is the support of a vector in  $\ker(A)$ .

Let  $\mathcal{T}(\ker(A))$  denote the tropicalization of the kernel of  $A$ . This tropical linear space is a balanced fan of dimension  $m-d$  in  $\mathbb{R}^m$ . It is also known as the *Bergman fan* of

$M^*(A)$ , and it admits various fan structures [11, 16]. Ardila and Klivans [1] showed that the chains in the geometric lattice of  $M^*(A)$  endow the tropical linear space  $\mathcal{T}(\ker(A))$  with the structure of a simplicial fan. The cones in this fan are  $\text{span}(e_{J_1}, e_{J_2}, \dots, e_{J_r})$  where  $\mathcal{J} = \{J_1 \subset J_2 \subset \dots \subset J_r\}$  runs over all chains of flats of  $M^*(A)$ . Such a cone is maximal when  $r = m - d - 1$ . Given any such maximal chain and any index  $i \in [m]$ , we associate with them the following  $m \times m$  matrix:

$$M(A, \mathcal{J}, i) := (A^T, e_{J_1}, e_{J_2}, \dots, e_{J_{m-d-1}}, e_i).$$

Its determinant is a linear expression in the Plücker coordinates of the row span  $\xi$  of  $A$ :

$$\det(M(A, \mathcal{J}, i)) = \xi \wedge e_{J_1} \wedge e_{J_2} \wedge \dots \wedge e_{J_{m-d-1}} \wedge e_i.$$

**Definition 4.3** Let  $A$  and  $A'$  be matrices representing points  $\xi$  and  $\xi'$  in  $G(d, \{[m]\})$ . These points belong to the same *tropical matroid stratum* if they have the same dual matroid, i.e.,

$$M^*(A) = M^*(A'),$$

and, in addition, for all  $i \in [m]$  and all maximal chains of flats  $\mathcal{J}$  in the above matroid, the determinants of the matrices  $M(A, \mathcal{J}, i)$  and  $M(A', \mathcal{J}, i)$  have the same sign.

**Remark 4.4** Dickenstein et al. [7] gave the following formula for the *tropical A-discriminant*:

$$\mathcal{T}(\Delta_A) = \mathcal{T}(\ker(A)) + \text{rowspan}(A). \quad (4.1)$$

This is a tropical cycle in  $\mathbb{R}^m$ , i.e. a polyhedral fan that is balanced relative to the multiplicities associated to its maximal cones. The dimension of  $\mathcal{T}(\Delta_A)$  equals  $m - 1$  whenever  $A$  is not defective. It is clear from the formula (4.1) that  $\mathcal{T}(\Delta_A)$  depends only on the subspace  $\xi = \text{rowspan}(A)$ , so it is a function of  $\xi \in G(d, \{[m]\})$ . The tropical matroid strata are the subsets of  $G(d, \{[m]\})$  throughout which the combinatorial type of (4.1) does not change.

**Example 4.5** We illustrate the definition of the tropical matroid strata by revisiting the formulas in (1.6) and Example 3.10. The Cayley matrix of the two sparse triangles equals

$$A = \text{Cay}(A_1, A_2) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & d_1 & 0 & 0 & d_2 & 0 \\ 0 & 0 & d_1 & 0 & 0 & d_2 \end{pmatrix}.$$

The matroid  $M^*(A)$  has rank 2, so every maximal chain of flats in  $M^*(A)$  consists of a single rank 1 flat. These flats are  $J_1 = \{1, 4\}$ ,  $J_2 = \{2, 5\}$ , and  $J_3 = \{3, 6\}$ . The  $6 \times 6$ -determinants  $\det(M(A, \mathcal{J}, i))$  obtained by augmenting  $A$  with one vector  $e_{J_k}$  and one unit vector  $e_i$  are 0,  $\pm d_1(d_1 - d_2)$ , or  $\pm d_2(d_1 - d_2)$ . This shows that  $d_1 \geq d_2 \geq 0$  and  $d_1 \geq 0 \geq d_2$  are tropical matroid strata, corresponding to (1.6) and to Example 3.10 with  $d_2$  replaced by  $-d_2$ .

**Remark 4.6** The verification that two configurations lie in the same tropical matroid stratum may involve a huge number of maximal flags if we use Definition 4.3 as it is. In practice, we can greatly reduce the number of signs of determinants to be checked, by utilizing a coarser fan structure on  $\mathcal{T}(\ker(A))$ . The coarsest fan structure is given by the *irreducible flats* and their *nested sets*, as explained in [11]. Rather than reviewing these combinatorial details for arbitrary matrices, we simply illustrate the resulting reduction in complexity when  $M^*(A)$  is the *uniform matroid*. This means that any  $d$  columns of  $A$  form a basis of  $\mathbb{R}^d$ . Then,  $M^*(A)$  has  $(m - d - 1)! \binom{m}{m-d-1}$  maximal flags  $J_1 \subset J_2 \subset \dots \subset J_{m-d-1}$  constructed as follows. Let  $I = \{i_1, \dots, i_{m-d-1}\}$  be an  $(m - d - 1)$ -subset of  $[m]$  and  $\sigma$  a permutation

of  $[m - d - 1]$ . Then, we set  $J_k := [m] \setminus \{i_{\sigma(1)}, \dots, i_{\sigma(k)}\}$ . It is clear that the sign of  $\det(A^T, e_{J_1}, e_{J_2}, \dots, e_{J_{m-d-1}}, e_k)$  is completely determined by the signs of the determinants

$$\det(A^T, e_{i_1}, e_{i_2}, \dots, e_{i_{m-d-1}}, e_k),$$

where  $i_1 < i_2 < \dots < i_{m-d-1}$ . Hence, we only need to check  $\binom{m}{m-d-1}$  conditions.

Recall that the  $A$ -discriminant cycle  $\tilde{\Delta}_A = \Delta_A^{i(A)}$  is effective of codimension 1, provided  $A$  is non-defective. The lattice index  $i(A)$  is the gcd of all maximal minors of  $A$ .

**Theorem 4.7** *The degree of the  $A$ -discriminant cycle is piecewise linear in the Plücker coordinates on  $G(d, \{[m]\})$ . It is linear on the tropical matroid strata. The formulas on maximal strata are unique modulo the linear forms obtained from the entries of  $\xi \wedge e_{[m]}$ .*

In both Theorem 1.1 and Theorem 4.7, the notion of “degree” allows for any grading that makes the respective discriminant homogeneous. For the mixed discriminant  $\Delta_{A_1, \dots, A_n}$  we are interested in the  $\mathbb{N}^n$ -degree. Theorem 1.1 will be derived as a corollary from Theorem 4.7.

*Proof of Theorem 4.7.* The uniqueness of the degree formula follows from our earlier remark that the entries of  $\xi \wedge e_{[m]}$  are the linear relations on the mixed Grassmannian  $G(d, \{[m]\})$ . We now show how tropical geometry leads to the desired piecewise linear formula.

From the representation of the tropical discriminant in (4.1), Dickenstein et al. [7, Theorem 5.2] derived the following formula for the initial monomial of the  $A$ -discriminant  $\Delta_A$  with respect to any generic weight vector  $\omega \in \mathbb{R}^m$ . The exponent of the variable  $x_i$  in the initial monomial  $\text{in}_\omega(\Delta_A)$  of the  $A$ -discriminant  $\Delta_A$  is equal to

$$\sum_{\mathcal{J} \in \mathcal{C}_{i, \omega}} |\det(A^T, e_{J_1}, \dots, e_{J_{m-d-1}}, e_i)|. \quad (4.2)$$

Here,  $\mathcal{C}_{i, \omega}$  is the set of maximal chains  $\mathcal{J}$  of  $M^*(A)$  such that the rowspan  $\xi$  of  $A$  has non-zero intersection with the relatively open cone  $\mathbb{R}_{>0}\{e_{J_1}, \dots, e_{J_{m-d-1}}, -e_i, -\omega\}$ .

It now suffices to prove the following statement: if two matrices  $A$  and  $A'$  lie in the same tropical matroid stratum, then there exist weight vectors  $\omega$  and  $\omega'$  such that  $\mathcal{C}_{i, \omega} = \mathcal{C}_{i, \omega'}$ . This ensures that the sum in (4.2), with the absolute value replaced with the appropriate sign, yields a linear function in the Plücker coordinates of  $\xi$  for the degree of  $\Delta_A$  and  $\Delta_{A'}$ .

The condition  $\mathcal{J} \in \mathcal{C}_{i, \omega}$  is equivalent to the weight vector  $\omega$  being in the cone

$$\mathbb{R}_{>0}\{e_{J_1}, \dots, e_{J_{m-d-1}}, -e_i\} + \xi.$$

Hence, it is convenient to work modulo  $\xi$ . This amounts to considering the exact sequence

$$0 \longrightarrow \mathbb{R}^d \xrightarrow{A^T} \mathbb{R}^k \xrightarrow{\beta} W \longrightarrow 0.$$

Choosing a basis for  $\ker(A)$ , we can identify  $W \simeq \mathbb{R}^{m-d}$ . The columns of the matrix  $\beta$  define a vector configuration  $B = \{b_1, \dots, b_m\} \subset \mathbb{R}^{m-d}$  called a *Gale dual configuration* of  $A$ . Projecting into  $W$ , we see that  $\mathcal{C}_{i, \omega}$  equals the set of all maximal chains  $J$  such that  $\beta(\omega)$  lies on the cone  $\mathbb{R}_{>0}\{\beta(e_{J_1}), \dots, \beta(e_{J_{m-d-1}}), -\beta(e_i)\}$ . It follows that

$$\mathcal{J} \in \mathcal{C}_{i, \omega} \quad \text{if and only if} \quad \beta(\omega) = \sum_{j=1}^m w_j b_j \in \mathbb{R}_{>0}\{\sigma_{J_1}, \dots, \sigma_{J_{m-d-1}}, -b_i\},$$

where  $\sigma_{\mathcal{J}} := \beta(e_{\mathcal{J}}) = \sum_{j \in \mathcal{J}} b_j$ .

We can also restate the definition of the tropical matroid strata in terms of Gale duals. Namely, there exists a non-zero constant  $c$ , depending only on  $d$ ,  $m$  and our choice of Gale dual  $B$ , such that, given a maximal chain of flats  $\mathcal{J}$  in the matroid  $M^*(A)$ , we have:

$$\det(A^T, e_{J_1}, e_{J_2}, \dots, e_{J_{m-d-1}}, e_i) = c \cdot \det(\sigma_{J_1}, \sigma_{J_2}, \dots, \sigma_{J_{m-d-1}}, b_i). \quad (4.3)$$

Hence, the tropical matroid strata in  $G(d, \{[m]\})$  are determined by the signs of the determinant on the right-hand side of (4.3). If  $\mathcal{J} \in \mathcal{C}_{i,\omega}$  for generic  $\omega \in \mathbb{R}^m$ , then the vectors  $\{\sigma_{J_1}, \dots, \sigma_{J_{m-d-1}}, b_i\}$  in  $\mathbb{R}^{m-d}$  are linearly independent. Let  $M(\mathcal{J}, B, i)$  be the matrix whose columns are these vectors. Then,  $\mathcal{J} \in \mathcal{C}_{i,\omega}$  if and only if the vector  $x = M(\mathcal{J}, B, i)^{-1}\beta(\omega)$  has positive entries. By Cramer's rule, those entries are

$$\begin{cases} x_k = \frac{\det(\sigma_{J_1}, \dots, \sigma_{J_{k-1}}, \beta(\omega), \sigma_{J_{k+1}}, \dots, \sigma_{J_{m-d-1}}, -b_i)}{\det(M(\mathcal{J}, B, i))} & \text{for } 0 \leq k < m-d, \\ x_m = \frac{\det(\sigma_{J_1}, \sigma_{J_2}, \dots, \sigma_{J_{m-d-1}}, \beta(\omega))}{\det(M(\mathcal{J}, B, i))}. \end{cases} \quad (4.4)$$

Suppose now that  $A$  and  $A'$  are two configurations in the same tropical matroid stratum. Let  $B$  and  $B'$  be their Gale duals. Then  $M^*(A) = M^*(A')$  and the denominators  $\det(M(\mathcal{J}, B, i))$  and  $\det(M(\mathcal{J}, B', i))$  in (4.4) have the same signs. On the other hand, let us consider the oriented hyperplane arrangement in  $\mathbb{R}^{m-d}$  consisting of the hyperplanes  $\mathcal{H}_{\mathcal{J}, B, k, i} = \langle \sigma_{J_1}, \dots, \sigma_{J_{k-1}}, \sigma_{J_{k+1}}, \dots, \sigma_{J_{m-d-1}}, b_i \rangle$ , for  $1 \leq k \leq m-d-1$ ,  $i \notin J_{m-d-1}$ , as well as the hyperplane  $\mathcal{H}_{\mathcal{J}} = \langle \sigma_{J_1}, \dots, \sigma_{J_{m-d-1}} \rangle$ , for all maximal chains  $\mathcal{J} \in M^*(A)$  such that  $\sigma_{J_1}, \dots, \sigma_{J_{m-d-1}}$  are linearly independent. The signs of the numerators in (4.4) are determined by the oriented hyperplane arrangement just defined. Since  $M^*(A) = M^*(A')$ , we can establish a correspondence between the cells of the complements of these arrangements that preserves the signs in (4.4) for both  $A$  and  $A'$ , given weights  $\omega$  and  $\omega'$  in corresponding cells. This means that  $\mathcal{C}_{i,\omega} = \mathcal{C}_{i,\omega'}$  as we wanted to show.  $\square$

We note that the conclusion of Theorem 4.7 is also valid on tropical matroid strata where  $A$  is defective. In that case, the  $A$ -discriminant  $\Delta_A$  equals 1, and its degree is the zero vector. We end this section by showing how to obtain our main result on mixed discriminants.

*Proof of Theorem 1.1.* Suppose that  $A$  is the Cayley matrix of  $n$  configurations  $A_1, \dots, A_n$  and let  $\mathcal{I} = \{I_1, \dots, I_n\}$  be the associated partition of  $[m]$ . It follows from (4.2) that

$$\deg_{A_k}(\Delta_A) = \sum_{i \in I_k} \sum_{J \in \mathcal{C}_{i,\omega}} |\det(A^T, e_{J_1}, \dots, e_{J_{m-d-1}}, e_i)|.$$

By the same argument as in the proof of Theorem 4.7, we conclude that the above expression defines a fixed linear form on  $\wedge^d \mathbb{R}^m$  for all matrices  $A$  in a fixed tropical matroid stratum.  $\square$

In closing, we wish to reiterate that combining Theorem 1.1 with Rincón's results in [16] leads to powerful algorithms for computing piecewise polynomial degree formulas. Here is an example that illustrates this. We consider the  $n$ -dimensional version of the system (1.5):

$$f_i = c_{i0} + c_{i1}x_1^{d_i} + c_{i2}x_2^{d_i} + \dots + c_{in}x_n^{d_i} \quad \text{for } i = 1, 2, \dots, n,$$

where  $0 \leq d_1 \leq d_2 \leq \dots \leq d_n$  are coprime integers. The Cayley matrix  $A$  has  $2n$  rows and  $n^2 + n$  columns. Using his software, Felipe Rincón computed the corresponding tropical discriminant for  $n \leq 4$ , while keeping the  $d_i$  as unknowns, and he found

$$\deg_{A_i}(\Delta_{A_1, \dots, A_n}) = d_1 \cdots d_{i-1} d_{i+1} \cdots d_n \cdot (d_i + (-n)d_1 + d_2 + d_3 + \dots + d_n).$$

Thus, we have a computational proof of this formula for  $n \leq 4$ , and it remains a conjecture for  $n \geq 5$ . This shows how the findings of this section may be used in experimental mathematics.

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