

Simulation of quasi-stationary distributions on countable spaces

Pablo Groisman* and Matthieu Jonckheere†

Abstract

Quasi-stationary distributions (QSD) have been widely studied since the pioneering work of Kolmogorov (1938), Yaglom (1947) and Sevastyanov (1951). They appear as a natural object when considering Markov processes that are certainly absorbed since they are invariant for the evolution of the distribution of the process conditioned on not being absorbed. They hence appropriately describe the state of the process at large times for non absorbed paths. Unlike invariant distributions for Markov processes, QSD are solutions of a non-linear equation and there can be 0, 1 or an infinity of them. Also, they cannot be obtained as Cesàro limits of Markovian dynamics. These facts make the computation of QSDs a nontrivial matter. We review different approximation methods for QSD that are useful for simulation purposes, mainly focused on Fleming-Viot dynamics. We also give some alternative proofs and extensions of known results.

Keywords: Quasi-stationary distributions. Fleming-Viot processes. Simulation.

AMS 2000 subject classification numbers: 60K35, 60J25.

1 Introduction

Let $Q = (q(x, y), x, y \in \Lambda \cup \{0\})$ be the rates matrix of a non-exploding continuous time Markov process $Z(\cdot, \mu)$, with initial condition μ on a countable state space $\Lambda \cup \{0\}$. We assume that Q is irreducible in Λ and think of 0 as an absorbing state. We also assume that absorption is certain. As usual we denote $q(x, x) = -\sum_{y \neq x} q(x, y)$.

Let τ^μ be the absorption time defined by $\tau^\mu = \inf\{t > 0: Z(t, \mu) = 0\}$ and $T_t\mu$ the evolution of the process $Z(\cdot, \mu)$ conditioned on survival

$$T_t\mu(x) := P(Z(t, \mu) = x \mid \tau^\mu > t) = \frac{\sum_{z \in \Lambda} \mu(z) P_t(z, x)}{1 - \sum_{z \in \Lambda} \mu(z) P_t(z, 0)}.$$

*Departamento de Matemática, Fac. Cs. Exactas y Naturales, Universidad de Buenos Aires and IMAS-CONICET. pgroisma@dm.uba.ar, <http://mate.dm.uba.ar/~pgroisma>.

†IMAS-CONICET. mjonckhe@dm.uba.ar, <http://matthieujonckheere.blogspot.com>.

Here $P_t = e^{tQ}$ is the semi-group associated with the transitions matrix Q . For μ such that $\sum_z \mu(z)|q(z, z)| < \infty$, using the Kolmogorov equations for Z we obtain the Kolmogorov equations for the conditioned evolution

$$\begin{aligned} \frac{d}{dt}T_t\mu(x) &= \sum_{y \in \Lambda} q(y, x) T_t\mu(y) + \sum_{y \in \Lambda} q(y, 0) T_t\mu(y) T_t\mu(x), & x \in \Lambda, \\ T_0(\mu)(x) &= \mu(x), & x \in \Lambda. \end{aligned} \tag{1.1}$$

In particular, we have the semi-group property $T_{s+t} = T_s(T_t)$. We call $(T_t\mu)$ the *conditioned evolution*. We can hence deduce from (1.1) that any limit point of $T_t\mu$ is a fixed point of $\{T_s, s \geq 0\}$.

Definition 1.1. *For a probability measure ν in Λ , we say that it is a quasi-stationary distribution (QSD) if and only if for any $t \geq 0$ we have $T_t\nu = \nu$.*

Evaluating (1.1) at $t = 0$, we obtain a non-linear equation for ν , i.e. ν is a QSD if and only if it verifies

$$0 = \sum_{y \in \Lambda} q(y, x) \nu(y) + \sum_{y \in \Lambda} q(y, 0) \nu(y) \nu(x). \tag{1.2}$$

Now, if ν is a QSD, it follows from the Markov property that there exists $\theta_\nu > 0$ such that

$$P(\tau^\nu > t) = \exp(-\theta_\nu t).$$

Hence, a necessary condition for the existence of a QSD is the existence of a state $x \in \Lambda$ and $\theta > 0$ such that $E(e^{\theta\tau^x}) < \infty$. Here we slightly abuse notation and write τ^x instead of τ^{δ_x} . Ferrari et. al (see below) showed in [27] that if $\Lambda = \mathbb{N}$ and for every $t > 0$, $\lim_{x \rightarrow \infty} P(\tau^x \leq t) = 0$, then this condition is also sufficient for the existence of a QSD (but does not imply uniqueness). Unfortunately, the additional condition $\lim_{x \rightarrow \infty} P(\tau^x \leq t) = 0$ is not just technical (see [40, p.9] for a simple counter-example). As a consequence, an absorbing process, even with integrable absorption times, might well have no QSD. It was also shown [30, 41, 42] that there are a lot of cases where an infinite number of QSD do exist. These include for instance birth and death process with constant or linear rates and sub-critical branching processes.

We now describe the dynamics of a process that is the cornerstone of several methods of both simulation and theoretical analysis of QSDs.

The μ -return process. Given a probability μ in Λ , consider a Markov process Z^μ with the following dynamics: it evolves according to the dynamics of Z until it hits 0, at which point, it instantaneously takes a new position in Λ with distribution μ . The transitions of this process are then given for $x \neq y$ by:

$$q^\mu(x, y) = q(x, y) + q(x, 0)\mu(y). \tag{1.3}$$

This process appears in the literature since the very beginning of the study of QSD ([8, 21]) but the approach was first discarded, arguing that there is a very weak relation

between the QSD of Q and the invariant distribution of Z^μ and that this last one can be changed into almost any distribution by a suitable choice of μ .

In [27], the authors considered the function that maps $\mu \mapsto \Phi(\mu)$, where $\Phi(\mu)$ is the invariant distribution of Z^μ . In that work, they realized that it was fruitful to study iterations of this map and they showed that ν is a QSD for Q if and only if it is a fixed point of Φ . More importantly, they obtain the following theorem.

Theorem 1.2 (Ferrari, Kesten, Martínez, Picco, [27]). *Assume $\Lambda = \mathbb{N}$ and that $P(\tau^x > t) \rightarrow 1$ as $x \rightarrow \infty$ for every $t > 0$. Then a necessary and sufficient condition for the existence of a QSD is*

$$E(e^{\theta\tau^x}) < \infty, \quad \text{for some } x \in \mathbb{N}, \theta > 0.$$

Their proof strategy is based on the fact that iterations of the map $\Phi^n(\mu)$ are expected to converge to a QSD. Similar approaches were then exploited by different authors and with different goals:

- The analysis of the dynamics of Z^μ is the key elements for the proof of existence and uniqueness of QSD under Doeblin conditions given by Jacka and Roberts [37].
- Asselah and Castell [3] use the functional Φ to prove existence of QSD for asymmetric conservative attractive particle systems in \mathbb{Z}^d . They work with Cesàro means of the iterations $\Phi^n(\mu)$ rather than with the sequence itself. They prove that limit points of the Cesàro means are QSD and they also used this fact to prove properties (regularity) of the QSD relying on a priori bounds.
- Aldous, Flannery and Palacios use also this idea to construct a method to simulate the QSD in finite spaces [1]. We give more details in Section 3.
- Fleming-Viot dynamics are also linked to the dynamics of Z^μ . In fact, the distribution of a tagged particle of a stationary Fleming-Viot system with N particles follows the dynamics Z^μ , with μ being the empirical distribution of the process (in this case μ depends on time and is random and correlated with the tagged particle). When FV is in equilibrium, the tagged particle is expected to converge to Z^ν when N goes to infinity, where ν is a QSD. In Section 2.3 we prove this fact for some models.
- Also [2, 7] deal with this map to approximate QSDs.

As it can be seen, the μ -return process has motivated a lot of work in the field of QSDs in general and in particular in the problem of their simulation, which is non trivial since they can not be simulated directly, using for example the rejection method.

In the sequel, we first review some results obtained in that direction in recent years. We mainly focus on Fleming-Viot (FV) dynamics, described in Section 2, as they constitute not only a method to practically simulate the QSD and a theoretical tool to study QSDs, but also an interesting particle system raising many stimulating questions.

We then provide a new proof for the convergence of Fleming-Viot dynamics to the conditioned evolution in compact time intervals and study the limiting behavior of a FV tagged particle. In Section 3, we explain the principles of two alternative methods for the computation of the QSD. Section 4 deals with related open problems.

An updated bibliography on QSDs can be found at

<http://www.maths.uq.edu.au/~pkp/papers/qlds/qlds.pdf>.

2 Fleming-Viot dynamics

We define the *Fleming-Viot* process driven by Z (or Q) as follows. It consists of N particles moving independently of the others as a continuous time Markov process with rates Q , until one attempts to jump to state 0. At that time, it immediately jumps back to Λ by choosing the position of one of the other particles chosen uniformly at random and starts afresh from this configuration. This process is Markov with state space Λ^N (here N is the number of particles) and its state at time $t \geq 0$ is denoted $\xi(t, \cdot)$. The variable $\xi(t, i)$ denotes the position of the i -th particle at time t . The generator \mathcal{L}^N acts on functions $f : \Lambda^N \rightarrow \mathbb{R}$ as follows

$$\mathcal{L}^N f(\xi) = \sum_{i=1}^N \sum_{x \in \Lambda} \left[q(\xi(i), x) + q(\xi(i), 0) \frac{\sum_{j \neq i}^N \mathbf{1}_{\{\xi(j)=x\}}}{N-1} \right] (f(\xi^{i,x}) - f(\xi)), \quad (2.1)$$

where $\xi^{i,x}(i) = x$, and for $j \neq i$, $\xi^{i,x}(j) = \xi(j)$. We use $\{S_t^N, t \geq 0\}$ for the semi-group of FV, i.e.

$$S_t^N f(\xi) := E(f(\xi(t)) | \xi(0) = \xi)$$

The well definiteness of the process (given that the driving process is well defined) is immediate. Nevertheless we refer the reader to [4, 5, 29] or Section 2.3 for (several) graphical constructions that are useful to infer different properties of the process.

These dynamics were originally introduced by Burdzy, Ingemar, Holyst and March in the context of Brownian motion in a bounded domain in [18], where it was conjectured that the empirical measure should converge to a deterministic solution of a PDE that represents the distribution of a Brownian particle conditioned on non-absorption.

As it is described here, the process is a variation of the original measure-valued process introduced by Fleming and Viot [31] (see also [26]). Many alternative dynamics have been later proposed and studied. Their common feature is that all the particles evolve independently according to a common driving process until a *selection mechanism* takes place, replacing particles having specific positions or potentials. In Moran systems for example, particles branch according to a state-dependent rate and are replaced according to a fitness functional of the positions. Brunet, Derrida and coauthors [13, 14, 15, 16] on the other hand, introduced a system where particles evolve driven by one dimensional Brownian Motion and branch with a fixed rate, at which time the left-most particle is instantaneously eliminated. Although these systems are simpler to analyze than FV (in fact much finer results have been obtained for this

model [9, 39]), they can only be defined for random walks in \mathbb{R}_+ (including Brownian Motion) as driving process.

Such models appear actually at the corner of several scientific fields. They were originally introduced in genetics to study the propagation of genes in a population. In a completely different context, they may represent the evolution of particles in a medium with an obstacle, and are linked with traveling waves equations in physics [13]. For more details, refer to [24] and the references therein.

Coming back to FV, a generic question is hence to understand for a given driving Markov process, how the empirical density of the FV process at fixed times or under its stationary regime (if it exists) approximates the conditioned evolution or the QSD of the driving process respectively. This line of research was pursued by several authors and the conjecture was settled for many cases of interest where the driving process *has a unique QSD*, like for jump processes with finite state spaces [4], chains with Doeblin conditions [29] and diffusion processes on bounded domains [10, 19, 33, 34].

An important and challenging problem then consists in studying the FV process when the driving process has an infinite number of QSDs. This question was recently answered for the case of a driving process given by sub-critical branching. In [5] it is shown that FV driven by this process is ergodic and that the empirical distribution of particles in equilibrium converges to the *minimal QSD* in the following sense.

Definition 2.1. *A QSD ν^* is called minimal if*

$$E(\tau^{\nu^*}) = \inf\{E(\tau^\nu) : \nu \text{ verifies (1.2)}\}.$$

This phenomenon is called *selection principle*, in the sense that FV “selects” the minimal QSD among all the QSD. This fact also gives evidence in favor of the minimal QSD to be interpreted as the one that represents the (quasi) stationary behavior of the process before absorption takes place.

Denote for $x \in \Lambda$ and $\xi \in \Lambda^N$

$$m(x, \xi) := \frac{1}{N} \sum_{i=1}^N \mathbf{1}\{\xi(i) = x\} \quad (2.2)$$

the proportion of ξ -particles at position x . We also think of $m(\cdot, \xi)$ as a probability measure in Λ . With a slight abuse of notation, for a measure γ^N in Λ^N we write $m(x, \gamma^N)$ for the random measure $m(x, \xi)$ where ξ is a random configuration of particles with distribution γ^N . Applying the generator of FV to the empirical distribution, we get

$$\mathcal{L}^N m(y, \cdot)(\xi) = \sum_{x \in \Lambda} q(x, y) m(x, \xi) + \frac{N}{N-1} \sum_{x \in \Lambda} q(x, 0) m(x, \xi) m(y, \xi), \quad (2.3)$$

which looks very similar to (1.1). This provides another convincing heuristic for the approximation of the conditioned evolution and QSD by FV.

It may be enlightening at this point to emphasize one of the main difficulties encountered when studying Fleming-Viot dynamics. Contrary to all the other mentioned

genetic interacting jumps particle models, the FV process is not attractive: natural partial orders are not preserved by trajectories of the processes started from ordered initial positions. This complicates drastically the analysis and impedes the construction of simple coupling and stochastic comparisons for proving convergence properties. Another consistent difficulty arises because of the stochastic dependence between the times at which the selection mechanism takes place and the trajectories of the particles. As a consequence, the well definiteness of the process for all times when it is driven by diffusions is a difficult problem¹ [10, 11, 33]. These difficulties can also be thought as somehow linked to the critical nature of those dynamics where the population size is kept constant.

In the rest of this section, we aim at discussing:

1. convergence of the empirical distribution for fixed time intervals,
2. existence of an invariant measure for fixed N and convergence towards this equilibrium.
3. convergence of the empirical distribution in equilibrium towards the minimal QSD (see (2.16) below);
4. the behavior of a tagged particle.

We provide a new proof of convergence with order $1/\sqrt{N}$ for the first item and establish via a simple coupling argument the limiting distribution of a tagged particle. We also review the recent results obtained for long-time behaviors of the empirical distribution.

2.1 Convergence to the conditioned evolution

In this section, we investigate the convergence of the empirical distribution for fixed time intervals. In [43], Villemonais proves a general result of convergence to the conditioned evolution with a rate of order $1/\sqrt{N}$. The proof in [43] is based on martingale arguments under the assumption that the number of jumps of the FV process does not explode in compact time intervals. We here propose a new proof (leading to the same speed of convergence) working under quite general technical conditions on the transition rates and based on different techniques. It uses in particular the method of supersolutions.

Theorem 2.2. *Assume $\sum_{y \in \Lambda} q(y, x) \leq C$ for all $x \in \Lambda \cup \{0\}$. Let μ be a probability in Λ such that $-\sum_z \mu(z)q(z, z) < \infty$, then*

$$\sup_{0 \leq t \leq T} E |m(x, \xi(t)) - T_t \mu(x)| \leq e^{7CT} \left(\frac{3}{\sqrt{N}} + \|\mu - m(\xi(0))\|_{TV} \right) \quad (2.4)$$

Remark 2.3. *Note that the technical condition $\sum_{y \in \Lambda} q(y, x) \leq C$ is valid for a wide class of processes. In particular, we do not assume that rates are uniformly bounded nor the condition $\sum_{y \in \Lambda} q(x, y) \leq C$.*

¹ In countable spaces however, this follows immediately but this fact illustrates the possible complications that can arise with this dynamics.

The proof of Theorem 2.2 is based on the following propositions. The first one is a useful decorrelation result proved in [4] and the second one proves the closeness between the semigroups associated with the conditional evolution of the driving process and the dynamics of the empirical measure of the FV process.

Proposition 2.4 (Proposition 2 of [4]). *Let $C_0 := \sup_{x \in \Lambda} q(x, 0)$. For each $t > 0$, and any $x, y \in \Lambda$*

$$\sup_{\xi \in \Lambda^N} |S_t^N(m(x, \cdot)m(y, \cdot))(\xi) - S_t^N m(x, \cdot)(\xi) S_t^N m(y, \cdot)(\xi)| \leq \frac{2e^{2C_0 t}}{N}. \quad (2.5)$$

Remark 2.5. *Proposition 2.4 is proved in [4] in fact for processes with bounded rates, but the proof can be extended to this case with no difficulty.*

Proposition 2.6. *Assume that $\sum_{y \in \Lambda} q(y, x) \leq C$ for all $x \in \Lambda \cup \{0\}$. If $-\sum_z \mu(z)q(z, z) < \infty$, then*

$$\sup_{0 \leq t \leq T} |Em(x, \xi(t)) - T_t \mu(x)| \leq e^{7CT} \left(\frac{1}{N} + \|\mu - m(\xi(0))\|_{TV} \right) \quad (2.6)$$

Proposition 2.6 is based on the following lemma.

Lemma 2.7. *Assume that $\sum_{y \in \Lambda} q(y, x) \leq C$ for all $x \in \Lambda \cup \{0\}$. Let $e: [0, T] \times \Lambda \rightarrow \mathbb{R}$ be a function such that for $x \in \Lambda$, $t \in [0, T]$*

$$\frac{\partial}{\partial t} e(t, x) \geq \sum_{z \in \Lambda} q(z, x)e(t, z) + c_1(t, x)e(t, x) + c_2(t, x) \sum_{z \in \Lambda} q(z, 0)e(t, z), \quad (2.7)$$

and $e(0, x) \geq 0$ for $x \in \Lambda$, with c_1 bounded on $[0, T]$ and c_2 bounded and nonnegative. Then $e(t, x) \geq 0$ for all $t \in [0, T]$, $x \in \Lambda$.

Proof. First, we can assume without loss of generality that $C + c_1 + c_2 < 0$ in $[0, T]$. If not, we consider $w(t, x) := \exp(-\lambda t)e(t, x)$, which verifies the same inequality with c_1 replaced by $c_1 - \lambda$. Second, we can also assume that we have strict inequalities in (2.7). If not we consider $e^\varepsilon(t, x) = e(t, x) + \varepsilon$ which verifies (2.7) with strict inequalities thanks to $C + c_1 + c_2 < 0$. Since $e^\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, if we prove $e^\varepsilon(t, x) \geq 0$ for all t, x, ε , this implies our conclusion.

So, assuming that the conclusion of the lemma is false, there is a first time t^* and a state x^* such that $e(t^*, x^*) = 0$. At that time we have

$$0 \geq \frac{\partial}{\partial t} e(t^*, x^*) > \sum_{z \neq x^*} q(z, x^*)e(t^*, z) + c_2(t^*, x^*) \sum_{z \in \Lambda} q(z, 0)e(t^*, z) \geq 0,$$

a contradiction that concludes the proof. ■

Proof of Proposition 2.6. Define

$$u(t, x) = S_t^N (m(x, \cdot)) (\xi), \quad \text{and} \quad v(t, x) = T_t \mu(x) \quad (2.8)$$

and let $\delta(t, x) = u(t, x) - v(t, x)$. Recall that

$$\frac{\partial}{\partial t} v(t, x) = \sum_{z \in \Lambda} q(z, x) v(t, z) + \sum_{z \in \Lambda} q(z, 0) v(t, z) v(t, x), \quad (2.9)$$

$$\frac{\partial}{\partial t} u(t, x) = \sum_{z \in \Lambda} q(z, x) u(t, z) + \sum_{z \in \Lambda} q(z, 0) u(t, z) u(t, x) + R(\xi; x, t), \quad (2.10)$$

where,

$$R(\xi; x, t) = \sum_{z \in \Lambda} q(z, 0) \left(\frac{N}{N-1} S_t^N (m(z, \cdot) m(x, \cdot)) (\xi) - S_t^N m(1, \cdot) (\xi) S_t^N m(x, \cdot) (\xi) \right). \quad (2.11)$$

Proposition 2.4 implies that

$$\sup_{\xi} |R(\xi; x, t)| \leq \frac{2C_0 e^{2C_0 t}}{N} + \frac{C_0}{N-1} \leq \frac{4C_0 e^{2C_0 t}}{N}. \quad (2.12)$$

Then, for $\delta(t, x)$ we get that:

$$\frac{\partial}{\partial t} \delta(t, x) = \sum_{z \in \Lambda} q(z, x) \delta(t, z) + \sum_{z \in \Lambda} q(z, 0) [u(t, z) \delta(t, x) + v(t, x) \delta(t, z)] + R(\xi; x, t). \quad (2.13)$$

Call $C_N := \frac{1}{N} + \|\mu - m(\xi(0))\|_{TV}$ and consider the *supersolution* given by $\bar{\delta}(t, x) = \bar{\delta}(t) := C_N \exp(7Ct)$. We have

$$\begin{aligned} & \sum_{z \in \Lambda} q(z, x) \bar{\delta}(t, z) + \sum_{z \in \Lambda} q(z, 0) [u(t, z) \bar{\delta}(t, x) + v(t, x) \bar{\delta}(t, z)] + R(\xi; x, t) = \\ & \bar{\delta}(t) \sum_{z \in \Lambda} q(z, x) + \bar{\delta}(t) \sum_{z \in \Lambda} q(z, 0) u(t, z) + \bar{\delta}(t) v(t, x) \sum_{z \in \Lambda} q(z, 0) + R(\xi; x, t) \leq \\ & C_N e^{7Ct} \left(\sum_{z \in \Lambda} q(z, x) + \sum_{z \in \Lambda} q(z, 0) u(t, z) + v(t, x) \sum_{z \in \Lambda} q(z, 0) \right) + \frac{4C_0 e^{2C_0 t}}{N} \leq \\ & C_N 7C e^{7Ct} = \frac{d}{dt} \bar{\delta}(t, x). \end{aligned} \quad (2.14)$$

In the last line we have used that $C_0 \leq C$ and $C_N \geq 1/N$. Hence the function $e(t, x) = \bar{\delta}(t, x) - \delta(t, x)$ verifies (2.7) with $c_1(t) = \sum_{z \in \Lambda} q(z, 0) u(t, z)$ and $c_2(t) = v(t, x)$. Lemma 2.7 then implies that

$$\delta(t, x) \leq e^{7Ct} \left(\frac{1}{N} + \|\mu - m(\xi(0))\|_{TV} \right), \quad x \in \Lambda, t \in [0, T].$$

Proceeding in the same way with $\tilde{e}(t, x) := \bar{\delta}(t, x) + \delta(t, x)$ instead of e we obtain the opposite inequality and hence

$$|\delta(t, x)| \leq e^{7Ct} \left(\frac{1}{N} + \|\mu - m(\xi(0))\|_{TV} \right), \quad x \in \Lambda, t \in [0, T]. \quad (2.15)$$

■

We can now proceed to the proof of the theorem.

Proof of Theorem 2.2. We apply the triangular inequality to get that for some $\kappa > 0$ and every $t \in [0, T]$

$$E [|m(x, \xi(t)) - T_t \mu(x)|] \leq E [|m(x, \xi(t)) - u(t, x)|] + |u(t, x) - v(t, x)|.$$

The control of the first term follows from Proposition 2.4 (with $y = x$)

$$\begin{aligned} E [|m(x, \xi(t)) - u(t, x)|] &\leq E \left[|m(x, \xi(t)) - u(t, x)|^2 \right]^{1/2} \\ &= |E(m(x, \xi(t))^2) - u(t, x)^2|^{1/2} \\ &\leq \sqrt{2/N} e^{C_0 t} \end{aligned}$$

while the second term follows from (2.15).

■

2.2 Long-time behavior

The motivation for studying FV steams in particular from the exciting possibility of approximation of QSD, which are in general difficult objects to simulate. The kind of result one expects concerning the long-time behavior are

1. For each $N \geq 1$, the Fleming-Viot process is ergodic, with invariant measure λ^N .
2. Let ν^* be the minimal QSD for the driving process, then

$$\lim_{N \rightarrow \infty} \int |m(x, \xi) - \nu^*(x)| d\lambda^N(\xi) = 0. \quad (2.16)$$

A unique QSD. For Markov jump processes, these results have been proven in the case of finite state space in [4] and for processes satisfying a Doeblin condition in [29]. For finite state space, ergodicity is immediate while it follows from a “perfect simulation” argument in [29]. To prove the convergence of the empirical distribution under the invariant measure of FV is a much more involved issue. In both [4, 29], the key element of the proof is a control of the correlations between particles. In the case of a finite state space, the control of the correlations for a fixed time is obtained uniformly on the set of initial distributions (see 2.4). This allows to extend this control under the stationary measure of FV. In [4], this step is done via a regeneration argument strongly connected to the Doeblin condition.

An infinity of QSD: the Galton-Watson case. Many processes do actually have an infinity of quasi-stationary measures. The convergence mentioned in (2.16) hence means that FV *selects* the minimal QSD. Proving these results becomes in this case much more challenging as one needs to prove not only tightness but also that the empirical measure of FV under the stationary measure belongs to the domain of attraction of the minimal QSD.

An important example of processes owning an infinity of QSD is the subcritical branching process. For a full description of the number of QSD for branching processes see [41], and for general birth and death processes see [20, 30, 38, 42].

Up to our knowledge, the only result that deals with convergence of FV in equilibrium when the QSD is not unique is the following one, which is proved in [5].

Theorem 2.8 (Asselah, Ferrari, Groisman, Jonckheere, [5]). *Consider a subcritical Galton- Watson process whose offspring law has some finite positive exponential moment. Let ν^* be the minimal QSD for the process conditioned on non-extinction. Then*

1. *For each $N \geq 1$ the associated N -particle Fleming-Viot process is ergodic with invariant measure λ^N .*
2. *The empirical measure of FV converges to a deterministic measure which is the minimal QSD of the driving process:*

$$\lim_{N \rightarrow \infty} \int |m(x, \xi) - \nu^*(x)| d\lambda^N(\xi) = 0. \quad (2.17)$$

There are many other open questions concerning the long-time behavior of FV for large N . We discuss them in Section 4.

2.3 The tagged particle limit

We now consider a particular particle (say particle 1) that we call the *tagged particle*. We show that under the invariant measure of FV, the tagged particle converges to a limiting process that has the QSD distribution as marginal and is absorbed at exponential times. At those times, it jumps back instantaneously to Λ according to the QSD. A similar result holds out of equilibrium, but in this case the “return“ jumps are according to the conditioned evolution. More precisely, the limiting process, $(Y(t, \mu), t \geq 0)$ is a continuous time non-homogeneous Markov jump process with initial condition μ and rates given by

$$q_t^\mu(x, y) = q(x, y) + q(x, 0)T_t\mu(y), \quad x, y \in \Lambda. \quad (2.18)$$

Observe that if μ is a QSD, then $q_t^\mu(x, y)$ is time-independent and also μ is invariant for (2.18). Moreover, in this case (2.18) are the rates of the μ - return process described in the introduction.

This problem has been previously addressed by Grigorescu and Kang [35] when the driving process is Brownian Motion in a bounded domain.

In order to state our result we need to consider the Skorohod space $D = D(\mathbb{R}^+, \Lambda)$ of right-continuous functions from $\mathbb{R}^+ \rightarrow \Lambda$ with finite left limits, equipped with the Skorohod topology [12]. We also introduce the notation ξ^γ for a FV process with generator (2.1) and initial condition distributed according to γ .

Proposition 2.9. *Assume that $C_0 := \sup_{x \in \Lambda} q(x, 0) < \infty$ and let γ^N be a sequence of measures in Λ^N such that*

$$\lim_{N \rightarrow \infty} \sup_{0 \leq t \leq T} E |m(x, \xi^{\gamma^N}(t)) - T_t \mu(x)| = 0 \quad \text{for all } x \in \Lambda. \quad (2.19)$$

Assume further that $\xi^{\gamma^N}(0, 1)$ converges in distribution to a random variable with distribution μ . Then the tagged particle $\xi^{\gamma^N}(\cdot, 1)$ converges in distribution to the process $Y(\cdot, \mu)$ as elements in D .

Remark 2.10. *Hence Proposition 2.9 holds for every jump Markov process that verifies (2.4) if $m(x, \xi^{\gamma^N}) \rightarrow \mu$, which include a large class of absorbing processes as shown by Theorem 2.2.*

As a consequence of Proposition 2.9 we obtain the following theorem.

Theorem 2.11. *Assume that for each $N \geq 1$, the FV process has an invariant measure λ^N and that for every x*

$$\lim_{N \rightarrow \infty} \int |m(x, \xi) - \nu(x)| d\lambda^N(\xi) = 0. \quad (2.20)$$

Then, under λ^N the tagged particle converges in distribution to the stationary process $Y(\cdot, \nu)$ as $N \rightarrow \infty$.

Remark 2.12. *Theorem 2.11 holds when one assumes convergence of FV in equilibrium. This is not known with the same generality that the fixed time case Theorem 2.2 but holds at least for the models of [4, 5, 29]. Of particular interest is the case of the driving process being a sub-critical branching process where an infinite number of QSD arise. In this case the results of [5] combined with Theorem 2.11 gives that the tagged particle in equilibrium converges to the stationary ν^* -return process, the probability ν^* being the minimal QSD.*

2.3.1 Graphical construction

The proof follows by coupling the trajectory of the limiting Markov process defined previously and the trajectory of a tagged particle of FV. To make the proof rigorous, we need to carefully build the coupling using a so called graphical construction. We first describe the construction for the FV process.

Recall that $C_0 = \sup_{x \in \Lambda} q(x, 0)$ is the (maximum) absorption rate, and let

$$\bar{q} := \sup_{x \in \Lambda} \sum_{y \in \Lambda \setminus \{x\}} q(x, y); \quad p(x, y) := \frac{q(x, y)}{\bar{q}}, \quad y \neq x; \quad p(x, x) := 1 - \sum_{y \in \Lambda \setminus \{x\}} p(x, y).$$

We assume for simplicity that $\bar{q} < \infty^2$. To each particle i , we associate two independent marked Poisson processes (ω_i^I, ω_i^V) on \mathbb{R}_+ , which we call respectively the *internal* and *voter* point processes, described as follows.

- For each i , the internal Poisson process has intensity \bar{q} . The marks are i.i.d uniform variables in $[0, 1]$. We use U_i^k for the mark of the k -th occurrence of ω_i^I . If the Poisson process rings at time t_k^i with a mark u_i^k whereas particle i is at x , then it jumps to y if and only if $u_i^k \in I(x, y)$, where for each $x \in \Lambda$, $\{I(x, y) : y \in \Lambda\}$ is any partition of $[0, 1]$ composed of Borel sets such that $|I(x, y)| = p(x, y)$. For details see [28, Chapter 1].
- The voter point process is also defined for each i . It has intensity C_0 and the marks are i.i.d. uniform variables on $[0, 1] \times [0, 1]$. We denote them by (B_i^k, V_i^k) . If ω_i^V rings at time t_k^i , with a mark (b_i^k, v_i^k) whereas particle i is at x , then it keeps its position if $b_i^k > \frac{q(x, 0)}{C_0}$. If not, it jumps to zero and instantaneously comes back to Λ by jumping to y if and only if $v_i^k \in J_i(t, y)$. Here $\{J_i(t, y) : y \in \Lambda\}$ is a (random) partition of $[0, 1]$ by Borel sets such that $|J_i(t, y)| = \frac{1}{N-1} \sum_{j \neq i} \mathbf{1}\{\xi(t, j) = y\}$. According to this procedure, particle i jumps to y with rate

$$\bar{q} \frac{q(\xi(t, i), y)}{\bar{q}} + C_0 \frac{q(\xi(t, i), 0)}{C_0} \frac{1}{N-1} \sum_{j \neq i} \mathbf{1}\{\xi(t, j) = y\},$$

which coincides with the rates of (2.1).

We call $\omega = ((\omega_i^I, \omega_i^V), i \in \{1, \dots, N\})$ an i.i.d. sequence of marked PP.

For any $s < t$ real numbers, we denote by $\omega_i[s, t)$ and $\omega_i[s, t]$ the projections of the marked-times in the time period $[s, t)$ and $[s, t]$, respectively.

We construct $\{\xi(t), t \geq 0\}$ in such a way that $\xi(t)$ is a function of the initial configuration ξ and the time-marks $\omega[0, t]$, $t \geq 0$. Fix an initial configuration $\xi \in \Lambda^N$, and $t > 0$. There is, almost surely, a finite number of time-marks within $[0, t]$, say K ; let $\{s^k, 0 \leq k \leq K\}$ be the ordered time-realizations together with an extra $s^0 = 0$. We build $\xi(\cdot)$ inductively as follows.

- At time $s^0 = 0$, the configuration is $\xi(0) = \xi$.
- Assume $\xi(s^k)$ is known. For $s \in [s^k, s^{k+1})$, set $\xi(s) = \xi(s^k)$. We describe now $\xi(s^{k+1})$.
 - If s^{k+1} corresponds to an internal time of particle i and mark u , we move particle i to y if and only if $u \in I(\xi(s^k, i), y)$.
 - If s^{k+1} corresponds to a voter time of particle i and mark (b, v) , we move particle i to the position y if and only if $b \leq \frac{q(x, 0)}{C_0}$ and $v \in J_i(t, y)$.

It is straightforward to check that $\{\xi(t), t \geq 0\}$, as constructed above, has generator given by (2.1), see [36].

²this is not needed but otherwise once has to define a marked Poisson process for each potential position of a particle which complicates the exposition.

2.3.2 The limiting process

We now construct jointly with FV the limiting process $\{Y(t, \mu), t \geq 0\}$ with initial condition μ as a deterministic function of (ω_1^I, ω_1^V) and a uniform random variable U . When is not necessary, we omit to display the dependence on μ and write just $Y(\cdot)$. Let $\{s_1^k, 0 \leq k \leq \tilde{K}\}$ be the ordered time realizations of (ω_1^I, ω_1^V) in $[0, t]$, with $s_1^0 = 0$.

- In order to couple this process with FV, we use the same variable U uniformly distributed in $[0, 1]$ to define

$$\xi(0, 1) = y \iff U \in J_1(0, y) \quad \text{and} \quad Y(0, \mu) = y \iff U \in \tilde{J}(0, y).$$

The sets $J_1(0, y)$ and $\tilde{J}(0, y)$ are defined below. To get the correct distribution for the initial configuration of FV (γ^N) we construct the configuration of the remaining particles with the conditional distribution $\gamma^N|_{\xi(0,1)}$ (the procedure used to do this is not relevant). Next we describe the evolution of the limiting process.

- Assume $Y(s_1^k)$ is known. For $s \in [s_1^k, s_1^{k+1})$, set $Y(s) = Y(s_1^k)$. We describe now $Y(s_1^{k+1})$.
 - If s_1^{k+1} corresponds to an internal time of particle 1 and mark u , then Y jumps to y if and only if $u \in I(Y(s_1^k), y)$.
 - If s_1^{k+1} corresponds to a voter time of particle 1 and mark (b, v) , then Y jumps to y if and only if $b \leq \frac{q(x, 0)}{C_0}$ and $v \in \tilde{J}(t, y)$. Here $\{\tilde{J}(t, y) : y \in \Lambda\}$ is a partition of $[0, 1]$ by Borel sets such that $|\tilde{J}(t, y)| = T_t \mu(y)$.

It is also straight-forward to see that $\{Y(t), t \geq 0\}$ has rates given by (2.18).

Observe that if $v_1^k \in \cup_{y \in \Lambda} [J_1(s_1^k, y) \cap \tilde{J}(s_1^k, y)]$ for every $0 \leq k \leq \tilde{K}$, then $\xi(\cdot, 1) = Y(\cdot)$ in $[0, t]$. So we construct $J_1(\cdot, y)$ and $\tilde{J}(\cdot, y)$ so that the intersection is large if the respective distributions are close. We consider a numbering of the elements of Λ ($y_l, l \geq 1$) and define for $t = 0$

$$J_1(0, y_l) = \left[\sum_{r=1}^{l-1} \mu_N(y_r), \sum_{r=1}^l \mu_N(y_r) \right),$$

$$\tilde{J}(0, y_l) = \left[\sum_{r=1}^{l-1} \mu(y_r), \sum_{r=1}^l \mu(y_r) \right).$$

Here μ_N is the distribution of $\xi(0, 1)$. For $t > 0$ these sets take the form

$$J_1(t, y_l) = \left[\sum_{r=1}^{l-1} \frac{\sum_{i>1} \mathbf{1}\{\xi(t, i) = y_r\}}{N-1}, \sum_{r=1}^l \frac{\sum_{i>1} \mathbf{1}\{\xi(t, i) = y_r\}}{N-1} \right),$$

$$\tilde{J}(t, y_l) = \left[\sum_{r=1}^{l-1} T_t \mu(y_r), \sum_{r=1}^l T_t \mu(y_r) \right).$$

We also define

$$\psi(t) := 1 - \mathbf{1}\{v_1^k \in \cup_{y \in \Lambda} [J(t_1^k, y) \cap \tilde{J}(t_1^k, y)] \text{ for every } k \text{ with } t_1^k \leq t\}.$$

Note that $\psi(t) = 1$ if the trajectories of both processes do not coincide on $[0, t]$. We have the following equation for the evolution of ψ .

Lemma 2.13.

$$\mathbb{P}(\psi(t) = 1) \leq \mathbb{P}(\psi(0) = 1) + C_0 \int_0^t E \left[\sum_{x \in \Lambda} |m(x, \xi^{\gamma^N}(s)) - T_s(\mu)(x)| \right] ds. \quad (2.21)$$

Proof. Since $\psi(s-)$ is $\sigma([0, s])$ -measurable, we have

$$\mathbb{P}(\psi(t) = 1) = \mathbb{P}(\psi(0) = 1) + \int_0^t E \left[\frac{d}{ds} \mathbb{P}(\psi(s) = 1 | \sigma(\omega[0, s])) \right] ds, \quad (2.22)$$

and

$$\begin{aligned} & \frac{d}{ds} \mathbb{P}(\psi(s) = 1 | \sigma(\omega[0, s])) \\ &= \sum_{y \in \Lambda} q(y, 0) \mathbf{1}\{Y(s-) = y, \psi(s-) = 0\} \sum_{x \in \Lambda} |m(x, \xi^{\gamma^N}(s-)) - T_s(\mu)(x)| \\ &\leq C_0 \sum_{x \in \Lambda} |m(x, \xi^{\gamma^N}(s-)) - T_s(\mu)(x)|. \end{aligned} \quad (2.23)$$

Observe that this also justifies the interchange of E and $\frac{d}{ds}$ in (2.22) and completes the proof. ■

Proof of Proposition 2.9. Recall that to prove convergence in \mathbb{D} , it is enough to show convergence in $\mathbb{D}_T = \mathbb{D}([0, T], \Lambda)$, the Skorohod space of càdlàg functions defined in $[0, T]$ with values in Λ , for every $T > 0$, cf. [12, Ch.3, Sec 16]. So, let $G_T: \mathbb{D}_T \rightarrow \mathbb{R}$ be a bounded, continuous function and define

$$A_N(T) := |G_T(\xi(\cdot, 1)) - G_T(Y(\cdot, \mu))|.$$

Using that when $\psi(t) = 0$, the trajectories of $\xi(\cdot, 1)$ and $Y(\cdot, \mu)$ coincide on $[0, t]$, we get that

$$\begin{aligned} E(A_N(T)) &\leq E(A_N(T) \mathbf{1}\{\psi(T) = 0\}) + E(A_N(T) \mathbf{1}\{\psi(T) = 1\}) \\ &\leq 2 \|G_T\|_\infty P(\psi(T) = 1) \\ &\leq 2 \|G_T\|_\infty \left(\sum_{x \in \Lambda} |\mu_N(x) - \mu(x)| + C_0 T \sup_{0 \leq t \leq T} \sum_{x \in \Lambda} E(|m(x, \xi^{\gamma^N}(t)) - T_t \mu(x)|) \right). \end{aligned}$$

The bound in the last line stems from Lemma 2.13. Now, since $\{T_t \mu, 0 \leq t \leq T\}$ is tight (as probability measures in Λ) and $m(\cdot, \gamma^N) \rightarrow \mu$, the convergence (2.19) implies

that $\{E(m(x, \xi^{\gamma^N}(t))), 0 \leq t \leq T, N \geq 1\}$ is also tight and hence, for any $\tilde{\varepsilon} > 0$, there exists a finite set Λ_F such that

$$\sup_{0 \leq t \leq T} \sum_{x \in \Lambda_F^c} \mu_N(x) + \mu(x) + E(m(x, \xi^{\gamma^N}(t))) + E(T_t \mu(x)) < \tilde{\varepsilon}.$$

Next we use again the pointwise convergence (2.19) for $x \in \Lambda_F$ and that $\mu_N \rightarrow \mu$ to get

$$\limsup_{N \rightarrow \infty} E(A_N(T)) \leq \tilde{\varepsilon}.$$

This concludes the proof. ■

Proof of Theorem 2.11. Since for each N , the measure λ^N is exchangeable and the empirical measure $m(\cdot, \lambda^N)$ distributed according to λ^N converges to ν in probability, we have that the tagged particle $\xi(1)$ in equilibrium converges in distribution to ν . So we can apply Proposition 2.9 with $\mu = \nu$. ■

3 Other ways to approximate QSDs

Since the QSD are eigenvalues of the transition matrices, in finite (small) spaces the computation of the (unique) QSD can be addressed with standard (or not so standard, but generic) methods. However this can not be done (at least a priori) when the state space is finite but huge or countable. In this section we introduce two alternatives to FV dynamics to simulate the QSD.

3.1 The method of Aldous-Flannery-Palacios

This method was proposed in [1] to simulate the quasi-stationary distribution of a finite Markov chain in discrete time. To simplify the exposition, in this section we consider Z_n as a discrete time Markov chain in a finite state space $\Lambda \cup \{0\}$ with transition matrix \hat{P} . We use \tilde{P} for the restriction of \hat{P} to Λ and we assume \tilde{P} is irreducible and aperiodic. Hence, from Perron-Frobenius theorem we have the existence of a unique quasi-stationary probability measure ν that verifies

$$\sum_{x \in \Lambda} \nu(x) \tilde{P}(x, y) = \lambda \nu(y), \quad y \in \Lambda. \quad (3.1)$$

So, in order to compute ν one has to solve (3.1), but this is unpractical if Λ is large. Also the acceptance-rejection method is not useful since $P(\tau > n)$ goes to zero exponentially fast.

The method of Aldous-Flannery-Palacios is motivated on the μ -return process described in the introduction, i.e. the renewal process Z_n^μ with transition matrix given by

$$\hat{P}^\mu(x, y) = \hat{P}(x, y) + \hat{P}(x, 0)\mu(y).$$

That, is Z_n^μ evolves as a Markov chain with transitions given by \hat{P} but when it attempts to jump to zero, it comes back instantaneously to Λ according to μ . This is an ergodic Markov chain with invariant measure $\Phi(\mu)$. If μ coincides with the QSD ν then it is invariant for \hat{P}^ν . The reciprocal also holds, i.e.: ν is a QSD if and only if $\Phi(\nu) = \nu$. This is the key fact used in [27] to prove their general existence result. The same fact is useful here to build a method approximating the QSD. The main idea is to replace ν in the “return” process (which is only used to define the dynamics at absorption times) by the empirical distribution of the history of the process.

Let M be the set of counting measures on Λ , and let (V_n, μ_n) be the $\Lambda \times M$ -valued Markov chain with transitions defined by

$$P(V_{n+1} = y, \mu_{n+1} = \mu + \delta_y | V_n = x, \mu_n = \mu) = \hat{P}(x, y) + \hat{P}(x, 0) \frac{\mu(y)}{|\mu|}, \quad (3.2)$$

where $|\mu|$ is the total mass of μ . In words, V_n evolves as a Markov chain with transition matrix \hat{P} , but when it attempts to jump to zero, it comes back to Λ instantaneously according to the empirical distribution of the history of the process μ .

The following theorem was obtained by Aldous, Flannery and Palacios concerning this process.

Theorem 3.1 (Aldous, Flannery and Palacios, [1]). *For the process defined by (3.2) we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mu_n = \nu.$$

The proof follows by embedding μ_n in the skeleton of a supercritical multitype branching process with offspring distribution given by $L(x, y) =$ number of visits to the state y , for the process with transition matrix P , started at x . Details can be found in [1].

This method, with improvements to gain efficiency was implemented by in [23, 25] to compute in particular the conditioned evolution and QSD of the contact process in different graphs. No proof of convergence are available in this case.

3.2 Supercritical multitype branching

The results of this section are the outcome of discussions with Pablo A. Ferrari.

Consider the restricted matrix $\tilde{Q} := (q(x, y) : x, y \in \Lambda)$ and assume for the moment that Λ is finite. Let Q_α be the matrix with entries

$$q_\alpha(x, y) = q(x, y) + \alpha \delta_{xy}, \quad x, y \in \Lambda.$$

Here δ_{xy} is Kronecker’s delta function. We consider a continuous-time multitype branching process as described by Athreya and Ney [6, Chapter V.7]. The set of types is Λ . An individual of type x lives for a mean one exponential time and then splits into a random offspring distribution $L(x) = (L(x, y), y \in \Lambda)$ on \mathbb{N}^Λ with means given by $\mathbf{m}_\alpha(x, y) = q_\alpha(x, y) + \delta_{xy}$. We use $X(t) = (X(t, x), x \in \Lambda)$ for the number of

individuals of each type at time $t \geq 0$. See [6] for details on this construction. It is well known ([6, p. 202]) that the expected number of individuals of type y when we start the process with a unique individual of type x is given by $E_x(X(t, y)) = e^{tQ_\alpha}(x, y)$ for all $x, y \in \Lambda$.

Since Λ is finite, Perron-Frobenius theory (see also [22]) implies the existence of a principal eigenvalue λ_α for Q_α (a real eigenvalue with real part larger than all other eigenvalues) and a positive left eigenvector ν , normalized to be a probability measure in Λ . Observe that λ_α is an eigenvalue of Q_α if and only if $\lambda_\alpha = \lambda + \alpha$, with λ eigenvalue of \tilde{Q} and hence λ_α is positive if α is large enough. For those values of α , the associated multitype branching process with mean offspring matrix \mathbf{m}_α is supercritical and the Kesten-Stigum theorem ([6], see also [32] for a conceptual proof) implies that

$$\lim_{t \rightarrow \infty} \frac{X(t)}{|X(t)|} = \nu,$$

almost surely on the event $\Omega_{\text{surv}} := \{|X(t)| > 0 \text{ for all } t > 0\}$. Now it is immediate that ν is a left eigenvector for \tilde{Q} with eigenvalue λ . Since $\sum_{y \in \Lambda} q(x, y) = -q(x, 0)$, summing over y on both sides of

$$\sum_{x \in \Lambda} \nu(x) q(x, y) = \lambda \nu(y),$$

we get that $\lambda = -\sum_{x \in \Lambda} \nu(x) q(x, 0)$ and hence ν verifies (1.2), which means that it is the unique QSD. This method is useful as a practical tool to simulate QSDs. It is valid for finite Λ and in any other case where a Kesten-Stigum type theorem is available. So, for a given absorbing process with rates matrix Q , it is important to understand whether the supercritical multitype branching process described above behaves as predicted by the Kesten-Stigum theorem.

4 Related open problems

The different methods for simulating QSDs mentioned previously still raise many challenging open problems. We only mention here a few of them.

Rates of convergence to equilibrium and error bounds for FV. Remark that Theorem 2.2 concerns rates of convergence of the conditioned evolution in compact time intervals. Concerning the simulation of a QSD, only convergence results have been obtained, but no rates. Also, the class of processes where these results have been proven is much smaller than the one for which Theorem 2.2 holds. This opens up several research directions:

- mixing times for FV for fixed N ;
- rates of convergence for the empirical measure in equilibrium towards the (minimal) QSD;

- scalings $(t_N)_N$ that guarantee that $m(\cdot, \xi(t_N))$ converges to the minimal QSD as $N \rightarrow \infty$.

Concerning this last point, let us emphasize that it can be seen from Theorem 2.2 that if $t_N \leq \varepsilon \log N$ and $m(\cdot, \xi(0)) \rightarrow \nu$ as $N \rightarrow \infty$, where ν is any QSD, then $m(\cdot, \xi(t_N)) \rightarrow \nu$. In particular if ν is not the minimal one, at this timescale we are approximating another QSD. This suggests that in a proper scaling any QSD can be approximated with the FV process. Thus, one has to be careful when choosing the scaling parameter not to go to the wrong limit.

The other eigenvectors. It has been shown that the FV dynamics allow the estimation of the maximal eigenvector of the infinitesimal generator associated with their driving process in many situations. One may naturally aims at showing similar results for other eigenvectors.

In [18], the authors are interested in estimating eigenvectors of the Laplacian on bounded domains. They propose to adapt the dynamics of the FV process introducing different types of particles and considering as driving process random walks or Brownian motions. It would be of great interest to prove that this procedure converges in this case and can be adapted for other driving processes.

Convergence of FV driven by drifted random walk. As it was pointed out before, the main questions related to FV are the ergodicity of the process for each N and the convergence of its empirical distribution. The case of FV driven by a drifted random walk is remarkably still an open problem. The difficulty stems from the fact that the driving process has an infinite number of QSDs together with the fact that it is not R -positive (see [41]).

The immortal particle for FV. Grigorescu and Kang [33] show that one can define the notion of immortal particle as the trajectory from which all particles proceed when time tends to infinity (in the context of diffusions processes). The convergence of the trajectory of the immortal particle towards a stationary regime (when it exists) has not been yet investigated.

Relation with N-BBM As alluded in the introduction, Brunet, Derrida and coauthors introduced a system where particles evolve driven by one dimensional Brownian Motion and branch with a fixed rate, at which time the left-most particle is instantaneously eliminated. Very precise quantitative informations have been investigated for these dynamics at various time scales [9, 39]. We conjecture that long time behaviors of this system properly renormalized are very similar to the one of the FV process. A systematic comparison of the empirical distributions and proof of convergence towards the minimal QSDs are open research avenues.

Number of jumps of FV driven by diffusions in compact time intervals

Though not included in the context of countable state space, we would like to mention the problem of the definition of the FV process for non-countable state space. As already underlined in the introduction, and defined as an open problem by K. Burdzy [17], it is not known in general under which conditions the number of jumps of the FV processes driven by diffusions on finite or infinite domain does not accumulate in compact times intervals.

In [10] and [33], some conditions on the domain were shown to be sufficient to impede this phenomenon from happening. Also [11] partly deals with this problem.

Acknowledgments. We would like to warmly thank Amine Asselah and Pablo Ferrari for our fruitful discussions and collaborations on this subject. PG and MJ are partially supported by UBACyT 20020090100208, ANPCyT PICT No. 2008-0315 and CONICET PIP 2010-0142 and 2009-0613 and MATHAMSUD's project "Stochastic structure of large interacting systems".

References

- [1] David Aldous, Barry Flannery, and José Luis Palacios. Two applications of urn processes: the fringe analysis of search trees and the simulation of quasi-stationary distributions of Markov chains. *Probab. Eng. Inf. Sci.*, 2(3):293–307, 1988.
- [2] Amine Asselah. On the Dirichlet problem for asymmetric zero-range process on increasing domains. *Probab. Theory Relat. Fields*, 131(1):62–96, 2005.
- [3] Amine Asselah and Fabienne Castell. Existence of quasi-stationary measures for asymmetric attractive particle systems on \mathbb{Z}^d . *Ann. Appl. Probab.*, 13(4):1569–1590, 2003.
- [4] Amine Asselah, Pablo A. Ferrari, and Pablo Groisman. Quasistationary distributions and Fleming-Viot processes in finite spaces. *J. Appl. Probab.*, 48(2):322–332, 2011.
- [5] Amine Asselah, Pablo A. Ferrari, Pablo Groisman, and Matthieu Jonckheere. Fleming-Viot selects the minimal QSD: The Galton-Watson case. *arXiv:1206.6114*, 2012.
- [6] K. B. Athreya and P. E. Ney. *Branching processes*. Dover Publications Inc., Mineola, NY, 2004. Reprint of the 1972 original [Springer, New York; MR0373040].
- [7] Andrew Barbour and Philip K. Pollet. Total variation approximation for quasi-equilibrium distributions. *J. Appl. Probab.*, 47(4):934–946, 2010.
- [8] M. S. Bartlett. *Stochastic population models in ecology and epidemiology*. Methuen's Monographs on Applied Probability and Statistics. Methuen & Co. Ltd., London, 1960.
- [9] J. Berestycki, N. Berestycki, and J. Schweinsberg. The genealogy of branching Brownian motion with absorption. *arXiv:1001.2337v2*, 2010.

- [10] M. Bieniek, K. Burdzy, and S. Finch. Non-extinction of a Fleming-Viot particle model. *arXiv:0905.1999v1*, 2009.
- [11] Mariusz Bieniek, Krzysztof Burdzy, and Soumik Pal. Extinction of Fleming-Viot-type particle systems with strong drift. *Electron. J. Probab.*, 17(11):1–15, 2011.
- [12] Patrick Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.
- [13] E. Brunet, B. Derrida, A. H. Mueller, and S. Munier. Noisy traveling waves: effect of selection on genealogies. *Europhys. Lett.*, 76(1):1–7, 2006.
- [14] É. Brunet, B. Derrida, A. H. Mueller, and S. Munier. Effect of selection on ancestry: an exactly soluble case and its phenomenological generalization. *Phys. Rev. E (3)*, 76(4):041104, 20, 2007.
- [15] Eric Brunet and Bernard Derrida. Shift in the velocity of a front due to a cutoff. *Phys. Rev. E (3)*, 56(3, part A):2597–2604, 1997.
- [16] Éric Brunet and Bernard Derrida. Effect of microscopic noise on front propagation. *J. Statist. Phys.*, 103(1-2):269–282, 2001.
- [17] Krzysztof Burdzy. My favorite open problems.
<http://www.math.washington.edu/~burdzy/open.pdf>.
- [18] Krzysztof Burdzy, Robert Holyst, David Ingerman, and Peter March. Configurational transition in a Fleming-Viot-type model and probabilistic interpretation of Laplacian eigenfunctions. *J. Phys. A, Math. Gen.*, 29(11):2633–2642, 1996.
- [19] Krzysztof Burdzy, Robert Holyst, and Peter March. A Fleming-Viot particle representation of the Dirichlet Laplacian. *Comm. Math. Phys.*, 214(3):679–703, 2000.
- [20] James A. Cavender. Quasi-stationary distributions of birth-and-death processes. *Adv. Appl. Probab.*, 10(3):570–586, 1978.
- [21] J. N. Darroch and E. Seneta. On quasi-stationary distributions in absorbing discrete-time finite Markov chains. *J. Appl. Probability*, 2:88–100, 1965.
- [22] J. N. Darroch and E. Seneta. On quasi-stationary distributions in absorbing continuous-time finite Markov chains. *J. Appl. Probability*, 4:192–196, 1967.
- [23] M.M. De Oliveira and R. Dickman. How to simulate the quasistationary state. *Physical Review E*, 71(1):016129, 2005.
- [24] Pierre Del Moral and L. Miclo. Particle approximations of Lyapunov exponents connected to Schrödinger operators and Feynman-Kac semigroups. *ESAIM Probab. Stat.*, 7:171–208, 2003.
- [25] R. Dickman and M. Martins de Oliveira. Quasi-stationary simulation of the contact process. *Physica A: Statistical Mechanics and its Applications*, 357(1):134–141, 2005.

- [26] A.M. Etheridge. *An introduction to superprocesses*. American Mathematical Society, 2000.
- [27] P. A. Ferrari, H. Kesten, S. Martinez, and P. Picco. Existence of quasi-stationary distributions. A renewal dynamical approach. *Ann. Probab.*, 23(2):501–521, 1995.
- [28] P.A. Ferrari and A. Galves. Construction of stochastic processes, coupling and regeneration. *XIII Escuela Venezolana de Matemática*, 2000.
- [29] Pablo A. Ferrari and Nevena Marić. Quasi stationary distributions and Fleming-Viot processes in countable spaces. *Electron. J. Probab.*, 12:no. 24, 684–702 (electronic), 2007.
- [30] Pablo A. Ferrari, Servet Martínez, and Pierre Picco. Existence of nontrivial quasi-stationary distributions in the birth-death chain. *Adv. in Appl. Probab.*, 24(4):795–813, 1992.
- [31] Wendell H. Fleming and Michel Viot. Some measure-valued Markov processes in population genetics theory. *Indiana Univ. Math. J.*, 28(5):817–843, 1979.
- [32] Hans-Otto Georgii and Ellen Baake. Supercritical multitype branching processes: the ancestral types of typical individuals. *Adv. in Appl. Probab.*, 35(4):1090–1110, 2003.
- [33] I. Grigorescu and M. Kang. Immortal particle for a catalytic branching process. *Probability Theory and Related Fields*, pages 1–29, 2011.
- [34] Ilie Grigorescu and Min Kang. Hydrodynamic limit for a Fleming-Viot type system. *Stochastic Process. Appl.*, 110(1):111–143, 2004.
- [35] Ilie Grigorescu and Min Kang. Tagged particle limit for a Fleming-Viot type system. *Electron. J. Probab.*, 11:no. 12, 311–331 (electronic), 2006.
- [36] T. E. Harris. Additive set-valued Markov processes and graphical methods. *Ann. Probability*, 6(3):355–378, 1978.
- [37] S. D. Jacka and G. O. Roberts. Weak convergence of conditioned processes on a countable state space. *J. Appl. Probab.*, 32(4):902–916, 1995.
- [38] Samuel Karlin and James McGregor. The classification of birth and death processes. *Trans. Amer. Math. Soc.*, 86:366–400, 1957.
- [39] Pascal Maillard. Branching brownian motion with selection of the n right-most particles: An approximate model, February 2012.
- [40] Sylvie Méléard and Denis Villemonais. Quasi-stationary distributions and population processes. *arXiv:1112.4732.*, 2011.
- [41] E. Seneta and D. Vere-Jones. On quasi-stationary distributions in discrete-time Markov chains with a denumerable infinity of states. *J. Appl. Probability*, 3:403–434, 1966.
- [42] Erik A. van Doorn. Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes. *Adv. in Appl. Probab.*, 23(4):683–700, 1991.

- [43] Denis Villemonais. Interacting particle processes and approximation of markov processes conditioned to not be killed. *arXiv:1106.0878v1.*, 2011.