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Article

On the Geometric Dimension of the Core of a TU-Game

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Abstract: In this paper we state a condition which characterizes the sub-class of TU-games (games with transferable utility) having non-empty core of dimension k, for any $0 \leq k \leq n-1$. It improves and generalizes the adding up (AU) property used by Brandenburger and Stuart for the case $k = 0$ (games with one-point core) to study biform games. It also embraces one of the two conditions stated by Zhao for the case $k = n-1$ (games having core with non-empty interior, relative to the set of pre-imputations) while studying some geometric properties of the core. The condition allows us to show that all the information about the geometric dimension of the core is contained in the vector of excesses associated to the nucleolus of the game. It also allows us to get some insight about the geometric properties of the cone of balanced games as well. In particular, we prove that all the games in the relative interior of each face of the cone have a core with the same geometric dimension. This fact is illustrated for the case of three-person games. We also present a couple of examples to show how the results of the paper can be used to deal with biform games from a new perspective.

Keywords: TU-games; core; geometric dimension; biform games

1. Introduction

The core of a game with transferable utility (TU-game) is the most appealing and widely studied solution concept for this class of games, although, for some games, it can be the empty set. The classical theorem of Bondareva [1] and Shapley [2] gives a necessary and sufficient condition for the non-emptiness of the core. In general, when the core is non-empty, it contains more than one point. Recently, Brandenburger and Stuart [3] worked with a condition that, whenever the the core of a game is non-empty, it is guaranteed that this set has only one element. This condition is then used extensively to study a rather new class of games, namely, biform games. On the other hand, Zhao [4] gives two necessary and sufficient conditions for the core of a game to have non-empty interior (relative to the set of pre-imputations). In this paper we provide a necessary and sufficient condition to guarantee that the core of a game has dimension k, for any $0 \leq k \leq n-1, n$ being the number of players in the game. For $k = n-1$, our condition is equivalent to Zhao's condition given in Theorem 1 of [4]. For $k < n-1$, the condition we state is strongly related to the existence of a balanced family of coalitions, other than N (N stands for the set of players in the game) with maximal worth (see Section 2). During the proof of Theorem 3 (Section 4), it will emerge that the geometric dimension of the core is determined, to a great extent, in the general case, by the structure of the family of coalition with maximum excess appearing in the ordered vector of excesses related to the nucleolus ([5]). This is a fact which is clearly stated by [4] for $k = n-1$, and which is proven in the general case in our Corollary 4 (Section 4). Our result also provides some insight about the geometry of the cone of balanced games. In particular, it allows us to show that all the games in the relative interior of each of the faces of that cone have core with the same dimension. The paper has the following organization: in the next section, we present some basic facts related to the theory of TU-games and the adding up property used by [3] as well. In Section 2, we tackle the case $k = 0$, namely, when the games have core with only one point which, therefore, coincides with the nucleolus of the game. The case $k = 0$ is then used as the first step of an inductive process which allows us to deal with the general situation in Section 4. Some of the results of this section are related to the work of [6]. In Section 5 we use the results of the previous section to bring a geometric description of the cone of balanced games in terms of the dimension of the core of the games in each of its faces. The case $n = 3$ is used to illustrate this approach. We also include a section to grasp how the results of the paper can be used to study biform games from a new perspective. We close the paper with some concluding where we outline some further lines of work.

2. Preliminaries

A TU-game is an ordered pair (N, v), where $N = \{1, 2, ..., n\}$ is a finite non-empty set, the set of *players*, and v is the characteristic function, which is a real valued function defined on the family $\mathcal{P}(N)$ of subsets of N, satisfying $v(\phi) = 0$. The elements of $\mathcal{P}(N)$ are the *coalitions*.

The set of *pre-imputations* is $E = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N)\}$, and the set of *imputations* is $A = \{x \in E : x_i \geq v(i) \text{ for all } i \in N\}$.

Let a game (N, v) be given. For any $x \in E$,$S \in \mathcal{P}(N)$,the *excess* of the coalition S with respect to x is $e(S, x) = v(S) - x(S)$, where, as usual, $x(S) = \sum_{i \in S} x_i$ if $S \neq \phi$ and 0 otherwise. The *core* of (N, v) is the set $C = \{x \in E : e(S, x) \leq 0 \text{ for all } S \in \mathcal{P}(N)\}$.

The core of a game may be the empty set. The Shapley-Bondareva theorem ([1],[2]) characterizes the sub-class of TU-games with non-empty core. In this result, the notion of a balanced family of coalitions plays a key role. A non-empty family of coalitions $\mathcal{B}$ is *balanced* if there exists a set of positive numbers $\lambda_{\mathcal{B}} = (\lambda_S)_{S \in \mathcal{B}}$, the *balancing weights*, such that $\sum\limits_{S \in \mathcal{B}(i)} \lambda_S = 1$ for all $i \in N$. Here, $\mathcal{B}(i) = \{S \in B : i \in S\}$. The quantity $w(\mathcal{B}, \lambda_{\mathcal{B}}, v) = \sum_{S \in \mathcal{B}} \lambda_S v(S)$ is the *worth* of the balanced family $\mathcal{B}$ with respect to the set of balancing weights $\lambda_{\mathcal{B}} = (\lambda_S)_{S \in \mathcal{B}}$. Balancedness can also be defined as follows. Let $\chi_S \in \mathbb{R}^n$ denote the n-dimensional vector defined by $(\chi_S)_i = 1$ if $i \in S$ and 0 if $i \notin S$ (the indicator vector of S). Then, a family $\mathcal{B}$ of coalitions is balanced if there exist positive balancing weights $\lambda_{\mathcal{B}} = (\lambda_S)_{S\in\mathcal{B}}$ such that $\sum_{S\in\mathcal{B}} \lambda_S \chi_S = \chi_N$. A minimal balanced family is one including no other proper balanced subfamily. Such a family always has a unique vector of balancing weights ([2]). In this case, we are going to use $w(\mathcal{B}, \lambda_{\mathcal{B}})$ simply to denote $w(\mathcal{B}, \lambda_{\mathcal{B}}, v)$.

A game (N, v) is balanced if $\sum_{S\in\mathcal{B}} \lambda_S v(S) \leq v(N)$ for any balanced family $\mathcal{B}$ with balancing weights $\lambda_{\mathcal{B}}$. Shapley-Bondareva's theorem states that the core of a $TU-$game is non-empty if and only if the game is balanced.

Usually, when the core is non-empty, it contains more than one point. However, in some situations, it is convenient to restrict oneself to work in the subclass of balanced games whose core has exactly one imputation. This happens, for instance, when studying biform as introduced by [3]. There, the authors use the Adding Up condition, which determines a subclass of balanced games having core with that property (See e.g. [7]). A game (N, v) possesses the *Adding Up* condition (AU) if

$$\sum_{i\in N}(v(N) - v(N\backslash\{i\})) = v(N).$$

This is only a sufficient condition that guarantees the single point property of the core (whenever this set is non-empty). In the following section, we extend the AU property and show that the extension is also necessary.

3. Games with a single-point core

To motivate our next definition, we note that $\mathcal{B} = \{N\backslash\{i\}\}_{i\in N}$ is a minimal balanced family of coalitions with cardinality n, whose indicator vectors $\chi_S, S \in \mathcal{B}$, are linearly independent. Furthermore, when the Adding Up property is valid, it satisfies that $w(\mathcal{B}, \lambda_{\mathcal{B}}, v) = v(N)$ for the unique collection of balancing weights $\lambda_{\mathcal{B}} = (\frac{1}{n-1})_{S\in\mathcal{B}}$ for $\mathcal{B}$. Given a family of coalitions $\mathcal{B}$ with m members, $M_{\mathcal{B}}$ will stand for an $m \times n$ matrix whose rows are the indicator vectors χ_S of the coalitions S belonging to $\mathcal{B}$.

Definition 1 Let a game (N, v) be given. A balanced family of coalitions $\mathcal{B}$ is determining in (N, v) if it satisfies the following two conditions:

$i)$ $rank(M_{\mathcal{B}}) = n$.

$ii)$ There is a collection of balancing weights $\lambda_{\mathcal{B}}$ for $\mathcal{B}$ such that $w(\mathcal{B}, \lambda_{\mathcal{B}}, v) \geq w(\mathcal{B}', \lambda_{\mathcal{B}'}, v)$ for any other balanced family $\mathcal{B}'$ with balancing weights given by $\lambda_{\mathcal{B}'}$.

When (N, v) is balanced, any determining family $\mathcal{B}$ satisfies $w(\mathcal{B}, \lambda_{\mathcal{B}}, v) = v(N)$.

Theorem 1 Let (N, v) be a TU-game. Then $|C| = 1$ if and only if there is a determining family $\mathcal{B}$ with $w(\mathcal{B}, \lambda_{\mathcal{B}}, v) = v(N)$ for some collection $\lambda_{\mathcal{B}}$ of balancing weights.

Proof Let us assume first that there exists a determining family $\mathcal{B}$, and a collection of balancing weights $\lambda_{\mathcal{B}}$ with $w(\mathcal{B}, \lambda_{\mathcal{B}}, v) = v(N)$. Then, the Shapley-Bondareva theorem guarantees the non-emptiness of the core. Moreover, if $x \in C$, then $x(S) = v(S)$ for all $S \in \mathcal{B}$. Since $\mathcal{B}$ is determining, the linear system

$$M_{\mathcal{B}} y = v_{\mathcal{B}},$$

where $v_{\mathcal{B}}$ is the vector $(v(S))_{S\in\mathcal{B}}$, has a unique solution. Thus, $|C| = 1$.

On the other hand, if $|C| = 1$, the game is balanced and $w(\mathcal{B}, \lambda_{\mathcal{B}}, v) \leq v(N)$ for any balanced family of coalitions $\mathcal{B}$ with balancing weights given by $\lambda_{\mathcal{B}}$. Moreover, the unique point in C is the nucleolus[1] $\hat{x}$ of (N, v). The nucleolus satisfies that $\max_S e(S, \hat{x}) = 0$. Moreover, from the characterization of the nucleolus given by [8], we have that $\mathcal{B}^* = \{S : e(S, \hat{x}) = 0\}$ is a balanced family of coalitions. For any other coalition S not in $\mathcal{B}^*, e(S, \hat{x}) < 0$. We claim that $\mathcal{B}^*$ is a determining family in (N, v). If rank of $M_{\mathcal{B}^*} < n$, there is $y \neq 0$ such that $M_{\mathcal{B}^*} y = 0$. Let $x(\varepsilon) = \hat{x} + \varepsilon y$. Then $M_{\mathcal{B}^*} x(\varepsilon) = v_{\mathcal{B}^*}$, so $e(S, x(\varepsilon)) = 0$ for all $S \in \mathcal{B}^*$. Furthermore, if ε is small enough, $e(S, x(\varepsilon)) < 0$ for all $S \notin \mathcal{B}^*$. Moreover, since for any collection $\lambda_{\mathcal{B}^*}$ of balancing weights for $\mathcal{B}^*$, it holds that

$$\lambda_{\mathcal{B}^*} M_{\mathcal{B}^*} y = \chi_N y = 0,$$

we have that $x(\varepsilon)$ is a pre-imputation, different from $\hat{x}$, belonging to C. But this contradicts the assumed cardinality for C. Then, $rank(M_{\mathcal{B}^*}) = n$. Finally, since

$$\begin{aligned} w(\mathcal{B}^*, \lambda_{\mathcal{B}^*}, v) &= \sum_{S \in \mathcal{B}^*} \lambda_S v(S) \\ &= \sum_{S \in \mathcal{B}^*} \lambda_S \hat{x}(S) = v(N), \end{aligned}$$

we conclude that $\mathcal{B}^*$ is a determining family of coalitions.■

Corollary 2 A sufficient condition for the core of a game (N, v) to have only one imputation is that there is a minimal balanced family $\mathcal{B}$ with $|\mathcal{B}| = n$ and $w(\mathcal{B}, \lambda_{\mathcal{B}}) = v(N)$.

The latter condition is not, however, a necessary condition as the following example shows.

Example 1 Let (N, v) a game with $N = \{1, 2, 3\}$ and $v(N) = 1, v(\{1, 3\}) = v(\{2, 3\} = 1, v(\{1, 2\}) = -1$, and $v(S) = 0$ otherwise. The only core imputation x in this game is $x = (0, 0, 1)$, and the only determining families are:

$\mathcal{B}_1 = \{\{1\}, \{2\}, \{1, 3\}, \{2, 3\}\}$ and $\mathcal{B}_2 = \{\{1\}, \{2\}, \{1, 3\}, \{2, 3\}, N\}$, none of them being a minimal balanced family of coalitions.

4. General case

Theorem 1 characterizes the class of balanced games with 0-dimensional core [2]. In order to bring a condition for the general case, we generalize the definition of determining family given in the previous section as follows.

Definition 2 Let a game (N, v) be given. A balanced family $\mathcal{B}$ is k-determining for the game (N, v) if it satisfies:

[1] To define the nucleolus of a game (N, v), we have to associate first, to each pre-imputation x, the 2^n-vector $\theta(x)$ whose entries are the quantities $(e(S, x))_{S \subseteq N}$ ordered in a non increasing order. $\theta(x)$ is the vector of excesses associated to x. The nucleolus of (N, v) is the pre-imputation $\hat{x}$ such that $\theta(\hat{x})$ is minimal in A with respect to the lexicographical order $\preceq_L$. $\theta(x) \prec_L \theta(y)$ if and only if there exists $1 \leq k < 2^n - 1$ such that $\theta_i(x) = \theta_i(y)$ for all $i \leq k$, and $\theta_{k+1}(x) < \theta_{k+1}(y)$. $\theta(x) =_L \theta(y)$ if and only if $\theta_i(x) = \theta_i(y)$ for all $1 \leq i \leq 2^n - 1$.

[2] We recall that the (geometric) dimension of a convex set is the smallest dimension of an affine subspace containing it.

$i)$ $rank(M_{\mathcal{B}}) = n - k$.

$ii)$ There is a collection of balancing weights $\lambda_{\mathcal{B}}$ for $\mathcal{B}$ such that $w(\mathcal{B}, \lambda_{\mathcal{B}}, v) \geq w(\mathcal{B}', \lambda_{\mathcal{B}'}, v)$ for any other balanced family $\mathcal{B}'$ with balancing weights given by $\lambda_{\mathcal{B}'}$.

$iii)$ $rank(M_{\mathcal{B}}) \geq rank(M_{\mathcal{B}'})$ for any other balanced family $\mathcal{B}'$ satisfying $w(\mathcal{B}', \lambda_{\mathcal{B}'}, v) = w(\mathcal{B}, \lambda_{\mathcal{B}}, v)$ for some set $\lambda_{\mathcal{B}'}$ of balancing weights.

Clearly, any determining family is a 0-determining family.

Theorem 3 Let a TU-game (N, v) be given. Then, the core of the game has dimension $k, 0 \leq k \leq n-1$, if and only if there is a k-determining family $\mathcal{B}$ with $w(\mathcal{B}, \lambda_{\mathcal{B}}, v) = v(N)$ for some collection $\lambda_{\mathcal{B}}$ of balancing weights.

Proof Theorem 1 proves the case $k = 0$. To complete the proof for the remaining values $0 < k \leq n-1$, we are going to use an inductive argument.

Let us assume that the claim has been proven for all $s < k \leq n-1$, and that there exists a k-determining family $\mathcal{B}$ satisfying $w(\mathcal{B}, \lambda_{\mathcal{B}}, v) = v(N)$ for some collection $\lambda_{\mathcal{B}}$ of balancing weights. Once more, because of $ii)$ of Definition 2, and the Shapley-Bondareva theorem, we get that $C \neq \phi$. We note that any core imputation $x \in C$ should satisfy $v(S) = x(S)$ for all $S \in \mathcal{B}$. Namely, it should be a solution of the linear system $M_{\mathcal{B}} y = v_{\mathcal{B}}$. But, since $rank(M_{\mathcal{B}}) = n - k$, the dimension of C is bounded by k. In the case that this dimension were $0 \leq s < k$, there would exist, because of the induction hypothesis, an s-determining family $\mathcal{B}'$ satisfying $w(\mathcal{B}', \lambda_{\mathcal{B}'}, v) = v(N) = w(\mathcal{B}, \lambda_{\mathcal{B}}, v)$ for some set $\lambda_{\mathcal{B}'}$ of balancing weights. However, since $rank(M_{\mathcal{B}'}) = n - s > n - k = rank(M_{\mathcal{B}})$, we get a contradiction with condition $iii)$ of Definition 2. This contradiction shows that the dimension of C should be k.

Conversely, let us assume now that C has dimension k. Then the game is balanced and $w(\mathcal{B}, \lambda_{\mathcal{B}}, v) \leq v(N)$ for any balanced family of coalitions $\mathcal{B}$ with balancing weights given by $\lambda_{\mathcal{B}}$. Like in the proof of Theorem 1, let $\hat{x}$ be the nucleolus of (N, v), and $\mathcal{B}^* = \{S : e(S, \hat{x}) = 0\}$, which is a balanced family of coalitions ([8]). For any coalition S outside of $\mathcal{B}^*, e(S, \hat{x}) < 0$. Let $n - \hat{k} = rank(M_{\mathcal{B}^*})$. We claim that $\mathcal{B}^*$ is a k-determining family indeed. In fact, since $w(\mathcal{B}^*, \lambda_{\mathcal{B}^*}, v) = v(N)$, for any s-determining family $\mathcal{B}$, it holds that $rank(M_{\mathcal{B}}) = n - s \geq n - \hat{k} = rank(M_{\mathcal{B}^*})$, and consequently, $s \leq \hat{k}$. On the other hand, $s \geq k$, or the core would have a dimension lesser than k, according to the induction hypothesis. Thus, $k \leq s \leq \hat{k}$. If $k < \hat{k}, rank(M_{\mathcal{B}^*}) < n - k$ and the dimension of $Kernel(M_{\mathcal{B}^*}) > k$. Let $\mathbf{N}(\varepsilon) = \{y \in Kernel(M_{\mathcal{B}^*}) : \|y\|_2 \leq \varepsilon\}$. For each $y \in \mathbf{N}(\varepsilon)$, let us consider the associated point $x(y) = \hat{x} + \varepsilon y$. Then $M_{\mathcal{B}^*} x(y) = v_{\mathcal{B}^*}$, so $e(S, x(y)) = 0$ for all $S \in \mathcal{B}^*$. Furthermore, if ε is small enough, $e(S, x(y)) < 0$ for all $S \notin \mathcal{B}^*$. Finally, because for any $y \in Kernel(M_{\mathcal{B}^*})$ it holds that $\chi_N y = 0$ (cfr. Section 3), it turns out that $x(y)$ is a pre-imputation belonging to C. Since a universal ε value can be found guaranteeing that $e(S, x(y)) < 0$ for all $S \notin \mathcal{B}^*$, $y \in \mathbf{N}(\varepsilon)$, we obtain that $\hat{x} + \mathbf{N}(\varepsilon)$, which is an affine manifold of dimension $\hat{k} > k$, is included in the core. But this is a contradiction with the assumed k value for the core dimension. This proves that $k = \hat{k} = s$ and thus, $\mathcal{B}^*$ is a k-determining family.■

Remark 1 The extreme case $k = n - 1$ also follows from Theorem 2 of [4], where he proves that a game (N, v) has a $(n-1)$-dimensional core if and only if the only balanced family of coalitions with maximal worth is $\mathcal{B} = \{N\}$, which in this case is a $(n-1)$-determining family. This condition implies that the non-equality of the first two components of the vector of excesses associated to the nucleolus is a necessary and sufficient condition for the core of a balanced game to have non-empty interior (relative to the set of pre-imputations). On the other hand, since $\{N\}$ is the only family which could be $(n-1)$-determining, Zhao's result turns to be equivalent to our statement in Theorem 2 for this particular case.

For $k = n-1$, Zhao's condition indicates that the information about the geometric dimension of the core is contained in the vector of excesses of the nucleolus of the game. Within the proof of Theorem 3 the properties of the family $\mathcal{B}^* = \{S : e(S, \hat{x}) = 0\}$ have played a key role suggesting that that relationship could be true for all the remaining cases. The next result highlights this fact.

Corollary 4 The geometric dimension of the core of a balanced TU game (N, v) is $k, 0 \leq k \leq n-1$, if and only if the $rank(M_{\mathcal{B}^*}) = n - k$.

Proof The last part of the proof of Theorem 3 includes the proof that if the core dimension is k, for some $0 \leq k \leq n-1$, then $rank(M_{\mathcal{B}^*}) = n - k$.

But, if $rank(M_{\mathcal{B}^*}) = n - k$, then, the geometric dimension of C cannot be greater than k. Moreover, if it were $k' < k$, a similar argument than that used during the proof of Theorem 3 would show that an affine manifold of dimension k is included in the core, providing a contradiction. Thus, the core dimension should be k.■

5. A geometric interpretation

Given a minimal balanced family of coalitions $\mathcal{B}$, let $\Lambda_{\mathcal{B}} = (\lambda_S)_{\phi \neq S \subseteq N}$ be the $(2^n - 1)$-vector where λ_S is the balancing weight of S if $S \in \mathcal{B}$ and $\lambda_S = 0$ if $S \notin \mathcal{B}$. Let $\delta_{\mathcal{B}} = \Lambda_{\mathcal{B}} - \lambda_{\{N\}}$. Thus, $\delta_{\mathcal{B}}$ differs from $\lambda_{\mathcal{B}}$ only in the entry indexed by N, taking the value $\delta_N = -1$ whenever $\mathcal{B} \neq \{N\}$ and $\delta_N = 0$ if $\mathcal{B} = \{N\}$. Let us denote with $\mathbb{V}^n$ the set of balanced TU-games with n-players, and with $\mathbb{B}$ the cone generated by the family $\{\delta_{\mathcal{B}} : \mathcal{B} \neq \{N\}$ is a minimal balanced family$\}$. Then, $\mathbb{V}^n = \mathbb{B}^*$, where $\mathbb{B}^* = \{y \in \mathbb{R}^{2^n-1} : \langle y, \delta \rangle \leq 0$ for all $\delta \in \mathbb{B}\}$ is the polar cone of $\mathbb{B}$. It is well-known that each generating vector $\delta_{\mathcal{B}}$ of $\mathbb{B}$ determines a $(2^n - 2)$-dimensional face of $\mathbb{V}^n$ ([2]). Theorem 3 allows us to characterize, in terms of the dimension of the core, all the members in the relative interior of each of these faces as well as in the relative interior of any other face of $\mathbb{V}^n$.

Theorem 5 Let N be given and let $\mathcal{B} \neq \{N\}$ be a minimal balanced family. Then, the dimension of the core of any game in the relative interior of the face of $\mathbb{V}^n$ determined by $\delta_{\mathcal{B}}$ is $n - rank(M_{\mathcal{B}})$.

Proof We recall that for any v in the relative interior of the face of $\mathbb{V}^n$ determined by $\delta_{\mathcal{B}}$, $\langle v, \delta_{\mathcal{B}} \rangle = 0$ and $\langle v, \delta_{\mathcal{B}'} \rangle < 0$ for any other $\mathcal{B}' \neq \mathcal{B}$. Thus, $\mathcal{B}$ is the only family different from $\{N\}$ with maximal worth. Therefore, it is a k-determining family for v, and according to Theorem 3, the dimension of its core is $k = n - rank(M_{\mathcal{B}})$. ■

Remark 2 A similar result to that stated in Theorem 5 is also true when the face of $\mathbb{V}^n$ considered is $\mathbb{V}^n$ itself. In this case, all the games in the relative interior of $\mathbb{V}^n$ have $(n-1)$ dimension core, as follows from Corollary 4, and the fact that, for all the games in this set, $\mathcal{B}^* = \{N\}$.

The next result enlighten the fact that all the games in the relative interior of any face of $\mathbb{V}^n$, and not only for those faces determined by minimal balanced families of coalitions, have core with the same geometric dimension.

Proposition 6 Let $\mathbb{W}$ be a lower dimension face of $\mathbb{V}^n$. Then, there exists $0 \leq k \leq n-1$ such that, for any game v in the relative interior of $\mathbb{W}$, the dimension of the core of v is k.

Proof Given $\mathbb{W}$ there exists a finite collection $\mathcal{B}^1, .., \mathcal{B}^s$ of minimal balanced families such that, for any v in the relative interior of $\mathbb{W}$, $\langle v, \delta_{\mathcal{B}^i}\rangle = 0$ for all $i = 1, ..s$, and $\langle v, \delta_{\mathcal{B}'}\rangle < 0$ for any other minimal balanced family $\mathcal{B}' \neq \mathcal{B}^i, i = 1, ..., s$. Let $\mathcal{B} = \bigcup_{i=1}^{s} \mathcal{B}^i$, which is also a balanced family. Since any set of balancing weights for $\mathcal{B}$ is a convex combination of the sets of balancing weights of the minimal balanced families $\mathcal{B}^i, i = 1, ..., s$, it follows that $\langle v, \delta_{\mathcal{B}^i}\rangle = 0$ for all v in the relative interior of $\mathbb{W}$. Consequently, $v(N) = w(\mathcal{B}, \lambda_{\mathcal{B}}, v) \geq w(\mathcal{B}', \lambda_{\mathcal{B}'}, v)$ for any other balanced family $\mathcal{B}'$ with set of balancing weights given by $\lambda_{\mathcal{B}'}$. On the other hand, if $\mathcal{B}'$ is any balanced family with $w(\mathcal{B}', \lambda_{\mathcal{B}'}, v) = w(\mathcal{B}, \lambda_{\mathcal{B}}, v)$, then $\mathcal{B}' = \bigcup_{i\in S'} \mathcal{B}^i$, where $S' \subseteq \{1, ..., s\}$. Therefore, $rank(M_{\mathcal{B}}) \geq rank(M_{\mathcal{B}'})$ and thus, if $rank(M_{\mathcal{B}}) = n-k$, $\mathcal{B}$ turns to be a k-determining family for any game v in the relative interior of $\mathbb{W}$. But this implies that all the games in this set have a core with the same dimension k.■

Example 1 Three person-games

This case can be analyzed exhaustively. We recall that the only minimal balanced families in this case are, apart from $\{N\}, \mathcal{B}^0 = \{\{1\}, \{2\}, \{3\}\}, \mathcal{B}^1 = \{\{1\}, \{2, 3\}\}, \mathcal{B}^2 = \{\{2\}, \{1, 3\}\}, \mathcal{B}^3 = \{\{3\}, \{1, 2\}\}$ and $\mathcal{B}^4 = \{\{12\}, \{1, 3\}, \{2, 3\}\}$. According to Remark 2, a game is in the relative interior of the cone of balanced 3-person games $\mathbb{V}^3$ if and only if its core has dimension two. On the other extreme, the only minimal-dimensional face of $\mathbb{V}^3$ is the 3-dimensional face determined by all the minimal balanced families together (or its corresponding $\delta' s$ vectors) and it embraces all essential games ($v(S) = \sum_{i\in S} v(\{i\})$ for all $\phi \neq S$) having 0-dimensional core. The cone $\mathbb{V}^3$ has five 6-dimensional faces determined by the minimal balanced families. The relative interior of the face corresponding to $\mathcal{B}^0$ is the set of all balanced games for which $v(\{1\})+v(\{2\})+v(\{3\}) = v(\{1, 2, 3\})$ (games for which $|\mathbf{A}| = 1$). Therefore, all these games also have a 0-dimensional core, although this is not the only 6-dimensional face sharing this property. In fact, all the games in the relative interior of the face corresponding to $\mathcal{B}^4$ have a one-point core too, and all of them are superadditive balanced games. In these two cases, all the faces, and not only their relative interiors, are composed for games with a one-point core. This core dimensional coincidence between the games in the relative interior of the faces associated to $\mathcal{B}^0$ and $\mathcal{B}^4$ follows from the general fact that a family $\mathcal{B}^C$, whose members are the complements of the members of a minimal balanced family $\mathcal{B}$, is also a minimal balanced family, and hence, $rank(M_{\mathcal{B}}) = rank(M_{\mathcal{B}^C})$. Consequently, balanced games in the relative interior of the faces associated to $\mathcal{B}$ and $\mathcal{B}^C$ have core with the same dimension.

The face corresponding to $\mathcal{B}^1$ (a behavior shared with $\mathcal{B}^2$ and $\mathcal{B}^3$) includes superadditive and non-superadditive games all of them having 1-dimensional core. However, the superadditivity condition can be violated only by coalitions $\{1, 2\}$ or $\{1, 3\}$ but never by coalition $\{2, 3\}$.

There are also nine 5-dimensional faces in $\mathbb{V}^3$, all of them embracing games with one-point core. The games in the relative interior of the faces associated to $\mathcal{B}^1 \cup \mathcal{B}^2, \mathcal{B}^1 \cup \mathcal{B}^3$ and $\mathcal{B}^2 \cup \mathcal{B}^3$ are not superadditive games, being $\{1, 2\}, \{1, 3\}$ and $\{2, 3\}$ the only coalitions violating the superadditivity condition in each case respectively. The games in the relative interior of the 5-dimensional face determined by $\mathcal{B}^1 \cup \mathcal{B}^4$ are superadditive games such that, apart from the constrain $v(\{2, 3\}) + v(\{1\}) = v(N)$ imposed by $\mathcal{B}^1$, they also satisfy the convexity constraint given by $v(\{1, 2, 3\}) = v(\{1, 2\}) + v(\{1, 3\}) - v(\{1\})$. The relative interior of the faces determined by $\mathcal{B}^2 \cup \mathcal{B}^4$ and $\mathcal{B}^3 \cup \mathcal{B}^4$ admit a similar description to that given for $\mathcal{B}^1 \cup \mathcal{B}^4$. The games in the relative interior of the face associated to $\mathcal{B}^0 \cup \mathcal{B}^1$ (and similarly those prescribed by $\mathcal{B}^0 \cup \mathcal{B}^2$ and $\mathcal{B}^0 \cup \mathcal{B}^3$) are balanced games satisfying the following two condition $v(\{1\}) + v(\{2\}) + v(\{3\}) = v(\{1,2,3\})$ and $v(\{2\}) + v(\{3\}) = v(\{2,3\})$ as well. To complete the description, we mention that the cone has three 4-dimensional faces. The games in the relative interior of the face determined by $\mathcal{B}^0 \cup \mathcal{B}^2 \cup \mathcal{B}^3$ are games with one-point set of imputation satisfying the following convexity constraint: $v(\{1,2,3\}) = v(\{1,2\}) + v(\{1,3\}) - v(\{1\})$ like the games in the face determined by $\mathcal{B}^1 \cup \mathcal{B}^4$. The other two 4-dimensional faces, namely, those determined by $\mathcal{B}^0 \cup \mathcal{B}^1 \cup \mathcal{B}^2$ and $\mathcal{B}^0 \cup \mathcal{B}^1 \cup \mathcal{B}^3$ admit a similar description to that given for the face determined by $\mathcal{B}^0 \cup \mathcal{B}^2 \cup \mathcal{B}^3$.

6. Biform games

In this section we focus on biform games, as studied by [3]. A biform game is a two-stage model. In the first stage, each one of the n players of the game selects a (non-cooperative) strategy determining a profile of strategies. Depending on this profile of strategies, a cooperative TU-game is prescribed to be played in the second stage of the game. Then, a payoff to each of the players is assigned, payoff which is strongly dependent on and related to the solution of the cooperative game played. Biform games are very flexible models capable to deal with a broad range of situations. However, in order to get positive mathematical results, some structure has to be assumed on the family of games to be played in the second stage of the game. One of the main results of [3] provides a necessary and sufficient condition for a profile of strategies be a Nash equilibrium in a biform game satisfying the Adding Up property, the No Externality condition and the balancedness property of the cooperative game associated to each profile of strategies. This latter condition, along with the Adding Up property, locates all the cooperative games associated with the profile of strategies in the same face of the cone of the balanced cooperative TU-games. Moreover, the Adding Up property guarantees that all the games in that face have core with just one element. Our purpose here is illustrate, with a couple of examples, what happens if the biform game prescribes that all the TU-cooperative games belong to the same face, but different from that specified by the Adding Up property, in the cone $\mathbb{V}^n$ of balanced cooperative games. In particular, we are going to consider the case in which all the cooperative games associated with the profile of strategies have one-dimensional core. We are going to work with a simpler version of n-biform games than the original of [3]. To this end let us consider n non-empty finite sets $S^1, ..., S^n$, the sets of individual non-cooperative strategies. Then, a n-biform game is a pair (S, V) where $S = (S^1, ..., S^n)$ is the set of profiles of strategies and $V : S \to \mathbb{V}^n$ is just a map. As usual, given a profile of strategies $s \in S$, a player i, and a strategy $r^i \in S^i$,(r^i, s^{-i}) denotes the profile of strategies obtained from s by changing the strategy s^i of the i-th player in s by the strategy r^i. Let $P : S \to \mathbb{R}^n$ be a function which, for each profile of strategies $s \in S$, selects an element $P(s) = (P_1(s), ..., P_n(s))$ in the core of the cooperative game $V(s)$. A profile of strategies $s \in S$ is a Nash equilibrium for the biform game (S, V) related to the procedure P (a P-Nash equilibrium) if and only if for each $i = 1, ..., n$, and for each $r^i \in S^i, P_i(s) \geq P_i(r^i, s^{-i})$. A biform game (S, V) satisfies the *No Externality* (NE) condition if for each $i = 1, .., n, s \in S$ and $r^i \in S^i, V(r^i, s^{-i})(N \backslash \{i\}) \geq V(s)(N \backslash \{i\})$. Given a biform game (N, V) satisfying AU and NE, a profile of strategies s is a Nash equilibrium if and only if $V(s)(N) \geq V(r^i, s^{-i})(N)$ for every $r^i \in S^i$ (Lemma 5.2 of [3]).

We point out that our definition of biform games implies that all the games $V(s), s \in S$, have non-empty core. Moreover, when AU is present, all the games involved in the biform game have one-point core. In this case, the procedure P is univocally defined as that assigning to each player, for each $s \in S$, the utility obtained for each player in the unique point of the core of $V(s)$.

Example 2

In this example, we consider a coordination game like Example 5.2 studied by [3]. This is a three person game in which each player has two non-cooperative strategies Yes ($\boldsymbol{Y}$) and No ($\boldsymbol{N}$). There are also two 2×2 matrices whose entries are 3-person TU-games. Then, player 1 chooses the row, player 2 chooses the column, and player 3 chooses the matrix. The figure below depicts the situation.

	$\boldsymbol{N}$	$\boldsymbol{Y}$
$\boldsymbol{N}$	$V(\boldsymbol{N},\boldsymbol{N},\boldsymbol{N})$	$V(\boldsymbol{N},\boldsymbol{Y},\boldsymbol{N})$
$\boldsymbol{Y}$	$V(\boldsymbol{Y},\boldsymbol{N},\boldsymbol{N})$	$V(\boldsymbol{Y},\boldsymbol{Y},\boldsymbol{N})$

$\boldsymbol{N}$

	$\boldsymbol{N}$	$\boldsymbol{Y}$
$\boldsymbol{N}$	$V(\boldsymbol{N},\boldsymbol{N},\boldsymbol{Y})$	$V(\boldsymbol{N},\boldsymbol{Y},\boldsymbol{Y})$
$\boldsymbol{Y}$	$V(\boldsymbol{Y},\boldsymbol{N},\boldsymbol{Y})$	$V(\boldsymbol{Y},\boldsymbol{Y},\boldsymbol{Y})$

$\boldsymbol{Y}$

We say that the biform game (N, V) satisfies condition $\mathcal{B}^3$ if for each $s \in S, V(s)$ belongs to the relative interior of the face of $\mathbb{V}^3$ determined by $\mathcal{B}^3 = \{\{3\},\{1,2\}\}$. For simplicity, we will also assume that the cooperative game associated to each profile of strategies $s \in S$ is superadditive. Moreover, we will assume that both marginal contributions $V(s)(\{2,3\}) - V(s)(\{3\})$ and $V(s)(\{2,3\}) - V(s)(\{3\})\}$ are positive. Then, for each profile of strategies $s \in S$, the core of the game $V(s)$ is given by

$$\begin{aligned} C(s) &= \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_3 = V(s)(\{3\}), \\ x_1 + x_2 &= V(s)(\{1,2\}), \\ x_1 &\geq \max\{V(s)(\{1,3\}) - V(s)(\{3\}), V(s)(\{1\})\}, \\ x_2 &\geq \max\{V(s)(\{2,3\}) - V(s)(\{3\})\}, V(s)(\{2\})\}\}, \end{aligned}$$

with at least one of the two last inequalities being strict. We will also assume that

$$\max\{V(s)(\{1,3\}) - V(s)(\{3\}), V(s)(\{1\})\} = V(s)(\{1,3\}) - V(s)(\{3\}),$$

and that

$$\max\{V(s)(\{2,3\}) - V(s)(\{3\}), V(s)(\{2\})\} = V(s)(\{2,3\}) - V(s)(\{3\}).$$

From

$$\begin{aligned} 2(V(s)(\{1,2\}) + V(s)(\{3\}) &= 2V(s)(N) \\ &> V(s)(\{1,3\}) + V(s)(\{2,3\}) + V(s)(\{1,2\}), \end{aligned}$$

it follows that

$$\begin{aligned} V(s)(\{1,2\}) &> V(s)(\{1,3\}) + V(s)(\{2,3\}) - 2V(s)(\{3\}) \\ &> 0. \end{aligned}$$

Let

$$\lambda(s) = \frac{V(s)(\{1,2\})}{V(s)(\{1,3\}) + V(s)(\{2,3\}) - 2V(s)(\{3\})}.$$

We then define the procedure P as

$$P(s) = (\lambda(s)(V(s)(\{1,3\}) - V(s)(\{1\})), \lambda(s)(V(s)(\{2,3\}) - V(s)(\{1\})), V(s)(\{3\})), \quad (1)$$

for all $s \in S$. The procedure P, for players 1 and 2, splits the value $V(s)(\{1,2\})$ proportional to the marginal value of these players when they join to player 3.

A particular case, when for each $s \in S, V(s)$ is a $0-1$ normalized game ($V(s)(\{i\}) = 0$ for all $i, V(s)(N) = 1$), reveals clearly the fact that, somehow, when condition $\mathcal{B}^3$ is present, coalitions $\{1,2\}$ and $\{3\}$ play the game without too much interaction.

In fact, when the procedure P is given by (1), $s = (s_1, s_2, s_3) \in S$ is a Nash equilibrium for (N, V) if and only if (s_1, s_2) is an equilibrium point for the zero sum two person game having the function $A(\hat{s}_1, \hat{s}_2) = -\frac{V(s_1,\hat{s}_2,\hat{s}_3)(1,2)}{V(s_1,\hat{s}_2,\hat{s}_3)(1,3)}$ as the payoff function for player 1.

To see this claim, we first note that, under the conditions of the theorem, for all $s \in S, V(s)(1,2) = V(s)(N) = 1$. It is also easy to see that, for each pair of profile of strategies $s, \hat{s}$, $P_1(s) \geq P_1(\hat{s})$ if and only if

$$-V(\hat{s})(1,3)V(s)(2,3) + V(\hat{s})(2,3)V(s)(1,3) \geq 0,$$

and similarly, $P_2(s) \geq P_2(\hat{s})$ if and only if

$$-V(\hat{s})(1,3)V(s)(2,3) + V(\hat{s})(2,3)V(s)(1,3) \leq 0.$$

Now, let us assume that $s = (s_1, s_2, s_3)$ is a Nash equilibrium for (N, V). Then, for all $\hat{s}_1$ we have that $P_1(s) \geq P_1(\hat{s}_1, s^{-1})$ and so,

$$\frac{V(s)(1,3)}{V(s)(2,3)} \geq \frac{V(\hat{s}_1, s^{-1})(1,3)}{V(\hat{s}_1, s^{-1})(2,3)}. \quad (2)$$

Similarly, from the fact that $P_2(s) \geq P_2(s_2, s^{-2})$ we get that

$$\frac{V(s)(1,3)}{V(s)(2,3)} \leq \frac{V(\hat{s}_2, s^{-2})(1,3)}{V(\hat{s}_2, s^{-2})(2,3)}. \quad (3)$$

But both inequalities, (2) and (3), imply that (s_1, s_2) is an equilibrium point for the zero sum game played by players 1 and 2 in which the former has the payoff function $A(s_1, s_2) = -\frac{V(s)(1,3)}{V(s)(2,3)}$.

Conversely, suppose that for some profile of strategies $s = (s_1, s_2, s_3) \in S$, the pair of strategies (s_1, s_2) is also an equilibrium point for the zero sum game already described. Then, both inequalities (2) and (3) hold. From them, we get that $P_1(s) \geq P(\hat{s}_1, s^{-1})$ for all $\hat{s}_1$ and $P_2(s) \geq P_2(\hat{s}_2, s^{-2})$ for all $\hat{s}_2$. On the other hand, since for any profile of strategies $s, P_3(s) = 0$, we also have that $P_3(s) \geq P_3(\hat{s}_3, s^{-3})$ for all $\hat{s}_3$. So, $s = (s_1, s_2, s_3)$ is a Nash equilibrium for the game (N, V).

Example 3 In this example, we consider a four dimensional version of the coordination game studied in Example 3. There are four players, each one with two non-cooperative strategies $\boldsymbol{Y}$ and $\boldsymbol{N}$. Then, for each $i = 1, 2, 3, 4$, $S^i = \{\boldsymbol{Y}, \boldsymbol{N}\}$ and the set of joint non-cooperative strategies is $S = S^1 \times S^2 \times S^3 \times S^4$. Given a strategy $s \in S$, a four-person TU-game $V(s)$ is prescribed to be played. The purpose of this example is to analyze a biform game for which all the games to be played in the second stage have one-dimensional core. Furthermore, like we did in Example 2, we will also assume that all of the cooperative games have a common structure.

According to the classification given by Shapley [2] of the minimal balanced families of coalitions in a four-person game, only $\mathcal{B}^3 = \{\{1,2\},\{3\},\{4\}\}$ and $\mathcal{B}^4 = \{\{1,2,3\},\{1,2,4\},\{3,4\}\}$ have $rank(M_{\mathcal{B}^3}) = rank(M_{\mathcal{B}^4}) = 3$. We will analyze here the case related to $\mathcal{B}^3$ to show that it behaves quite in the same way as Example 2 does. We say that a four-person biform game (N,V) satisfies condition $\mathcal{B}^3$ if for each $s \in S, V(s)$ belongs to the relative interior of the face of $\mathbb{V}^4$ determined by $\mathcal{B}^3$. Thus, in each case, and for any $s \in S, V(s)$ has one-dimensional core. We will also assume that each game $V(s)$ is in $0-1$ normalization.

Given a four-person biform game (N,V) satisfying condition $\mathcal{B}^3$, for each profile of strategies $s \in S$, the core of the game $V(s)$ is given by

$$\begin{aligned} C(s) &= \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^3 : x_3 = 0, x_4 = 0), \\ x_1 + x_2 &= V(s)(\{1,2\}) = 1, \\ x_1 &\geq \max\{V(s)(\{1,3\}), V(s)(\{1,4\}), V(s)(\{1,3,4\}), 0\}, \\ x_2 &\geq \max\{V(s)(\{2,3\}), V(s)(\{2,4\}), V(s)(\{2,3,4\}), 0\}. \end{aligned}$$

We will also assume that

$$\begin{aligned} \max\{V(s)(\{1,3\}), V(s)(\{1,4\}), V(s)(\{1,3,4\}), 0\} &= V(s)(\{1,3\}) \\ &> 0, \end{aligned}$$

and

$$\begin{aligned} \max\{V(s)(\{1,3\}), V(s)(\{1,4\}), V(s)(\{1,3,4\}), 0\} &= V(s)(\{2,3\}) \\ &> 0. \end{aligned}$$

Since for each $s \in S$,

$$\frac{1}{2}(V(s)(\{1,2\}) + V(s)(\{1,3\}) + V(s)(\{2,3\})) + V(s)(\{4\}) < 1,$$

we conclude that always

$$0 < V(s)(\{1,3\}) + V(s)(\{2,3\}) < V(s)(\{1,2\}).$$

Then, if

$$\lambda(s) = \frac{V(s)(\{1,2\})}{V(s)(\{1,3\}) + V(s)(\{2,3\})},$$

we define the procedure P as

$$P(s) = (\lambda(s)V(s)(\{1,3\}), \lambda(s)V(s)(\{2,3\}), 0, 0).$$

This procedure is very similar to that defined by (1) in Example 3, and with similar arguments to those used there, we can prove that $s = (s_1,s_2,s_3,s_4) \in S$ is a Nash equilibrium for the game (N,V) if and only if (s_1,s_2) is an equilibrium point for the zero sum two person game having the function $A(\hat{s}_1,\hat{s}_2) = -\frac{V(s_1,\hat{s}_2,\hat{s}_3,\hat{s}_4)(1,2)}{V(s_1,\hat{s}_2,\hat{s}_3,\hat{s}_4)(1,3)}$ as the payoff function for player 1.

7. Conclusions

In this paper we have given a geometric description of the cone of n-person balanced TU-games. We think that this geometric approach opens the possibility to study several game theoretic problems from a different point of view. We mention a couple of examples to illustrate this point. Like we did in Section 6, a better understanding of the set of balanced TU-games can be useful to endow biform games with a definite structure and to study, for instance, some stability properties of the solutions. On the other hand, when dealing with a game with empty core, a standard procedure is to modify the characteristic function of the game somehow to obtain a new game with non-empty core. The family of strong ε-cores ([9]) is, perhaps, the most relevant example of this approach to study non-balanced TU-games. The aspiration core ([10], [11]) is another key example of this methodology. The geometric characterization we present here could help to get a better knowledge about how these procedures "project" non-balanced games on the cone of balanced games. It could also help to define new more appropriate procedures.

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References

1. Bondareva, O.N. Some applications of linear programming methods to the theory of cooperative games. *Problemi Kibernitiki* **1963**, *10*, 119-139.
2. Shapley, L.S. On balanced sets and core. *Naval Research Logistic Quarterly* **1967**, *14*, 453-460.
3. Brandenburger, A.; Stuart, F. Biform games. *Management Science* **2007**, *53*, 537-549.
4. Zhao, J. The relative interior of base polyhedron and the core. *Economic Theory* **2001**, *18*, 635-648.
5. Schmeidler, D. The nucleolus of a characteristic function game. *SIAM Journal on Applied Mathematics* **1969**, *17*, 1163-1170.
6. Dragan, I.; Potter, J.; Tijs, S. Superadditivity for solutions of coalitional games. *Libertas Mathematica* **1989**, *9*, 101-110.
7. Moulin, H. Cooperative Microeconomics: A Game-Theoretic Introduction. Princeton University Press **1995**.
8. Kohlberg, E. On the nucleolus of a characteristic function game. *SIAM Journal on Applied Mathematics* **1971**, *20*, 62-67.
9. Shapley, L.; Shubik, M. Quasi-cores in a monetary economy with non-convex preferences. *Econometrica* **1966**, *34*, 805-827.
10. Bennett, E. The aspiration approach to predicting coalition formation and payoff distributions in sidepayment games. *International Journal of Game Theory* **1983**, *12*, 1-28.
11. Bejan, C.; Gomez, J. C. Generalizing the axiomatization of the core. *International Journal of Game Theory* **2012**, *DOI: 10.1007/s00182-011-0316-4*.