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Quantization of Newton's Gravity

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Abstract

In this work we will use a recently developed non relativistic (NR) quantization methodology that successfully overcomes troubles with infinities that plague non-renormalizable quantum field theories (QFTs). The ensuing methodology is here applied to Newton's gravitation potential. We employ here the concomitant mathematical apparatus to formulate the NR QFT discussed in the well known classical text-book by Fetter and Walecka. We emphasize the fact that we speak of non relativistic QFT. This is so because we appeal to Newton's gravitational potential, while in a relativistic QFT one does not employ potentials. Our main protagonist is the notion of propagator. This notion is of the essence in non relativistic quantum field theory (NR-QFT). Indeed, propagators are indispensable tools for both nuclear physics and condensed matter theory, among other disciplines. In the present work we deal with propagators for both fermions and bosons.

Keywords

Non-Relativistic Quantum Field Theory, Newton's Gravity, Schwartz' Distributions

1. Introduction

1.1. Preliminaries

In this work we will use a recently developed non relativistic quantization methodology that successfully overcomes all troubles of non-renormalizable QFT [1]. The essential result of such procedures is that we can dispense with renormalization and counter-terms. The reader can consult the recent references [1] [2] [3] [4] [5]. The proofs given there are conclusive.

The above claims are validated because infinities in Feynman diagrams, that arise in the convolution of quantum propagators (QP), disappear if one 1) represents QP by ultra-hyperfinctions (a generalization of Schwartz' distributions) and follows this technique with an appropriate Laurent expansion. The facts 1) and 2) above are clearly explained, with all kind of details, in [1] [2] [3] [4] [5]. Accordingly, no more mathematical aspects of the procedure need to be given in this paper.

The techniques of [1] [2] [3] [4] [5] are here applied to Newton's gravitation potential. We strongly emphasize the fact that, since we will be inserting a gravitational potential into a Schrödinger Equation (SE), the ensuing discussion is per force non-relativistic, and as such is the character of SE.

1.2. Organizing Our Material

In Section 2 we revisit Newton's gravity. Section 3 is devoted to an explicit display of results belonging to [6], concerning non relativistic quantum field theory (NR-QFT). In Section 4 we apply the results of Sections 2 and 3 so as to obtain the N-QFT of Newton's gravity. We discuss, as examples, the calculation of the self-energy for fermions and of the dressed propagator for both, bosons and fermions, to first order in perturbation theory. Some conclusions are drawn in Section 5.

2. Newton's Gravity

As stated above, r^{-1} is viewed here as

$$r^{-1}=\frac{1}{2}\left[(r-i0)^{-1}+(r+i0)^{-1}\right]=PV\frac{1}{r}. \tag{2.1}$$

Remember also that

$$\delta(r)=0. \tag{2.2}$$

We need now the Fourier transform of r^{-1}. We have

$$\begin{aligned}&\int r^{-1}\mathrm{e}^{ik\cdot x}\mathrm{d}^3x\\&=\lim_{\epsilon\to 0}\left[\int\limits_0^\infty\int\limits_0^{\frac{\pi}{2}}\int\limits_0^{2\pi}\mathrm{e}^{i(k+i\epsilon)r\cos\theta}r\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi+\int\limits_0^\infty\int\limits_{\frac{\pi}{2}}^{\pi}\int\limits_0^{2\pi}\mathrm{e}^{i(k-i\epsilon)r\cos\theta}r\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi\right].\end{aligned} \tag{2.3}$$

Integrating over ϕ one finds

$$\lim_{\epsilon\to 0}\left[2\pi\int\limits_0^\infty\int\limits_0^{\frac{\pi}{2}}\mathrm{e}^{i(k+i\epsilon)r\cos\theta}r\sin\theta\mathrm{d}r\mathrm{d}\theta+2\pi\int\limits_0^\infty\int\limits_{\frac{\pi}{2}}^{\pi}\mathrm{e}^{i(k-i\epsilon)r\cos\theta}r\sin\theta\mathrm{d}r\mathrm{d}\theta\right]. \tag{2.4}$$

Evaluating now for θ we reach

$$\lim_{\epsilon\to 0}2\pi\left\{\int\limits_0^\infty\left[\frac{\mathrm{e}^{i(k+i\epsilon)r}}{i(k+i\epsilon)}-\frac{\mathrm{e}^{i(k-i\epsilon)r}}{i(k-i\epsilon)}\right]\mathrm{d}r\right\}. \tag{2.5}$$

Finally, dealing with the variable r we arrive at

$$2\pi \lim_{\epsilon\to 0}\left[\frac{1}{(k+i\epsilon)^2}+\frac{1}{(k-i\epsilon)^2}\right]. \tag{2.6}$$

As an example, consider now the anti transform of $4\pi k^{-2}$ and verify that it is $PV\frac{1}{r}$.

$$2\pi\left[\frac{1}{(k+i0)^2}+\frac{1}{(k-i0)^2}\right]=4\pi PV\frac{1}{k^2}\equiv 4\pi k^{-2}. \tag{2.7}$$

One has

$$\begin{aligned}&\frac{4\pi}{(2\pi)^3}\int k^{-2}\mathrm{e}^{-ik\cdot x}\mathrm{d}^3k\\ &=\lim_{\epsilon\to 0}\left[\frac{1}{2\pi^2}\int_0^\infty\int_0^{\frac{\pi}{2}}\mathrm{e}^{-i(r-i\epsilon)k\cos\theta}\sin\theta\mathrm{d}k\mathrm{d}\theta+\frac{1}{2\pi^2}\int_0^\infty\int_{\frac{\pi}{2}}^{\pi}\mathrm{e}^{-i(r+i\epsilon)k\cos\theta}\sin\theta\mathrm{d}k\mathrm{d}\theta\right],\end{aligned} \tag{2.8}$$

so that

$$-\lim_{\epsilon\to 0}\frac{1}{\pi}\left\{\int_0^\infty\left[\frac{\mathrm{e}^{-i(r-i\epsilon)k}}{ik(r-i\epsilon)}-\frac{\mathrm{e}^{i(r+i\epsilon)k}}{ik(r+i\epsilon)}\right]\mathrm{d}k\right\}, \tag{2.9}$$

or

$$\frac{i}{\pi}PV\frac{1}{r}\int_{\infty}^{\infty}PV\frac{1}{k}\mathrm{e}^{-ikr}\mathrm{d}k=PV\frac{1}{r}, \tag{2.10}$$

where we used (see Ref. [7])

$$\int_{\infty}^{\infty}PV\frac{1}{k}\mathrm{e}^{-ikx}\mathrm{d}k=\frac{\pi}{i}Sgn(x), \tag{2.11}$$

together with $Sgn(r)=1$, where $Sgn(x)$ is the function sign of *x*.

3. Materials Needed from Fetter and Walecka's Book

3.1. Self Energies

The energy that a particle gains as the result of environment-modifications that it itself generates is called a self-energy Σ. This quantity denotes the contribution to the particle's effective mass due to interactions particle-surrounding medium (SM). Consider the particular (and common) condensed matter scenario: electrons moving in a material. Σ represents there the potential felt by a given electron due to the SM's interactions with it. Given that electrons repel each other, a moving electron does polarize the electrons in its vicinity, This, in turn, changes the potential of the moving electron fields. Such effects necessarily involve self-energy.

3.2. Fermion Dressed Propagators

The dressed propagator is defined to be the two-point function to all orders of the perturbation expansion. It changes the bare mass to the physical mass. We will use this notion here. For an accessible discussion of the concept we recommend the book [8]. In Fetter and Walecka's (FW) [6] one, this idea is comprehensively discussed for a fermion's NR QFT. In the case of free fermions, FW defined the following (current) propagator

$$iG^0_{\alpha\beta}\left(\boldsymbol{x},t;\boldsymbol{x}',t'\right)=\langle 0|T\left[\psi_\alpha\left(\boldsymbol{x},t\right)\psi^+_\beta\left(\boldsymbol{x}',t'\right)\right]|0\rangle. \tag{3.1}$$

One has

$$\begin{aligned}&iG^0_{\alpha\beta}\left(\boldsymbol{x},t;\boldsymbol{x}',t'\right)\\&=\frac{\delta_{\alpha\beta}}{(2\pi)^3}\int \mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\mathrm{e}^{-\omega_k(t-t')}\left[\Theta(t-t')\Theta(k-k_F)-\Theta(t'-t)\Theta(k_F-k)\right]\mathrm{d}^3k.\end{aligned} \tag{3.2}$$

Θ is the Heaviside's step function. We appeal now to the very well known relation

$$\Theta(t-t')=-\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{\mathrm{e}^{-i\omega(t-t')}}{\omega+i0}\mathrm{d}\omega, \tag{3.3}$$

and find

$$\begin{aligned}&iG^0_{\alpha\beta}\left(\boldsymbol{x},t;\boldsymbol{x}',t'\right)\\&=\frac{\delta_{\alpha\beta}}{(2\pi)^3}\int\int\limits_{-\infty}^{\infty}\mathrm{e}^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\mathrm{e}^{-\omega_k(t-t')}\left[\frac{\Theta(k-k_F)}{\omega-\omega_k+i0}-\frac{\Theta(k_F-k)}{\omega-\omega_k-i0}\right]\mathrm{d}^3k\mathrm{d}\omega.\end{aligned} \tag{3.4}$$

Thus, the pertinent expression in momentum space reads

$$\hat{G}^0_{F\alpha\beta}\left(\boldsymbol{k},\omega\right)=\delta_{\alpha\beta}\left[\frac{\Theta(k-k_F)}{\omega-\omega_k+i0}+\frac{\Theta(k_F-k)}{\omega-\omega_k-i0}\right], \tag{3.5}$$

with

$$\frac{1}{\omega-\omega_k\pm i0}=PV\frac{1}{\omega-\omega_k}\mp i\pi\delta(\omega-\omega_k), \tag{3.6}$$

where $k=|\boldsymbol{k}|$ and $\omega_k=\sqrt{k^2/2m}$ We already stated above that PV signifies "principal value of a function". The system's interaction's Hamiltonian is defined by a two-body V_F potential such that

$$V_F\left(\boldsymbol{x}_1-\boldsymbol{x}_2\right)=V_F\left(|\boldsymbol{x}_1-\boldsymbol{x}_2|\right)\mathbf{1}(1)\mathbf{1}(2), \tag{3.7}$$

where $\mathbf{1}$ is the unity matrix. The dressed propagator here verifies

$$\hat{G}_{F\alpha\beta}=\delta_{\alpha\beta}\hat{G}_F, \tag{3.8}$$

so that the dressed propagator becomes diagonal. Then, ($\hat{G}^0_F\left(\boldsymbol{k},\omega\right)\equiv\hat{G}^0_F(k)$)

$$\hat{G}_F(k)=\hat{G}^0_F(k)+\hat{G}^0_F(k)\Sigma_F(k)\hat{G}^0_F(k), \tag{3.9}$$

with $\Sigma_F(k)$ the self-energy. We can pass now to its perturbative expansion at first order

$$\Sigma^{(1)}_F(k)\equiv\Sigma^{(1)}(\boldsymbol{k})=\frac{n}{\hbar}\hat{V}(0)-\frac{1}{(2\pi)^3\hbar}\int\hat{V}_F\left(\boldsymbol{k}-\boldsymbol{k}'\right)\Theta\left(k_F-k'\right)\mathrm{d}^3k', \tag{3.10}$$

with $n=N/V$ and

$$\hat{V}_F(\boldsymbol{k}) = \int V_F(\boldsymbol{x}) \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \mathrm{d}^3 x. \tag{3.11}$$

Consequently (up to first order),

$$\hat{G}_F^{(1)}(k) = \hat{G}_F^0(k) + \hat{G}_F^0(k) \Sigma_F^{(1)}(k) \hat{G}_F^0(k). \tag{3.12}$$

3.3. Bosons' Dressed Propagators from FW's Book

For free bosons FW introduce the propagator in momentum space as

$$iG^0(\boldsymbol{x}, t; \boldsymbol{x}', t') = \langle 0 | T[\phi(\boldsymbol{x}, t) \phi^+(\boldsymbol{x}', t')] | 0 \rangle. \tag{3.13}$$

It reads

$$\hat{G}_B^0(k) = \frac{1}{k_0 - \omega_k + i0}, \tag{3.14}$$

with $\omega_k = \sqrt{k^2/2m}$. One has then

$$\hat{G}_B(k) = -(2\pi)^4 n_0 i\delta(k_0, k) + \hat{G}'_B(k), \tag{3.15}$$

where the primed part refers to the noncondensate ($n_0 = N_0/V$)

$$\hat{G}_B(k) = -(2\pi)^4 n_0 i\delta(k_0, k) + \hat{G}_B^0(k) + \hat{G}_B'^{(1)}(k), \tag{3.16}$$

$$\hat{G}_B'^{(1)}(k) = \frac{n_0}{h} \hat{G}_B^0(k) \left[\hat{V}_B(0) + \hat{V}_B(\boldsymbol{k}) \right] \hat{G}_B^0(k), \tag{3.17}$$

and

$$\hat{V}_B(\boldsymbol{k}) = \int V_B(\boldsymbol{x}) \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \mathrm{d}^3 x. \tag{3.18}$$

4. Non-Relativistic QFT of Newton's Gravity

4.1. Fermions

We wish to calculate $\Sigma^{(1)}$ for the potential $-\frac{Gm^2}{r}$.

$$V_F(r) = -\frac{Gm^2}{r}. \tag{4.1}$$

One has

$$\hat{V}_F(k) = \int V_F(x) \mathrm{e}^{ik\cdot x} \mathrm{d}^3 x, \tag{4.2}$$

and then

$$\hat{V}_F(k) = -\frac{4\pi Gm^2}{k^2}, \tag{4.3}$$

with

$$\hat{V}_F(0) = 0. \tag{4.4}$$

Starting here, a father lengthy manipulation leads to

$$-4\pi Gm^2 \int \frac{\Theta(k_F - k')}{|\boldsymbol{k} - \boldsymbol{k}'|^2} \mathrm{d}^3 k' = -8\pi^2 Gm^2 \frac{k_F^2 - k^2}{2k} \ln\left(\frac{k_F + k}{k_F - k}\right)^2, \tag{4.5}$$

so that the self energy reads

$$\Sigma^{(1)}(\boldsymbol{k}) = \frac{Gm^2}{\pi\hbar}\frac{k_F^2 - k^2}{2k}\ln\left(\frac{k_F + k}{k_F - k}\right)^2. \tag{4.6}$$

Accordingly, one writes for the dressed propagator

$$\hat{G}_F^{(1)}(k) = \hat{G}_F^0(k) + \frac{Gm^2}{\pi\hbar}\frac{k_F^2 - k^2}{2k}\ln\left(\frac{k_F + k}{k_F - k}\right)^2\left[\hat{G}_F^0(k)\right]^2, \tag{4.7}$$

noting that

$$\hat{G}_F^0(k) = \hat{G}_F^0(\boldsymbol{k},\omega) = \left[\frac{\Theta(k - k_F)}{\omega - \omega_k + i0} + \frac{\Theta(k_F - k)}{\omega - \omega_k - i0}\right]. \tag{4.8}$$

We recall at this stage that, in Ref. [9], it was been proved that

$$PV\frac{1}{x^n}\delta^{(m)}(x) = \frac{(-1)^n}{2}\frac{m!}{(m+n)!}\cdot\delta^{(m+n)}(x). \tag{4.9}$$

Then, using the result

$$PV\frac{1}{x^n}PV\frac{1}{x^m} = PV\frac{1}{x^{(n+m)}}, \tag{4.10}$$

we reach

$$\frac{1}{\omega - \omega_k - i0}\frac{1}{\omega - \omega_k - i0} = \frac{1}{(\omega - \omega_k - i0)^2}, \tag{4.11}$$

so that

$$\left[\hat{G}_F^0(\boldsymbol{k},\omega)\right]^2 = \left[\frac{\Theta(k - k_F)}{(\omega - \omega_k + i0)^2} + \frac{\Theta(k_F - k)}{(\omega - \omega_k - i0)^2}\right], \tag{4.12}$$

If $V \to \infty$, $k_F \to \infty$, n finite, we find

$$\int\frac{1}{|\boldsymbol{k} - \boldsymbol{k}'|^2}\mathrm{d}^3k' = 0, \tag{4.13}$$

so that

$$\Sigma_F^{(1)}(k) \simeq 0, \tag{4.14}$$

and thus

$$\hat{G}_F^{(1)}(k) \simeq \hat{G}_F^0(k). \tag{4.15}$$

4.2. Bosons' Potential $V_B(\boldsymbol{r})$

We calculate now the dressed propagator for

$$V_B(r) = -\frac{Gm^2}{r}. \tag{4.16}$$

Since

$$\hat{V}_B(k) = -\frac{4\pi Gm^2}{k^2}, \tag{4.17}$$

one has

$$\hat{V}_B(0) = 0. \tag{4.18}$$

For this result, we have used the relation of [7] concerning the regularization of integrals that depend upon a power of x. Thus, for the dressed propagator we find, up to first order

$$\hat{G}'^{(1)}_B(k) = -\frac{n_0}{h}\frac{4\pi G m^2}{k^2}\left[\hat{G}^0_B(k)\right]^2. \tag{4.19}$$

We must proceed from here as we did for the fermion case to obtain

$$\left[\hat{G}^0_B(k)\right]^2 = \frac{1}{(k_0 - \omega_k + i0)^2}, \tag{4.20}$$

and we obtain for the dressed propagator the relation

$$\hat{G}_B(k) = -in_0 2\pi^4 \delta(k_0, k) + \hat{G}^0_B(k) - \frac{n_0}{h}\frac{4\pi G m^2}{k^2}\left[\hat{G}^0_B(k)\right]^2. \tag{4.21}$$

5. Conclusions

We have here applied a recently developed non relativistic quantization methodology [2] [9] [10] [11] [12] to Newton's gravitation potential.

- We emphasize that our methodology successfully tackles all renormalization issues. We made full use ultra-hyperfunctions' theory, in particular the results reported in [2].
- With such tools we have been able to construct a non-relativistic quantum field theory (NR QFT) of Newton's gravitation (NG).
- This was done for pairs of fermions or bosons that interact between themselves via NG.
- Our manipulations were based on the results of the classical book [6].
- As special examples, we have obtained the dressed propagators for both types of particles, up to first order in perturbation theory, and also the fermions' self-energy.
- The examples indicate that we have indeed constructed, both for fermions and bosons, a viable non-relativistic quantum field theory of gravitation.
- Remark that we were here concerned only with Newton's gravitation.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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