

More on vacua of exotic massive 3D gravity

Gaston Giribet¹ and Julio Oliva²

¹*Departamento de Física, Universidad de Buenos Aires FCEyN-UBA
and IFIBA-CONICET Ciudad Universitaria, pabellón 1,
1428 Buenos Aires, Argentina*

²*Departamento de Física, Universidad de Concepción Casilla 160-C, 4070386 Concepción, Chile*



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In a recent paper [M. Chernicoff *et al.*, *J. High Energy Phys.* **08** (2018) 087], we explored the space of solutions of the exotic massive 3D gravity theory proposed in [M. Özkan *et al.*, *J. High Energy Phys.* **08** (2018) 035]. We showed that the theory admits a rich space of vacua, including asymptotically anti de-Sitter (AdS) geometries obeying different types of boundary conditions. The examples include black holes dressed with low decaying gravitons. Based on what happens in other theories of massive gravity, we conjectured that such geometries appear on a curve of the parameter space (chiral curve) where the exotic massive gravity on AdS with sufficiently strong boundary conditions results dual to a 2D chiral conformal field theory. Here, we show that this expectation is consistent with the conserved charges and thermodynamical properties of the black holes of the theory, which have recently been computed [Mann *et al.*, Energy of asymptotically AdS black holes in exotic massive gravity and its log-extension, [arXiv:1812.09525](https://arxiv.org/abs/1812.09525)]. When the boundary conditions are relaxed relative to the standard Brown-Henneaux boundary conditions, the theory exhibits solutions consistent with the definition of the so-called log-gravity. The asymptotic behavior of these solutions presents a logarithmic term in the Fefferman-Graham expansion that, nonetheless, is compatible with the AdS asymptotic symmetries. This long range interaction is due to a mode of the massive gravity that becomes massless precisely on the chiral curve. We construct exact solutions exhibiting this behavior, which admit to be interpreted as fully backreacting gravitational waves propagating on an extremal black hole, and carrying nonvanishing gravitational energy. We also discuss other vacua of the theory, such as warped-AdS₃ black holes, gravitational waves on such backgrounds, AdS₂ × S¹ spaces, and black holes in dS₃ space.

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I. INTRODUCTION

Exotic massive gravity (EMG) is a parity-odd theory describing a propagating massive spin-2 fields in three dimensions (3D) [1]. It generalizes topologically massive gravity theory (TMG) [2] supplementing it with third- and fourth-order terms. EMG can be regarded as the next-to-simplest example of an infinite sequence of consistent 3D models obeying what has been called *the third-way mechanism* [3], i.e., a mechanism by means of which classical metric theories with no action can still satisfy the Bianchi identities on-shell without imposing additional constraints. The simplest example of such a theory is the so-called minimal massive gravity (MMG), proposed in [4]; EMG theory is its natural higher-order generalization.

In the metric formalism, EMG is defined by the field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = T_{\mu\nu} \quad (1)$$

where the Cotton tensor $C_{\mu\nu}$ is

$$C_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu}^{\alpha\beta}\nabla_{\alpha}R_{\beta\nu} + \frac{1}{2}\varepsilon_{\nu}^{\alpha\beta}\nabla_{\alpha}R_{\beta\mu}, \quad (2)$$

and where the tensor $T_{\mu\nu}$ is

$$T_{\mu\nu} = \frac{1}{m^2}H_{\mu\nu} - \frac{1}{m^4}L_{\mu\nu} \quad (3)$$

with

$$H_{\mu\nu} = \varepsilon_{\mu}^{\alpha\beta}\nabla_{\alpha}C_{\nu\beta}, \quad L_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu}^{\alpha\beta}\varepsilon_{\nu}^{\gamma\sigma}C_{\alpha\gamma}C_{\beta\sigma}. \quad (4)$$

μ and m are two arbitrary coupling constants of mass dimension 1, and Λ is the cosmological constant.

The covariant divergence of $H_{\mu\nu}$ and $L_{\mu\nu}$ does not vanish identically, and this is where the third-way mechanism plays a role: Despite the tensor $T_{\mu\nu}$ not being covariantly

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conserved identically, one can prove that in virtue of the identities

$$\begin{aligned} & \varepsilon_\mu^{\alpha\beta} C_\alpha^\sigma \left(R_{\beta\sigma} - \frac{1}{2} R g_{\beta\sigma} - \frac{1}{m^2} H_{\beta\sigma} \right) \\ &= \varepsilon_\mu^{\alpha\beta} C_\alpha^\sigma \left(-\Lambda g_{\beta\sigma} - \frac{1}{m^4} L_{\beta\sigma} \right) = 0 \end{aligned} \quad (5)$$

the conservation condition $\nabla_\nu T^\nu{}_\mu = 0$ does hold on-shell without imposing incompatible constraints. This is the type of theories recently studied in [3,5], for which the *matter* contribution $T_{\mu\nu}$ that supplements TMG field equations satisfies the Bianchi identities as a consequence of the gravitational field equation itself.

In a recent paper [6], we explored the space of solutions to the field equations (1)–(4) and we showed that it contains interesting geometries, including asymptotically AdS_3 black holes and gravitational waves. We also showed that, on a particular curve of the parameter space, the theory also admits logarithmically decaying solutions in AdS_3 . These are analogous to the solutions of TMG that appear at the so-called chiral point [7]. Therefore, we conjectured that, on a special curve of the parameter space [see (23) below], EMG on AdS_3 with sufficiently strong boundary conditions is dual to a 2D chiral conformal field theory (CFT_2). Here, we will show that this conjecture is consistent with the conserved charges and thermodynamical properties of the black holes of the theory [8]. In Sec. II, we will study the simplest black hole solution of the theory, the Bañados-Teitelboim-Zanelli (BTZ) solution, which describes black holes in AdS_3 space. We will review the main properties of BTZ, now seen as a solution to EMG, and in particular its conserved charges. By studying the black hole thermodynamical properties, and assuming the existence of a CFT_2 dual description, in Sec. III we will obtain the central charges of the boundary theory. Then, in Sec. IV, we will study the special properties the theory exhibits when formulated on the aforementioned special curve of the parameter space, to which hereafter we will refer to as the *chiral curve*. On this curve, the left-moving central charge of the dual CFT_2 vanishes and this is taken as evidence that, provided sufficiently strong AdS_3 boundary conditions are prescribed, the dual CFT_2 turns out to be chiral. In contrast, when weakened AdS_3 boundary conditions are considered, the dual CFT_2 has both left- and right-moving dynamics, and this is in correspondence with additional normalizable massless fields appearing in the bulk. In Sec. V, we will discuss explicit examples of these geometries, which describe gravitational waves propagating on extremal black holes. Sections VI–X will be dedicated to study other vacua of the theory, including massive gravitational waves in AdS_3 and other deformations of the extremal BTZ solution, black holes in warped- AdS_3 (WAdS_3) space, evanescent gravitational waves on WAdS_3 space, $\text{AdS}_2 \times S^1$ spaces, and hairy black holes in both AdS_3 and dS_3 space.

II. BTZ BLACK HOLES

As other *bona fide* massive modifications to Einstein theory in 3D, EMG admits the BTZ black hole as exact solution. Its metric takes the form [9]

$$ds_{M,J}^2 = -N^2(r)dt^2 + \frac{dr^2}{N^2(r)} + r^2(N^\varphi(r)dt + d\varphi)^2 \quad (6)$$

with

$$N^2(r) = \frac{r^2}{\ell^2} - 8GM + \frac{16G^2J^2}{r^2}, \quad N^\varphi(r) = -\frac{4GJ}{r^2} \quad (7)$$

where $t \in \mathbb{R}$, $r \in \mathbb{R}_{\geq 0}$, $0 \leq \varphi \leq 2\pi$, and where $\ell^2 = -1/\Lambda$. M and J are two arbitrary integration constants. For generic M , the space (6)–(7) has isometry $\mathbb{R} \times SO(2)$. This geometry represents a black hole in AdS_3 space, provided $M \geq 0$ and $|J| \leq M\ell$. The horizon radii are

$$r_\pm = \sqrt{2G\ell(\ell M + J)} \pm \sqrt{2G\ell(\ell M - J)}, \quad (8)$$

with $0 \leq r_- \leq r_+$. This means

$$M = \frac{r_+^2 + r_-^2}{8G\ell^2}, \quad J = \frac{r_+ r_-}{4G\ell}. \quad (9)$$

Being a 3D Einstein spaces with negative cosmological constant, BTZ black holes are locally equivalent to AdS_3 , and they can be constructed by making global identifications on the universal covering of the maximally symmetric geometry [10]. It is convenient to define the geometric temperatures

$$T_\pm = \frac{r_+ \pm r_-}{2\pi\ell^2} \quad (10)$$

which are the inverse of periods of the geometry [10]. In terms of these, the Hawking temperature is given by

$$T_H = 2 \frac{T_+ T_-}{T_+ + T_-} = \frac{r_+^2 - r_-^2}{2\pi\ell^2 r_+}. \quad (11)$$

The conserved charges of the BTZ solution in EMG were recently computed in [8] (see also [11]), being the energy

$$Q[\partial_t] = \frac{1}{8G\ell^2} \left[\left(1 - \frac{1}{m^2\ell^2} \right) (r_+^2 + r_-^2) - \frac{2}{\mu\ell} r_+ r_- \right] \quad (12)$$

and the angular momentum

$$Q[\partial_\varphi] = \frac{1}{8G\ell} \left[\left(1 - \frac{1}{m^2\ell^2} \right) 2r_+ r_- - \frac{1}{\mu\ell} (r_+^2 + r_-^2) \right]. \quad (13)$$

These are the conserved charges associated to the Killing vectors ∂_t and ∂_φ , respectively. With them, one obtains the black hole entropy

$$S_{\text{BH}} = \frac{\pi}{2G} \left[\left(1 - \frac{1}{m^2 \ell^2} \right) r_+ - \frac{1}{\mu \ell} r_- \right] \quad (14)$$

which follows from the first principle of black hole thermodynamics

$$dQ[\partial_t] = T_{\text{H}} dS_{\text{BH}} - \Omega dQ[\partial_\phi] \quad (15)$$

with the horizon angular velocity $\Omega = N^\phi(r_+) = -r_-/r_+$. Notice that, while for $m^2 = \mu = \infty$ (14) obeys the Bekenstein-Hawking area law, the presence of higher-derivative terms introduce corrections to it. This is, of course, a common feature of higher-curvature gravity.

III. CENTRAL CHARGE

As for other 3D models, we can rewrite the entropy (14) in a suggestive form: A Cardy-type formula of the dual CFT_2 can be written in terms of the geometric temperatures (10) and the entropy (14), namely [12]

$$S_{\text{BH}} = \frac{\pi^2 \ell}{3} (c_+ T_+ + c_- T_-) \quad (16)$$

which is obeyed for the following values of the central charge(s)

$$c_\pm = \frac{3\ell}{2G} \left(1 - \frac{1}{m^2 \ell^2} \mp \frac{1}{\mu \ell} \right). \quad (17)$$

This, of course, agrees with the right-moving and left-moving central charges of TMG in the limit $m^2 \rightarrow \infty$.

Now, let us convince ourselves of (17) being the correct result: We can write the operators L_0 and \bar{L}_0 , which generate the $SO(2) \times \mathbb{R}$ piece of the global conformal group at the cylindrical boundary ($r = \infty$). These operators would correspond to the zero-modes of the Virasoro operators L_n , \bar{L}_n that generate the whole asymptotic symmetry algebra [13]. These are related to the conserved charges (12)–(13) as follows

$$L_0 + \bar{L}_0 = \ell Q[\partial_t], \quad L_0 - \bar{L}_0 = Q[\partial_\phi]. \quad (18)$$

Denoting $L_0^+ \equiv L_0$ and $L_0^- \equiv \bar{L}_0$, we can write

$$L_0^\pm = \frac{c_\pm (r_+ \pm r_-)^2}{24 \ell^2}. \quad (19)$$

Notice that these values are consistent with (17) and with the vacuum (vac) of the theory: In fact, global AdS_3 space corresponds to (6)–(7) with $r_+^2 = -\ell^2$ and $r_- = 0$. Then, the values of the charges (12)–(13) corresponding to the vacuum geometry are

$$Q^{(\text{vac})}[\partial_t] = \frac{1}{8G} \left(\frac{1}{m^2 \ell^2} - 1 \right), \quad Q^{(\text{vac})}[\partial_\phi] = \frac{1}{8G\mu}, \quad (20)$$

and this implies that, for the vacuum,

$$L_0^{\pm(\text{vac})} = -\frac{c_\pm}{24}. \quad (21)$$

Moreover, notice that with (19) we can alternatively write the Cardy formula as follows

$$S_{\text{CFT}} = 4\pi \sqrt{L_0^+ \frac{c_+}{24}} + 4\pi \sqrt{L_0^- \frac{c_-}{24}} \quad (22)$$

which, again, reproduces the black hole entropy (14). In other words, we get $S_{\text{BH}} = S_{\text{CFT}}$ and thus (17) seem to be the correct values of the central charges.

Conformal and diffeomorphism anomalies in the dual CFT_2 pose restrictions on the parameters of the bulk theory for the latter to have chances of being well defined [14]: On the one hand, assuming the existence of a dual CFT_2 with an $SL(2, \mathbb{C})$ invariant vacuum, the values of the central charges c_\pm imply that the theory cannot be defined for continuous values of $3\ell(m^2 \ell^2 - 1)/(2Gm^2 \ell^2)$. This follows from the Zamolodchikov c -theorem in the dual CFT_2 . On the other hand, as in TMG, the diffeomorphism anomaly in the dual CFT_2 is controlled by the number $c_+ - c_- = -3/(G\mu)$, which has to take values $1/(8G\mu) \in \mathbb{Z}$ for the theory to be modular invariant.

IV. CHIRAL CURVE

In [6], we were particularly interested in the curve of the parameter space of EMG defined by the relation

$$\mu = \frac{m^2 \ell}{1 - m^2 \ell^2}, \quad \Lambda = -\frac{1}{\ell^2}. \quad (23)$$

Hereafter, we will refer to this as the chiral curve, as the conjecture in [6] was that EMG with couplings obeying (23) and sufficiently strong boundary conditions turns out to be dual to a chiral CFT_2 . Indeed, when the values of the central charge (17) are evaluated on (23), we obtain

$$c_- = 0, \quad c_+ = \frac{3\ell}{G} \left(1 - \frac{1}{m^2 \ell^2} \right), \quad (24)$$

which is a necessary condition for chirality [7]. This is related to the fact that logarithmically decaying modes in EMG appear at (23), and chirality demands these modes to be decoupled. The picture might be similar to what happens in TMG. I.e., at the point $c_- = 0$, two theories exist: while the theory with standard boundary conditions in AdS_3 [13] results dual to a chiral CFT_2 , the theory defined with

weaker boundary conditions is presumably dual to a logarithmic CFT₂ [15].

The theory exhibits other special features at $c_- = 0$. At this point, all BTZ solutions satisfy the extremal condition

$$Q[\partial_\varphi] = \ell Q[\partial_t], \quad (25)$$

regardless the values of M and J . This suggests a high degeneracy of the spectrum. Also at this point, the solutions with $r_- = r_+$ (i.e., $|J| = \ell M$) present vanishing entropy, $S_{\text{BH}} = 0$. We will see in the next section that, in addition to BTZ black holes, at $c_- = 0$ the theory presents a large family of asymptotically AdS₃ solutions with $\mathbb{R} \times SO(3)$ isometries, vanishing entropy, and nonvanishing gravitational energy.

V. CHIRAL WAVES ON BLACK HOLES

The metric of the extremal BTZ solution takes the form

$$ds_{M,\mp\ell M}^2 = -\frac{(r^2 - r_+^2)^2}{\ell^2 r^2} dt^2 + \frac{\ell^2 r^2}{(r^2 - r_+^2)^2} dr^2 + \frac{r^2}{\ell^2} \left(\ell d\varphi \pm \frac{r_+^2}{r^2} dt \right)^2, \quad (26)$$

where $r_+^2 = r_-^2 = 4\ell^2 GM$. Starting from this geometry, a new family of nonlocally AdS₃ solutions to EMG can be constructed by means of a Kerr-Schild transformation: At the chiral point (23), the following deformation of the extremal BTZ is also a solution

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = ds_{M,-\ell M}^2 + L(t, r, \varphi) (dt + \ell d\varphi)^2, \quad (27)$$

with

$$L(t, r, \varphi) = A(x^+) \log(r^2 - r_+^2) + B(x^+) (r^2 - r_+^2)^{\frac{1+m^2\ell^2}{2}} + C(x^+) (r^2 - r_+^2) + D(x^+) \quad (28)$$

where $x^\pm = t \pm \ell\varphi$, and where A, B, C , and D are arbitrary functions that depend only on x^+ ; namely, they satisfy

$$\partial_- A = 0, \quad \partial_- B = 0, \quad \partial_- C = 0, \quad \partial_- D = 0, \quad (29)$$

with $\partial_\pm = \partial/\partial x^\pm$.

We can now analyze all the contributions appearing in (28) and study their physical meaning: Let us start with the A -modes, which accompany the logarithm in (28). This logarithmically growing potential is compatible with the AdS₃ boundary conditions

$$g_{tt} \simeq \frac{r^2}{\ell^2} + \mathcal{O}(\log r), \quad g_{t\varphi} \simeq \mathcal{O}(\log r), \quad g_{\varphi\varphi} \simeq r^2 + \mathcal{O}(\log r), \\ g_{rr} \simeq \frac{\ell^2}{r^2} + \mathcal{O}(1/r^4), \quad g_{r\varphi} \simeq \mathcal{O}(1), \quad g_{rt} \simeq \mathcal{O}(1), \quad (30)$$

which are the boundary conditions proposed in [16] in the context of log-gravity [15]. Periodicity of φ implies functions A to be periodic under $t \rightarrow t + 2\pi\ell$. This yields the mode expansion

$$A(x^+) = \sum_{n=0}^{\infty} (\zeta_n e^{in(t/\ell+\varphi)} + \zeta_n^* e^{-in(t/\ell+\varphi)}). \quad (31)$$

The periodicity in t is well understood, since ℓ is the time that takes for a massive or massless mode in AdS₃ to go toward the boundary and bounce back due to the gravitational potential. The long range interaction represented by the A -mode is due to the fact that this mode is actually a mode of the massive theory that becomes massless precisely on the curve (23). In fact, it satisfies the fourth-order massless wave equation

$$\square_A^2 e^{in(t/\ell+\varphi)} \log(r^2 - r_+^2) = 0, \quad (32)$$

where $n \in \mathbb{Z}$ and where \square_A is the Laplacian operator corresponding to the metric (27)–(28) with a given function A . The fact that the A -mode is massless, in the sense of obeying (32), does not mean that the full nonlinear solution has vanishing gravitational energy. In fact, we will see below that the A -mode does contribute to the gravitational energy of the geometry.

The B -mode also satisfies a wave equation: it obeys the massive second order equation

$$(\square_A - \kappa^2) e^{in(t/\ell+\varphi)} (r^2 - r_+^2)^{\frac{1+m^2\ell^2}{2}} = 0, \quad \text{with} \\ \kappa^2 \ell^2 = (m^2 \ell^2 + 3)(m^2 \ell^2 + 1). \quad (33)$$

However, unlike the A -mode, the B -mode does not represent a massive mode in AdS₃ as it grows like $\sim r^{1+m^2\ell^2}$ and thus violates the asymptotic (30). In other words, its linearized version would correspond to a non-normalizable state in AdS₃. To understand the origin of the B -mode, we can notice that in the limit $m^2 \rightarrow 0$ Eq. (23) implies $\mu = 0$, and this is the point of the parameter space where the field equations are satisfied by any conformally flat metric. This is because at $m^2 = \mu = 0$ all the terms in the field equations contain the Cotton tensor and its derivatives. In that limit, the B -mode behaves at large distance like $\sim r$, and this is the typical behavior of the mode of conformal gravity [17]. We will discuss nonlinear solutions associated to this mode in Sec. X.

While the A - and B -modes are lawful modes of the massive theory, the C - and the D -modes correspond to general relativity (GR) modes and they can be regarded as

pure gauge. Therefore, we can set $C = D = 0$. When $m^2\ell^2 = 1$, the B - and the C -modes degenerate. The solutions with $B = C = 0$ and generic A and D represent asymptotically AdS₃ solutions, which nonetheless are not locally equivalent to AdS₃ and are not conformally flat. In the context of AdS/CFT correspondence, function A acts as a source of a field in the dual CFT₂, while it determines the boundary condition for the bulk field configuration.

The particular case $M = J = 0$ of solution (27)–(28) corresponds to the so-called AdS-waves, which are special example of Siklos spacetimes [18,19]; see also [20,21]. Metrics (27)–(28) thus generalize the TMG wave solutions of [18], and then can be interpreted as nonlinear gravitational waves on the extremal BTZ geometry. Function A determines the profile of the wave. Remarkably, Eq. (32) is also obeyed if one replaces the Laplacian operator \square_A by the Laplacian of the undeformed BTZ solution, $\square_{A=0}$. This is the case even when the two operators do not identically coincide. Therefore, (27)–(28) can either be interpreted as gravitational waves propagating in stealth mode on the black hole background, or as nonlinear waves propagating on their own backreacting geometries.

As explained above, the periodicity in t is easily explained because ℓ is the characteristic timescale in AdS space. That is, the period of the wave is determined by the AdS radius itself and not by the rotation period of the black hole. What the black hole angular momentum does determine is the growth of the radial dependence close to the horizon. It produces a divergence in the angular and temporal components of the metric at $r = r_+$, and this ultimately determines the direction of the rotation: The wave turns out to be corotating with the black hole.

Solutions (27)–(28) also come to generalize the log-gravity solution presented in [22], which corresponds to the particular case $A = \text{const}$. The gravitational energy and angular momentum of (27)–(28) can be computed, yielding

$$\begin{aligned} Q[\partial_t] &= \frac{1}{4\pi G\ell} \left(1 + \frac{1}{m^2\ell^2}\right) \int_0^{2\pi\ell} A(s) ds, \\ Q[\partial_\varphi] &= -\ell Q[\partial_t], \end{aligned} \quad (34)$$

which reduces to the result of [8] in the particular case $\partial_\pm A = 0$.

VI. MASSIVE WAVES IN THE MIRROR

A mirror image of the family of solutions (27)–(28) can be constructed by means of a similar Kerr-Schild ansatz but considering as a seed the extremal BTZ with the opposite angular momentum. This anticlockwise version of (27)–(28), however, does not behave in the same way as the theory is not parity invariant. In fact, the following deformation of the $M = +J/\ell$ black hole is also a solution of EMG when (23) holds,

$$ds^2 = ds_{M,+ \ell M}^2 + L(t, r, \varphi)(dt - \ell d\varphi)^2, \quad (35)$$

with

$$\begin{aligned} L(t, r, \varphi) &= A(x^-)(r^2 - r_+^2) \log(r^2 - r_+^2) \\ &\quad + B(x^-)(r^2 - r_+^2)^{\frac{1-m^2\ell^2}{2}} + C(x^-)(r^2 - r_+^2) \\ &\quad + D(x^-), \end{aligned} \quad (36)$$

and where the functions now satisfy

$$\partial_+ A = 0, \quad \partial_+ B = 0, \quad \partial_+ C = 0, \quad \partial_+ D = 0. \quad (37)$$

Again, at $m^2 = 0$ the B -mode corresponds to the conformal mode, and at $m^2\ell^2 = 1$ the B -mode degenerates with the C -mode. Within the range $0 \leq m^2\ell^2 < 1$ the B -mode blows up at infinity. In contrast, when $m^2\ell^2 > 1$, the B -modes becomes compatible with the Brown-Henneaux AdS₃ boundary conditions [13]. Indeed, provided $A = C = 0$, the solution (35) obeys

$$\begin{aligned} g_{tt} &\simeq \frac{r^2}{\ell^2} + \mathcal{O}(1), \quad g_{t\varphi} \simeq \mathcal{O}(1), \quad g_{\varphi\varphi} \simeq r^2 + \mathcal{O}(1), \\ g_{rr} &\simeq \frac{\ell^2}{r^2} + \mathcal{O}(1/r^4), \quad g_{r\varphi} \simeq \mathcal{O}(1/r^3), \quad g_{rt} \simeq \mathcal{O}(1/r^3), \end{aligned} \quad (38)$$

and the B -mode now does represent a massive mode in AdS₃. It satisfies the wave equation

$$\begin{aligned} (\square_A - \kappa^2)e^{in(t/\ell - \varphi)}(r^2 - r_+^2)^{\frac{1-m^2\ell^2}{2}} &= 0, \quad \text{with} \\ \kappa^2\ell^2 &= (m^2\ell^2 - 3)(m^2\ell^2 - 1); \end{aligned} \quad (39)$$

it decays rapidly enough at large distance and acts as a source in the dual CFT₂.

VII. WARPED BLACK HOLES

Now, let us discuss black holes in warped deformations of AdS₃ space (WAdS₃). These solutions were found in [23] in the context of TMG, and they were exhaustively studied in [24]. The metric of these black holes is given by

$$\begin{aligned} ds_{r_+, r_-}^2 &= dt^2 + \frac{\ell^2}{\nu^2 + 3} \frac{dr^2}{(r - r_+)(r - r_-)} \\ &\quad - 2 \left(\nu r + \frac{1}{2} \sqrt{r_+ r_- (\nu^2 + 3)} \right) dt d\varphi \\ &\quad + \frac{r}{4} \left(3(\nu^2 - 1)r + (\nu^2 + 3)(r_+ + r_-) \right. \\ &\quad \left. + 4\nu \sqrt{r_+ r_- (\nu^2 + 3)} \right) d\phi^2 \end{aligned} \quad (40)$$

where $t \in \mathbb{R}$, $r \in \mathbb{R}_{\geq 0}$, and $0 \leq \phi \leq 2\pi$; ℓ is a real parameter related to the Gaussian curvature of the

manifold; and r_{\pm} are two integration constants taken to satisfy $0 \geq r_- \geq r_+$. r_+ represents the outer horizon of the black hole, and r_- represents its inner horizon.

WAdS₃ black holes (40) are locally equivalent to the three-dimensional section of the Gödel cosmological solution to Einstein equations. This is locally equivalent to either squashed ($\nu^2 < 1$) or stretched ($\nu^2 > 1$) deformations of AdS₃, with the parameter ν controlling such deformation. $\nu^2 = 1$ corresponds to locally AdS₃ (undeformed) spaces. $\nu = 0$ corresponds to another interesting case that we will discuss below: it is locally AdS₂ \times S¹. In fact, space (40) and its equivalents can be written as a Hopf fibration over AdS₂ or over its Euclidean version \mathbb{H}_2^+ , with the fiber being \mathbb{R} . $\nu = 0$ is a kind of extreme limit.

Black holes (40) are solutions to EMG field equations provided

$$\mu = \frac{\ell^3 m^4 \nu}{3(\nu^4 - \nu^2(1 + \ell^2 m^2) + \ell^4 m^4/9)},$$

$$\nu^6 - 2\nu^4 + \nu^2 + m^4 \ell^2(1 + \Lambda \ell^2)/9 = 0. \quad (41)$$

They have nonvanishing gravitational energy, which is given by

$$Q[\partial_t] = \frac{(\nu^2 + 3)(3\nu^2 + \ell^2 m^2)}{24Gm^2 \ell^3} \times \left(r_+ + r_- - \frac{1}{\nu} \sqrt{(\nu^2 + 3)r_- r_+} \right). \quad (42)$$

Notice that this formula generalizes the TMG formula to which it reduces in the limit $m^2 \rightarrow \infty$.

VIII. EVANESCENT GRAVITONS IN WAdS₃

EMG field equations admits further deformations of (40). In particular, it admits solutions that describe evanescent gravitational waves on WAdS₃ spaces. To see this, let us consider the case $r_{\pm} = 0$, whose metric is

$$ds_{0,0}^2 = dt^2 + \frac{2dr^2}{(\nu^2 + 3)r^2} - 2\nu r dt d\phi + \frac{3}{4} r^2 (\nu^2 - 1) d\phi^2, \quad (43)$$

where we are using the convention $\ell \equiv 1$ for short. Field equations (1)–(4), provided equations (41) hold, admits the following solution

$$ds^2 = ds_{0,0}^2 + e^{-\omega t} r^{-\frac{2\nu\omega}{\nu^2+3}} A(x^-) (dx^-)^2, \quad (44)$$

where A is an arbitrary periodic function of x^- , with

$$x^{\pm} = \phi \pm \frac{2}{(\nu^2 + 3)r}, \quad (45)$$

and where ω is a parameter given by the polynomial equation

$$(5\nu^2 + 6\nu\omega + \omega^2 + 3)m^4 - 3\nu(\nu + \omega)(5\nu^2 + 6\nu\omega + \omega^2 + 3)m^2 - 27\nu^2(\nu^2 - 1)(\nu^2 - 3\nu\omega - \omega^2 - 3) = 0. \quad (46)$$

This solution generalizes one of the TMG solutions found in [25]. In fact, notice that in the limit $m^2 \rightarrow \infty$, (46) reduces to the TMG condition $\omega = -3 \pm \sqrt{4\nu^2 - 3}$ found therein. Provided $\omega > 0$, solution (44) describes an evanescent gravitational wave on the massless WAdS₃ black hole geometry. As the log-gravity waves discussed in Sec. V, waves (44) concentrate close to the horizon (in this case, at $r = 0$), where the metric components g_{tt} , $g_{\phi t}$, and $g_{\phi\phi}$ diverge (assuming $\omega\nu > 0$, which is necessary for the solution to be asymptotically WAdS₃). This wave satisfies the massive equation

$$(\square_A - \kappa^2) e^{-\omega t + i\nu x^-} r^{-\frac{2\nu\omega}{\nu^2+3}} = 0, \quad \text{with } \kappa^2 = \omega(\omega - 2\nu), \quad (47)$$

$n \in \mathbb{Z}$. \square_A stands here for the Laplacian corresponding to metric (44); nevertheless, the same equation holds if one replaces \square_A by $\square_{A=0}$. This follows from the arbitrariness of A , what makes the *ansatz* to be linear in the function of x^- . Perturbation (44) carries no gravitational energy.

IX. ADS₂ \times S¹ VACUA

EMG generalizes TMG and, consequently, contains conformal gravity as a particular case: The latter corresponds to the limit $\mu \rightarrow 0$, $m^2 \rightarrow \infty$. In particular, AdS₂ \times S¹ and $\mathbb{H}_2^+ \times \mathbb{R}$ spaces appear as solutions of EMG only in that limit. Let us briefly discuss these geometries and relate them with the ones discussed above: In fact, an ingenious trick to work out the AdS₂ \times S¹ case is to think of it as the $\nu \rightarrow 0$ limit of the WAdS₃ space. From (41), one observes that when ν goes to zero μ also goes to zero, in such a way that

$$\frac{\mu}{\nu} = \frac{3}{\ell}, \quad \ell^2 = -\frac{1}{\Lambda}. \quad (48)$$

We can think of the large m^2 limit. In that case, the left- and right-moving central charges for asymptotically WAdS₃ spaces are [26]

$$c_- = \frac{4\nu\ell}{G(\nu^2 + 3)}, \quad c_+ = \frac{(5\nu^2 + 3)\ell}{G\nu(\nu^2 + 3)}. \quad (49)$$

Notice that these expressions are consistent with (17) in the AdS₃ limit $\nu \rightarrow 1$: Considering $\mu\ell = 3$ and $m^2 = \infty$, one obtains $c_- = \ell/G$ and $c_+ = 2\ell/G$, in accordance with (17). In the AdS₂ limit $\nu \rightarrow 0$, on the other hand, one gets

$$c_- = 0, \quad c_+ = \frac{3}{G\mu} \equiv 24k, \quad (50)$$

where the Chern-Simons (CS) level k relates to the TMG coupling constant by $k/(4\pi) = 1/(32\pi G\mu)$, which is the correct normalization of the gravitational CS action

$$I_{\text{CS}} = \frac{k}{4\pi} \int d^3x \varepsilon^{\mu\nu\rho} \left(\Gamma_{\mu\alpha}^\beta \partial_\nu \Gamma_{\rho\beta}^\alpha + \frac{2}{3} \Gamma_{\mu\alpha}^\eta \Gamma_{\nu\beta}^\alpha \Gamma_{\rho\eta}^\beta \right) \quad (51)$$

that yields the field equation $(1/\mu)C_{\mu\nu} = 0$, in agreement with the $\mu \rightarrow 0$ limit of (1). It is worth noticing that (50) exactly reproduces the result of [27] for \mathbb{H}_2^+ holography. Furthermore, one can easily verify that the nonperturbative states discussed in [27] are actually diffeomorphic to the $\nu \rightarrow 0$ limit of the metric (40), which have vanishing conserved charges.

X. HAIRY BLACK HOLES IN (A)DS

The theory (1) can also be defined in the limit $m^2 \rightarrow 0$, $\mu \rightarrow 0$, in which the Einstein tensor disappears from the field equations. This limit, however, is subtle; it has to be taken in the right order for the consistency equations $\nabla_\mu T^\mu_\nu = 0$ to hold. When the Einstein tensor is absent, the field equations are satisfied for all conformally flat spaces. This is because the Cotton tensor is zero if and only if a metric is conformally equivalent to flat space, at least locally. A very interesting case of conformally flat space is the following [17]

$$ds^2 = -\frac{(r-r_+)(r-r_-)}{\ell^2} dt^2 + \frac{\ell^2 dr^2}{(r-r_+)(r-r_-)} + r^2 d\varphi^2 \quad (52)$$

with $t \in \mathbb{R}$, $r \in \mathbb{R}_{\geq 0}$, and $0 \leq \varphi \leq 2\pi$. This space, provided either r_+ or r_- is positive, represents a static black hole. For $\ell^2 > 0$, (52) is asymptotically AdS₃, in the sense of the weakened boundary conditions studied in [28]. In particular, this implies the behavior

$$g_{tt} \simeq -\frac{r^2}{\ell^2} + \mathcal{O}(r). \quad (53)$$

This linear term in r , together with its analogs in the angular components of the metric, correspond to the conformal gravity mode that, as we discussed in Sec. V, appears in the $m^2 \rightarrow 0$ limit. Despite this weakened asymptotics, solution (52) has finite conserved charges: Defining

$$M \equiv -\frac{r_+ r_-}{8G\ell^2}, \quad N = \frac{(r_+ + r_-)^2}{32G\ell^2} \quad (54)$$

we obtain that the mass and the angular momentum are given by

$$Q[\partial_t] = -\frac{M+N}{m^2 \ell^2} = -\frac{(r_+ - r_-)^2}{32Gm^2 \ell^4},$$

$$Q[\partial_\varphi] = -\frac{M+N}{\mu} = -\frac{(r_+ - r_-)^2}{32G\mu \ell^2}, \quad (55)$$

respectively. These values for the charges can be obtained from the results (12) and (13) in a simple way: One considers in (52) the coordinate changes $r \rightarrow r - (r_+ + r_-)/2$, and then takes in (12)–(13) the limit $m^2 \rightarrow 0$, $\mu \rightarrow 0$.

Assuming $r_+ \geq r_- \geq 0$, for which two horizons exist, the temperature of the black hole takes the form

$$T_{\text{H}} = \frac{r_+ - r_-}{4\pi \ell^2}, \quad (56)$$

which vanishes in the extremal limit $r_+ = r_-$, for which the near horizon geometry renders AdS₂ × S¹. This means that the entropy of the black hole is proportional to the difference between the outer and inner horizons. More precisely,

$$S_{\text{BH}} = -\frac{\pi(r_+ - r_-)}{4Gm^2 \ell^2}, \quad (57)$$

which follows from the first principle of the black hole thermodynamics. The negative sign in (57) and in the mass $Q[\partial_t]$ demands to perform $m \rightarrow im$. Remarkably, this entropy can be obtained from (56) by using Cardy formula: By considering the large $1/m^2$, $1/\mu$ limit in (17), we actually obtain

$$S_{\text{BH}} = \frac{\pi^2 \ell}{3} (c_+ + c_-) T_{\text{H}} = 4\pi \sqrt{L_0^+ \frac{c_+}{24}} + 4\pi \sqrt{L_0^- \frac{c_-}{24}}, \quad (58)$$

where $L_0^+ + L_0^- = \ell Q[\partial_t]$, $L_0^+ - L_0^- = Q[\partial_\varphi]$, and $c_+ + c_- = -3/(Gm^2 \ell)$, $c_+ - c_- = -3/(G\mu)$. This is a case in which the Cardy formula is shown to hold for a solution with nonconstant curvature invariants—unlike the cases of BTZ and WAdS₃ black holes—in this sense, this case is more general than the analysis of [29]; see also [30].

This black hole solution generalizes the static BTZ black hole solution, which corresponds to the special case $r_+ = -r_- \geq 0$ (i.e., $N = 0$). In (52), N is a sort of *hair*. This hair is, however, of gravitational origin. One can easily verify that the value of N explicitly appears in the scalar curvature and it makes the black hole with $N \neq 0$ to have a curvature singularity at $r = 0$. It is also worth mentioning that a rotating generalization of (52) also exists [28]. More interestingly, the solution above may also describe a static asymptotically de Sitter (dS) black hole, which follows from considering in (52) the change $\ell^2 \rightarrow -\ell^2$. In that case, r_- gives the location of the black hole horizon, while r_+ is the radius of the cosmological horizon, which turns out to be in thermal equilibrium with the black hole. This also

represents a solution to field equations (1) in the limit $m^2 \rightarrow 0$, $\mu \rightarrow 0$.

XI. CONCLUDING REMARKS

In this paper, we have continued the study of the vacua of the exotic massive 3D gravity theory [1] initiated in Ref. [6]. We mainly focused our attention on asymptotically AdS₃, Warped-AdS₃, and AdS₂ × S¹ geometries. This includes black holes and nonlinear gravitational waves. In the case of asymptotically AdS₃ geometries, we obtained the values of the central charge of the dual CFT₂ for the Cardy formula (22) to reproduce the entropy of BTZ black holes of EMG. The theory, however, contains a much more rich bestiary of black holes with different asymptotics [6]; this includes warped-WAdS₃ black holes, which we also discussed, together with Lifshitz black holes. In those cases, a similar type of holography inspired computation is in principle possible. Of particular interest is the case of Lifshitz black holes, for which EMG enables an arbitrary value of dynamical exponent z . It would be interesting to verify that a computation similar to the one in [31] can be done in EMG; see also [32]. This would first demand to compute the conserved charges in nonconstant curvature, asymptotically Lifshitz spaces for such a third-way gravity model.

We also discussed deformations of AdS₃ and WAdS₃ black holes. In the case of AdS₃ space, we constructed a family of logarithmically decaying gravitons on the extremal BTZ black holes. These solutions are analogous

to the log-gravity solutions that TMG exhibits at the chiral point. In the case of WAdS₃ spaces, we gave explicit examples of evanescent gravitational waves propagating on massless black holes. We also discussed the conformal gravity limit of the theory and its locally AdS₂ × S¹ solutions, recovering some results of [27] as a limit of the WAdS₃/CFT₂ symmetry analysis. Finally, we showed that, in a particular limit of the parameter space, EMG admits asymptotically hairy (A)dS₃ black holes with nonvanishing conserved charges and nontrivial thermodynamical properties. The fact that all these holographically well-motivated geometries appear in different corners of the space of solutions of EMG manifestly shows that this theory is a very nice toy model to explore holography beyond AdS. The theory, however, has peculiar features that make its holographic application knotty: The lack of an action in its metric formulation and the subtle problem of how to couple matter in a consistent way are two features that require further analysis. In relation to that, it would be interesting to investigate further the definition of EMG in the Chern-Simons like formulation, for which an action in terms of the dreibein and the spin connection is available.

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