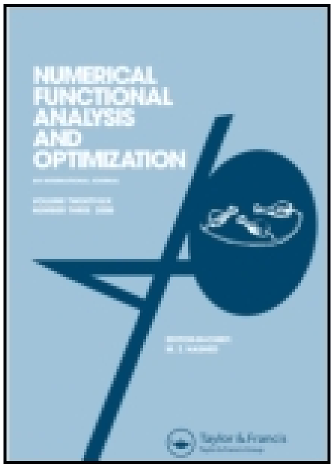

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EXTENDED BEST POLYNOMIAL APPROXIMATION OPERATOR IN ORLICZ SPACES

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□ *In this article we consider the best polynomial approximation operator, defined in an Orlicz space $L^{\Phi}(B)$, and its extension to $L^{\varphi}(B)$ where φ is the derivative function of Φ. A characterization of these operators and several properties are obtained.*

Keywords Best polynomial Φ-approximation operators; Extended best polynomial approximation from L^{Φ} to L^{φ}; Orlicz spaces.

Mathematics Subject Classification 41A10; 41A50; 41A45.

1. INTRODUCTION

In this article we set $\Im$ for the class of all continuous and nondecreasing functions φ defined for all real number $t \geq 0$, with $\varphi(0^+) = 0$, $\varphi(t) \to \infty$ as $t \to \infty$ and $\varphi(t) > 0$ for $x > 0$. We also assume a Δ_2 condition for the functions φ, which means that there exists a constant $\Lambda = \Lambda_\varphi > 0$ such that $\varphi(2a) \leq \Lambda\varphi(a)$ for all $a \geq 0$.

Now, given $\varphi \in \Im$, we consider $\Phi(x) = \int_0^x \varphi(t)\,dt$. Observe that $\Phi : [0,\infty) \to [0,\infty)$ is a convex function such that $\Phi(a) = 0$ iff $a = 0$. For such a function Φ, we have $\frac{\Phi(x)}{x} \to 0$ as $x \to 0$ and $\frac{\Phi(x)}{x} \to \infty$ as $x \to \infty$, and, according to [9], a function with this property is called an N function. Observe that the function φ satisfies a Δ_2 condition if and only if the function Φ satisfies a Δ_2 condition.

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If $\varphi \in \Im$, then it satisfies a Δ_2 condition. Thus, the next inequality holds

$$\frac{1}{2}(\varphi(a)+\varphi(b)) \leq \varphi(a+b) \leq \Lambda_\varphi(\varphi(a)+\varphi(b)) \tag{1.1}$$

for every $a, b \geq 0$.

Also, note that the Δ_2 condition on Φ implies

$$\frac{x}{2\Lambda_\varphi}\varphi(x) \leq \Phi(x) \leq x\varphi(x), \tag{1.2}$$

for every $x \geq 0$.

Let B be a bounded measurable set in $\mathbb{R}^n$. If $\varphi \in \Im$, we denote by $L^\varphi(B)$ the class of all Lebesgue measurable functions f defined on $\mathbb{R}^n$ such that $\int_B \varphi(t|f|)\,dx < \infty$ for some $t > 0$ and where dx denotes the Lebesgue measure on $\mathbb{R}^n$. Note that as $\varphi \in \Im$ and it satisfies a Δ_2 condition then $L^\varphi(B)$ is the space of all measurable functions f defined on $\mathbb{R}^n$ such that $\int_B \varphi(|f|)\,dx < \infty$. For the convex function Φ, $L^\Phi(B)$ is the classic Orlicz space that was very well studied in [9] and [14].

Let Π^m be the space of algebraic polynomials, defined on $\mathbb{R}^n$, of a degree at most m. Then a polynomial $P \in \Pi^m$ is called a best approximation of $f \in L^\Phi(B)$ if and only if

$$\int_B \Phi(|f-P|)\,dx = \inf_{Q\in\Pi^m}\int_B \Phi(|f-Q|)\,dx. \tag{1.3}$$

Definition 1. For $f \in L^\Phi(B)$, we set $\mu_\Phi(f)$ for the set of all polynomials P that satisfy (1.3).

In the following, we also refer to $\mu_\Phi(f)$ as the multivalued operator defined for functions in $L^\Phi(B)$ and images on Π^m.

In this article, we study the nature of this best polynomial approximation for functions in $L^\Phi(B)$ and we extend, in a continuous way, the definition of best polynomial approximation for functions belonging to $L^\varphi(B)$ where $\varphi = \Phi'$. These results extend those obtained in [3] for the L^p case.

We point out that the conditional expectation is the most well-known example of an extension of the best approximation operator. Originally, we think of the conditional expectation as the projection of a function $f \in L^2$ on a probability space $(\Omega, \mathcal{A}, P)$ onto the subspace of $\mathcal{A}_0$ measurable functions which are in L^2, where $\mathcal{A}_0$ denotes a sub sigma-algebra of $\mathcal{A}$. Using the fact that this projection is a monotone operator we can extend this best approximation operator from L^2 to L^1 and thus we obtain the well known conditional expectation operator. For $\Phi(t) = t^p$, $1 < p < \infty$, a similar best approximation operator is considered in L^p and then extended to L^{p-1}, see [11]. In that article, the approximation class is the set of all the $\mathscr{A}_0$ measurable functions in L^p, where now $\mathscr{A}_0$ is a sub-sigma lattice of $\mathscr{A}$. This best approximation operator and other classical operators in harmonic analysis are also considered in a general Orlicz spaces L^Φ, see [10], [1] or [8]. For the special case of $\mathscr{A}_0 = \{\emptyset, \Omega\}$, that is, when the approximation class is the set of constant functions in Ω, the extension of the best approximation operator was detailed studied in several articles, see [12], [5] and [6]. Also, the extension of the best approximation operator in L^Φ for a general sub sigma-lattice $\mathscr{A}_0$ was treated in [2]. In all of these cases, the monotonicity of the best approximation operator was strongly used in the space where it was originally defined. If the approximation class is the algebraic polynomials, we lose the monotonicity of the best approximation operator and the extension has to be treated in a different way. For the L^2 case see [13], and for the L^p case we refer to [3].

In Section 2, we define the best polynomial approximation operator for each $f \in L^\Phi(B)$ and we characterize this best approximation in a similar way as has been done in [7] for functions of $L^\Phi(B)$ in the case that the approximation class is a lattice instead of the space of polynomials. We also get a strong type inequality for $f \in L^\Phi(B)$ which generalizes Theorem 2.1 in [4], where the extended best polynomial approximation operator is considered for functions in $L^p(B)$. In Section 3, we use this inequality to extend the best polynomial approximation from $L^\Phi(B)$ to $L^\varphi(B)$, where $\varphi = \Phi'$. This is done in an easier way than the one developed in [3], where the existence of the extension is proved without using the inequality in Theorem 2.4. At the end of this section, we prove the uniqueness and a continuity property for the extended best polynomial approximation of $f \in L^\varphi(B)$ for a strictly increasing functions $\varphi \in \Im$.

2. EXISTENCE AND UNIQUENESS OF THE BEST POLYNOMIAL APPROXIMATION OPERATOR IN $L^\Phi(B)$

For $P \in \Pi^m$, we set $\|P\|_\infty = \max_{x\in B} |P(x)|$ and $\|P\|_1 = \int_B |P|\, dx$.

We begin with the existence of the best polynomial approximation operator of functions in $L^\Phi(B)$. We start with the next lemma.

Lemma 2.1. *Let $\varphi \in \Im$, $\Phi(x) = \int_0^x \varphi(t)\, dt$ and let P_n be a sequence in Π^m, such that there exists a constant C that satisfies $\int_B \Phi(|P_n|)\, dx \leq C$. Then, the sequence P_n is uniformly bounded.*

Proof. From Jensen's inequality, we have

$$|B|\Phi\left(\frac{1}{|B|}\int_B |P_n|\, dx\right) \leq \int_B \Phi(|P_n|)\, dx \leq C. \tag{2.1}$$

Then, since $\|P\|_1$ is equivalent to $\|P\|_\infty$, for $P \in \Pi^m$ and using the Δ_2 condition on Φ, we obtain

$$\Phi(\|P_n\|_\infty) \leq M,$$

for some constant M. Then, as $\Phi(x)$ goes to ∞ when x goes to ∞ the lemma follows. □

The next two theorems follow standard techniques. However, for the sake of completeness detailed proofs of them are included.

Theorem 2.2. *Let $\varphi \in \Im$, $\Phi(x) = \int_0^x \varphi(t)\,dt$ and let $f \in L^\Phi(B)$. Then, there exists $P \in \Pi^m$ such that*

$$\int_B \Phi(|f - P|)\,dx = \inf_{Q \in \Pi^m} \int_B \Phi(|f - Q|)\,dx.$$

Proof. Let $I = \inf_{Q \in \Pi^m} \int_B \Phi(|f - Q|)\,dx$, then there exists a sequence $\{P_n\}_{n \in \mathbb{N}} \subset \Pi^m$ such that

$$\int_B \Phi(|f - P_n|)\,dx \to I \quad \text{as } n \to \infty. \tag{2.2}$$

Due to the monotonicity and convexity of Φ on $[0, \infty)$, we get

$$\Phi\left(\frac{|P_n|}{2}\right) \leq \Phi\left(\frac{1}{2}|P_n - f| + \frac{|f|}{2}\right) \leq \frac{1}{2}\Phi(|P_n - f|) + \frac{1}{2}\Phi(|f|).$$

Thus,

$$\int_B \Phi\left(\frac{|P_n|}{2}\right)\,dx \leq \frac{1}{2}\int_B \Phi(|P_n - f|)\,dx + \frac{1}{2}\int_B \Phi(|f|)\,dx,$$

and then

$$2\int_B \Phi\left(\frac{|P_n|}{2}\right)\,dx \leq \int_B \Phi(|f|)\,dx + I + 1. \tag{2.3}$$

Now, Lema 2.1 implies $\|P_n\|_\infty \leq K$. Hence, there exists a subsequence $\{P_{n_k}\} \subseteq \{P_n\}_{\{n \in \mathbb{N}\}}$ such that $\{P_{n_k}\}$ converges uniformly on Π^m.

Let $P = \lim_{n_k \to \infty} P_{n_k}$. Since Φ satisfies the Δ_2 condition, we have

$$\Phi(|f - P_{n_k}|) \leq \Lambda_\Phi(\Phi(|f|) + \Phi(|P_{n_k}|)) \leq \Lambda_\Phi(\Phi(|f|) + \Phi(K)).$$

Then, by the Lebesgue Dominated Convergence Theorem, we have $I = \int_B \Phi(|f - P|)\,dx$. □

The next theorem gives a characterization of the best polynomial approximation of functions in $L^{\Phi}(B)$.

Theorem 2.3. *Let* $\varphi \in \Im$, $\Phi(x) = \int_0^x \varphi(t)\,dt$ *and let* $f \in L^{\Phi}(B)$. *Then* $P \in \Pi^m$ *is in* $\mu_\Phi(f)$ *if and only if*

$$\int_B \varphi(|f-P|)\mathrm{sgn}(f-P)Q\,dx = 0, \tag{2.4}$$

for every $Q \in \Pi^m$.

Proof. For P in $\mu_\Phi(f)$ and $Q \in \Pi^m$ we set

$$F_Q(\varepsilon) = \int_B \Phi(|f-P+\varepsilon Q|)\,dx.$$

Next, we prove that F_Q is a convex function defined on $[0,\infty)$. For $a, b \geq 0$ such that $a+b=1$, we have

$$\begin{aligned} F_Q(a\epsilon_1 + b\epsilon_2) &= \int_B \Phi(|(a+b)(f-P) + (a\epsilon_1 + b\epsilon_2)Q|)\,dx \\ &\leq \int_B \Phi(a|(f-P)+\epsilon_1 Q| + b|(f-P)+\epsilon_2 Q|)\,dx \\ &\leq \int_B a\Phi(|(f-P)|+\epsilon_1 Q|)\,dx + \int_B b\Phi(|(f-P)|+\epsilon_2 Q|)\,dx \\ &= aF_Q(\epsilon_1) + bF_Q(\epsilon_2), \end{aligned}$$

for every $\epsilon_1,\ \epsilon_2 \geq 0$. Then

$$F_Q(0) = \min_{[0,\infty)} F_Q(\epsilon), \tag{2.5}$$

and this identity holds if and only if $0 \leq F_Q'(0^+)$.

Now, using the Mean Value Theorem we have

$$\frac{|\Phi(|f-P+\epsilon Q|) - \Phi(|f-P|)|}{\epsilon|Q|} \leq |Q|(\varphi(|f-P|) + \varphi(|Q|)),$$

for $0 \leq \epsilon \leq 1$.

Then, since $|Q|(\varphi(|f-P|)+\varphi(|Q|))$ is an integrable function, we are allowed to differentiate inside the integral in the formula of $F_Q(\epsilon)$, and therefore

$$0 \leq F_Q'(0^+) = \int_B \varphi(|f-P|)\mathrm{sgn}(f-P)Q\,dx, \tag{2.6}$$

for any $Q \in \Pi^m$.

Now for any polynomial $Q \in \Pi^m$, we take the polynomial $-Q$ in (2.6) and this completes the proof. □

The following result, similar to Theorem 2.1 in [4], provides us an inequality that we will need below.

Theorem 2.4. *Let $\varphi \in \Im$, $\Phi(x) = \int_0^x \varphi(t)\,dt$ and let $f \in L^\varphi(B)$. Suppose the polynomial $P \in \Pi^m$ satisfies*

$$\int_B \varphi(|f-P|)\operatorname{sgn}(f-P)Q\,dx = 0, \tag{2.7}$$

for every $Q \in \Pi^m$. Then

$$\int_B \varphi(|P|)|Q|\,dx \le 5\Lambda_\varphi \int_B \varphi(|f|)|Q|\,dx, \tag{2.8}$$

for every $Q \in \Pi^m$ satisfying $\operatorname{sgn}(Q(t)P(t)) = (-1)^\eta$ at any $t \in B$ such that $Q(t)P(t) \neq 0$ and where $\eta = 0$ or $\eta = 1$.

Proof. Suppose first let $Q \in \Pi^m$ such that $Q(x)P(x) > 0$.

Let $N = \{x \in B : f(x) > P(x)\}$ and $L = \{x \in B : f(x) \le P(x)\}$.

Then

$$\begin{aligned} 0 &= \int_{N\cup L} \varphi(|f-P|)\operatorname{sgn}(f-P)Q\,dx \\ &= \int_N \varphi(|f-P|)\operatorname{sgn}(f-P)Q\,dx + \int_L \varphi(|f-P|)\operatorname{sgn}(f-P)Q\,dx \end{aligned}$$

Thus,

$$\int_N \varphi(|f-P|)\,Q\,dx = \int_L \varphi(|f-P|)\,Q\,dx. \tag{2.9}$$

Let $H(x) = \varphi(|P(x)-f(x)|)Q(x)$ and consider the sets

$$\begin{aligned} &U_1 = N \cap \{x \in B : P(x) \ge 0\}, \quad U_2 = N \cap \{x \in B : P(x) < 0\}, \\ &U_3 = L \cap \{x \in B : P(x) \ge 0\}, \quad U_4 = L \cap \{x \in B : P(x) < 0\}. \end{aligned}$$

Then by (2.9), we get

$$\int_{U_1\cup U2} H\,dx = \int_{U_3\cup U4} H\,dx,$$

and therefore

$$\int_{U_1} H\,dx - \int_{U_4} H\,dx = \int_{U_3} H\,dx - \int_{U_2} H\,dx. \tag{2.10}$$

Due to the monotonicity of φ, we have

$$\int_B \varphi(|P|)|Q|\,dx \le \int_B \varphi(|P-f| + |f|)|Q|\,dx,$$

and using (1.1) we get

$$\begin{aligned}\int_B \varphi(|P-f|+|f|)|Q|\,dx &\le \Lambda_\varphi \int_B \varphi(|P-f|)|Q|\,dx + \Lambda_\varphi \int_B \varphi(|f|)|Q|\,dx \\ &= \Lambda_\varphi \int_{\bigcup_{i=1}^4 U_i} |H|\,dx + \Lambda_\varphi \int_B \varphi(|f|)|Q|\,dx \\ &= \Lambda_\varphi \sum_{i=1}^{4} \int_{U_i} |H|dx + \Lambda_\varphi \int_B \varphi(|f|)|Q|dx = \Lambda_\varphi(I_1 + I_2).\end{aligned}$$

Now, we will find an upper bound of $I_1 = \sum_{i=1}^4 \int_{U_i} |H|\,dx$.

Note that we have $|P-f| \le |f|$ on U_1 and U_4. Next, since the monotonicity of φ, we obtain

$$\begin{aligned}\int_{U_1\cup U_4} |H|\,dx &= \int_{U_1} |H|\,dx + \int_{U_4} |H|\,dx \\ &\le \int_{U_1} \varphi(|f|)|Q|\,dx + \int_{U_4} \varphi(|f|)|Q|\,dx \le 2\int_B \varphi(|f|)|Q|\,dx. \end{aligned} \tag{2.11}$$

Since $\operatorname{sgn} Q = \operatorname{sgn} P$, from (2.10) and (2.11), we get

$$\begin{aligned}\int_{U_2} |H|\,dx + \int_{U_3} |H|\,dx &= \int_{U_2} (-H)\,dx + \int_{U_3} H\,dx \\ &= \int_{U_1} H\,dx - \int_{U_4} H\,dx = \int_{U_1} |H|\,dx + \int_{U_4} |H|\,dx \\ &= \int_{U_1\cup U_4} |H|\,dx \le 2\int_B \varphi(|f|)|Q|\,dx. \end{aligned} \tag{2.12}$$

Therefore, $I_1 \le 4\int_B \varphi(|f|)|Q|\,dx$ and

$$\int_B \varphi(|P|)|Q|\,dx \le 5\Lambda_\varphi \int_B \varphi(|f|)|Q|\,dx. \tag{2.13}$$

Now if $Q \in \Pi^m$ satisfies $Q(x)P(x) < 0$, we proceed in an analogous way to obtain (2.12), then

$$\begin{aligned}\int_{U_2} |H|\,dx + \int_{U_3} |H|\,dx &= \int_{U_2} H\,dx - \int_{U_3} H\,dx \\ &= -\int_{U_1} H\,dx + \int_{U_4} H\,dx = \int_{U_1} |H|\,dx + \int_{U_4} |H|\,dx \\ &= \int_{U_1 \cup U_4} |H|\,dx \le 2\int_B \varphi(|f|)|Q|\,dx,\end{aligned}$$

and thus

$$\int_B \varphi(|P|)|Q|\,dx \le 5\Lambda_\varphi \int_B \varphi(|f|)|Q|\,dx \tag{2.14}$$

for $Q \in \Pi^m$ such that $Q(x)P(x) < 0$.

Finally, (2.8) follows from (2.13) and (2.14). □

The next corollary will be useful in the following.

Corollary 2.5. *Let $\varphi \in \Im$, $\Phi(x) = \int_0^x \varphi(t)\,dt$ and let $f \in L^\Phi(B)$. If P is the best polynomial approximation of $f \in L^\Phi(B)$, then*

$$\int_B \varphi(|P|)|P|\,dx \le 5\Lambda_\varphi \|P\|_\infty \int_B \varphi(|f|)\,dx. \tag{2.15}$$

Proof. It follows for $Q = P$ in (2.8) of Theorem 2.4 and employing $|P| \le \|P\|_\infty$. □

Remark 2.6. In order to obtain Theorem 2.4, we have used that the polynomial P is a solution of (2.7) for f in $L^\varphi(B)$. Thus, the inequality (2.15) holds for any polynomial P that satisfies identity (2.7) and f belonging to $L^\varphi(B)$.

3. EXTENSION OF THE BEST POLYNOMIAL APPROXIMATION TO $L^\varphi(B)$

In order to get a continuous extension of $\mu_\Phi(f)$ for functions in the bigger space $L^\varphi(B)$, we need the following auxiliary results. Throughout this section, we will consider $\varphi \in \Im$ and $\Phi(x) = \int_0^x \varphi(t)\,dt$.

Lemma 3.1. *Let f_n be a sequence in $L^\Phi(B)$ such that there exists a constant C that satisfies $\int_B \varphi(|f_n|)\,dx \le C$. Then $\{\|P\|_\infty : P \in \mu_\Phi(f_n), n = 1, 2, \ldots\}$ is bounded.*

Proof. Using Corollary 2.5, we have

$$\int_B \varphi(|P|)|P|\,dx \le 5\Lambda_\varphi \|P\|_\infty \int_B \varphi(|f_n|)\,dx \le 5C\Lambda_\varphi \|P\|_\infty, \tag{3.1}$$

for each $P \in \mu_\Phi(f_n)$ and for every all n. Thus, using (1.2) we get

$$\int_B \Phi(|P|)\,dx \le 5\Lambda_\varphi C\|P\|_\infty.$$

Then, from Jensen's inequality, we obtain

$$|B|\Phi\left(\frac{1}{|B|}\int_B |P|\,dx\right) \le \int_B \Phi(|P|)\,dx.$$

Now, since $\|P\|_1$ is a norm which is equivalent to $\|P\|_\infty$, for $P \in \Pi^m$, we obtain for a suitable constant K,

$$\Phi\left(\frac{K}{|B|}\|P\|_\infty\right) \le 5\Lambda_\varphi^2 \frac{C}{|B|}\|P\|_\infty.$$

Thus, taking into account that $\frac{\Phi(x)}{x}$ goes to ∞ as x tends to ∞ the lemma is proved. □

Lemma 3.2. *Let f_n, f be functions in $L^\varphi(B)$ such that*

$$\int_B \varphi(|f_n - f|)\,dx \to 0 \tag{3.2}$$

as $n \to \infty$.

Also let g_n, g be measurable functions such that $|g_n| \le C$ for all n and $g_n \to g$ a.e. for $x \in \{f \ne 0\}$, as $n \to \infty$. Then there exists a subsequence n_k such that

$$\int_B \varphi(|f_{n_k}|)g_{n_k}\,dx \to \int_B \varphi(|f|)g\,dx \tag{3.3}$$

as $k \to \infty$.

Proof. Since φ is a non-decreasing function and $\varphi(x) > 0$ for $x > 0$, there exists a subsequence f_{n_k} which converges to f a.e. We will now use the sequence $\varphi(|f_n|)$ that has equiabsolutely continuous integrals. That means, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_E \varphi(|f_n|)\,dx \le \varepsilon$, for any $E \subset B$, $|E| \le \delta$, and for every n. This fact follows at once from $\int_B \varphi(|f - f_n|)\,dx \to 0$, and

$$\int_E \varphi(|f_n|) \le \Lambda_\varphi \int_B \varphi(|f - f_n|)\,dx + \Lambda_\varphi \int_E \varphi(|f|)\,dx.$$

Now, by Egorov's theorem, given $\delta > 0$ there exists $F \subset B$, $|B - F| < \delta$ such that the subsequence $\varphi(|f_{n_k}|)g_{n_k}$ uniformly converges to $\varphi(|f|)g$ on F. Then

$$\begin{aligned} &\int_B \varphi(|f_{n_k}|)g_{n_k}\,dx - \int_B \varphi(|f|)g\,dx \\ &\quad = \int_{B-F} (\varphi(|f_{n_k}|)g_{n_k} - \varphi(|f|)g)\,dx + \int_F (\varphi(|f_{n_k}|)g_{n_k} - \varphi(|f|)g)\,dx \\ &\quad = I_k + J_k. \end{aligned}$$

Now, using the uniform convergence of the sequence on F we have that J_k goes to 0 as k goes to ∞. On the other hand, since we are dealing with equiabsolutely continuous integrals we get $|I_k| < \varepsilon$ for every k. □

Theorem 3.3. *If $f \in L^{\varphi}(B)$, then there exists $P \in \Pi^m$ such that*

$$\int_B \varphi(|f - P|)\mathrm{sgn}(f - P)Q\,dx = 0, \tag{3.4}$$

for every $Q \in \Pi^m$.

And

$$\int_B \Phi(|P|)\,dx \le K\|P\|_\infty \int_B \varphi(|f|)\,dx, \tag{3.5}$$

for a suitable constant K.

Proof. Set the sequence of functions $f_n = min(max(f, -n), n)$ which are in $L^{\Phi}(B)$. Then, by Theorem 2.3, there exists $P_n \in \mu_\Phi(f_n)$ such that

$$\int_B \varphi(|f_n - P_n|)\mathrm{sgn}(f_n - P_n)Q\,dx = 0, \tag{3.6}$$

for every $Q \in \Pi^m$.

Observe that $\int_B \varphi(|f_n - f|)\,dx \to 0$, as $n \to \infty$. Now, by Lemma 3.1, the sequence $\|P_n\|_\infty$ is bounded. Then there exists a subsequence P_{n_k} which uniformly converges on B to a polynomial $P \in \Pi^m$. Thus, by Lemma 3.2, we get

$$\begin{aligned} 0 &= \lim_{k\to\infty} \int_B \varphi(|f_{n_k} - P_{n_k}|)\mathrm{sgn}(f_{n_k} - P_{n_k})Q\,dx \\ &= \int_B \varphi(|f - P|)\mathrm{sgn}(f - P)Q\,dx, \end{aligned}$$

for every $Q \in \Pi^m$.

Now, by Remark 2.6 and (1.2), we also get

$$\int_B \Phi(|P|)\, dx \le \int_B \varphi(|P|)\, |P|\, dx \le 5\Lambda_\varphi \|P\|_\infty \int_B \varphi(|f|)\, dx,$$

and the proof is completed. □

Now Theorem 3.3 allows us to extend the definition of the best approximation operator for functions in $L^\varphi(B)$.

Definition 2. For $f \in L^\varphi(B)$, we set $\mu_\varphi(f)$ for the set of polynomials $P \in \Pi^m$ that satisfies (3.4) and we refer to this set as the extended best approximation operator.

Next, we list some properties of this best approximation operator.

Theorem 3.4. *If Φ is a strictly convex function, then there exists a unique extended best polynomial approximation for every $f \in L^\varphi(B)$.*

Proof. For $f \in L^\varphi(B)$ we consider $P_1, P_2 \in \mu_\varphi(f)$, $P_1 \ne P_2$, then

$$\int_B \varphi(|f - P_1|)\mathrm{sgn}(f - P_1) Q\, dx = \int_B \varphi(|f - P_2|)\mathrm{sgn}(f - P_2) Q\, dx = 0, \tag{3.7}$$

for every $Q \in \Pi^m$.

Set the polynomial $R = P_1 - P_2 \in \Pi^m$ and the pairwise disjoint sets

$$\widetilde{A} = \{x \in B : P_2(x) > P_1(x)\}$$
$$\widetilde{B} = \{x \in B : P_1(x) > P_2(x)\}$$
$$\widetilde{C} = \{x \in B : P_1(x) = P_2(x)\}$$

then $\widetilde{A} \cup \widetilde{B} \cup \widetilde{C} = B$ and $|\widetilde{C}| = 0$.

Since Φ is a strictly convex function, we have that $\varphi(|x|)\mathrm{sgn}(x)$ is a strictly increasing function. Consider $R < 0$ and $f - P_2 < f - P_1$ on the set $\widetilde{A}$, then

$$\varphi(|f - P_2|)\mathrm{sgn}(f - P_2) < \varphi(|f - P_1|)\mathrm{sgn}(f - P_1),$$

and thus

$$\varphi(|f - P_1|)\mathrm{sgn}(f - P_1) R < \varphi(|f - P_2|)\mathrm{sgn}(f - P_2) R.$$

Hence,

$$\int_{\widetilde{A}} \varphi(|f-P_1|)\mathrm{sgn}(f-P_1)R\,dx \le \int_{\widetilde{A}} \varphi(|f-P_2|)\mathrm{sgn}(f-P_2)R\,dx. \tag{3.8}$$

Analogously, if $R > 0$ and $f - P_1 < f - P_2$ on the set $\widetilde{B}$, then

$$\varphi(|f-P_1|)\mathrm{sgn}(f-P_1)R < \varphi(|f-P_2|)\mathrm{sgn}(f-P_2)R.$$

Therefore,

$$\int_{\widetilde{B}} \varphi(|f-P_1|)\mathrm{sgn}(f-P_1)R\,dx \le \int_{\widetilde{B}} \varphi(|f-P_2|)\mathrm{sgn}(f-P_2)R\,dx. \tag{3.9}$$

Now, since P_1 and P_2 are continuous functions and $P_1 \neq P_2$ on B, then $|\widetilde{A}| > 0$ or $|\widetilde{B}| > 0$. Thus, at least one of the inequalities (3.8) or (3.9) must be strict.

From (3.7), (3.8) and (3.9) we get

$$\begin{aligned} 0 &= \int_B \varphi(|f-P_1|)\mathrm{sgn}(f-P_1)R\,dx \\ &= \int_{\widetilde{A}} \varphi(|f-P_1|)\mathrm{sgn}(f-P_1)R\,dx + \int_{\widetilde{B}} \varphi(|f-P_1|)\mathrm{sgn}(f-P_1)R\,dx \\ &< \int_{\widetilde{A}} \varphi(|f-P_2|)\mathrm{sgn}(f-P_2)R\,dx + \int_{\widetilde{B}} \varphi(|f-P_2|)\mathrm{sgn}(f-P_2)R\,dx \\ &= \int_B \varphi(|f-P_2|)\mathrm{sgn}(f-P_2)R\,dx = 0, \end{aligned}$$

which is a contradiction, and the proof is completed. □

Proposition 3.5. *For any $f \in L^{\varphi}(B)$ it satisfies $\mu_\varphi(f+P) = \mu_\varphi(f) + P$ for all $P \in \Pi^m$.*

Proof. It follows directly from the definition of the extended best approximation operator $\mu_\varphi(f)$. □

Theorem 3.6. *Let Φ be a strictly convex function and $h_n, h \in L^{\varphi}(B)$ such that*

$$\int_B \varphi(|h_n - h|)\,dx \to 0 \quad \text{as } n \to \infty. \tag{3.10}$$

Then $\mu_\varphi(h_n) \to \mu_\varphi(h)$ as $n \to \infty$.

Proof. Set $P_n = \mu_\varphi(h_n)$. By inequality (3.5), the sequence P_n is uniformly bounded. We consider a subsequence P_{n_k} which converges to a polynomial P. Now, we select a subsequence of h_{n_k}, which will be also called by h_{n_k}, that converges to h a.e; we also have, for any $Q \in \Pi^m$,

$$\int_B \varphi(|h_{n_k} - P_{n_k}|)\mathrm{sgn}(h_{n_k} - P_{n_k})Q\, dx = 0. \tag{3.11}$$

Now, by Lemma 3.2, we get

$$\int_B \varphi(|h - P|)\mathrm{sgn}(h - P)Q\, dx = 0, \tag{3.12}$$

and taking into account Theorem 3.4, $P = \mu_\varphi(f)$ and the whole sequence P_n converges to P. Thus the proof is completed. □

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