TENSOR PRODUCT OF MODULES OVER A LIE CONFORMAL ALGEBRA

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Abstract. We find a necessary and sufficient condition for the existence of the tensor product of modules over a Lie conformal algebra. We provide two algebraic constructions of the tensor product. We show the relation between tensor product and conformal linear maps. We prove commutativity of the tensor product.

1. Introduction

Since the pioneering papers [BPZ] and [Bo], there has been a great deal of work towards understanding of the algebraic structure underlying the notion of the operator product expansion (OPE) of chiral fields of a conformal field theory. The singular part of the OPE encodes the commutation relations of fields, which leads to the notion of a Lie conformal algebra [K].

In the past few years a structure theory [DK], representation theory [CK, CKW] and cohomology theory [BKV] of finite Lie conformal algebras has been developed.

In this work we define and construct the tensor product of modules over Lie conformal algebras. We translate the ideas on the tensor product of modules over vertex algebras, presented in [L1], to the case of Lie conformal algebras. In this way, we answer an open question suggested by V. Kac during a graduate course at MIT in 1997, where he presented the problem and the result that should be obtained for the modules over the Virasoro conformal algebra. It has been a necessary and missing part of the theory in the last 20 years.

We introduce the notion of intertwining operator for the case of modules over Lie conformal algebras and we define the tensor product by the universal property.

We find a necessary and sufficient condition for the existence of the tensor product of modules over a Lie conformal algebra. We provide two algebraic constructions of the tensor product. We show the relation between tensor product and conformal linear maps, and we prove commutativity of the tensor product.

It is important to point out that Y. Z. Huang communicated that our tensor product presented in [L1] is not the same as the one defined by Huang and Lepowsky [HL1]-[HL5]. But, we believe that our definition and construction can be changed in order to obtain an algebraic construction of their tensor product. The proof of associativity of the tensor product of vertex algebras presented in [L1], is probably wrong since it is based on the wrong proof given in [DLM].

In [L2], we extend the results in this work to the case of H-pseudoalgebras (see [BDK]).

In section 2, we present the basic definitions and notations. In section 3, we find a necessary and sufficient condition for the existence of the tensor product of modules over a Lie conformal algebra, that we called the *kernel intertwining operator full equality condition*, and we present the first construction.

In section 4, we present the relation between the tensor product and the conformal analog of the Hom functor. This relationship obtained in Theorem 4.2, provides the motivation for

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the definition of intertwining operators. In section 5, we prove the commutativity of the tensor product.

In section 6, we present a second construction, based on the ideas for the tensor product of modules over a vertex operator algebra in [Li]. The idea is simple, for two finite conformal R-modules M and N, we take certain finite submodule of $\mathrm{Chom}(M, N^{*_c}) \simeq M^{*_c} \otimes N^{*_c}$ (see Proposition 4.1), called $\Delta(M, N^{*_c})$. Then, the conformal dual $\big(\Delta(M, N^{*_c})\big)^{*_c}$ is the tensor product $M \underset{R}{\otimes} N$.

In section 7, we try to compute the tensor product of finite irreducible conformal modules over the Virasoro conformal algebra.

Finally, for me it is important to point out that in 2010, we obtained the second construction. In 2011 we found the kernel condition, but due to personal problems we abandoned this paper until January 2014, when we found the first construction of the tensor product. Therefore, it was basically finished in 2014. In march 2020, we resumed this work, producing this final version.

Unless otherwise specified, all vector spaces, linear maps and tensor products are considered over $\mathbb{C}$.

2. Definitions and notation

In order to make a self-contained paper, in this section we present the notion of Lie conformal algebra and their modules, intertwining operators and tensor product.

A *Lie conformal algebra* R is a $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map $R\otimes R \longrightarrow \mathbb{C}[\lambda]\otimes R$, $a \otimes b \mapsto [a_\lambda b]$, called the λ-bracket, satisfying the following axioms $(a, b, c \in R)$:

- *Sesquilinearity*: $\quad [(\partial a)_\lambda b] = -\lambda [a_\lambda b], \qquad [a_\lambda (\partial b)] = (\lambda + \partial)[a_\lambda b],$
- *Skew-commutativity*: $\quad [a_\lambda b] = -[a_{-\partial-\lambda} b],$
- *Jacobi identity*: $\quad [a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + [b_\mu [a_\lambda c]].$

In all the expressions in this work, and specially in the skew-commutativity, the powers of λ, $-\partial - \lambda$, etc. are moved to the left. A Lie conformal algebra is called *finite* if it has finite rank as $\mathbb{C}[\partial]$-module. The notions of homomorphism, ideal and subalgebras of a Lie conformal algebra are defined in the usual way.

A *module* M over a Lie conformal algebra R is a $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map $R \otimes M \longrightarrow \mathbb{C}[[\lambda]] \otimes M$, $a \otimes v \mapsto a_\lambda^M v$, called the λ-action, satisfying the following axioms $(a,\, b \in R,\ v \in M)$:

- *Sesquilinearity*: $\quad (\partial a)_\lambda^M v = -\lambda\, a_\lambda^M v, \qquad a_\lambda^M (\partial^M v) = (\lambda + \partial^M)\, a_\lambda^M v,$
- *Jacobi identity*: $\quad a_\lambda^M (b_\mu^M v) = [a_\lambda b]_{\lambda+\mu}^M v + b_\mu^M (a_\lambda^M v).$

A module is called *conformal* if $a_\lambda^M v \in \mathbb{C}[\lambda] \otimes M$. A module is called *finite* if it has finite rank as $\mathbb{C}[\partial]$-module. The notions of homomorphism, and submodules are defined in the usual way. From now on, we shall simply use $a_\lambda v$ instead of $a_\lambda^M v$, if the situation is clear. Similarly, we use ∂ instead of ∂^M.

Now, we introduce the notions of intertwining operators and tensor product of conformal modules. One of the motivation is given by the similar notions for modules over a vertex algebra presented in [L1] and [Li].

Definition 2.1. Let M, N and W be three conformal R-modules. An *intertwining operator* of type $\binom{W}{M\,,\,N}$ is a $\mathbb{C}$-bilinear map

$$\begin{aligned} I_\lambda \,:\, M \times N &\longrightarrow \mathbb{C}[\lambda\,] \otimes W \\ (u\,,v) \;\; &\longmapsto I_\lambda(u\,,v) = \sum_{n\in\mathbb{Z}_+} \frac{\lambda^n}{n!}\; I_{(n)}(u\,,v) \end{aligned}$$

satisfying the following conditions:

- *Translation - Derivation*: For all $u \in M$ and $v \in N$

$$I_\lambda(\partial\, u, v) = -\lambda\, I_\lambda(u,v), \quad \text{and} \;\; \partial\,(I_\lambda(u,v)) = I_\lambda(\partial\, u, v) + I_\lambda(u, \partial\, v). \tag{2.1}$$

- *Jacobi identity*: For all $a \in V, u \in M$ and $v \in N$

$$a_\lambda I_\gamma(u\,,v) = I_{\lambda+\gamma}\,(a_\lambda u\,,v) + I_\gamma(u\,,a_\lambda v)\,. \tag{2.2}$$

Observe that the notion of intertwining operator is a generalization of the notion of conformal R-module, so that for a conformal R-module $(M, \overset{M}{\lambda})$, the bilinear map $\overset{M}{\lambda}$ is an intertwining operator of type $\binom{M}{R\,,\,M}$. We denote by $\binom{W}{M\,,\,N}$ the vector space of all intertwining operators of the indicated type.

Definition 2.2. Let M and N be two conformal R-modules. A pair $(M \underset{R}{\otimes} N, F_\lambda)$, which consists of a conformal R-module $M \underset{R}{\otimes} N$ and an intertwining operator F_λ of type $\binom{M \otimes_R N}{M\,,\,N}$, is called a *tensor product* for the ordered pair (M, N) if the following universal property holds: For any conformal R-module W and any intertwining operator I_λ of type $\binom{W}{M\,,\,N}$ there exists a unique R-homomorphism φ from $M \underset{R}{\otimes} N$ to W such that $I_\lambda = \varphi \circ F_\lambda$, where φ is extended canonically to a linear map from $\mathbb{C}[\lambda\,] \otimes (M \underset{R}{\otimes} N)$ to $\mathbb{C}[\lambda\,] \otimes W$.

Just as in the classical algebra theory, it follows from the universal property that if there exists a tensor product for the ordered pair (M, N), then it is unique up to an R-module isomorphism. Namely, the pair (W, G_λ) is another tensor product if and only if there exists an R-module isomorphism $\phi : M \underset{R}{\otimes} N \to W$ such that $G_\lambda = \phi \circ F_\lambda$.

3. First construction of the tensor product

First of all, we need some definitions in order to find necessary and sufficient conditions for the existence of the tensor product.

Definition 3.1. (a) Let M and N be two conformal R-modules. We define the *kernel of the intertwining operators from the pair* (M,N) as follows

$$Ker\binom{\cdot}{M,N} := \Big\{(u,v)\in M\times N \mid \exists\, l_{u,v}\in\mathbb{Z}_+ \text{ such that } I_{(n)}(u,v)=0$$
$$\text{for all } n\geq l_{u,v},\ \text{for all conformal } R\text{-modules } W, \text{ and for all } I_\lambda\in\tbinom{W}{M,N}\Big\}$$

(b) We say that the pair (M,N) satisfies the *kernel intertwining operator full equality condition* if

$$M\times N = Ker\binom{\cdot}{M,N}.$$

Proposition 3.2. *(Necessary condition) If the tensor product* $(M\underset{R}{\otimes}N, F_\lambda)$ *exists, then the pair* (M,N) *satisfies the kernel intertwining operator full equality condition.*

Proof. Fix $(u,v)\in M\times N$ and using that $F_\lambda(u,v)\in(M\underset{R}{\otimes}N)[\lambda]$, we have that there exists $N\in\mathbb{Z}_+$ (depending on u and v) such that $F_{(n)}(u,v)=0$ for all $n\geq N$. Since for any conformal R-module W and any intertwining operator I_λ of type $\binom{W}{M\ ,\ N}$ there exists a unique R-homomorphism φ from $M\underset{R}{\otimes}N$ to W such that $I_\lambda=\varphi\circ F_\lambda$, where φ is extended canonically to a linear map from $(M\underset{R}{\otimes}N)[\lambda]$ to $W[\lambda]$, it follows that $I_{(n)}(u,v)=0$ for all $n\geq N$. □

Theorem 3.3. *(Sufficient condition) If the pair* (M,N) *satisfies the kernel intertwining operator full equality condition, then the tensor product* $(M\underset{R}{\otimes}N, F_\lambda)$ *exists.*

We shall do the construction of the tensor product in the rest of this section, and we shall see that it is also a sufficient condition for the existence of the tensor product.

The essential idea is to consider "strings over $\mathbb{Z}_+$" for each pair $(u,v)\in M\times N$ and then take the quotient by all the necessary conditions in order to get an intertwining operator by taking the generating series of these strings (cf. [Li]).

Therefore, let M and N be two conformal R-modules, and set

$$F_0(M,N)=\mathbb{C}[t]\otimes M\otimes N.$$

As usual in conformal algebra theory, it is more clear to work with generating series in order to manipulate string of vectors. For this reason we introduce the following very important notation for any $u\in M$ and $v\in N$ (cf. formula (5.2.2) in [Li]):

$$u\underset{\gamma}{\otimes}v := \sum_{n\in\mathbb{Z}_+}\frac{\gamma^n}{n!}\,(t^n\otimes u\otimes v).$$

Now, we want to take the necessary quotients in order to obtain that $\underset{\gamma}{\otimes}$ is an intertwining operator. In particular, it should satisfies the Jacobi identity (2.2), and this is the motivation for the following λ-action of R on $F_0(M,N)$ (for $a\in R, u\in M, v\in N$):

$$a_\lambda(u\underset{\gamma}{\otimes}v)=(a_\lambda u)\underset{\lambda+\gamma}{\otimes}v+u\underset{\gamma}{\otimes}(a_\lambda v). \tag{3.1}$$

Motivated by the derivation property (2.1), we define the $\mathbb{C}[\partial]$-module structure on $F_0(M,N)$, by the linearly extended map given on generators by $\partial(u\underset{\gamma}{\otimes}v)=\partial u\underset{\gamma}{\otimes}v+u\underset{\gamma}{\otimes}\partial v$.

Recall the standard notation for the λ-action that produce the (n)-operators:

$$a_\lambda w := \sum_{n\in\mathbb{Z}_+} \frac{\lambda^n}{n!}\, a_{(n)}w.$$

If we consider the coefficients in λ and γ, the λ-action (3.1), corresponds to:

$$a_{(m)}(t^n \otimes u \otimes v) = \sum_{i=0}^{m} \binom{m}{i} t^{n+m-i} \otimes a_{(i)}u \otimes v + t^n \otimes u \otimes a_{(m)}v, \tag{3.2}$$

for $m, n \in \mathbb{Z}_+$, $a \in R, u \in M, v \in N$, similar to the action in the tensor product of modules over a vertex algebra [L1].

Proposition 3.4. *Under the above defined λ-action, $F_0(M,N)$ is an R-module.*

Proof. The sesquilinearity follows by straightforward computations. Now, we prove the Jacobi identity. For $a, b \in R$, $u \in M$ and $v \in N$, we have

$$\begin{aligned} a_\lambda\big(b_\gamma(u \underset{\mu}{\otimes} v)\big) &= a_\lambda\big(b_\gamma u \underset{\mu+\gamma}{\otimes} v + u \underset{\mu}{\otimes} b_\gamma v\big) = a_\lambda(b_\gamma u) \underset{\mu+\gamma+\lambda}{\otimes} v + (b_\gamma u) \underset{\mu+\gamma}{\otimes} (a_\lambda v) \\ &\qquad + (a_\lambda u) \underset{\mu+\lambda}{\otimes} (b_\gamma v) + u \underset{\mu}{\otimes} a_\lambda(b_\gamma v), \end{aligned}$$

and

$$b_\gamma\big(a_\lambda(u \underset{\mu}{\otimes} v)\big) = b_\gamma(a_\lambda u) \underset{\mu+\gamma+\lambda}{\otimes} v + (a_\lambda u) \underset{\mu+\lambda}{\otimes} (b_\gamma v) + (b_\gamma u) \underset{\mu+\gamma}{\otimes} (a_\lambda v) + u \underset{\mu}{\otimes} b_\gamma(a_\lambda v).$$

Therefore, we obtain

$$\begin{aligned} [a_\lambda, b_\gamma]\big(u \underset{\mu}{\otimes} v\big) &= [a_\lambda, b_\gamma]\, u \underset{\mu+\gamma+\lambda}{\otimes} v + u \underset{\mu}{\otimes} [a_\lambda, b_\gamma]\, v = [a_\lambda b]_{\lambda+\gamma} u \underset{\mu+\gamma+\lambda}{\otimes} v + u \underset{\mu}{\otimes} [a_\lambda b]_{\lambda+\gamma} v \\ &= [a_\lambda b]_{\lambda+\gamma}\big(u \underset{\mu}{\otimes} v\big), \end{aligned}$$

finishing the proof. □

Let J_0 be the R-submodule of $F_0(M,N)$ generated by the following subspace:

$$\mathbb{C}\text{−span}\left\{ t^n \otimes u \otimes v \in F_0(M,N) \mid I_{(n)}(u,v) = 0 \text{ for all } R\text{-modules } W, \text{ and for all } I_\lambda \in \binom{W}{M,N} \right\}$$

Since $M \times N = Ker\,\binom{\cdot}{M,N}$, then for every $(u,v) \in M \times N$, there exists $l \in \mathbb{N}$ such that $I_{(n)}(u,v) = 0$ for all $n \geq l$, for all R-modules W, and for all $I_\lambda \in \binom{W}{M\,,\,N}$.

Now, we take

$$F_1(M,N) = F_0(M,N)/J_0.$$

We will still use the notation $u \underset{\gamma}{\otimes} v$ and $t^n \otimes u \otimes v$ for elements in the quotient space $F_1(M,N)[[\gamma]]$ and $F_1(M,N)$ respectively. We have the following important result:

Proposition 3.5. *For any $a \in R$ and any $t^n \otimes u \otimes v \in F_1(M,N)$, we have*

(a) $a_\lambda(t^n \otimes u \otimes v)$ involves only finitely many positive powers of λ.

(b) $u \underset{\gamma}{\otimes} v$ involves only finitely many positive powers of γ.

(c) The map ∂ is well defined in $F_1(M,N)$, that is $\partial(J_0) \subseteq J_0$.

Proof. (a) We fix $a \in R$ and $t^n \otimes u \otimes v \in F_1(M,N)$. Recall formula (3.2):

$$a_{(m)}(t^n \otimes u \otimes v) = \sum_{i=0}^{m} \binom{m}{i} t^{m+n-i} \otimes a_{(i)}u \otimes v + t^n \otimes u \otimes a_{(m)}v.$$

Recall that for $a \in R$ and $u \in M$, there exists $l_{a,u} \in \mathbb{N}$ such that $a_{(k)}u = 0$ for all $k \geq l_{a,u}$. Observe that in the second term, $a_{(m)}v = 0$ for a sufficiently large m, and in the first term, the finite sum is independent of m for all $m \geq l_{a,u}$. Then for each element $a_{(i)}u \otimes v$ (with $i \leq l_{a,u}$), there exists a power of t such that $t^l \otimes a_{(i)}u \otimes v = 0$ if l is large enough. Therefore, for a large enough m the result is proved.

(b) It is immediate from the definition of J_0 and the kernel condition that is assume for the pair (M,N).

(c) Suppose $t^n \otimes u \otimes v \in J_0$, then $\partial(t^n \otimes u \otimes v) = t^n \otimes \partial u \otimes v + t^n \otimes u \otimes \partial v$. Using (2.1), we have $I_{(n)}((\partial u, v) + (u, \partial v)) = \partial\ I_{(n)}(u,v) = 0$, finishing the proof. $\square$

Now, an easy computation shows that the subspace generated by all the coefficients in the elements of the form (for $u \in M, v \in N$)

$$\partial u \underset{\gamma}{\otimes} v + \gamma(u \underset{\gamma}{\otimes} v),$$

is R-invariant, and motivated by the translation property in the definition of intertwining operators, let J_1 be this R-submodule of $F_1(M,N)$. By straightforward computations, it is easy to see that J_1 is invariant by ∂. We define

$$M \underset{R}{\otimes} N = F_1(M,N)/J_1.$$

Then, we obtain

Proposition 3.6. *The space $M \underset{R}{\otimes} N$ is an R-module, and $\underset{\gamma}{\otimes}$ is an intertwining operator of type $\binom{M \underset{R}{\otimes} N}{M,N}$.*

Combining Proposition 3.5, Proposition 3.6 and the definition of $M \underset{R}{\otimes} N$, we have the following result:

Theorem 3.7. *If the pair (M,N) satisfies the kernel intertwining operator full equality condition, then the pair $(M \underset{R}{\otimes} N, \underset{\gamma}{\otimes})$ is a tensor product of the pair (M,N).*

4. Relation between Chom and tensor product

In this section we introduce the notion of conformal linear map, producing the "*Hom*" functor for conformal modules over a Lie conformal algebra, called Chom (see [K] and [BKL]). Then, we show the intimate relationship between Chom and the tensor product.

Given two $\mathbb{C}[\partial]$-modules U and V, a *conformal linear map* from U to V is a $\mathbb{C}$-linear map $a : U \to \mathbb{C}[\lambda] \otimes V$, denoted by $a_\lambda : U \to V$, such that

$$a_\lambda\, \partial^U = (\lambda + \partial^V)\ a_\lambda.$$

The vector space of all such maps, denoted by $\mathrm{Chom}(U,V)$, is a $\mathbb{C}[\partial]$-module with

$$(\partial a)_\lambda := -\lambda a_\lambda.$$

Let U and V be two modules over a Lie conformal algebra R. Then, the $\mathbb{C}[\partial]$-module $\mathrm{Chom}(U,V)$ has an R-module structure defined by

$$(a_\lambda\varphi)_\mu u = a_\lambda^V(\varphi_{\mu-\lambda}u) - \varphi_{\mu-\lambda}(a_\lambda^U u),$$

where $a \in R$, $\varphi \in \mathrm{Chom}(U,V)$ and $u \in U$. It is clear that the R-module $\mathrm{Chom}(U,V)$ is conformal iff both U and V are finite conformal R-modules. Observe that one can define the *conformal dual* of U as the R-module $U^{*c} = \mathrm{Chom}(U,\mathbb{C})$, where $\mathbb{C}$ is viewed as the trivial R-module and $\mathbb{C}[\partial]$-module. We also define the tensor product $U \otimes V$ of R-modules as the ordinary tensor product over $\mathbb{C}$ with $\mathbb{C}[\partial]$-module structure ($u \in U, v \in V$):

$$\partial\,(u\otimes v) \;=\; \partial\, u\otimes v + u\otimes \partial\, v$$

and λ-action defined by ($r \in R$):

$$r_\lambda(u\otimes v) \;=\; r_\lambda u\otimes v + u\otimes r_\lambda v.$$

Proposition 4.1. [BKL] *Let U and V be two R-modules. Suppose that U has finite rank as a $\mathbb{C}[\partial]$-module. Then $U^{*c}\otimes V \simeq$ Chom(U,V) as R-modules, with the identification $(f\otimes v)_\lambda(u) = f_{\lambda+\partial^V}(u)\,v$, $f\in U^{*c}$, $u \in U$ and $v\in V$.*

Now, we present the main result of this section. The following theorem provides the motivation for the definition of intertwining operator.

Theorem 4.2. *Let M, N and W be conformal R-modules.*

(a) For any intertwining operator I_λ of type $\binom{W}{M,N}$, there exists a unique R-homomorphism ψ form M to Chom(N,W) such that

$$I_\lambda(u,v) = \big[\psi(u)\big]_\lambda(v),$$

for $u \in M$ and $v\in N$.

(b) We have the following linear isomorphisms:

$$\binom{W}{M\,,\,N} \simeq Hom_R(M \underset{R}{\otimes} N, W) \simeq Hom_R(M, \mathrm{Chom}(N,W)).$$

Proof. (a) First, let us see that $\psi(u) \in\mathrm{Chom}(N,W)$ iff $I_\gamma(u,\partial\, v) = (\gamma+\partial)I_\gamma(u,v)$:

$$I_\gamma(u,\partial v) = \big[\psi(u)\big]_\gamma(\partial v) = (\gamma+\partial)\big[\psi(u)\big]_\gamma(v) = (\gamma+\partial)I_\gamma(u,v). \tag{4.1}$$

Now, let us see that ψ is a $\mathbb{C}[\partial]$-homomorphism iff I_γ satisfies the translation property:

$$\big[\psi(\partial u)\big]_\gamma(v) = I_\gamma(\partial u, v) = -\gamma I_\gamma(u,v) = -\gamma\big[\psi(u)\big]_\gamma(v) = \big[\partial\cdot\psi(u)\big]_\gamma(v).$$

Combining this with (4.1), we get the derivation property of I_γ. Finally, we prove that ψ is an R-homomorphism iff I_γ satisfies the Jacobi identity:

$$\begin{aligned} I_\gamma(a_\lambda u, v) &= \big[\psi(a_\lambda u)\big]_\gamma(v) = \big[a_\lambda\big(\psi(u)\big)\big]_\gamma(v) = a_\lambda\Big(\big[\psi(u)\big]_{\gamma-\lambda}(v)\Big) - \big[\psi(u)\big]_{\gamma-\lambda}(a_\lambda v)\\ &= a_\lambda\Big(I_{\gamma-\lambda}(u,v)\Big) - I_{\gamma-\lambda}(u, a_\lambda v), \end{aligned}$$

finishing the proof of (a).

(b) The first isomorphism follows by the universal property the defines the tensor product. Finally, the isomorphism $\binom{W}{M\,,N} \simeq Hom_R(M,\mathrm{Chom}(N,W))$ is given by part (a). $\square$

5. Commutativity of the tensor product

For any interwining operator I_γ of type $\binom{W}{M\,,\,N}$, the *transpose* of I_γ is defined as

$$(I^t)_\gamma(v,u) := I_{-\gamma-\partial}(u,v),$$

for any $u\in M, v\in N$.

Theorem 5.1. *Let M,N and W be conformal R-modules.*
(a) The transpose of an intertwining operator of type $\binom{W}{M\,,\,N}$ is an intertwining operator of type $\binom{W}{N\,,\,M}$.
(b) If $(M\underset{R}{\otimes}N, \underset{\gamma}{\otimes})$ is a tensor product of the pair (M,N), then $(M\underset{R}{\otimes}N, (\underset{\gamma}{\otimes})^t)$ is a tensor product of the pair (N,M).
(c) The map

$$\begin{aligned} \tau\,: M\underset{R}{\otimes}N &\longrightarrow N\underset{R}{\otimes}M \\ u\underset{\gamma}{\otimes}v &\longmapsto v\underset{-\gamma-\partial}{\otimes}u \end{aligned}$$

is an R-isomorphism.

Proof. (a) First, we prove the translation-derivation properties:

$$(I^t)_\gamma(\partial v,u) = I_{-\gamma-\partial}(u,\partial v) = -\gamma\, I_{-\gamma-\partial}(u,v) = -\gamma\,(I^t)_\gamma(v,u),$$

and

$$\partial\,((I^t)_\gamma(v,u)) = \partial\,(I_{-\gamma-\partial}(u,v)) = I_{-\gamma-\partial}(\partial u,v) + I_{-\gamma-\partial}(u,\partial v) = (I^t)_\gamma(v,\partial u) + (I^t)_\gamma(\partial v,u).$$

And now, the Jacobi identity (recall that the powers of $(-\gamma-\partial)$ must be moved to the left):

$$\begin{aligned} a_\lambda\big((I^t)_\gamma(v,u)\big) &= a_\lambda\big(I_{-\gamma-\partial}(u,v)\big) = I_{-\gamma-\partial}(a_\lambda u,v) + I_{-\lambda-\gamma-\partial}(u,a_\lambda v) \\ &= (I^t)_\gamma(v,a_\lambda u) + (I^t)_{\lambda+\gamma}(a_\lambda v,u). \end{aligned}$$

(b) Let W be any conformal R-module, and $I_\lambda\in\binom{W}{N\,,\,M}$. Then, it is easy to see that there is an R-homomorphism ψ form $M\underset{R}{\otimes}N$ to W such that $I_\lambda=\psi\circ(\underset{\lambda}{\otimes})^t$ if and only if $(I^t)_\lambda = \psi\circ\underset{\lambda}{\otimes}$, finishing (b).

(c) Let $F_\gamma: M\times N\to\mathbb{C}[\gamma]\otimes(N\underset{R}{\otimes}M)$ be given by $F_\gamma(u,v) = v\underset{-\gamma-\partial}{\otimes}u$, for $u\in M$ and $v\in N$. Since $F_\gamma = (\underset{\gamma}{\otimes})^t$, using (a), we get that F_γ is an intertwining operator of type $\binom{N\underset{R}{\otimes}M}{M\,,\,N}$. Then, there is a unique R-homomorphism $\tau: M\underset{R}{\otimes}N\to N\underset{R}{\otimes}M$, with $\tau(u\underset{\gamma}{\otimes}v) = F_\gamma(u,v) = v\underset{-\gamma-\partial}{\otimes}u$. Similarly, interchanging the roles of M and N, we obtain an R-homomorphism $\tilde\tau: N\underset{R}{\otimes}M\to M\underset{R}{\otimes}N$, with $\tilde\tau(v\underset{\gamma}{\otimes}u) = u\underset{-\gamma-\partial}{\otimes}v$. Both composite of these maps are obviously identity maps, and so τ is an R-isomorphism. ∎

6. Second construction of the tensor product

If we restrict the definition of tensor product to finite conformal modules, there exists an alternative construction that follows the ideas for the tensor product of modules over a vertex operator algebra in Chapter 7 of [Li] (similar to the ideas in [HL1]-[HL5]). In our case, the idea is very simple: for two finite conformal R-modules M and N, we suppose that there exists a unique nontrivial finite submodule of $\mathrm{Chom}(M, N^{*_c}) \simeq M^{*_c} \otimes N^{*_c}$ (see Proposition 4.1), called $\Delta(M, N^{*_c})$. Then, the conformal dual $\big(\Delta(M, N^{*_c})\big)^{*_c}$ is the tensor product $M \underset{R}{\otimes} N$.

In the construction of the tensor product of modules over a vertex operator algebra in Chapter 7 of [Li], Haisheng Li takes (what in our case is) the *maximal* (finite) submodule of $\mathrm{Chom}(M, N^{*_c})$.

We shall need the following basic result.

Proposition 6.1. *Let M and N be finite conformal R-modules.*
(a) Let $T : M \longrightarrow N$ be an R-homomorphism. The map $T^{*_c} : N^{*_c} \longrightarrow M^{*_c}$ defined by

$$\Big[T^{*_c}(f)\Big]_\lambda(u) = f_\lambda(T(u))$$

for $u \in M$ and $f \in N^{*_c}$, is an R-homomorphism.
(b) The map $\varphi : M \longrightarrow (M^{*_c})^{*_c}$ defined by

$$\Big[\varphi(u)\Big]_\lambda(f) = f_{-\lambda}(u)$$

for $u \in M$ and $f \in M^{*_c}$, is an R-isomorphism.

Proof. (a) First, we prove that T^{*_c} is well defined:

$$\Big[T^{*_c}(f)\Big]_\lambda(\partial u) = f_\lambda(T(\partial u)) = f_\lambda(\partial T(u)) = \lambda f_\lambda(T(u)) = \lambda\Big[T^{*_c}(f)\Big]_\lambda(u).$$

Now, we prove that T^{*_c} is an R-homomorphism, using

$$\Big[T^{*_c}(a_\gamma f)\Big]_\lambda(u) = (a_\gamma f)_\lambda(T(u)) = -f_{\lambda-\gamma}(a_\gamma(T(u))) = -f_{\lambda-\gamma}(T(a_\gamma u)),$$

and

$$\Big[a_\gamma\big(T^{*_c}(f)\big)\Big]_\lambda(u) = -\Big[T^{*_c}(f)\Big]_{\lambda-\gamma}(a_\gamma u) = -f_{\lambda-\gamma}(T(a_\gamma u)).$$

(b) First, we prove that φ is well defined:

$$\Big[\varphi(u)\Big]_\lambda(\partial f) = (\partial f)_{-\lambda}(u) = \lambda f_{-\lambda}(u) = \lambda\Big[\varphi(u)\Big]_\lambda(f).$$

Now, we prove that φ is an R-homomorphism, using

$$\Big[\varphi(a_\gamma u)\Big]_\lambda(f) = f_{-\lambda}(a_\gamma u),$$

and

$$\Big[a_\gamma \varphi(u)\Big]_\lambda(f) = -\Big[\varphi(u)\Big]_{\lambda-\gamma}(a_\gamma f) = -(a_\gamma f)_{\gamma-\lambda}(u) = f_{-\lambda}(a_\gamma u).$$

Suppose that $M=\oplus_{i=1}^{n}\mathbb{C}[\partial]\,u_i$, then $M^{*c}=\oplus_{i=1}^{n}\mathbb{C}[\partial]\,u_i^*$ with $(u_i^*)_\lambda(u_j)=\delta_{ij}$, and $(M^{*c})^{*c}=\oplus_{i=1}^{n}\mathbb{C}[\partial]\,(u_i^*)^*$ with $[(u_i^*)^*]_\lambda(u_j^*)=\delta_{ij}$. Observe that φ satisfies $\big[\varphi(u_i)\big]_\lambda(u_j^*)=(u_j^*)_{-\lambda}(u_i)=\delta_{ij}$, proving the isomorphism. □

For any interwining operator I_λ of type $\binom{W}{M\,,\,N}$, the *adjoint* of I_λ is defined by

$$\Big[(I^*)_\lambda(u,f)\Big]_\mu(v):=-f_{\mu-\lambda}\big(I_\lambda(u,v)\big),$$

for any $u\in M, v\in N$ and $f\in W^{*c}$.

Proposition 6.2. *Let M,N and W be conformal R-modules. The transpose of an intertwining operator of type $\binom{W}{M\,,\,N}$ is an intertwining operator of type $\binom{N^{*c}}{M\,,\,W^{*c}}$.*

Proof. The translation-derivation properties are obtained by the following identities:

$$\Big[(I^*)_\lambda(\partial u,f)\Big]_\mu(v)=-f_{\mu-\lambda}\big(I_\lambda(\partial u,v)\big)=\Big[-\lambda\,\big((I^*)_\lambda(u,f)\big)\Big]_\mu(v),$$

and

$$\begin{aligned}\Big[(I^*)_\lambda(u,\partial f)\Big]_\mu(v)&=-(\partial f)_{\mu-\lambda}\big(I_\lambda(u,v)\big)=-(\lambda-\mu)\,f_{\mu-\lambda}\big(I_\lambda(u,v)\big)\\&=\Big[(\lambda-\mu)\,(I^*)_\lambda(u,f)\Big]_\mu(v)=\Big[(\lambda+\partial)\,\big((I^*)_\lambda(u,f)\big)\Big]_\mu(v).\end{aligned}$$

The Jacobi identity is obtained by the following identities:

$$\begin{aligned}\Big[a_\gamma\Big((I^*)_\lambda(u,f)\Big)\Big]_\mu(v)&=-\Big[(I^*)_\lambda(u,f)\Big]_{\mu-\gamma}(a_\gamma v)=f_{\mu-\gamma-\lambda}\Big(I_\lambda(u,a_\gamma v)\Big),\\ \Big[(I^*)_{\gamma+\lambda}(a_\gamma u,f)\Big]_\mu(v)&=-f_{\mu-\gamma-\lambda}\Big(I_{\gamma+\lambda}(a_\gamma u,v)\Big),\\ \Big[(I^*)_\lambda(u,a_\gamma f)\Big]_\mu(v)&=-(a_\gamma f)_{\mu-\lambda}\Big(I_\lambda(u,v)\Big)=f_{\mu-\lambda-\gamma}\Big(a_\gamma\big(I_\lambda(u,v)\big)\Big).\end{aligned}$$

□

Now, we restrict the definition of tensor product to the category of *finite* conformal modules. Let M and N be two finite conformal R-modules. **Suppose that there exists a unique nontrivial finite submodule of** $\mathrm{Chom}(M,N^{*c})\simeq M^{*c}\otimes N^{*c}$ (see Proposition 4.1), and denote it as $\Delta(M,N^{*c})$. Observe that $\Delta(M,N^{*c})$ must satisfy a non trivial property, that could probably be improved. Now, we define F as the natural intertwining operator of type

$$\binom{N^{*c}}{\Delta(M,N^{*c})\,,\,M}=Hom_{R}\big(\Delta(M,N^{*c}),\mathrm{Chom}(M,N^{*c})\big)$$

given by the inclusion, that is

$$F_\lambda(f,u)=f_\lambda(u).$$

Theorem 6.3. $\Big(\big(\Delta(M,N^{*c})\big)^{*c},(F^t)^*\Big)$ *is a tensor product for the pair (M,N).*

Proof. Observe that F^t is an intertwining operator of type $\binom{N^{*c}}{M,\,\Delta(M,N^{*c})}$ and $(F^t)^*$ is an intertwining operator of type $\binom{(\Delta(M,N^{*c}))^{*c}}{M\quad,\quad N}$, using Proposition 6.1(b).

Let W be any finite conformal R-module, and let I be any intertwining operator of type $\binom{W}{M\,,\,N}$. It follows by Theorem 5.1 (a) and Proposition 6.2, that $(I^*)^t$ is an intertwining operator of type

$$\binom{N^{*c}}{W^{*c}\,,\,M}=Hom_{R}\big(W^{*c},\mathrm{Chom}(M,N^{*c})\big).$$

Then, there exists a (unique) R-homomorphism ψ from W^{*_c} to $\mathrm{Chom}(M, N^{*_c})$ such that

$$\Big[(I^*)^t\Big]_\lambda(g,u) = \big[\psi(g)\big]_\lambda(u), \tag{6.1}$$

for any $u \in M$ and $g \in W^{*_c}$. It follows from the definition of $\Delta(M, N^{*_c})$ that ψ is an R-homomorphism from W^{*_c} to $\Delta(M, N^{*_c})$. Therefore, using Proposition 6.1(a), we obtain an R-homomorphism ψ^{*_c} from $(\Delta(M, N^{*_c}))^{*_c}$ to W (using that $W \simeq (W^{*_c})^{*_c}$, see Proposition 6.1(b)). Now, we have to prove that this map satisfies the universal property, that is

$$I_\lambda(u,v) = \Big[\psi^{*_c} \circ (F^t)^*\Big]_\lambda(u,v),$$

for any $u \in M$ and $v \in N$.

First, we compute $(F^t)^*$. Observe that

$$(F^t)_\lambda(u,f) = F_{-\lambda-\partial^{N^{*_c}}}(f,u) = f_{-\lambda-\partial^{N^{*_c}}}(u),$$

for $u \in M$ and $f \in \Delta(M, N^{*_c})$. Then, for $u \in M$, $v \in N \simeq (N^{*_c})^{*_c}$, and $g \in \Delta(M, N^{*_c})$, we have

$$\begin{aligned}\Big[\big[(F^t)^*\big]_\lambda(u,v)\Big]_\mu(g) &= -v_{\mu-\lambda}\Big((F^t)_\lambda(u,g)\Big) = -\Big[(F^t)_\lambda(u,g)\Big]_{\lambda-\mu}(v) \\ &= -\Big(g_{-\lambda-\partial^{N^{*_c}}}(u)\Big)_{\lambda-\mu}(v) = -\Big(g_{-\mu}(u)\Big)_{\lambda-\mu}(v).\end{aligned} \tag{6.2}$$

Now, for $u \in M$ and $v \in N$, we have that $\Big[\psi^{*_c} \circ (F^t)^*\Big]_\lambda(u,v) \in (W^{*_c})^{*_c} \simeq W$. Then, using the definition of ψ^{*_c} in Proposition 6.1(a), (6.2) and (6.1), for $f \in W^{*_c}$, we have

$$\begin{aligned}\Big[\Big[\psi^{*_c} \circ (F^t)^*\Big]_\lambda(u,v)\Big]_\mu(f) &= \Big[\psi^{*_c}\Big(\big[(F^t)^*\big]_\lambda(u,v)\Big)\Big]_\mu(f) = \Big[\big[(F^t)^*\big]_\lambda(u,v)\Big)\Big]_\mu(\psi(f)) \\ &= -\Big((\psi(f))_{-\mu}(u)\Big)_{\lambda-\mu}(v) = -\Big(\big[(I^*)^t\big]_{-\mu}(f,u)\Big)_{\lambda-\mu}(v) \\ &= -\Big[(I^*)_{\mu-\partial^{N^{*_c}}}(u,f)\Big]_{\lambda-\mu}(v) = -\Big[(I^*)_\lambda(u,f)\Big]_{\lambda-\mu}(v) \\ &= f_{-\mu}\big(I_\lambda(u,v)\big) = \Big[I_\lambda(u,v)\Big]_\mu(f),\end{aligned}$$

finishing the proof. □

Recall that $\mathrm{Chom}(M, N^{*_c}) \simeq M^{*_c} \otimes N^{*_c}$. Now, we take generators of them: $M^{*_c} = \mathbb{C}[\partial] \otimes M_0^{*_c}$ and $N^{*_c} = \mathbb{C}[\partial] \otimes N_0^{*_c}$. If the submodule generated by $M_0^{*_c} \otimes N_0^{*_c}$ is a **finite** submodule of $M^{*_c} \otimes N^{*_c}$, then we conjecture that the conformal dual of it should be the tensor product of M and N.

We think that there should be a simpler construction of the tensor product of finite modules.

7. Tensor product of modules over the Virasoro conformal algebra

In this section, we try to compute the tensor product of irreducible conformal modules over the Virasoro conformal algebra.

Let us consider the example of the Virasoro conformal algebra: we define $Vir = \mathbb{C}[\partial]\, L$, with λ-bracket defined on generator, by $[L_\lambda L] = (2\lambda + \partial)\, L$. There exists a family of finite conformal modules over the Virasoro algebra given by $M_{\Delta,\alpha} = \mathbb{C}[\partial]\, m_{\Delta,\alpha}$, where $\Delta, \alpha \in \mathbb{C}$, with λ-action

$$L_\lambda m_{\Delta,\alpha} = (\Delta\lambda + \partial + \alpha)\, m_{\Delta,\alpha}.$$

It was proved in [CK] (see [K]), that all the finite irreducible conformal modules over Vir are given by $M_{\Delta,\alpha}$ with $\Delta \neq 0$, and $(\partial + \alpha)\, M_{0,\alpha}$ is a nontrivial submodule of $M_{0,\alpha}$.

In 1997, during a graduate course at MIT, Victor Kac explained that there should be a tensor product of conformal modules, that should be closed for finite modules, and for the modules $M_{\Delta,\alpha}$ over the Virasoro conformal algebra, it should be:

$$M_{\Delta,\alpha} \underset{Vir}{\otimes} M_{\Delta',\alpha'} = M_{\Delta+\Delta'-1,\, \alpha+\alpha'}. \tag{7.1}$$

A simple computation shows that

$$Hom_{Vir}\big(M_{\Delta,\alpha}, \mathrm{Chom}\big(M_{\Delta',\alpha'}, M_{\widetilde{\Delta},\widetilde{\alpha}}\big)\big) = \{\, f \mid [f(m_{\Delta,\alpha})]_\mu(m_{\Delta',\alpha'}) = c\, m_{\Delta+\Delta'-1,\, \alpha+\alpha'} \text{ with } c \in \mathbb{C}\}$$

if $\widetilde{\Delta} = \Delta + \Delta' - 1$ and $\widetilde{\alpha} = \alpha + \alpha'$; and it is zero otherwise. On the other hand, it is easy to see that

$$Hom_{Vir}\big(M_{\bar{\Delta},\bar{\alpha}}, M_{\widetilde{\Delta},\widetilde{\alpha}}\big) = \begin{cases} 0 & \text{if } \bar{\Delta} \neq \widetilde{\Delta} \text{ or } \bar{\alpha} \neq \widetilde{\alpha} \\ \{\, f \mid f(m_{\bar{\Delta},\bar{\alpha}}) = c\, m_{\widetilde{\Delta},\widetilde{\alpha}} \text{ with } c \in \mathbb{C}\} & \text{if } \bar{\Delta} = \widetilde{\Delta} \text{ and } \bar{\alpha} = \widetilde{\alpha}. \end{cases}$$

Therefore, we obtain

$$Hom_{Vir}\big(M_{\Delta,\alpha} \underset{Vir}{\otimes} M_{\Delta',\alpha'}, M_{\widetilde{\Delta},\widetilde{\alpha}}\big) \simeq Hom_{Vir}\big(M_{\Delta+\Delta'-1,\, \alpha+\alpha'}, M_{\widetilde{\Delta},\widetilde{\alpha}}\big),$$

and this shows that $M_{\Delta+\Delta'-1,\, \alpha+\alpha'}$ satisfies the universal property of the tensor product $M_{\Delta,\alpha} \underset{Vir}{\otimes} M_{\Delta',\alpha'}$ in the special case when "W" is $M_{\widetilde{\Delta},\widetilde{\alpha}}$.

We were not able to describe

$$Hom_{Vir}\big(M_{\Delta,\alpha}, \mathrm{Chom}\big(M_{\Delta',\alpha'}, W\big)\big)$$

for all finite conformal Vir-module W. But, we will show how to apply the first construction in this case, under certain strong assumption, to prove (7.1).

Suppose that $I_{(n)}(m_{\Delta,\alpha}, m_{\Delta',\alpha'}) = 0$ for all $n \geq 1$, and all intertwining operators I of type $\binom{W}{M_{\bar{\Delta},\bar{\alpha}}, M_{\widetilde{\Delta},\widetilde{\alpha}}}$. Therefore, $l_{m_{\Delta,\alpha}, m_{\Delta',\alpha'}} = 1$ in the definition of the kernel of intertwining operators, producing that in the quotient by J_0 we have

$$t^n \otimes m_{\Delta,\alpha} \otimes m_{\Delta',\alpha'} = 0 \qquad \text{for all } n \geq 1. \tag{7.2}$$

The second quotient by J_1 corresponds to the identity

$$t^n \otimes \partial u \otimes v = -n\, t^{n-1} \otimes u \otimes v, \tag{7.3}$$

for all $n \geq 0$, $u \in M_{\Delta,\alpha}$ and $v \in M_{\Delta',\alpha'}$. Combining (7.2) and (7.3), we obtain that

$$t^k \otimes \partial^l\, m_{\Delta,\alpha} \otimes m_{\Delta',\alpha'} = \delta_{k,l}\, (-1)^k\, k!\, (t^0 \otimes m_{\Delta,\alpha} \otimes m_{\Delta',\alpha'}),$$

for all $k, l \geq 0$. And, in general:

$$t^n \otimes \partial^k\, m_{\Delta,\alpha} \otimes \partial^l\, m_{\Delta',\alpha'} = \begin{cases} 0 & \text{if } n < k \\ (-1)^k\, k!\, (t^{n-k} \otimes m_{\Delta,\alpha} \otimes \partial^l\, m_{\Delta',\alpha'}) & \text{if } n \geq k. \end{cases}$$

Therefore, it is enough to consider elements of the form $t^n \otimes m_{\Delta,\alpha} \otimes \partial^l m_{\Delta',\alpha'}$. Now, using the $\mathbb{C}[\partial]$-module structure, we have that

$$t^n \otimes m_{\Delta,\alpha} \otimes \partial^l m_{\Delta',\alpha'} = (\partial^{\otimes} - 1 \otimes \partial \otimes 1)^l (t^n \otimes m_{\Delta,\alpha} \otimes m_{\Delta',\alpha'})$$
$$= \sum_{i=0}^{l} \binom{l}{i} (\partial^{\otimes})^{l-i} (-1)^i (t^n \otimes \partial^i m_{\Delta,\alpha} \otimes m_{\Delta',\alpha'}),$$

where $\partial^{\otimes} = 1 \otimes \partial \otimes 1 + 1 \otimes 1 \otimes \partial$. Hence, we obtain that $t^n \otimes m_{\Delta,\alpha} \otimes \partial^l m_{\Delta',\alpha'} = 0$ if $n > l$, and

$$t^n \otimes m_{\Delta,\alpha} \otimes \partial^l m_{\Delta',\alpha'} = \tfrac{l!}{(l-n)!} (\partial^{\otimes})^{l-n} (t^0 \otimes m_{\Delta,\alpha} \otimes m_{\Delta',\alpha'}),$$

for all $0 \leq n \leq l$, or equivalently

$$t^n \otimes m_{\Delta,\alpha} \otimes \partial^{n+i} m_{\Delta',\alpha'} = \tfrac{(n+i)!}{i!} (\partial^{\otimes})^i (t^0 \otimes m_{\Delta,\alpha} \otimes m_{\Delta',\alpha'}),$$

for all $n, i \geq 0$. Therefore, we have proved that $M_{\Delta,\alpha} \underset{Vir}{\otimes} M_{\Delta',\alpha'}$ is generated over $\mathbb{C}$ by the elements of the form $(\partial^{\otimes})^i (t^0 \otimes m_{\Delta,\alpha} \otimes m_{\Delta',\alpha'})$, with $i \geq 0$. Finally, using (3.2), we have that

$$L_\lambda (t^0 \otimes m_{\Delta,\alpha} \otimes m_{\Delta',\alpha'}) = \big((\Delta + \Delta' - 1)\lambda + \partial^{\otimes} + \alpha + \alpha'\big)(t^0 \otimes m_{\Delta,\alpha} \otimes m_{\Delta',\alpha'}).$$

proving (under a strong assumption) that

$$M_{\Delta,\alpha} \underset{Vir}{\otimes} M_{\Delta',\alpha'} = M_{\Delta+\Delta'-1,\, \alpha+\alpha'}.$$

Now, we will try to apply the second construction to prove (7.1). We shall need the following result.

Proposition 7.1. $\big(M_{\Delta,\alpha}\big)^{*_c} \simeq \ M_{1-\Delta,-\alpha}$.

Proof. We take $(M_{\Delta,\alpha})^{*_c} = \mathbb{C}[\partial]\, m^*_{\Delta,\alpha}$, with $m^*_{\Delta,\alpha}$ defined by $(m^*_{\Delta,\alpha})_\mu (p(\partial)\, m_{\Delta,\alpha}) = p(\mu)$. Then

$$\Big[L_\lambda m^*_{\Delta,\alpha}\Big]_\mu (m_{\Delta,\alpha}) = -(m^*_{\Delta,\alpha})_{\mu-\lambda}(L_\lambda m_{\Delta,\alpha}) = -(m^*_{\Delta,\alpha})_{\mu-\lambda}\big((\Delta\lambda + \partial + \alpha)\, m_{\Delta,\alpha}\big)$$
$$= -(\Delta\lambda + \mu - \lambda + \alpha)(m^*_{\Delta,\alpha})_{\mu-\lambda}(m_{\Delta,\alpha}) = (1-\Delta)\lambda - \mu - \alpha$$
$$= \Big[\big((1-\Delta)\lambda + \partial - \alpha\big)\, m^*_{\Delta,\alpha}\Big]_\mu (m_{\Delta,\alpha}),$$

finishing the proof. □

Observe that in general $\mathbb{C}[\partial]\,(m_{\Delta,\alpha} \otimes m_{\Delta',\alpha'})$ is a finite submodule of $M_{\Delta,\alpha} \otimes M_{\Delta',\alpha'}$, and using that $L_\lambda (m_{\Delta,\alpha} \otimes m_{\Delta',\alpha'}) = \big((\Delta + \Delta')\lambda + \partial^{\otimes} + \alpha + \alpha'\big)\,(m_{\Delta,\alpha} \otimes m_{\Delta',\alpha'})$, we have that

$$M_{\Delta+\Delta',\alpha+\alpha'} \simeq \mathbb{C}[\partial]\,(m_{\Delta,\alpha} \otimes m_{\Delta',\alpha'}). \tag{7.4}$$

Using Proposition 7.1, we obtain

$$\big(M_{\Delta,\alpha}\big)^{*_c} \otimes \big(M_{\Delta',\alpha'}\big)^{*_c} \simeq \ M_{1-\Delta,-\alpha} \otimes M_{1-\Delta',-\alpha'}$$

We conjecture that for certain values of Δ and Δ', the module $\mathbb{C}[\partial]\,(m_{1-\Delta,-\alpha} \otimes m_{1-\Delta',-\alpha'})$ is the unique finite submodule of $M_{1-\Delta,-\alpha} \otimes M_{1-\Delta',-\alpha'}$. Now, using (7.4), we have

$$M_{2-\Delta-\Delta',-\alpha-\alpha'} \simeq \mathbb{C}[\partial]\,(m_{1-\Delta,-\alpha} \otimes m_{1-\Delta',-\alpha'}).$$

Therefore, using Theorem 6.3 and Proposition 7.1, the conformal dual

$$(M_{2-\Delta-\Delta',-\alpha-\alpha'})^{*_c} \simeq M_{\Delta+\Delta'-1,\alpha+\alpha'}$$

is the tensor product $M_{\Delta,\alpha} \underset{Vir}{\otimes} M_{\Delta',\alpha'}$, obtaining (7.1).

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