An approach to Quantum Conformal Algebra

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Abstract

We aim to explore if inside a quantum vertex algebras, we can find the right notion of a quantum conformal algebra.

1 Introduction

Since the pioneering papers [BPZ, Bo1], there has been a great deal of work towards understanding of the algebraic structure underlying the notion of the operator product expansion (OPE) of chiral fields of a conformal field theory. The singular part of the OPE encodes the commutation relations of fields, which leads to the notion of a conformal algebra [K1].

In [BK], they develop foundation of the theory of field algebras, which are a "non-commutative version" of a vertex algebra. Among other results they show that inside certain field algebras, more precisely strong field algebras (where the n-product axiom holds) we have a conformal algebra and a diferential algebra toghehter with certain compatibility equations, and conversely, having this two structures plus those equations we can recover a strong field algebra. One of these equations is the conformal analog of the Jacobi Identiy. They call a conformal algebra satisfying this equation *Leibnitz* conformal algebra.

A definition of a quantum vertex algebra, which is a deformation of a vertex algebra, was introduced by Etingof and Kazhdan in 1998,[EK]. Roughly speaking, a quantum vertex algebra is a braided state-field correspondence which satisfies associativity and braided locality axioms. Such braiding is a one-parameter braiding with coefficients in Laurent series.

Recently in [DGK], they developed a structure theory of quantum vertex algebras, parallel to that of vertex algebras. In particular, they introduce braided n-products for a braided state-field correspondence and prove for quantum vertex algebras a version of the Borcherds identity.

Following [BK], in this article, we try to determine the quantum analog of the notion of conformal algebra inside a quantum vertex algebra V. For this purpose, we introduced new products parametrized by Laurent polinomials f, and we showed that all this products are determined by those corresponding

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$f = 1$ and $f = z^{-1}$. The case $f = 1$ coincides with the λ-product defining a conformal algebra([K1],[BK]). This allows us to deal with the coefficients of the braiding in V. An important remark is that V together with the λ-product is no longer a Leibnitz conformal algebra, since due to the braiding, the analog of the Jacoby identity involves not only the products corresponding to $f = 1$ (as in [BK]), but those of $f = z^{-1}$. We translate to this language the hexagon axiom, quasi-associativity and associativity relations, and the braided skew-symmetry in a quantum vertex algebra, and all this allows us to give an equivalent definition of quantum vertex algebra and present a candidate of a quantum conformal algebra.

The article is organized as follows. In Section 2 we review all the definitions and basic notion of field algebras and braided field algebras. In Section 3 we introduce the (λ, f)-product and prove some of its properties and we finish the section proving in Theorem 3 that shows that having a strong braided field algebra is the same of having a conformal algebra, a differential algebra with unit with some compatibility equations. In Section 4, we translate the hexagon axiom, quasi-associativity, and associativity relations, and the braided skew-symmetry in a quantum vertex algebra, we give an equivalent definition of quantum vertex algebra and present a candidate of a quantum conformal algebra.

2 Preliminaries

In this section review some basic definitions followig [BK],[DGK]. Throughout the paper all vector spaces, tensor products,etc are over a field $\mathbb{K}$ of characteristic zero, unless otherwise specified.

2.1 Calculus of formal distribution

Given a vector space V, we let $V[[z, z^{-1}]]$ be the space of formal power series with coefficients in V; they are called *formal distributions*. A *qauntum field* over V is a formal distribution $a(z) \in (\mathrm{End}V)[[z, z^{-1}]]$ with coefficients in End$V$, such that $a(z)v \in V((z))$ for every $v \in V$. Hereafter $V((z)) = V[[z]][z^{-1}]$ stands for the space of Laurent series with coefficients in V.

Throughout the article $\iota_{z,w}$ (resp $\iota_{w,z}$) denotes the geometric series expansion in the domain $|z| > |w|$ (resp $|w| > |z|$), namely we set for $n \in \mathbb{Z}$,

$$\iota_{z,w}(z+w)^n = \sum_{l\in\mathbb{Z}_+} \binom{n}{l} z^{n-l} w^l$$

where

$$\binom{n}{l} = \frac{n(n-1)\cdots(n-l+1)}{l!}.$$

For an arbitrary formal distribution $a(z)$, we have

$$\mathrm{Res}_z(a(z)) = a_{-1}, \tag{1}$$

which is the coefficient of z^{-1}. Denote by gl$f(V)$ the space of all End$V$-valued fields. We also need the *Taylor's Formula* (cf. Proposition 2.4,[K1]), namely,

$$\iota_{z,w}a(z+w) = \sum_{j\in\mathbb{Z}_+} \frac{\partial_z^j}{j!} a(z)w^j = e^{w\partial_z}a(w). \tag{2}$$

For each $n \in \mathbb{Z}$ one defines the n-th *product* of fields $a(z)$ and $b(z)$ by the following formula:

$$a(z)_{(n)}b(z) = \mathrm{Res}_x(a(x)b(z)\iota_{x,z}(x-z)^n - b(z)a(x)\iota_{z,x}(x-z)^n). \tag{3}$$

Denote by

$$a(z)_+ = \sum_{j\le -1} a_{(j)}z^{-j-1}, \qquad a(z)_- = \sum_{j\ge 0} a_{(j)}z^{-j-1}.$$

2.2 Conformal algebras and Field Algebras

In this subsection we recall the definition of a field algebra, conformal algebras and its properties following [BK]

A *state-field correspondence* on a pointed vector space $(V, |0\rangle)$ is a linear map $Y : V \otimes V \to V((z))$, $a \otimes b \to Y(z)(a \otimes b)$ satisfying

(i) (vacuum axioms)$Y(z)(|0\rangle \otimes a) = a$, $Y(z)(a \otimes |0\rangle) \in a + V[[z]]z$;

(ii) (translation covariance)$[T, Y(z)](a \otimes b) = \partial_z Y(z)(a \otimes b)$,

(iii) $$Y(z)(Ta \otimes b) = \partial_z Y(z)(a \otimes b),$$

where $T(a) := \partial_z(Y(z)(a \otimes |0\rangle)) \mid_{z=0}= a_{(-2)}|0\rangle$, is called the translation operator.

Note that we will also denote by Y the map $Y : V \to \mathrm{End}V[[z, z^{-1}]]$, $a \mapsto Y(a, z) = \sum_{k\in\mathbb{Z}} a_{(k)}z^{-k-1}$, such that $Y(a, z)b = Y(z)(a \otimes b)$.

Note that $Y(a, z)$ is a quantum field, i.e $Y(a, z)b \in V((z))$ for any $b \in V$.

The following results, proved in [BK], will be usefull in the sequel.

Proposition 1. *(cf. [BK], Prop.2.7). Given $Y : V \otimes V \to V((z))$ satisfaying conditions (i) and (ii) above, we have:*

(a) $Y(z)(a \otimes |0\rangle)) = e^{zT}a$;

(b) $e^{wT}Y(z)(1 \otimes e^{-wT}) = \iota_{z,w}Y(z+w)$.

If, moreover, Y is a state-field correspondence, then

(c) $Y(z)(e^{wT} \otimes 1) = \iota_{z,w}Y(z+w)$.

Given a state field correspondence Y, define

$$Y^{op}(z)(u \otimes v) = e^{zT} Y(-z)(v \otimes u). \tag{4}$$

Then Y^{op} is also a state-field correspondence, called the *opposite* to Y. (cf. [BK], Prop 2.8).

Let $(V, |0\rangle)$ be a pointed vector space and let Y be a state-field correspondence. Recall that Y satisfies the n-th *product axiom* if for all $a, b \in V$ and $n \in \mathbb{Z}$

$$Y(z)(a_{(n)}b, z) = Y(z)_{(n)} Y(z)(a \otimes b). \tag{5}$$

We say that Y satisfies the *associativity axiom* if for all $a, b, c \in V$, there exists $N \gg 0$ such that

$$\begin{aligned}(z-w)^N Y(-w)((Y(z) \otimes 1))(a \otimes b \otimes c)\\ = (z-w)^N \iota_{z,w} Y(z-w)(1 \otimes Y(-w))(a \otimes b \otimes c).\end{aligned} \tag{6}$$

Let $(V, |0\rangle)$ be a pointed vector space. As in [BK], a *field algebra* $(V, |0\rangle, Y)$ is a state-field correspondence Y for $(V, |0\rangle)$ satisfying the associativity axiom (6). A *strong field algebra* $(V, |0\rangle, Y)$ is a state-field correspondence Y satisfying the n-th product axiom (5).

Let $(V, |0\rangle)$ be a pointed vector space and let Y be a state-field correspondence. For $a, b \in V$, [BK] defined the λ-*product* given by

$$a_\lambda b = \mathrm{Res}_z e^{\lambda z} Y(z)(a \otimes b) = \sum_{n \geq 0} \frac{\lambda^n}{n!} a_{(n)} b. \tag{7}$$

and the $\cdot$-*product* on V, which is denote as

$$a \cdot b = \mathrm{Res}_z z^{-1} Y(z)(a \otimes b) = a_{(-1)} b. \tag{8}$$

The vacuum axioms for Y implies

$$|0\rangle \cdot a = a = a \cdot |0\rangle, \tag{9}$$

while the translation invariance axioms imply

$$T(a \cdot b) = T(a) \cdot b + a \cdot T(b), \tag{10}$$

and

$$T(a_\lambda b) = (Ta)_\lambda b + a_\lambda (Tb), \qquad (Ta)_\lambda b = -\lambda a_\lambda b \tag{11}$$

for all $a, b \in V$. Notice that from these equations we can derive that $T(|0\rangle) = 0$ and $|0\rangle_\lambda a = 0 = a_\lambda |0\rangle$ for $a \in V$.

Conversely, if we are given a linear operator T, a λ-product and a $\cdot$-product on $(V, |0\rangle)$, satisfying the above properties (9)-(11), we can reconstruct the state-field correspondence Y by the formulas

$$Y(a, z)_+ b = (e^{zT} a).b, \qquad Y(a, z)_- b = (a_{-\partial_z} b)(z^{-1}), \tag{12}$$

where $Y(a,z) = Y(a,z)_+ + Y(a,z)_-$.

A $\mathbb{K}[T]$-module V, equipped with a linear map $V \otimes V \to \mathbb{K} \otimes V$, $a \otimes b \to a_\lambda b$, satisfying (11) is called a ($\mathbb{K}[T]$)-*conformal algebra*. On the other hand with respect to the $\cdot$-product, V is a ($\mathbb{K}[T]$)-*differential algebra* (i.e an algebra with derivation T) with a unit $|0\rangle$.

Summarizing, (Cf. [BK], Lemma 4.1), we have that, giving a state-field correspondence on a pointed vector space $(V, |0\rangle)$ is equivalent to provide V with a structure of a $\mathbb{K}$[T]-conformal algebra and a structure of a $\mathbb{K}[T]$-differential algebra with a unit $|0\rangle$.

Now, recall the following results. Later on, we will prove some analogous result for the braided environment.

Lemma 1. *([BK], Lemma 4.2) Let $(V, |0\rangle)$ be a pointed vector space and let Y be a state-field correspondence. Fix $a, b, c \in V$. Then the collection of n-th product identities $Y(z)(a_{(n)}b \otimes c, z) = (Y(z)_{(n)}Y(z))(a \otimes b \otimes c)$ (for $n \geq 0$) implies*

$$(a_\lambda b)_{\lambda+\mu} = a_\lambda(b_\mu c) - b_\mu(a_\lambda c), \tag{13}$$

$$a_\lambda(b.c) = (a_\lambda b).c + b.(a_\lambda c) + \int_0^\lambda (a_\lambda b)_\mu c \, d\mu. \tag{14}$$

The (-1)-st product identity $Y(z)(a_{(-1)}b \otimes c) = (Y(z)_{(-1)}Y(z))(a \otimes b \otimes c)$ implies

$$(a \cdot b)_\lambda c = (e^{T\partial_\lambda} a).(b_\lambda c) + (e^{T\partial_\lambda} b).(a_\lambda c) + \int_0^\lambda b_\mu(a_{\lambda-\mu} c) \, d\mu, \tag{15}$$

$$(a.b).c - a.(b.c) = \left(\int_0^T d\lambda \, a \right) \cdot (b_\lambda c) + \left(\int_0^T d\lambda \, b \right)_{.} (a_\lambda c). \tag{16}$$

Identity (13) is called the (left) *Jacobi identity*. A conformal algebra satisfying this identity for all $a, b, c \in V$ is called a (left) *Leibnitz conformal algebra*. Equation (14) is known as the "non-commutative" Wick formula, while (16) is called the *quasi-associativity* formula.

Finally, we also recall the following result.

Theorem 1. *([BK], Theorem 4.4) Giving a strong field algebra structure on a pointed vector space $(V, |0\rangle)$ is the same as providing V with a structure of Leibnitz $\mathbb{K}[T]$-conformal algebra and a structure of a $\mathbb{K}[T]$-differential algebra with a unit $|0\rangle$, satisfying (14)-(16).*

Recall also the following result.

Theorem 2. *([BK], Theorem 6.3) A vertex algebra is the same as a field algebra $(V, |0\rangle, Y)$ for which $Y = Y^{op}$.*

Therefore we may assume this as a definition of vertex algebra.

2.3 Braided Field Algebras

We will follow the notation and presentation introduced in [DGK].

Throughout the rest of the paper we shall work over the algebra $\mathbb{K}[[h]]$ of formal series in the variable h, and all the algebraic structures that we will consider are modules over $\mathbb{K}[[h]]$.

A *topologically free* $\mathbb{K}[[h]]$*-module* is isomorphic to $W[[h]]$ for some $\mathbb{K}$-vector space W.

Note that $W[[h]] \ncong W \otimes \mathbb{K}[[h]]$, unless W is finite-dimensional over $\mathbb{K}$, and that the tensor product $U[[h]] \otimes_{\mathbb{K}[[h]]} W[[h]]$ of topologically free $\mathbb{K}[[h]]$-modules is not topologically free, unless one of U and W are finite dimensional. For any vector space U and W, the *completed* tensor product by

$$U[[h]]\hat{\otimes}_{\mathbb{K}[[h]]}W[[h]] := (U \otimes W)[[h]] \tag{17}$$

This is a completion in h-adic topology of $U[[h]] \otimes_{\mathbb{K}[[h]]} W[[h]]$.

Given a topologically free $\mathbb{K}[[h]]$-module V, we let

$$V_h((z)) = \left\{a(z) \in V[[z, z^{-1}]] \mid a(z) \in V((z)) \bmod h^M \text{ for every } M \in \mathbb{Z}_{\geq 0}\right\}. \tag{18}$$

Namely, expanding $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$, we ask that

$$\lim_{n\to+\infty} a_{(n)} = 0$$

in h-adic topology.

Let V be a topologically free $\mathbb{K}[[h]]$-module. Following [DGK], we call $\mathrm{End}_{\mathbb{K}[[h]]}V$*-valued quantum field* an $\mathrm{End}_{\mathbb{K}[[h]]}V$-valued formal distribution $a(z)$ such that $a(z)b \in V_h((z))$ for any $b \in V$.

Later on, we will need the following lemmas, proved in [DGK](cf. Lemma 3.2 and 3.3).

Lemma 2. *Let $|0\rangle \in V$ and $T : V \to V$ be a $\mathbb{K}[[h]]$-linear map such that $T(|0\rangle) = 0$. Then for any $End_{\mathbb{K}[[h]]}V$-valued quantum field $a(z)$ such that $[T, a(z)] = \partial_z a(z)$ (translation covariance), we have*

$$a(z)|0\rangle = e^{zT}a = \sum_{k\geq 0} \frac{T^k a}{k!}\, z^k, \tag{19}$$

where $a = Res_z z^{-1}a(z)|0\rangle$.

Lemma 3. *Let $T : V \to V$ be a $\mathbb{K}[[h]]$-linear map and let $a(z)$ be an $End_{\mathbb{K}[[h]]}V$-valued quantum field such that $[T, a(z)] = \partial_z a(z)$. We have*

$$e^{wT}a(z)e^{-wT} = \iota_{z,w}a(z + w). \tag{20}$$

Let V be a topologically free $\mathbb{K}[[h]]$-module, with a given non-zero vector $|0\rangle \in V$ (vacuum vector) and a $\mathbb{K}[[h]]$-linear map $T : V \to V$ such that $T(|0\rangle) = 0$ (translation operator). Again, following [DGK],

(a) *A topological state-field correspondence* on V is a linear map

$$Y : V\hat{\otimes}V \to V_h((z)), \tag{21}$$

satisfying

(i) (vacuum axioms) $Y(z)(|0\rangle \otimes v) = v$ and

$$Y(z)(v \otimes |0\rangle) \in v + V[[z]]z, \text{ for all } z \in V;$$

(ii) (*translation covariance*)

$$\partial_z Y(z) = TY(z) - Y(z)(1 \otimes T) = Y(z)(T \otimes 1), \tag{22}$$

(b) A *braiding* on V is a $\mathbb{K}[[h]]-$linear map

$$\mathcal{S}(z) : V\hat{\otimes}V \to V\hat{\otimes}V\hat{\otimes}(K((z))[[h]]) \tag{23}$$

such that $\mathcal{S} = 1 + O(h)$.

A *braided state-field correspondance* is a quintuple $(V, |0\rangle, T, Y, \mathcal{S})$ where Y is a topological state-field correspondance and S is a braiding as above.

We will use the following standard notation: given $n \geq 2$ and $i, j \in \{1, \cdots, n\}$, we let

$$S^{i,j}(z) : V^{\widehat{\otimes} n} \to V^{\widehat{\otimes} n}\hat{\otimes}(\mathbb{K}((z))[[h]]), \tag{24}$$

act in the i-th and j-th factors (in this order) of $V^{\widehat{\otimes}_n}$, leaving the other factors unchanged.

A *braided vertex algebra* is a quintuple $(V, |0\rangle, T, Y, \mathcal{S})$ where Y is a topological state-field correspondance and S is a braiding as above, satisfying the following $\mathcal{S}$-*locality*: for every $a, b \in V$ and $M \in \mathbb{Z}_{\geq 0}$, there exists $N = N(a, b, M) \geq 0$ such that

$$\begin{aligned}(z-w)^N Y(z)(1 \otimes Y(w))\mathcal{S}^{12}(z-w)(a \otimes b \otimes c)& \\ = (z-w)^N Y(w)(1 \otimes Y(z))(b \otimes a \otimes c),&\end{aligned} \tag{25}$$

where this equality holds mod h^M, for all $c \in V$.

Again, given a topological state-field correspondence Y, set

$$Y^{op}(z)(u \otimes v) = e^{zT} Y(-z)(v \otimes u). \tag{26}$$

It was shown in [DGK], Lemma 3.6, that in a braided vertex algebra V we have

$$Y(z)\mathcal{S}(z)(a \otimes b) = Y^{op}(z)(a \otimes b) \tag{27}$$

for all $a, b \in V$.

After the proof of this result,(cf. Remark 3.7, [DGK]) they point out that it is enough to have the $\mathcal{S}$-locality (25) holding just for $c = |0\rangle$, to prove that $YS = Y^{op}$ in a braided vertex algebra. We will use this remark later.

We recall at this point two important Propositions for our sequel.

Proposition 2. *([EK], Prop. 1.1) Let V be a braided vertex algebra. for every $a, b, c \in V$ and $M \in \mathbb{Z}_{\geq 0}$, there exists $N \geq 0$ such that*

$$\iota_{z,w}((z+w)^N Y(z+w)(1 \otimes Y(w))\mathcal{S}^{23}(w)\mathcal{S}^{13}(z+w)(a \otimes b \otimes c))$$
$$= (z+w)^N Y(w)\mathcal{S}(w)(Y(z) \otimes 1)(a \otimes b \otimes c) \quad mod\, h^M. \tag{28}$$

Proposition 3. *([DGK], Proposition 3.9) Let $(V, |0\rangle, T, Y, \mathcal{S})$ be a braided vertex algebra. Extend $Y(z)$ to a map $V \hat{\otimes} V \hat{\otimes} (\mathbb{K}((z))[[h]])$ in the obvious way. Then, modulo $Ker Y(z)$, we have*

(a) $\mathcal{S}(|0\rangle \otimes a) \equiv |0\rangle$, *and* $\mathcal{S}(z)(|0\rangle \otimes a) \equiv |0\rangle \otimes a$;

(b) $[T \otimes 1, \mathcal{S}(z)] \equiv -\partial_z \mathcal{S}(z)$ *(left shift condition);*

(c) $[1 \otimes T, \mathcal{S}(z)] \equiv \partial_z \mathcal{S}(z)$ *(right shift condition);*

(d) $[T \otimes 1 + 1 \otimes T, \mathcal{S}(z)] \equiv 0$ *;*

(e) $\mathcal{S}(z)\mathcal{S}^{21}(-z) = 1$ *(unitary).*

Moreover, we have the quantum Yang-Baxter equation:

(f) $\mathcal{S}^{12}(z_1 - z_2)\mathcal{S}^{13}(z_1 - z_3)\mathcal{S}^{23}(z_2 - z_3) \equiv \mathcal{S}^{23}(z_2 - z_3)\mathcal{S}^{13}(z_2 - z_3)\mathcal{S}^{12}(z_1 - z_2)$, *modulo* $Ker(Y(z_1)(1 \otimes Y(z_2))(1^{\otimes 2} \otimes Y(z_3)(- \otimes - \otimes - \otimes |0\rangle)))$.

3 On the structure of braided state-field correspondence

As in [BK], we aim to show that there are, inside certain braided vertex algebras, a "braided conformal algebra" and a "differential algebra" satisfying some family of equation. Conversely, we will show that given such structures under some nice conditions, we can give some reconstruction theorem.

Let $(V, |0\rangle, T, Y, \mathcal{S})$ be a braided-state field correspondence. For $n \in \mathbb{Z}$, the *quantum n-product* $Y(z)^{\mathcal{S}}_{(n)}Y(z)$ is defined as

$$\begin{aligned}(Y(z)^{\mathcal{S}}_{(n)}Y(z))(a\otimes b\otimes c) = Res_x(\iota_{x,z}(x-z)^n Y(x)(1\otimes Y(z))(a\otimes b\otimes c)\\ -\iota_{z,x}(x-z)^n Y(z)(1\otimes Y(x))S^{12}(z-x)(b\otimes a\otimes c)).\end{aligned} \tag{29}$$

Now, we have the following result.

Lemma 4. *Given $(V, |0\rangle, T, Y, \mathcal{S})$ a braided state-field correspondence satisfying the equations*

$$[T\otimes 1,\, \mathcal{S}(z)] = -\partial_z S(z), \tag{30}$$

$$[1\otimes T, S(z)] = \partial_z \mathcal{S}(z). \tag{31}$$

The quantum n-product (29) satisfies the following equation

$$\partial_z(Y(a,z)^{\mathcal{S}}_n Y(b,z)) = (\partial_z Y(a,z))^{\mathcal{S}}_n Y(b,z) + Y(a,z)^{\mathcal{S}}_n(\partial_z Y(b,z)). \tag{32}$$

Proof. Applying the definition of quantum n-product (29), using integration by parts and translation covariance (22), the LHS becomes

$$
\begin{aligned}
&Res_x \iota_{x,z} \partial_z ((x-z)^n Y(x)(1 \otimes Y(z)))(a \otimes b \otimes c) \\
&-Res_x \iota_{z,x} \partial_z ((x-z)^n Y(z)(1 \otimes Y(x)) S^{12}(z-x))(b \otimes a \otimes c) \\
&= Res_x \iota_{x,z} \partial_z (x-z)^n Y(x)(1 \otimes Y(z))(a \otimes b \otimes c) \\
&+Res_x \iota_{x,z} (x-z)^n Y(x)(1 \otimes \partial_z)(1 \otimes Y(z))(a \otimes b \otimes c) \\
&-Res_x \iota_{z,x} \partial_z (x-z)^n Y(z)(1 \otimes Y(x)) S^{12}(z-x)(b \otimes a \otimes c) \\
&-Res_x \iota_{z,x} (x-z)^n \partial_z Y(z)(1 \otimes Y(x)) S^{12}(z-x)(b \otimes a \otimes c) \\
&-Res_x \iota_{z,x} (x-z)^n Y(z)(1 \otimes Y(x)) \partial_z S^{12}(z-x)(b \otimes a \otimes c) \\
&= -Res_x \iota_{x,z} \partial_x (x-z)^n Y(x)(1 \otimes Y(z))(a \otimes b \otimes c) \\
&+Res_x \iota_{x,z} (x-z)^n Y(x)(1 \otimes Y(z))(1 \otimes T \otimes 1)(a \otimes b \otimes c) \\
&+Res_x \iota_{z,x} \partial_x (x-z)^n Y(z)(1 \otimes Y(x)) S^{12}(z-x)(b \otimes a \otimes c) \\
&-Res_x \iota_{z,x} (x-z)^n Y(z)(T \otimes 1)(1 \otimes Y(x)) S^{12}(z-x)(b \otimes a \otimes c) \\
&+Res_x \iota_{z,x} (x-z)^n Y(z)(1 \otimes Y(x)) \partial_x S^{12}(z-x)(b \otimes a \otimes c) \\
&= Res_x \iota_{x,z} (x-z)^n Y(x)(T \otimes 1)(1 \otimes Y(z))(a \otimes b \otimes c) \\
&+Res_x \iota_{x,z} (x-z)^n Y(x)(1 \otimes Y(z)(1 \otimes T \otimes 1)(a \otimes b \otimes c) \\
&-Res_x \iota_{z,x} (x-z)^n Y(z)(1 \otimes Y(x))(1 \otimes T \otimes 1) S^{12}(z-x)(b \otimes a \otimes c) \\
&-Res_x \iota_{z,x} (x-z)^n Y(z)(1 \otimes Y(x))(T \otimes 1 \otimes 1) S^{12}(z-x)(b \otimes a \otimes c).
\end{aligned}
\tag{33}
$$

On the other hand using translation covariance, RHS becomes

$$
\begin{aligned}
&Res_x \iota_{x,z} (x-z)^n Y(x)(T \otimes 1)(1 \otimes Y(z))(a \otimes b \otimes c) \\
&-Res_x \iota_{z,x} (x-z)^n Y(z)(1 \otimes Y(x)) \mathcal{S}^{12}(z-x)(1 \otimes T \otimes 1)(b \otimes a \otimes c) \\
&+Res_x \iota_{x,z} (x-z)^n Y(x)(1 \otimes Y(z))(1 \otimes T \otimes 1)(a \otimes b \otimes c) \\
&-Res_x \iota_{z,x} (x-z)^n Y(z)(1 \otimes Y(x)) \mathcal{S}^{12}(z-x)(T \otimes 1 \otimes 1)(b \otimes a \otimes c).
\end{aligned}
\tag{34}
$$

Due to equations (30) and (31) we get

$$
\begin{aligned}
&(T \otimes 1 \otimes 1)\mathcal{S}^{12}(z-x)(b \otimes a \otimes c) = \\
&\qquad \mathcal{S}^{12}(z-x)(T \otimes 1 \otimes 1)(b \otimes a \otimes c) + \partial_x \mathcal{S}^{12}(z-x)(b \otimes a \otimes c),
\end{aligned}
\tag{35}
$$

and

$$
\begin{aligned}
&(1 \otimes T \otimes 1)\mathcal{S}^{12}(z-x)(b \otimes a \otimes c) = \\
&\qquad \mathcal{S}^{12}(z-x)(1 \otimes T \otimes 1)(b \otimes a \otimes c) - \partial_x \mathcal{S}^{12}(z-x)(b \otimes a \otimes c).
\end{aligned}
\tag{36}
$$

Applying equations (35) and (36) to RHS, we get

$$\begin{aligned}
&Res_x\iota_{x,z}(x-z)^nY(x)(T\otimes 1)(1\otimes Y(z))(a\otimes b\otimes c)\\
&-Res_x\iota_{z,x}(x-z)^nY(z)(1\otimes Y(x))(1\otimes T\otimes 1)\mathcal{S}^{12}(z-x)(b\otimes a\otimes c)\\
&+Res_x\iota_{z,x}(x-z)^nY(z)(1\otimes Y(x))\partial_x\mathcal{S}^{12}(z-x)(b\otimes a\otimes c)\\
&+Res_x\iota_{x,z}(x-z)^nY(x)(1\otimes Y(z))(1\otimes T\otimes 1)(a\otimes b\otimes c)\\
&-Res_x\iota_{z,x}(x-z)^nY(z)(1\otimes Y(x))(T\otimes 1\otimes 1)\mathcal{S}^{12}(z-x)(b\otimes a\otimes c)\\
&-Res_x\iota_{z,x}(x-z)^nY(z)(1\otimes Y(x))\partial_x\mathcal{S}^{12}(z-x)(b\otimes a\otimes c)\\
&=Res_x\iota_{x,z}(x-z)^nY(x)(T\otimes 1)(1\otimes Y(z))(a\otimes b\otimes c)\\
&-Res_x\iota_{z,x}(x-z)^nY(z)(1\otimes Y(x))(1\otimes T\otimes 1)\mathcal{S}^{12}(z-x)(b\otimes a\otimes c)\\
&+Res_x\iota_{x,z}(x-z)^nY(x)(1\otimes Y(z))(1\otimes T\otimes 1)(a\otimes b\otimes c)\\
&-Res_x\iota_{z,x}(x-z)^nY(z)(1\otimes Y(x))(T\otimes 1\otimes 1)\mathcal{S}^{12}(z-x)(b\otimes a\otimes c).
\end{aligned} \tag{37}$$

Then equations (33) and (37) are equal, therefore the claim follows. □

Remark 1. Recall that, as we quote in Proposition 3, it was shown by [DGK] that in a braided vertex algebra, equations (30) and (31) hold mod Ker$Y$. In [EK], condition (30) is asked as part of the definition of a braided vertex operator algebra. In this context, asking (31) is equivalent to ask T to be a derivation of a braided vertex operator algebra. It is shown in [Li], that if in addition we ask the undelying field algebra to be *non-degenerate* (cf. definition 5.12, [Li]), we have that (31) holds in a braided vertex algebra where the associativity relation (6) holds (cf. [EK]).

Let $(V,|0\rangle,T,Y,\mathcal{S})$ be a braided-state field correspondence. Y satisfies the *quantum* n-th *product identities* if for all $a,b,c\in V$ and $n\in\mathbb{Z}$

$$Y(z)^{\mathcal{S}}_{(n)}Y(z)(a\otimes b\otimes c)=Y(z)(a^{\mathcal{S}}_{(n)}b\otimes c), \tag{38}$$

where

$$a^{\mathcal{S}}_{(n)}b=\mathrm{Res}(z^nY(z)S(z)(a\otimes b)). \tag{39}$$

Y satisfies the *associativity relation* if for any $a,b,c\in V$ and $M\in\mathbb{Z}_{\geq 0}$ there exists $N\in\mathbb{Z}_{\geq 0}$ such that

$$\begin{aligned}
&\iota_{z,w}(z+w)^NY(z+w)((1\otimes Y(w)))(a\otimes b\otimes c)\\
&=(z+w)^NY(w)(Y(z)\otimes 1)(a\otimes b\otimes c)\bmod h^M,
\end{aligned} \tag{40}$$

Let $(V,|0\rangle)$ be a pointed vector space. We define a *braided field algebra* $(V,|0\rangle,Y,T,\mathcal{S})$ is a braided state-field correspondence Y satisfying the associativity relation (6). We also introduce a *strong braided field algebra* $(V,|0\rangle,Y,T,\mathcal{S})$

as a state-field correspondence Y satisfying the quantum n-th product identities (38). This are the braided versions of field algebra and strong field algebra introduced by [BK].

Let $(V,|0\rangle,T,\mathcal{S})$ be a braided-state field correspondence. For $a,b\in V, f\in\mathbb{K}((z))[[h]]$, we define the (λ,f)-*product* by the

$$a_{(\lambda,f)}b=\mathrm{Res}_z e^{\lambda z}f(z)Y(z)(a\otimes b)=\sum_{n\in Z_{\geq 0},\,\text{finite}}\ \sum_{i\in\mathbb{Z}}f_i(h)\,a_{(n+i)}b\,\frac{\lambda^n}{n!}\in V\otimes\mathbb{K}[\lambda][[h]] \tag{41}$$

where $f(z)=\sum_{i\in\mathbb{Z}}f_i(h)z^i,\ f_i(h)\in\mathbb{K}[[h]]$. Note that $f_i(h)=0$ for $i<<0$.

Remark 2. If in adition we ask V to have a structure of $\mathbb{K}((z))$-module structure, more precisely $z^k(a_{(n)}b)=a_{(n+k)}b$, this (λ,f)-product resembles the operations introduced in [GKK]. Instead, we are asking V to have a braiding that involves some elements of $\mathbb{K}((z))[[h]]$.

We have the following useful Lemma.

Lemma 5. *Given $(V,|0\rangle,T,\mathcal{S})$ be a braided-state field correspondence, we have*

(a) $a_{(\lambda,z^mf)}b=\partial_\lambda^m a_{(\lambda,f)}b$ for $m\geq 0$, and $f\in\mathbb{K}((z))[[h]]$. In particular, $a_{(\lambda,z^m)}b=\partial_\lambda^m a_{(\lambda,1)}b$ for $m\geq 0$,

(b) $a_{(\lambda,z^{-k})}b=((\lambda+T)^{(k-1)}a)_{(\lambda,z^{-1})}b$, for $k\geq 1$,

Proof. Let $f(z)=\sum_i f_i(h)z^i$, item (a) follows from the definition of (λ,f)-product:

$$\begin{aligned}
a_{(\lambda,z^mf)}b &= \mathrm{Res}_z e^{\lambda z}z^m f(z)Y(z)(a\otimes b)\\
&= \sum_i f_i(h)\mathrm{Res}_z\sum_{k\geq 0}\lambda^k/k!\sum_{j\in\mathbb{Z}}a_{(j)}b\,z^{-j-1+k+m+i}\\
&= \sum_i f_i(h)\sum_{k\geq 0}\lambda^{k-m}/(k-m)!\,a_{(k+i)}b\\
&= \partial_\lambda^m a_{(\lambda,f)}b.
\end{aligned}$$

Applying definition of (λ,f)-product and using integration by parts and translation covariance we get item (b), namely:

$$\begin{aligned}
a_{(\lambda,z^{-k})}b &= a_{(\lambda,\,(-\partial)_z^{(k-1)}z^{-1})}b\\
&= \mathrm{Res}_z e^{\lambda z}(-\partial)_z^{(k-1)}z^{-1}Y(z)(a\otimes b)\\
&= \mathrm{Res}_z z^{-1}\partial_z^{(k-1)}e^{\lambda z}Y(z)(a\otimes b)\\
&= \mathrm{Res}_z z^{-1}e^{\lambda z}\sum_{r=0}^{k-1}\lambda^{(r)}Y(z)(T^{(k-1-r)}a\otimes b)\\
&= \mathrm{Res}_z z^{-1}e^{\lambda z}Y(z)((\lambda+T)^{(k-1)}a\otimes b)\\
&= ((\lambda+T)^{(k-1)}a)_{(\lambda,z^{-1})}b.
\end{aligned}$$

□

Note that if $f = 1$ in (41), we recover the λ-product introduced in (7) for a state-field correspondence. We will denote $a_{(\lambda,1)} = a_\lambda b$. Observe also that, due to the Lemma above, any (λ, f)-product can be written in terms of the λ-product and the (λ, z^{-1})-product.

The vacuum axioms for Y imply that,

$$|0\rangle_{(\lambda,z^{-1})}a = a = a_{(\lambda,z^{-1})}|0\rangle, \tag{42}$$

while the translation invariance axioms show that,

$$T(a_{(\lambda,f)}b) = T(a)_{(\lambda,f)}b + a_{(\lambda,f)}T(b) \tag{43}$$

and

$$T(a)_{(\lambda,f)}b = -\lambda a_{(\lambda,f)}b - a_{(\lambda,f')}b \tag{44}$$

for all $a, b \in V$ and $f \in \mathbb{K}((z))$. Note that, when $f = 1$ in (43) and (44), we recover equation (11).

Conversely, if we are given a pointed topologically free $\mathbb{K}[[h]]$-module $(V, |0\rangle)$, togheter with a $\mathbb{K}[[h]]$- linear map T, a braiding $\mathcal{S}$, a $(\lambda, 1)$-product and a (λ, z^{-1})-product on V satisfying the properties (42)-(44), we can reconstruct the braided state-field correspondence Y by the formulas:

$$Y(a,z)_+b = (e^{zT}a)_{(\lambda,z^{-1})}b|_{\lambda=0}, \qquad Y(a,z)_-b = (a_{(-\partial_z,1)}b)(z^{-1})), \tag{45}$$

where $Y(a,z) = Y(a,z)_+ + Y(a,z)_-$.

We will need the following Lemma.

Lemma 6. *We have that*

$$a_{(\lambda,f^{(l)})}b = ((-\lambda - T)^l a)_{(\lambda,f)}b,$$

for all a and $b \in V$ and $l \geq 0$. Here and further $f^{(l)}(z) = \partial_z^l f(z)$.

Proof. Straightforward using (44). □

For the following Proposition it will be useful to introduce the following notation:

$$a_{(\cdot,f)}b := a_{(\lambda,z^{-1}f)}b|_{\lambda=0} = \mathrm{Res}_z z^{-1}f(z)Y(z)(a\otimes b) = \sum_{i\in\mathbb{Z}} f_i(h)\, a_{(i-1)}b, \tag{46}$$

for $a, b \in V, f \in \mathbb{K}((z))[[h]]$, $f(z) = \sum_{i\in\mathbb{Z}} f_i(h)z^i$, $f_i(h) \in \mathbb{K}[[h]]$. Note that in the case $f = 1$ we obtain the $\cdot-$product in [BK], (cf. (8)), namely

$$a_{(\cdot,1)}b = a_{(\lambda,z^{-1})}b|_{\lambda=0} = a\cdot b,$$

since it is easy to show that

$$a_{(\lambda,z^{-1})}b = a.b + \int_0^\lambda a_\mu b\, d\mu \tag{47}$$

.

Whith all this, we can state the following result.

Proposition 4. *Let $(V, |0\rangle, T, Y, \mathcal{S})$ be a braided state field correspondence such $\mathcal{S}$-locality holds for $c = |0\rangle$. Then the collection of the n-th quantum product identities (29) for $n \geq -1$ implies:*

$$(a_{-\alpha-T}b)_{\alpha+\beta}c = -b_\alpha(a_{(\beta)}c) + \sum_{i=1}^r \sum_{l\geq 0}(-1)^l a^i_{(\beta,(f_i(z))^{(l)})}(b^i_{(\alpha,x^l)}c), \tag{48}$$

$$\begin{aligned}(a_{-\lambda}b).c \quad &= \quad -b_{(\lambda-T)}(a.c) + \sum_{i=1}^r \sum_{l\geq 0}(-1)^l a^i_{(\cdot,(f_i(z))^{(l)})}(b^i_{(\lambda-T,x^l)}c)\\ &+ \quad \int_0^{T-\lambda}(a_{-\lambda}b)_\mu c\, d\mu,\end{aligned} \tag{49}$$

$$\begin{aligned}(a.b)_\lambda c \quad &= \quad (e^{T\partial_\lambda}b).(a_\lambda c) - \int_0^{-T}(a_{-\mu-T}b)_\lambda c\, d\mu + \sum_{i=1}^r[(e^{T\partial_\lambda}a^i)_{(\cdot,f_i(z))}(b^i_\lambda c)\\ &- \quad \sum_{l\geq 0}\int_0^\lambda a^i_{(\mu,(f_i(z))^{(l)})}(b^i_{(\lambda-\mu,x^l)}c)d\mu],\end{aligned} \tag{50}$$

$$\begin{aligned}(a\cdot b).c \quad &= \quad b.(a.c) + Res_z\left(\int_0^T d_\lambda b\right)_. (b_\lambda c) - \int_0^{-T}(a_{\mu-T}b).c\, d\mu\\ &+ \quad \sum_{i=1}^r \sum_{l\geq 0}\left(\int_0^T d_\lambda a^i\right)_. (b^i_{\lambda,\, D_l f_i(z)}c)\\ &+ \quad \sum_{i=1}^r \sum_{m,\, l\geq 0}(-1)^l a^i_{(\cdot,z^{m+1})}(b^i_{(.,\, D_l(f_i(z))z^{-m})}c),\end{aligned} \tag{51}$$

where $D_l = z^l\partial_z^{(l)}$ and $\mathcal{S}(x)(a\otimes b) = \sum_{i=0}^r f_i(z)a^i\otimes b^i$.

Proof. Recall that the fact that the $\mathcal{S}$-locality holds for $c = |0\rangle$, implies that $Y(z)S(z) = Y^{op}(z)$. Applying definitions of λ-product and the definition of Y^{op}, due to Lemma 27 we get

$$\begin{aligned} a_\lambda b &= \mathrm{Res}_z e^{\lambda z} Y(z)(a\otimes b) \\ &= \mathrm{Res}_z e^{(\lambda+T)z} Y^{op}(-z)(b\otimes a) \\ &= -\mathrm{Res}_z e^{-(\lambda+T)z} Y(z)S(z)(b\otimes a) \\ &:= -b^{\mathcal{S}}_{-(\lambda+T)} a. \end{aligned} \tag{52}$$

The collection of n-th product identities (38) together (52) are equivalent to:

$$\begin{aligned} Y(a_{-\lambda} b, z)c &= -Y(b^{\mathcal{S}}_{(\lambda-T)} a, z)c \\ &= -\sum_{n\geq 0} Y(z)(b^{\mathcal{S}}_{(n)} a\otimes c)\frac{(\lambda-T)^n}{n!} \\ &= -\sum_{n\geq 0} Y(z)^{\mathcal{S}}_{(n)} Y(z)(b\otimes a\otimes c)\frac{(\lambda-T)^n}{n!} \\ &= -\sum_{n\geq 0}[\mathrm{Res}_x \iota_{x,z}(x-z)^n Y(x)(1\otimes Y(z))(b\otimes a\otimes c) \\ &+ \iota_{z,x}(x-z)^n Y(z)(1\otimes Y(x))\mathcal{S}^{12}(z-x)(a\otimes b\otimes c)]\frac{(\lambda-T)^n}{n!} \\ &= -\mathrm{Res}_x e^{(\lambda-T)(x-z)} Y(x)(1\otimes Y(z))(b\otimes a\otimes c) \\ &+ \mathrm{Res}_x e^{(\lambda-T)(x-z)} \sum_{i=1}^r e^{-x\partial_z}(f_i(z))Y(z)(1\otimes Y(x))(a^i\otimes b^i\otimes c) \\ &= -e^{(-\lambda+T)z}[b_{(\lambda-T)}(Y(a,z)c) \\ &+ \sum_{i=1}^r\sum_{l\geq 0}(-\partial_z)^{(l)}(f_i(z))Y(a^i,z)(b^i_{(\lambda-T,x^l)}c)]. \end{aligned} \tag{53}$$

Taking $\mathrm{Res}_z e^{(\lambda+\mu)z}$, and changing $\lambda - T$ by α and $\mu + T$ by β, we obtain (48). Taking $\mathrm{Res}_z z^{-1}$ in (53) and using $e^{(-\lambda+T)z}z^{-1} = z^{-1} + \int_0^{-\lambda+T} e^{\mu z}d\mu$, we get

$$\begin{aligned} (a_{-\lambda} b).c &= -b_{(\lambda-T)}(a.c) - \int_0^{T-\lambda} b_{(\lambda-T)}(a_\mu c)\, d\mu \\ &+ \sum_{i=1}^r\sum_{l\geq 0}(-1)^l[a^i_{(\cdot,(f_i(z))^{(l)})}(b^i_{(\lambda-T,x^l)}c) \\ &+ \int_0^{T-\lambda} a^i_{(\mu,(f_i(z))^{(l)})}(b^i_{(\lambda-T,x^l)}c)\, d\mu]. \end{aligned} \tag{54}$$

This, together with (48), implies (49) (after the substitution $\mu' = \lambda + \mu - T$).

Applying definitions of (-1)-st product and Y^{op}, due to (27) we get

$$
\begin{aligned}
a.b &= \mathrm{Res}_z z^{-1} Y(z)(a\otimes b)\\
&= \mathrm{Res}_z z^{-1} e^{zT} Y^{op}(-z)(b\otimes a)\\
&= \mathrm{Res}_z z^{-1} e^{-zT} Y(z)S(z)(b\otimes a)\\
&= \mathrm{Res}_z z^{-1} Y(z)S(z)(b\otimes a) + \int_0^{-T} e^{\mu z} Y(z)S(z)(b\otimes a) d\mu\\
&= b^{\mathcal{S}}_{\cdot} a + \int_0^{-T} b^{\mathcal{S}}_{\mu} a\, d\mu. \qquad (55)
\end{aligned}
$$

Then this equation together the quantum (-1)- product we get

$$
\begin{aligned}
Y(a\cdot b, z)c &= Y(z)(b^{\mathcal{S}}_{(-1)} a\otimes c) + \int_0^{-T} Y(z)(b^{\mathcal{S}}_{\mu} a\otimes c)\, d\mu\\
&= (Y(z)^{\mathcal{S}}_{(-1)} Y(z))(b\otimes a\otimes c) + \int_0^{-T} Y(z)(b^{\mathcal{S}}_{\mu} a\otimes c)\, d\mu\\
&= Y(b,z)_+ Y(a,z)c + \sum_{i=1}^{r} Y(a^i, z)(b^i_{-\partial_z} c)(z^{-1} f_i(z))\\
&+ \int_0^{-T} Y(b^{\mathcal{S}}_{\mu} a, z)c\, d\mu. \qquad (56)
\end{aligned}
$$

Taking $\mathrm{Res}_z e^{\lambda z}$ and using integration by parts, we get:

$$
\begin{aligned}
(a.b)_\lambda &= \mathrm{Res}_z (e^{T\partial_\lambda} e^{\lambda z} b).(a_\lambda c) + \mathrm{Res}_z \sum_{i=1}^{r} Y(a^i, z)(b^i_{\lambda-\partial_z} c(e^{\lambda z} z^{-1} f_i(z)))\\
&+ \int_0^{-T} (b^{\mathcal{S}}_{\mu} a)_\lambda c\, d\mu\\
&= (e^{T\partial_\lambda} b).(a_\lambda c) + \mathrm{Res}_z \sum_{i=1}^{r} Y(a^i, z)(b^i_{\lambda-\partial_z} c(z^{-1} f_i(z) + \int_0^{\lambda} f_i(z) e^{\mu z}\, d\mu)\\
&- \int_0^{-T} (a_{-\mu-T} b)_\lambda c\, d\mu\\
&= (e^{T\partial_\lambda} b).(a_\lambda c) - \int_0^{-T} (a_{-\mu-T} b)_\lambda c\, d\mu + \sum_{i=1}^{r} [(e^{T\partial_\lambda} a^i)_{(\cdot, f_i(z))}(b^i_\lambda c)\\
&+ \sum_{l\geq 0} \int_0^{\lambda} a^i_{(\mu,(f_i(z))^{(l)})}(b^i_{(\lambda-\mu, x^l)} c)d\mu]. \qquad (57)
\end{aligned}
$$

Due to (56) and taking $\text{Res}_z z^{-1}$, we get

$$\begin{aligned}
(a.b).c &= \text{Res}_z z^{-1} Y(b,z)_+ Y(a,z)_+ c + \text{Res}_z z^{-1} Y(b,z)_+ Y(a,z)_- c \\
&+ \text{Res}_z z^{-1} \int_0^{-T} Y(b^{\mathcal{S}}_{\mu} a, z) c \, d\mu \\
&+ \sum_{i=1}^{r} \sum_{l \geq 0} (-1)^l \text{Res}_z z^{-1} Y(a^i, z)(f_i(z))^{(l)} (\partial_z)^l Y(b^i, z)_- c \\
&= b.(a.c) + \text{Res}_z z^{-1})((e^{zT} - 1)b).(Y(a,z)_- c) + \int_0^{-T} (b^{\mathcal{S}}_{\mu} a).c \, d\mu \\
&+ \sum_{i=1}^{r} \sum_{l \geq 0} \text{Res}_z z^{-1}((e^{zT} - 1)a^i).((f_i(z))^{(l)} (\partial_z)^l Y(b^i, z)_- c) \\
&+ \sum_{i=1}^{r} \sum_{l \geq 0} \text{Res}_z z^{-1} Y(a^i, z)_- ((f_i(z))^{(l)} (\partial_z)^l Y(b^i, z)_- c) \\
&= b.(a.c) + \text{Res}_z \left(\int_0^T e^{\lambda z} d_\lambda b \right)_. (Y(b,z)_- c) - \int_0^{-T} (a_{\mu - T} b).c \, d\mu \\
&+ \sum_{i=1}^{r} \sum_{l \geq 0} \text{Res}_z \left(\int_0^T e^{\lambda z} d_\lambda a^i \right)_. (f_i(z))^{(l)} (\partial_z)^l Y(b^i, z)_- c) \\
&+ \sum_{i=1}^{r} \sum_{m,l \geq 0} (-1)^l a^i_{(\cdot, z^{m+1})} (b^i_{(., D_l(f_i(z)) z^{-m})} c),
\end{aligned} \tag{58}$$

which proves (51). □

Note that V together with the λ-product is what [BK] called *conformal algebra*, and V with (λ, z^{-1})-product is also a $\mathbb{K}[T]$-differential algebra with unit due to (42) and (43).

An important remark is that V togheter with the λ-product is no longer a Leibnitz conformal algebra, since due to the braiding, the analog of the Jacoby identity involves (λ, z^{-1})-products.

With this in mind, we can prove the following

Theorem 3. *Giving a braided state field correspondence $(V, |0\rangle, Y, \mathcal{S})$, satisfying the $\mathcal{S}$-locality for $|0\rangle$ and the axiom of quantum (n)-product (29) implies to provide V with a structure of a conformal algebra and a structure of a $\mathbb{K}[T]$-differential algebra with a unit $|0\rangle$, satisfying (48)-(51).*

Conversely, given V a topologically free $\mathbb{K}[[h]]$-module, a $\mathbb{K}[[h]]$- linear map T and a braiding $\mathcal{S}$. Assume that V has a structure of conformal algebra and a structure of a $\mathbb{K}[T]$-differential algebra with a unit $|0\rangle$, satisfying (48)-(51) and $\mathcal{S}$ satisfies (30)- (31), then $(V, |0\rangle, Y, \mathcal{S})$, is a braided state field correspondence satisfying the axiom of quantum (n)-product, namely a strong braided field algebra.

Proof. If $(V, |0\rangle, Y, \mathcal{S})$ is a braided state field algebra satisfying the axiom of quantum n-product, then by the above discussion we can define a (λ, f)-product on V satisfying all the requirement. Conversely, given a (λ, f)-product we define a braided state field correspondence Y by (45). In the proof of Lemma 4, we have seen that the equations (48)-(49) are equivalent to the identities

$$\mathrm{Res}_z(Y(b_n^{\mathcal{S}}a, z) - Y(b, z)_{(n)}^{\mathcal{S}}Y(a, z))F = 0, \qquad a, b \in V, n \geq 0,\ F = e^{\lambda z} \text{ or } z^{-1}, \tag{59}$$

while the equations (50)-(51) are equivalent to the identities

$$\mathrm{Res}_z \sum_{k\geq 0}[(-\partial_z)^{(k)}Y(b_{(k-1)}^{\mathcal{S}}a, z) - \sum_{j=0}^{k}(-1)^k\partial_z^{(k-j)}Y(b, z)_{(k-1)}^{\mathcal{S}}\partial_z^{(j)}Y(a, z)]F = 0, \tag{60}$$

for $a, b \in V$ and $F = e^{\lambda z}$ or z^{-1}.

Due to Lemma 4 and using translation invariance of Y, this identity is equivalent to

$$\mathrm{Res}_z \sum_{k\geq 0}[Y(b_{(k-1)}^{\mathcal{S}}a, z) - Y(b, z)_{(k-1)}^{\mathcal{S}}Y(a, z)](\partial_z)^{(k)}F = 0, \tag{61}$$

$a, b \in V$, $F = e^{\lambda z}$ or z^{-1}.

Using the translation invariance of Y and integration by parts, we see that identity (59) holds also with F replaced with $\partial_z F$. Hence equations (59) and (61) hold for all $F = z^l, l < 0$. For $F = e^{\lambda z}$, taking coefficients at power of λ shows that they are satisfied also for $F = z^l, l \leq 0$. This implies the n-th quantum product axioms for $n \geq -1$. The the proof remains the same that proof of Theorem 4.4 [BK]. ☐

4 Quantum conformal algebra

In this section, based on what we have seen in Section 3, we aim to give a definition of braided conformal algebra. Until now, we didn't ask any furhter structure for the braiding S besides (30) and (31). We have the following results.

Proposition 5. *If the hexagon relation*

$$\mathcal{S}(x)(Y(z) \otimes 1) = (Y(z) \otimes 1)\mathcal{S}^{23}(x)\iota_{x,z}\mathcal{S}^{13}(x + z) \tag{62}$$

holds in a braided state field correspondence, then we have that:

$$\begin{aligned} \mathcal{S}(x)(a_{(\lambda,f)}b \otimes c) &= \sum_{l\geq 0}\partial_\lambda^l((\cdot_{(\lambda,f)}\cdot) \otimes 1)\mathcal{S}^{23}(x)\partial_x^{(l)}\mathcal{S}^{13}(x)(a \otimes b \otimes c) \quad (63)\\ &= e^{\partial_\lambda\partial_{x_1}}((\cdot_{(\lambda,f)}\cdot) \otimes 1)\mathcal{S}^{23}(x)\mathcal{S}^{13}(x_1)(a \otimes b \otimes c)|_{x_1=x} \quad (64)\end{aligned}$$

for all $a, b, c \in V$.

Proof. Applying the definition of (λ, f)-product and using the hexagon relation (62), definition of S, Taylor expansion and change of variables we get,

$$\begin{aligned}
\mathcal{S}(x)(a_{(\lambda,f)}b\otimes c) &= \mathcal{S}(x)\mathrm{Res}_z e^{\lambda z} f(z)(Y(z)\otimes 1)(a\otimes b\otimes c)\\
&= \mathrm{Res}_z e^{\lambda z} f(z)\mathcal{S}(x)(Y(z)\otimes 1)(a\otimes b\otimes c)\\
&= \mathrm{Res}_z e^{\lambda z} f(z)(Y(z)\otimes 1)\mathcal{S}^{23}(x)\iota_{x,z}\mathcal{S}^{13}(x+z)(a\otimes b\otimes c)\\
&= \sum_{i,j\in\mathbb{Z}} \mathrm{Res}_z e^{\lambda z} f(z)(Y(z)\otimes 1)h_i(x)\iota_{x,z}g_j(x+z)(a^{(j)}\otimes b^{(i)}\otimes (c^{(j)})^{(i)})\\
&= \sum_{i,j\in\mathbb{Z}} h_i(x)\mathrm{Res}_z e^{\lambda z} f(z)(e^{z\partial_x}g_j(x))(Y(a^{(j)},z)b^{(i)}\otimes (c^{(j)})^{(i)})\\
&= \sum_{i,j,m,r\in\mathbb{Z}}\sum_{k,l\in\mathbb{Z}_{\geq 0}} h_i(x)\lambda^{(k)} f_r\,\mathrm{Res}_z g_j^{(l)}(x)a^{(j)}_{(m)}b^{(i)}\otimes (c^{(j)})^{(i)}z^{-m-1+k+l+r}\\
&= \sum_{i,j,r\in\mathbb{Z}}\sum_{k,l\in\mathbb{Z}_{\geq 0}} h_i(x)\lambda^{(k)} f_r g_j^{(l)}(x)a^{(j)}_{(k+l+r)}b^{(i)}\otimes (c^{(j)})^{(i)}\\
&= \sum_{i,j,r\in\mathbb{Z}}\sum_{k\geq l\in\mathbb{Z}_{\geq 0}} h_i(x)\lambda^{(k-l)} f_r g_j^{(l)}(x)a^{(j)}_{(k+r)}b^{(i)}\otimes (c^{(j)})^{(i)}\\
&= \sum_{l\geq 0}\partial_\lambda^l(\cdot_{(\lambda,f)}\cdot)\mathcal{S}^{23}(x)\partial_x^{(l)}\mathcal{S}^{13}(x)(a\otimes b\otimes c), \qquad (65)
\end{aligned}$$

where $S^{23}(x)(a\otimes b\otimes c) = \sum_i h_i(x)a\otimes b^i\otimes c^i$ and $S^{13}(x)(a\otimes b\otimes c) = \sum_j g_j(x)a^j\otimes b\otimes c^j$. □

Similarly, we have the following results.

Proposition 6. *If the associativity relation holds, namely, there exists $N\in\mathbb{Z}_{\geq 0}$ such that*

$$\iota_{z,w}(z+w)^N Y(z+w)((1\otimes Y(w)))(a\otimes b\otimes c) = (z+w)^N Y(w)(Y(z)\otimes 1)(a\otimes b\otimes c)$$

$mod\, h^M$, for any $a,b,c\in V$ and $M\in\mathbb{Z}_{\geq 0}$ in a (braided) state field correspondence, then

$$\partial_\lambda^N a_\lambda(b_\mu c) = \partial_\lambda^N (a_\lambda b)_{\lambda+\mu}c, \qquad (66)$$

$mod\, h^M$, for all $a,b,c\in V$.

Proof. Changing $z+w$ by x in the associativity relation we have

$$x^N Y(x)(1\otimes Y(w))(a\otimes b\otimes c) = x^N Y(w)\iota_{x,w}(Y(x-w)\otimes 1)(a\otimes b\otimes c). \qquad (67)$$

Taking $\mathrm{Res}_x\mathrm{Res}_w e^{\lambda x}e^{\mu w}$ to the LHS of (67) and using Lemma 5 (a), we have

$$\begin{aligned}
&\mathrm{Res}_x\mathrm{Res}_w e^{\lambda x}e^{\mu w}\, x^N Y(x)((1\otimes Y(w)))(a\otimes b\otimes c)\\
&\quad = (a_{(\lambda,x^N)}(b_\mu c)) = \partial_\lambda^N(a_\lambda(b_\mu c)).
\end{aligned} \qquad (68)$$

Now, taking $\mathrm{Res}_x\mathrm{Res}_w e^{\lambda x}e^{\mu w}$ to the RHS of (67), using Taylor's formula (2), translation covariance and Lemma 5 (a),

$$
\begin{aligned}
&\mathrm{Res}_x\mathrm{Res}_w e^{\lambda x}e^{\mu w}\, x^N Y(w)\iota_{x,w}(Y(x-w)\otimes 1)(a\otimes b\otimes c)\\
&\qquad = \mathrm{Res}_x\mathrm{Res}_w e^{\lambda x}e^{\mu w}\, x^N Y(w)e^{-w\partial_x}(Y(x)\otimes 1)(a\otimes b\otimes c)\\
&\qquad = \mathrm{Res}_x\mathrm{Res}_w e^{\lambda x}e^{\mu w}\, x^N Y(w)(Y(x)\otimes 1)((e^{-wT}a)\otimes b\otimes c)\\
&\qquad = \mathrm{Res}_w e^{\mu w} Y((e^{-wT}a)_{(\lambda,x^N)}b,w)c \qquad\qquad (69)\\
&\qquad = \mathrm{Res}_w e^{\mu w}\partial_\lambda^N Y((e^{-wT}a)_\lambda b,w)c\\
&\qquad = \partial_\lambda^N \mathrm{Res}_w e^{\mu w}e^{w\lambda}Y(a_\lambda b,w)c\\
&\qquad = \partial_\lambda^N (a_\lambda b)_{\lambda+\mu}c.
\end{aligned}
$$

Equating (68) and (69), we finish the proof. □

Now, we will show a similar resut but for the quasi-associativity (70).

Proposition 7. *Let V be a braided state field correspondance. Suppose that for every a, b, $c\in V$ and $M\in\mathbb{Z}_{\geq 0}$ there exists $N\geq 0$ such that*

$$
\begin{aligned}
&\iota_{z,w}((z+w)^N Y(z+w)(1\otimes Y(w))\mathcal{S}^{23}(w)\mathcal{S}^{13}(z+w)(a\otimes b\otimes c))\\
&\qquad = (z+w)^N Y(w)\mathcal{S}(w)(Y(z)\otimes 1)(a\otimes b\otimes c) \quad mod\, h^M. \qquad (70)
\end{aligned}
$$

holds $mod h^M$. Then

$$
\sum_{i,j}\partial_\sigma^N a^j_{(\sigma,g_j)}(b^i_{(-\lambda+\mu,h_i)}(c^j)^i)|_{\sigma=\lambda} = (\partial_\lambda+\partial_\mu)^N\sum_r (a_\lambda b)^r_{(\mu,f_r)}c^r \quad mod\, h^M, \qquad (71)
$$

where

$$
\begin{aligned}
S(x)(a_\lambda b)\otimes c &= \sum_r f_r(x)(a_\lambda b)^r\otimes c^r),\\
S^{13}(x)(a\otimes b\otimes c) &= \sum_j g_j(x)(a^j\otimes b\otimes c^j),\\
S^{23}(x)(a^j\otimes b\otimes c^j) &= \sum_i h_i(x)(a^j\otimes b^i\otimes (c^j)^i).
\end{aligned}
$$

Proof. Taking $\mathrm{Res}_x\mathrm{Res}_w e^{\lambda z}e^{\mu w}$ to the LHS of (70), using Taylor's formula (2) and integration by parts, we have

$$
\begin{aligned}
&\mathrm{Res}_z\mathrm{Res}_w e^{\lambda z}e^{\mu w}\iota_{z,w}((z+w)^N Y(z+w)(1\otimes Y(w))\mathcal{S}^{23}(w)\mathcal{S}^{13}(z+w)(a\otimes b\otimes c)) =\\
&\quad = \mathrm{Res}_z\mathrm{Res}_w e^{\lambda z}e^{\mu w}(z+w)^N e^{w\partial_z}(Y(z))(1\otimes Y(w))\mathcal{S}^{23}(w)e^{w\partial_z}(\mathcal{S}^{13}(z))(a\otimes b\otimes c))\\
&\quad = \sum_{i,j}\mathrm{Res}_z\mathrm{Res}_w e^{-w\partial_z}(e^{\lambda z}(z+w)^N)e^{\mu w}Y(z))(1\otimes Y(w))g_j(z)h_i(w)(a^j\otimes b^i\otimes (c^j)^i). \qquad (72)
\end{aligned}
$$

It is straightforward that $e^{-w\partial_z}(z+w)^N = z^N$ and $e^{-w\partial_z}e^{\lambda z} = e^{-w\lambda}e^{\lambda z}$, thus

$$
\begin{aligned}
&\sum_{i,j} \mathrm{Res}_z \mathrm{Res}_w e^{-w\partial_z}\left(e^{\lambda z}(z+w)^N\right) e^{\mu w} Y(z)(1\otimes Y(w)) g_j(z) h_i(w)(a^j \otimes b^i \otimes (c^j)^i) \\
&= \sum_{i,j} \mathrm{Res}_z \mathrm{Res}_w e^{-w\lambda} e^{\lambda z}(z)^N e^{\mu w} Y(z)(1\otimes Y(w)) g_j(z) h_i(w)(a^j \otimes b^i \otimes (c^j)^i) \\
&= \sum_{i,j} a^j_{(\lambda, x^N g_j)}(b^i_{(-\lambda+\mu, h_i)}(c^j)^i) \\
&= \sum_{i,j} \partial_\sigma^N a^j_{(\sigma, g_j)}(b^i_{(-\lambda+\mu, h_i)}(c^j)^i)|_{\sigma=\lambda}.
\end{aligned} \tag{73}
$$

In the last equality we used Lemma 5(a). Now, lets take residues in the RHS of (70) and use Lemma 5(a)again. Thus

$$
\begin{aligned}
&\mathrm{Res}_z \mathrm{Res}_w e^{\lambda z} e^{\mu w}(z+w)^N Y(w)\mathcal{S}(w)(Y(z)\otimes 1)(a\otimes b\otimes c) \\
&= \sum_r \sum_{k=0}^N \binom{N}{k} \mathrm{Res}_w e^{\mu w} f_r(w) w^{N-k} (a_{(\lambda, z^k)} b)^r \otimes c^r \\
&= \sum_r \sum_{k=0}^N \binom{N}{k} (a_{(\lambda, z^k)} b)^r_{(\mu, f_r(w) w^{N-k})} \otimes c^r \\
&= (\partial_\lambda + \partial_\mu)^N \sum_r (a_\lambda b)^r_{(\mu, f_r)} c^r.
\end{aligned} \tag{74}
$$

Equating modh^M, we have the desired result. ☐

Finaly, let us translate the condition $Y(z)S(z) = Y^{op}(z)$ to the (λ, f)-product.

Proposition 8. *Suppose we have a state-field correspondance V where $Y(z)S(z) = Y^{op}(z)$ holds. Then, for a and $b \in V$,*

$$
-\,b_{-\lambda-T}\,a = \sum_i a^i_{(\lambda, f_i)} b^i, \tag{75}
$$

where $\mathcal{S}(z)(a\otimes b) = \sum_i f_i(z) a^i \otimes b^i$

Proof. We have that

$$
\begin{aligned}
\mathrm{Res}_z e^{\lambda z} Y(z) S(z)(a\otimes b) &= \sum_i \mathrm{Res}_z e^{\lambda z} f_i(z) Y(a^i, z) b^i \\
&= \sum_i a^i_{(\lambda, f_i)} b^i.
\end{aligned} \tag{76}
$$

On the other hand, using $Y(z)S(z) = Y^{op}(z)$,

$$\begin{aligned} \mathrm{Res}_z e^{\lambda z} Y(z)S(z)(a \otimes b) &= \mathrm{Res}_z e^{\lambda z} Y^{op}(z)(a \otimes b) \\ &= \mathrm{Res}_z e^{\lambda z} e^{Tz} Y(-z)(b \otimes a) \\ &= -\mathrm{Res}_z e^{(-\lambda - T)z} Y(z)(b \otimes a) \\ &= -b_{-\lambda - T}\, a, \end{aligned} \tag{77}$$

finishing the proof. □

A braided vertex algebra where the associativity relation holds, is called *quantum vertex algebra.* (Cf. Definition 3.12, [DGK]). In the Characterization Theorem (cf. Theorem 5.13,[DGK]) they proved, among other equivalences, that a quantum vertex algebra is a braided state field correspondence such that the associativity relation and $YS = Y^{op}$ holds. We have shown in the discussion before Lemma 6, combined with the fact that all (λ, f) products can be rewritten in terms of λ-products and (λ, z^{-1})-products, that having a braided state field correspondence is the same of having topologically free $\mathbb{K}[[h]]$-module V, together with a $\mathbb{K}[[h]]$-linear map $T : V \to V$, a distinguished vector $|0\rangle$, a braiding $\mathcal{S}$ on V and linear maps $(\lambda, f) : V \otimes V \to \mathbb{K}[\lambda][[h]], a \otimes b \to a_{(\lambda,f)}b$ for $f \in \mathbb{K}((Z))[[h]]$, such that

$$|0\rangle_{(\lambda,z^{-1})}a = a = a_{(\lambda,z^{-1})}|0\rangle, \tag{78}$$

$$T(a_{(\lambda,f)}b) = T(a)_{(\lambda,f)}b + a_{(\lambda,f)}T(b) \tag{79}$$

and

$$T(a)_{(\lambda,f)}b = -\lambda a_{(\lambda,f)}b - a_{(\lambda,f')}b \tag{80}$$

for all $a, b \in V$. Combining this with Porposition 6 and Proposition 8 we have the following.

Theorem 4. *Let V be topologically free $\mathbb{K}[[h]]$-module, together with a $\mathbb{K}[[h]]$-linear map $T : V \to V$, a distinguished vector $|0\rangle$, a braiding $\mathcal{S}$ on V. Define in V linear maps $(\lambda, f) : V \otimes V \to \mathbb{K}[\lambda][[h]], a \otimes b \to a_{(\lambda,f)}b$ for $f \in \mathbb{K}((z))[[h]]$, such that the equation above hold. Let Y be a topological state-field correspondence.*

The following statements are equivalent:

(i) $(V, T, |0\rangle, Y, \mathcal{S})$ is a quantum vertex algebra.

(ii)$(V, T, |0\rangle, (\cdot_{(\lambda,f)}\cdot), \mathcal{S})$ satisfies the equations:

$$|0\rangle_{(\lambda,z^{-1})}a = a = a_{(\lambda,z^{-1})}|0\rangle, \tag{81}$$

$$T(a_{(\lambda,f)}b) = T(a)_{(\lambda,f)}b + a_{(\lambda,f)}T(b) \tag{82}$$

and

$$T(a)_{(\lambda,f)}b = -\lambda a_{(\lambda,f)}b - a_{(\lambda,f')}b \tag{83}$$

for all $a, b \in V$, and

$$-\, b_{-\lambda-T}\, a = \sum_i a^i_{(\lambda,f_i)}b^i, \tag{84}$$

where $\mathcal{S}(z)(a\otimes b)=\sum_i f_i(z)a^i\otimes b^i$, *and there exists* $N>>0$ *such that*

$$\partial_\lambda^N a_\lambda(b_\mu c)=\partial_\lambda^N(a_\lambda b)_{\lambda+\mu}c, \tag{85}$$

$mod\, h^M$, *for all* $a,b,c\in V$.

If was proved in Proposition 3.13 in [DGK] that if a braided vertex algebra satisfies the hexagon relation then the associativity relation holds.

Assume that we have a braided vertex algebra V and the hexagon relation holds, thus we have a quantum vertex algebra. I we also ask in V the condition

$$[T\otimes 1,\mathcal{S}(x)]=-\partial_x\mathcal{S}(x)\quad\text{and}\quad[1\otimes T,\mathcal{S}(x)]=\partial_x\mathcal{S}(x),$$

(which hold, for instance, in what [EK] called non-degenerate quantum vertex algebra), and consider here the λ-product above, we showed that (V,T,S) together with the λ-product is a conformal algebra (in the sense of [BK]), sitting inside our quantum vertex algebra such that (64) holds. All these, leads us to the following definition.

Definition 1. A *quantum conformal algebra* is a topologically free $\mathbb{K}[[h]]$-module V, together with a $\mathbb{K}[[h]]$-linear map $T:V\to V$, a braiding $\mathcal{S}$ on V and a linear map $\lambda: V\otimes V\to\mathbb{K}[\lambda], a\otimes b\to a_\lambda b$ such that:$(a,b,c\in V)$

(i) $[T\otimes 1,\mathcal{S}(x)]=-\partial_x\mathcal{S}(x)$ (left shift condition);

(ii) $[1\otimes T,\mathcal{S}(x)]=\partial_x\mathcal{S}(x)$ (right shift condition);

(iii) $T(a_\lambda b)=(Ta)_\lambda b+a_\lambda(Tb),\qquad (Ta)_\lambda b=-\lambda a_\lambda b$;

(iv) $\mathcal{S}(x)(a_\lambda b\otimes c)=e^{\partial_\lambda\partial_{x_1}}((\cdot_\lambda\cdot)\otimes 1)\mathcal{S}^{23}(x)\mathcal{S}^{13}(x_1)(a\otimes b\otimes c)|_{x_1=x}$, (hexagon relation).

Moreover if we ask

(iv) $a_\lambda(b_\mu c)=(a_\lambda b)_{\lambda+\mu}c$,

we call V *associative* quantum conformal algebra.

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