LIPSCHITZ CONTINUITY OF MINIMIZERS IN A PROBLEM WITH NONSTANDARD GROWTH

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To our dear friend Sandro Salsa on the occasion of his 70th birthday

Abstract. In this paper we obtain the Lipschitz continuity of nonnegative local minimizers of the functional $J(v) = \int_\Omega \big(F(x,v,\nabla v) + \lambda(x)\chi_{\{v>0\}}\big)\,dx$, under nonstandard growth conditions of the energy function $F(x,s,\eta)$ and $0 < \lambda_{\min} \le \lambda(x) \le \lambda_{\max} < \infty$. This is the optimal regularity for the problem. Our results generalize the ones we obtained in the case of the inhomogeneous $p(x)$-Laplacian in our previous work [17].

Nonnegative local minimizers u satisfy in their positivity set a general nonlinear degenerate/singular equation $\mathrm{div} A(x,u,\nabla u) = B(x,u,\nabla u)$ of nonstandard growth type. As a by-product of our study, we obtain several results for this equation that are of independent interest.

1. Introduction

In this paper we study the regularity properties of nonnegative, local minimizers of the functional

$$J(v) = \int_\Omega \big(F(x,v,\nabla v) + \lambda(x)\chi_{\{v>0\}}\big)\,dx, \tag{1.1}$$

under nonstandard growth conditions of the energy function $F(x,s,\eta)$ and $0 < \lambda_{\min} \le \lambda(x) \le \lambda_{\max} < \infty$.

There has been a great deal of interest in these type of problems. Their study started with the seminal paper of Alt and Caffarelli [2] where the case $F(x,s,\eta) = \frac{1}{2}|\eta|^2$ was considered. Later on, [3] considered the case $F(x,s,\eta) = G(|\eta|^2)$ under uniform ellipticity assumptions. The general power case $F(x,s,\eta) = \frac{1}{p}|\eta|^p$ with $1 < p < \infty$ was studied in [8], and $F(x,s,\eta) = G(|\eta|)$ with G convex under the assumption that G' satisfies Lieberman's condition namely, $G''(t) \sim G'(t)/t$, was analyzed in [19]. The linear inhomogeneous case $F(x,s,\eta) = \frac{1}{2}|\eta|^2 + f(x)s$ was addressed in [12] and [15].

The minimization problem for the functional (1.1) with $F(x,s,\eta) = \frac{1}{p(x)}|\eta|^{p(x)}$ was first considered in [6] for $p(x) \ge 2$ and then, in [16] and [17] in the inhomogeneous case $F(x,s,\eta) = \frac{1}{p(x)}|\eta|^{p(x)} + f(x)s$, for $1 < p(x) < \infty$ and $f \in L^\infty(\Omega)$. In [17], among other results, we proved that nonnegative local minimizers u are locally Lipschitz continuous and satisfy

$$\Delta_{p(x)}u := \mathrm{div}(|\nabla u(x)|^{p(x)-2}\nabla u) = f \quad \text{in } \{u > 0\}.$$

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The operator $\Delta_{p(x)}$, called the $p(x)$-Laplacian, extends the Laplacian, where $p(x) \equiv 2$ and the p-Laplacian, where $p(x) \equiv p$. This is a prototype operator with nonstandard growth. The functional setting for the study of this type of operators are the variable exponent Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$.

Functionals and PDEs with nonstandard growth have a wide range of applications, such as the modelling of non-Newtonian fluids, as for instance, electrorheological [21] or thermorheological fluids [4]. Other areas of application include non-linear elasticity [24], image reconstruction [1, 7], the modelling of electric conductors [25], as well as processes of filtration of gases in non-homogeneous porous media [5].

As far as we know, no result on the minimization of (1.1) with $F(x, s, \eta)$ a general function with nonstandard growth has been obtained.

The main purpose of our work is to prove the local Lipschitz continuity of nonnegative local minimizers of such an energy. We stress that this is the optimal regularity since it is known from the particular cases refered to above that the gradient of a minimizer u jumps across $\Omega \cap \partial\{u > 0\}$.

We prove that nonnegative minimizers of (1.1) are solutions to the associated equation in their positivity sets. That is, a local minimizer $u \geq 0$ satisfies

$$\text{div}A(x, u, \nabla u) = B(x, u, \nabla u) \tag{1.2}$$

in $\{u > 0\}$, where

$$A(x, s, \eta) = \nabla_\eta F(x, s, \eta), \qquad B(x, s, \eta) = F_s(x, s, \eta).$$

Under our assumptions, the governing equation (1.2) is given by $A(x, s, \eta)$ satisfying

$$\lambda_0 |\eta|^{p(x)-2}|\xi|^2 \leq \sum_{i,j} \frac{\partial A_i}{\partial \eta_j}(x, s, \eta)\xi_i\xi_j \leq \Lambda_0|\eta|^{p(x)-2}|\xi|^2,$$

and has a right hand side given by $B(x, s, \eta) \not\equiv 0$ of $p(x)$-type growth in η. This equation is singular in the regions where $1 < p(x) < 2$ and degenerate in the ones where $p(x) > 2$.

Our study thus presents new features, needed in order to overcome the deep technical difficulties arising due to the nonlinear degenerate/singular nature and the x and s dependence of this general operator associated to our energy functional (1.1).

The first part of the paper is devoted to the study of equation (1.2) in a domain Ω, under nonstandard growth conditions of $p(x)$-type. We prove existence results, a comparison principle, a uniqueness result, a maximum principle and other local L^∞ bounds of solutions of this equation. These delicate results are of independent interest.

Some of these results are obtained under the growth assumption (3.14). We remark that this hypothesis on the functions A and B allows to consider very general equations. This condition not only enables us to get the inequality in Proposition 3.3 that is a main tool for all the proofs in the paper, but also it is invariant under rescalings. All these results are included in Section 3.

In the second part of the paper we deal with the minimization problem for the functional (1.1). In fact, in Section 4 we first get an existence result for minimizers. We also prove nonnegativity and boundedness, under suitable assumptions. Then, we prove the local Hölder and Lipschitz continuity of nonnegative local minimizers (Theorems 4.3 and 4.5).

The proofs in Section 4 involve delicate rescalings. One of the main difficulties this problem presents is that it is not invariant under the rescaling $u(x) \longmapsto \frac{u(tx)}{k}$, if $t \neq k$ —rescaling that is a crucial tool in dealing with this type of problems. The rescaled functionals lose the uniform properties and nontrivial modifications are needed to get through the proofs. Even after these modifications, there is in general no limit equation for the rescaled problems due to the growth we are allowing to the function $B(x,s,\eta)$. Novel arguments are used to complete the proof of Theorem 4.4. In fact, we are able to show that, although there is in general no limit equation for the rescaled problems, there is a limit function and it satisfies Harnack's inequality (see (4.58)).

A thorough follow up of the dependence of the bounds found in Section 3 with respect to the structural conditions on F, A and B is of most importance as well.

Let us point out that the results in the paper are new even in the case $p(x) \equiv p$ constant.

Finally, in Section 5 we present some examples of functionals (1.1) where our results can be used. Our examples include functionals (1.1) involving energy functions of the form

$$F(x,s,\eta) = a(x,s)\frac{|\eta|^{p(x)}}{p(x)} + f(x,s).$$

A possible example of admissible functions $a(x,s), f(x,s)$ is given by

$$a(x,s) = a_0(x)(1+s)^{-q(x)}, \qquad a_0(x) > 0,\ 0 < q(x) \le q_0(x),$$

for s in the range where the nonnegative local minimizer takes values, $q_0(x)$ a function depending on $p(x)$ and

$$f(x,s) = b(x)|s|^{\tau(x)}, \qquad b(x) \ge 0,\ \tau(x) \ge 2,$$

with $\tau(x)$ satisfying (2.7).

Our results also apply to functionals (1.1) involving energy functions of the form

$$F(x,s,\eta) = G(x,\eta) + f(x,s).$$

Some admissible $G(x,\eta), f(x,s)$ are

$$G(x,\eta) = a(x)\widetilde{G}\big(|\eta|^{p(x)}\big) \qquad a(x) > 0, \widetilde{G}'' \ge 0,$$

$$G(x,\eta) = \widetilde{A}(x)\eta\cdot\eta|\eta|^{p(x)-2} \qquad \widetilde{A}(x) \in \mathbb{R}^{N\times N} \text{ uniformly elliptic},$$

$$f(x,s) = g(x)s.$$

Also,

$$F(x,s,\eta) = a_1(x)F_1(x,s,\eta) + a_2(x)F_2(x,s,\eta), \qquad a_i(x) > 0,$$

is an admissible function if both $F_1(x,s,\eta)$ and $F_2(x,s,\eta)$ are admissible.

We begin our paper with a section where we state the hypotheses on F, A, B, λ and $p(x)$ that will be used throught the article. And we end it with an Appendix where we state some properties of the function spaces $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$ where the problem is well posed.

1.1. **Preliminaries on Lebesgue and Sobolev spaces with variable exponent.** Let $p : \Omega \to [1,\infty)$ be a measurable bounded function, called a variable exponent on Ω and denote $p_{\max} = \operatorname{esssup} p(x)$ and $p_{\min} = \operatorname{essinf} p(x)$. We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u : \Omega \to \mathbb{R}$ for which the modular $\varrho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\,dx$ is finite. We define the Luxemburg norm on this space by

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \varrho_{p(\cdot)}(u/\lambda) \le 1\}.$$

This norm makes $L^{p(\cdot)}(\Omega)$ a Banach space.

There holds the following relation between $\varrho_{p(\cdot)}(u)$ and $\|u\|_{L^{p(\cdot)}}$:

$$\min\Big\{\Big(\int_\Omega |u|^{p(x)}\,dx\Big)^{1/p_{\min}},\Big(\int_\Omega |u|^{p(x)}\,dx\Big)^{1/p_{\max}}\Big\}\le \|u\|_{L^{p(\cdot)}(\Omega)}$$
$$\le \max\Big\{\Big(\int_\Omega |u|^{p(x)}\,dx\Big)^{1/p_{\min}},\Big(\int_\Omega |u|^{p(x)}\,dx\Big)^{1/p_{\max}}\Big\}.$$

Moreover, the dual of $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$ with $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$.

Let $W^{1,p(\cdot)}(\Omega)$ denote the space of measurable functions u such that u and the distributional derivative ∇u are in $L^{p(\cdot)}(\Omega)$. The norm

$$\|u\|_{1,p(\cdot)} := \|u\|_{p(\cdot)} + \||\nabla u|\|_{p(\cdot)}$$

makes $W^{1,p(\cdot)}(\Omega)$ a Banach space.

The space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the closure of the $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

For the sake of completeness we include in an Appendix at the end of the paper some additional results on these spaces that are used throughout the paper.

1.2. **Notation.**

- N spatial dimension
- $|S|$ N-dimensional Lebesgue measure of the set S
- $B_r(x_0)$ open ball of radius r and center x_0
- B_r open ball of radius r and center 0
- χ_S characteristic function of the set S
- $u^+=\max(u,0)$, $u^-=\max(-u,0)$
- $\langle\,\xi\,,\,\eta\,\rangle$ and $\xi\cdot\eta$ both denote scalar product in $\mathbb{R}^N$

2. Assumptions

In this section we collect all the assumptions that will be made along the paper.

Throughout the paper Ω will denote a C^1 bounded domain in $\mathbb{R}^N$. In addition, the following assumptions will be made:

2.1. **Assumptions on $p(x)$.**

We assume that the function $p(x)$ is measurable in Ω and verifies

$$1<p_{\min}\le p(x)\le p_{\max}<\infty, \qquad x\in\Omega.$$

We assume further that $p(x)$ is Lipschitz continuous in Ω and we denote by L the Lipschitz constant of $p(x)$, namely, $\|\nabla p\|_{L^\infty(\Omega)}\le L$.

When we are restricted to a ball B_r we use p_r^- and p_r^+ to denote the infimum and the supremum of $p(x)$ over B_r.

2.2. **Assumptions on $\lambda(x)$.**

We assume that the function $\lambda(x)$ is measurable in Ω and verifies

$$0<\lambda_{\min}\le \lambda(x)\le \lambda_{\max}<\infty, \qquad x\in\Omega.$$

2.3. **Assumptions on F.** We assume that F is measurable in $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$, and for every $x \in \overline{\Omega}$, $F(x, \cdot, \cdot) \in C^1(\mathbb{R} \times \mathbb{R}^N) \cap C^2(\mathbb{R} \times \mathbb{R}^N \setminus \{0\})$.

We denote $A(x,s,\eta) = \nabla_\eta F(x,s,\eta)$ and $B(x,s,\eta) = F_s(x,s,\eta)$.

2.4. **Assumptions on A.** We assume that $A \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$ and for every $x \in \overline{\Omega}$, $A(x, \cdot, \cdot) \in C^1(\mathbb{R} \times \mathbb{R}^N \setminus \{0\}, \mathbb{R}^N)$. Moreover, there exist positive constants λ_0 and Λ_0, and $\beta \in (0,1)$ such that for every $x, x_1, x_2 \in \overline{\Omega}$, $s, s_1, s_2 \in \mathbb{R}$, $\eta \in \mathbb{R}^N \setminus \{0\}$ and $\xi \in \mathbb{R}^N$, the following conditions are satisfied:

$$A(x,s,0) = 0, \tag{2.1}$$

$$\sum_{i,j} \frac{\partial A_i}{\partial \eta_j}(x,s,\eta)\xi_i\xi_j \geq \lambda_0 |\eta|^{p(x)-2}|\xi|^2, \tag{2.2}$$

$$\sum_{i,j} \Big| \frac{\partial A_i}{\partial \eta_j}(x,s,\eta) \Big| \leq \Lambda_0 |\eta|^{p(x)-2}, \tag{2.3}$$

$$\big|A(x_1,s,\eta) - A(x_2,s,\eta)\big| \leq \Lambda_0 |x_1 - x_2|^\beta \big(|\eta|^{p(x_1)-1} + |\eta|^{p(x_2)-1}\big)\big(1 + \big|\log|\eta|\big|\big), \tag{2.4}$$

$$\big|A(x,s_1,\eta) - A(x,s_2,\eta)\big| \leq \Lambda_0 |s_1 - s_2||\eta|^{p(x)-1}. \tag{2.5}$$

2.5. **Assumptions on B.** We assume that B is measurable in $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$ and for every $x \in \overline{\Omega}$, $B(x, \cdot, \cdot) \in C^1(\mathbb{R} \times \mathbb{R}^N)$, and for every $(x,s,\eta) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$,

$$\big|B(x,s,\eta)\big| \leq \Lambda_0 \big(1 + |\eta|^{p(x)} + |s|^{\tau(x)}\big), \tag{2.6}$$

where Λ_0 is as in the assumptions on A and

$$\begin{aligned} &\tau(x) \geq p(x) \quad \text{and} \quad \tau \in C(\overline{\Omega}), \\ &\tau(x) \leq p^*(x) = \frac{Np(x)}{N - p(x)} \text{ if } p_{\max} < N, \\ &\tau(x) \text{ arbitrary if } p_{\min} > N, \\ &\tau(x) = p(x) \text{ if } p_{\min} \leq N \leq p_{\max}. \end{aligned} \tag{2.7}$$

Remark 2.1. From (2.1) and (2.3) we get

$$|A_i(x,s,\eta)| = |A_i(x,s,\eta) - A_i(x,s,0)| = \Big| \int_0^1 \sum_j \frac{\partial A_i}{\partial \eta_j}(x,s,t\eta)\eta_j \, dt, \Big| \leq \bar{\alpha}(p_{\min})\Lambda_0 |\eta|^{p(x)-1},$$

so that

$$|A(x,s,\eta)| \leq \bar{\alpha}(p_{\min}) N \Lambda_0 |\eta|^{p(x)-1}. \tag{2.8}$$

From (2.1) and (2.2) we have

$$A(x,s,\eta) \cdot \eta = (A(x,s,\eta) - A(x,s,0)) \cdot \eta = \int_0^1 \sum_{ij} \frac{\partial A_i}{\partial \eta_j}(x,s,t\eta)\eta_j\eta_i \, dt,$$

so that

$$A(x,s,\eta) \cdot \eta \geq \alpha(p_{\max})\lambda_0 |\eta|^{p(x)}. \tag{2.9}$$

3. Existence, uniqueness and bounds of solutions to equation (1.2)

In this section we consider A and B as in Section 2 and we prove results for solutions of the equation

$$\text{div}A(x,u,\nabla u) = B(x,u,\nabla u) \quad \text{in} \quad \Omega. \tag{3.1}$$

Namely, existence, comparison principle, uniqueness, maximum principle and bounds of solutions.

Our first result is Proposition 3.1, were we prove existence of a solution to (3.1) with given boundary data. In order to prove the existence of a solution to (3.1) we show that, given $u \in W^{1,p(\cdot)}(\Omega)$, there exists a minimizer of the functional

$$\mathcal{J}_\Omega(v) = \int_\Omega F(x,v,\nabla v)\,dx \tag{3.2}$$

in $u + W_0^{1,p(\cdot)}(\Omega)$, where F is as in Section 2, $A(x,s,\eta) = \nabla_\eta F(x,s,\eta)$ and $B(x,s,\eta) = F_s(x,s,\eta)$.

Then, in Proposition 3.2 we get an existence result under a growth assumption on the function F stronger than (3.3) in Proposition 3.1, but without the small oscillation hypothesis there.

In Proposition 3.4 and Corollary 3.2 we prove comparison and uniqueness for this problem, assuming that condition (3.14) below holds. In Proposition 3.5 we prove that solutions to (3.1) with bounded boundary data are bounded and in Proposition 3.6 we prove a maximum principle for this problem, under suitable assumptions. In Proposition 3.7 we give another existence result of a bounded solution.

We start with the definition of solution to (3.1).

Definition 3.1. Let p, A and B be as in Section 2. We say that u is a solution to (3.1) if $u \in W^{1,p(\cdot)}(\Omega)$ and, for every $\varphi \in C_0^\infty(\Omega)$, there holds that

$$-\int_\Omega A(x,u,\nabla u)\cdot\nabla\varphi\,dx = \int_\Omega B(x,u,\nabla u)\varphi\,dx.$$

We are using that, under the conditions in (2.7), the embedding theorem (see Theorem A.5) applies.

Our first existence result is

Proposition 3.1. *Let p, F, A, B as in Section 2 and let $\Omega' \subset \Omega$ be a C^1 domain. Let $u \in W^{1,p(\cdot)}(\Omega')$ and let us call $p^+ = \sup_{\Omega'} p(x)$, $p^- = \inf_{\Omega'} p(x)$. Assume that there exist $\nu, c_1 \in \mathbb{R}_+$, $p_{\min} > \delta > 0$ and $g \in L^1(\Omega)$ such that*

$$F(x,s,\eta) \geq \nu|\eta|^{p(x)} - c_1|s|^{p(x)-\delta} - g(x) \quad \text{in} \quad \Omega. \tag{3.3}$$

Assume, moreover that $\delta > p^+ - p^-$ and that

$$F(x,s,\eta) \leq \nu^{-1}|\eta|^{p(x)} + c_1|s|^{\tau(x)} + g(x) \quad \text{in} \quad \Omega, \tag{3.4}$$

with τ satisfying (2.7).

Then, there exists a solution $v \in u + W_0^{1,p(\cdot)}(\Omega')$ to (3.1) in Ω'.

Moreover, $\|v\|_{W^{1,p(\cdot)}(\Omega')} \leq C$, for a constant C depending only $\|u\|_{W^{1,p(\cdot)}(\Omega')}$, $\|g\|_{L^1(\Omega')}$, $|\Omega'|$, $\text{diam}(\Omega')$, N, p^-, p^+, δ, L, ν, c_1, $||\tau||_{L^\infty(\Omega')}$ and the C^1 norm of $\partial\Omega'$.

Proof. We will show that there is a minimizer of $\mathcal{J}_{\Omega'}$ in $u + W_0^{1,p(\cdot)}(\Omega')$ where

$$\mathcal{J}_{\Omega'}(v) = \int_{\Omega'} F(x, v, \nabla v)\, dx.$$

This minimizer is a solution to the associated Euler-Lagrange equation (3.1) in Ω'.

We will use the embedding theorem (see Theorem A.5) that states that, under the conditions in (2.7), $W^{1,p(\cdot)}(\Omega') \hookrightarrow L^{\tau(\cdot)}(\Omega')$ continuously.

So, let v_n be a minimizing sequence. That is, $v_n \in u + W_0^{1,p(\cdot)}(\Omega')$ and

$$I = \lim_{n\to\infty} \mathcal{J}_{\Omega'}(v_n) = \inf_{u+W_0^{1,p(\cdot)}(\Omega')} \mathcal{J}_{\Omega'}(v) \leq \int_{\Omega'} F(x, u, \nabla u)\, dx.$$

Let us show that there is a constant $\kappa > 0$ such that $\|v_n\|_{L^{p(\cdot)}(\Omega')} \leq \kappa$. In fact, by (3.3), for n large,

$$\int_{\Omega'} |\nabla v_n|^{p(x)}\, dx \leq 1 + \int_{\Omega'} F(x, u, \nabla u)\, dx + \frac{c_1}{\nu} \int_{\Omega'} |v_n|^{p(x)-\delta}\, dx + \frac{1}{\nu} \int_{\Omega'} g(x)\, dx.$$

By Poincare's inequality (Theorem A.4)

$$\|v_n - u\|_{L^{p(\cdot)}(\Omega')} \leq C_{\Omega'} \|\nabla(v_n - u)\|_{L^{p(\cdot)}(\Omega')}.$$

Hence, recalling Proposition A.1,

$$\begin{aligned}
\|v_n\|_{L^{p(\cdot)}(\Omega')} &\leq \|u\|_{L^{p(\cdot)}(\Omega')} + C_{\Omega'} \big[\|\nabla v_n\|_{L^{p(\cdot)}(\Omega')} + \|\nabla u\|_{L^{p(\cdot)}(\Omega')}\big] \\
&\leq C\big[\|u\|_{W^{1,p(\cdot)}(\Omega')} + \max\big\{\big(\int_{\Omega'} |\nabla v_n|^{p(x)}\, dx\big)^{1/p^-}, \big(\int_{\Omega'} |\nabla v_n|^{p(x)}\, dx\big)^{1/p^+}\big\}\big] \\
&\leq \bar{C}\big[1 + \max\big\{\big(\int_{\Omega'} |v_n|^{p(x)-\delta}\, dx\big)^{1/p^-}, \big(\int_{\Omega'} |v_n|^{p(x)-\delta}\, dx\big)^{1/p^+}\big\}\big]
\end{aligned}$$

with $\bar{C}$ depending on $\|u\|_{W^{1,p(\cdot)}(\Omega')}, \|g\|_{L^1(\Omega')}, N, p^-$, p^+, δ, $|\Omega'|$, $\operatorname{diam}(\Omega')$, L, $||\tau||_{L^\infty(\Omega')}$, the C^1 norm of $\partial\Omega'$, and the constants in (3.3).

Observe that in case $u \equiv M$, there holds that $\int_{\Omega'} F(x, u, \nabla u)\, dx$ is bounded by a constant that depends only on M, $\|\tau\|_{L^\infty(\Omega')}$ and $|\Omega'|$. Hence, in that case $\bar{C}$ is independent of the regularity of Ω'.

Since we want to find a uniform bound of $\|v_n\|_{L^{p(\cdot)}(\Omega')}$, we may assume that this norm is larger than 1. Let q be the middle point of the interval $[p^+ - \delta, p^-]$. By Young's inequality with $r(x) = \frac{q}{p(x)-\delta}$,

$$\int_{\Omega'} |v_n|^{p(x)-\delta}\, dx \leq C_\varepsilon + \varepsilon \int_{\Omega'} |v_n|^q\, dx,$$

for $0 < \varepsilon < 1$ with C_ε depending only on $|\Omega'|, \varepsilon, p^-, p^+$ and δ. On the other hand, since $\|v_n\|_{L^q(\Omega')} \leq C\|v_n\|_{L^{p(\cdot)}(\Omega')}$ with C depending only on $|\Omega'|, p^-, p^+$ and δ,

$$\int_{\Omega'} |v_n|^q\, dx \leq \big(C\|v_n\|_{L^{p(\cdot)}(\Omega')}\big)^q.$$

So that

$$\begin{aligned}
\|v_n\|_{L^{p(\cdot)}(\Omega')} &\leq C\big[\widetilde{C}_\varepsilon + \varepsilon^{\frac{1}{p^+}} \big(\|v_n\|_{L^{p(\cdot)}(\Omega')}\big)^{\frac{q}{p^-}}\big] \\
&\leq C\big[\widetilde{C}_\varepsilon + \varepsilon^{\frac{1}{p^+}} \|v_n\|_{L^{p(\cdot)}(\Omega')}\big].
\end{aligned}$$

By choosing ε small enough, we find that

$$\|v_n\|_{L^{p(\cdot)}(\Omega')} \leq C \tag{3.5}$$

with C depending on $|\Omega'|$, $\operatorname{diam}(\Omega')$, $\|u\|_{W^{1,p(\cdot)}(\Omega')}, p^-, p^+$, N, δ, $\|g\|_{L^1(\Omega')}$, L, $||\tau||_{L^\infty(\Omega')}$, the C^1 norm of $\partial\Omega'$, ν and c_1.

From the computations above we find that $\int_{\Omega'} |v_n|^{p(x)-\delta}\,dx \le C_1$. So that we have that $I > -\infty$ and

$$\|\nabla v_n\|_{L^{p(\cdot)}(\Omega')} \le C_2, \tag{3.6}$$

with C_2 depending on $|\Omega'|$, $\operatorname{diam}(\Omega')$, $\|u\|_{W^{1,p(\cdot)}(\Omega')}, p^-, p^+$, N, δ, $\|g\|_{L^1(\Omega')}$, L, $||\tau||_{L^\infty(\Omega')}$, the C^1 norm of $\partial\Omega'$, ν and c_1.

From our comment above, we have that in case $u \equiv M$ in Ω', the constant C_2 is independent of the regularity of $\partial\Omega'$.

Let us proceed with the proof of the existence of a minimizer. By the estimates above, for a subsequence that we still call v_n, there holds that there exists $v \in u + W_0^{1,p(\cdot)}(\Omega')$, such that

$$v_n \rightharpoonup v \quad \text{in} \quad W^{1,p(\cdot)}(\Omega'), \qquad v_n \to v \quad \text{in} \quad L^{p^-}(\Omega') \quad \text{and almost everywhere,}$$

and such that the bounds (3.5) and (3.6) also hold for v.

By Egorov's Theorem, for every $\varepsilon > 0$ there exists Ω_ε such that $|\Omega' \setminus \Omega_\varepsilon| < \varepsilon$ and $v_n \to v$ uniformly in Ω_ε.

On the other hand, if we set $\Omega_K = \{x \in \Omega' \,/\, |v| + |\nabla v| \le K\}$, there holds that $|\Omega' \setminus \Omega_K| \to 0$ as $K \to \infty$.

Let $\Omega_{\varepsilon,K} = \Omega_\varepsilon \cap \Omega_K$. Then, $|\Omega' \setminus \Omega_{\varepsilon,K}| \to 0$ as $\varepsilon \to 0$ and $K \to \infty$.

There holds

$$\limsup_{n\to\infty} \int_{\Omega_{\varepsilon,K}} F(x, v_n, \nabla v_n)\,dx \le I + c_1 \int_{\Omega'\setminus\Omega_{\varepsilon,K}} |v|^{p(x)-\delta}\,dx + \int_{\Omega'\setminus\Omega_{\varepsilon,K}} g\,dx. \tag{3.7}$$

Let us prove that

$$\int_{\Omega_{\varepsilon,K}} F(x, v, \nabla v)\,dx \le I + c_1 \int_{\Omega'\setminus\Omega_{\varepsilon,K}} |v|^{p(x)-\delta}\,dx + \int_{\Omega'\setminus\Omega_{\varepsilon,K}} g\,dx.$$

In fact,

$$\int_{\Omega_{\varepsilon,K}} F(x, v_n, \nabla v_n)\,dx - \int_{\Omega_{\varepsilon,K}} F(x, v, \nabla v)\,dx = \int_{\Omega_{\varepsilon,K}} \big[F(x, v_n, \nabla v_n) - F(x, v_n, \nabla v)\big]\,dx$$
$$+ \int_{\Omega_{\varepsilon,K}} \big[F(x, v_n, \nabla v) - F(x, v, \nabla v)\big]\,dx = \mathcal{A} + \mathcal{B}.$$

On the one hand, $\mathcal{B} \to 0$ since $F(x, v_n, \nabla v) - F(x, v, \nabla v) \to 0$ uniformly in $\Omega_{\varepsilon,K}$ and it is uniformly bounded. On the other hand, by the convexity assumption on $F(x, s, \eta)$ with respect to η,

$$\mathcal{A} \ge \int_{\Omega_{\varepsilon,K}} A(x, v_n, \nabla v) \cdot (\nabla v_n - \nabla v)\,dx \to 0 \quad \text{as} \quad n \to \infty$$

since $A(x, v_n, \nabla v) \to A(x, v, \nabla v)$ uniformly in $\Omega_{\varepsilon,K}$, they are uniformly bounded and $\nabla v_n \rightharpoonup \nabla v$ weakly in $L^{p(\cdot)}(\Omega_{\varepsilon,K})$.

Hence, for every ε, K,

$$\int_{\Omega_{\varepsilon,K}} F(x, v, \nabla v)\,dx \le I + c_1 \int_{\Omega'\setminus\Omega_{\varepsilon,K}} |v|^{p(x)-\delta}\,dx + \int_{\Omega'\setminus\Omega_{\varepsilon,K}} g\,dx.$$

Now, by letting $\varepsilon \to 0$ and $K \to \infty$, we get

$$\int_{\Omega'} F(x, v, \nabla v)\, dx \le I,$$

and therefore, v is a minimizer of $\mathcal{J}_{\Omega'}$ in $u + W_0^{1,p(\cdot)}(\Omega')$ and a solution to (3.1). ☐

As a corollary of Proposition 3.1 we have the following existence result that will be used in the next section.

Corollary 3.1. *Let p, F, A, B as in Section 2 and let $\Omega' \subset \Omega$ be a C^1 domain. Let $u \in W^{1,p(\cdot)}(\Omega')$ and let us call $p^+ = \sup_{\Omega'} p(x)$, $p^- = \inf_{\Omega'} p(x)$. Assume that there exist $\nu, c_1 \in \mathbb{R}_+$ and $p_{\min} > \delta > 0$ such that*

$$F(x, s, \eta) \ge \nu |\eta|^{p(x)} - c_1\big(|s|^{p(x)-\delta} + 1\big) \quad in \quad \Omega. \tag{3.8}$$

Assume, moreover that $\delta > p^+ - p^-$ and that

$$F(x, s, \eta) \le \nu^{-1} |\eta|^{p(x)} + c_1\big(|s|^{\tau(x)} + 1\big) \quad in \quad \Omega, \tag{3.9}$$

with $\tau(x)$ satisfying (2.7).

Then, there exists a solution $v \in u + W_0^{1,p(\cdot)}(\Omega')$ to (3.1) in Ω' and $\|v\|_{W^{1,p(\cdot)}(\Omega')} \le C$, for a constant C depending only $\|u\|_{W^{1,p(\cdot)}(\Omega')}$, $|\Omega'|$, $\operatorname{diam}(\Omega')$, N, p^-, p^+, δ, L, ν, c_1, $||\tau||_{L^\infty(\Omega')}$ and the C^1 norm of $\partial\Omega'$.

With a stronger growth assumption on the s variable for the function $F(x, s, \eta)$ we get an existence result without the small oscillation assumption of the function p.

Proposition 3.2. *Let p, F, A, B as in Section 2 and let $\Omega' \subset \Omega$ be a C^1 domain. Let $u \in W^{1,p(\cdot)}(\Omega')$. Assume that there exist $\nu, c_1 \in \mathbb{R}_+$, $g \in L^1(\Omega)$ and $1 \le q < p_{\min}$ such that*

$$F(x, s, \eta) \ge \nu |\eta|^{p(x)} - c_1 |s|^q - g(x) \quad in \quad \Omega. \tag{3.10}$$

Assume, moreover that

$$F(x, s, \eta) \le \nu^{-1} |\eta|^{p(x)} + c_1 |s|^{\tau(x)} + g(x) \quad in \quad \Omega, \tag{3.11}$$

with τ satisfying (2.7).

Then, there exists a solution $v \in u + W_0^{1,p(\cdot)}(\Omega')$ to (3.1) in Ω' and $\|v\|_{W^{1,p(\cdot)}(\Omega')} \le C$, for a constant C depending only $\|u\|_{W^{1,p(\cdot)}(\Omega')}$, $\|g\|_{L^1(\Omega')}$, $|\Omega'|$, $\operatorname{diam}(\Omega')$, N, $p_{\min}$, $p_{\max}$, q, L, ν, c_1, $||\tau||_{L^\infty(\Omega')}$ and the C^1 norm of $\partial\Omega'$.

Proof. We proceed as in the proof of Proposition 3.1 and we prove that a minimizing sequence $\{v_n\}$ satisfies

$$\nu \int_{\Omega'} |\nabla v_n|^{p(x)}\, dx \le \int_{\Omega'} F(x, u, \nabla u) + 1 + \int_{\Omega'} g(x)\, dx + c_1 \int_{\Omega'} |v_n|^q\, dx. \tag{3.12}$$

We want to prove that there is a constant such that $\int_{\Omega'} |\nabla v_n|^{p(x)}\, dx \le C$. So, we can assume that $\int_{\Omega'} |\nabla v_n|^{p(x)}\, dx > 1$.

Thus,

$$\begin{aligned} \|v_n\|_{L^q(\Omega')} \le C\|v_n\|_{L^{p(\cdot)}(\Omega')} &\le C\big[\|u\|_{W^{1,p(\cdot)}(\Omega')} + \|\nabla v_n\|_{L^{p(\cdot)}(\Omega')}\big] \\ &\le C\Big[\|u\|_{W^{1,p(\cdot)}(\Omega')} + \Big(\int_{\Omega'} |\nabla v_n|^{p(x)}\, dx\Big)^{1/p_{\min}}\Big], \end{aligned}$$

where C depends on $q, p_{\min}, p_{\max}$, N, L and $|\Omega'|$, $\mathrm{diam}(\Omega')$. Hence, as $q < p_{\min}$,

$$\begin{aligned}\int_{\Omega'} |v_n|^q\,dx &\le C\Big(1 + \big(\int_{\Omega'} |\nabla v_n|^{p(x)}\,dx\big)^{q/p_{\min}}\Big)\\ &\le \widetilde{C} + \varepsilon\int_{\Omega'} |\nabla v_n|^{p(x)}\,dx\end{aligned} \tag{3.13}$$

with C depending only on $q, p_{\min}, p_{\max}, N, |\Omega'|, \mathrm{diam}(\Omega'), L, \|u\|_{W^{1,p(\cdot)}(\Omega')}$, and $\widetilde{C}$ depending on the same constants and also on ε.

Thus, by (3.12) and (3.13),

$$\int_{\Omega'} |\nabla v_n|^{p(x)}\,dx \le \widehat{C}$$

with $\widehat{C}$ depending only on $q, p_{\min}, p_{\max}, N, \nu, |\Omega'|, \mathrm{diam}(\Omega'), L, \int_{\Omega'} g(x)\,dx,\ c_1,\ ||\tau||_{L^\infty(\Omega')}$, the C^1 norm of $\partial\Omega'$ and $\|u\|_{W^{1,p(\cdot)}(\Omega')}$.

Now, as in the proof of Proposition 3.1, we get that there exists a subsequence that we still call $\{v_n\}$ and a function $v \in u + W_0^{1,p(\cdot)}(\Omega')$ such that

$$v_n \to v \quad \text{in} \quad L^{p_{\min}}(\Omega'), \qquad v_n \rightharpoonup v \quad \text{weakly in } W^{1,p(\cdot)}(\Omega').$$

Now, the proof follows as that of Proposition 3.1. □

We next prove a result valid for solutions of equation (3.1) that will be of use in the proofs of Hölder and Lipschitz continuity of minimizers of the energy functional (1.1)

Proposition 3.3. *Let p, F, A and B be as in Section 2. Assume moreover that*

$$2|A_s(x,s,\eta)\cdot\xi w| \le \frac{1}{2}\sum_{i,j}\frac{\partial A_i}{\partial\eta_j}(x,s,\eta)\xi_i\xi_j + B_s(x,s,\eta)w^2, \tag{3.14}$$

for every $(x,s,\eta) \in \overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N\setminus\{0\}$, $\xi\in\mathbb{R}^N$ and $w\in\mathbb{R}$.

Let $u \in W^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$ and let $v \in W^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$ be such that

$$\begin{cases}\mathrm{div}A(x,v,\nabla v) = B(x,v,\nabla v) & \text{in } \Omega,\\ v = u \quad \text{on } \partial\Omega.\end{cases} \tag{3.15}$$

Then,

$$\begin{aligned}&\int_\Omega \big(F(x,u,\nabla u) - F(x,v,\nabla v)\big)\,dx \ge\\ &\frac{1}{2}\alpha\lambda_0\Big(\int_{\Omega\cap\{p(x)\ge 2\}} |\nabla u - \nabla v|^{p(x)}\,dx + \int_{\Omega\cap\{p(x)<2\}}\Big(|\nabla u| + |\nabla v|\Big)^{p(x)-2}|\nabla u - \nabla v|^2\,dx\Big),\end{aligned} \tag{3.16}$$

where $\alpha = \alpha(p_{\min}, p_{\max})$ and λ_0 is as in (2.2).

Proof. For $0 \le \sigma \le 1$, let $u^\sigma = v + \sigma(u - v)$. Then, denoting $\nabla_\eta F = A$ and $F_s = B$, we obtain

$$\begin{aligned} &\int_\Omega \big(F(x,u,\nabla u) - F(x,v,\nabla v)\big)\,dx = \int_0^1 \int_\Omega A(x,u^\sigma,\nabla u^\sigma)\cdot\nabla(u^\sigma - v)\frac{1}{\sigma}\,dx\,d\sigma \\ &\quad + \int_0^1\int_\Omega B(x,u^\sigma,\nabla u^\sigma)(u^\sigma - v)\frac{1}{\sigma}\,dx\,d\sigma = \int_0^1\int_\Omega \big(A(x,u^\sigma,\nabla u^\sigma) - A(x,v,\nabla v)\big)\cdot\nabla(u^\sigma - v)\frac{1}{\sigma}\,dx\,d\sigma \\ &\quad + \int_0^1\int_\Omega \big(B(x,u^\sigma,\nabla u^\sigma) - B(x,v,\nabla v)\big)(u^\sigma - v)\frac{1}{\sigma}\,dx\,d\sigma = I + II, \end{aligned} \tag{3.17}$$

where we have used (3.15). Moreover,

$$\begin{aligned} I &= \int_0^1\int_0^1\int_{|\nabla v|\ge|\nabla u^\sigma|} A_s(x,u^{\sigma\tau},\nabla u^{\sigma\tau})\cdot\nabla(u^\sigma - v)(u^\sigma - v)\frac{1}{\sigma}\,dx\,d\sigma\,d\tau \\ &+ \int_0^1\int_0^1\int_{|\nabla v|<|\nabla u^\sigma|} A_s(x,u^{\sigma(1-\tau)},\nabla u^{\sigma(1-\tau)})\cdot\nabla(u^\sigma - v)(u^\sigma - v)\frac{1}{\sigma}\,dx\,d\sigma\,d\tau \\ &+ \int_0^1\int_0^1\int_{|\nabla v|\ge|\nabla u^\sigma|} \sum_{i,j}\frac{\partial A_i}{\partial\eta_j}(x,u^{\sigma\tau},\nabla u^{\sigma\tau})(u^\sigma - v)_{x_i}(u^\sigma - v)_{x_j}\frac{1}{\sigma}\,dx\,d\sigma\,d\tau \\ &+ \int_0^1\int_0^1\int_{|\nabla v|<|\nabla u^\sigma|} \sum_{i,j}\frac{\partial A_i}{\partial\eta_j}(x,u^{\sigma(1-\tau)},\nabla u^{\sigma(1-\tau)})(u^\sigma - v)_{x_i}(u^\sigma - v)_{x_j}\frac{1}{\sigma}\,dx\,d\sigma\,d\tau \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.18}$$

Now, using (2.2), and the inequality

$$|\eta' + t(\eta - \eta')| \ge \frac{1}{4}|\eta - \eta'|, \quad \text{for} \quad |\eta'| \ge |\eta|,\ 0 \le t \le \frac{1}{4}, \tag{3.19}$$

we get

$$\begin{aligned} I_3 + I_4 &\ge \int_0^1\int_0^1\int_{|\nabla v|\ge|\nabla u^\sigma|} \lambda_0|\nabla u^{\sigma\tau}|^{p(x)-2}|\nabla(u^\sigma - v)|^2\frac{1}{\sigma}\,dx\,d\sigma\,d\tau \\ &+ \int_0^1\int_0^1\int_{|\nabla v|<|\nabla u^\sigma|} \lambda_0|\nabla u^{\sigma(1-\tau)}|^{p(x)-2}|\nabla(u^\sigma - v)|^2\frac{1}{\sigma}\,dx\,d\sigma\,d\tau \\ &\ge \alpha\lambda_0\Big(\int_{\{p(x)\ge 2\}}|\nabla u - \nabla v|^{p(x)}\,dx + \int_{\{p(x)<2\}}\Big(|\nabla u| + |\nabla v|\Big)^{p(x)-2}|\nabla u - \nabla v|^2\,dx\Big), \end{aligned} \tag{3.20}$$

where $\alpha = \alpha(p_{\min}, p_{\max})$ and λ_0 is as in (2.2). On the other hand,

$$\begin{aligned} II &= \int_0^1\int_0^1\int_{|\nabla v|\ge|\nabla u^\sigma|} B_s(x,u^{\sigma\tau},\nabla u^{\sigma\tau})(u^\sigma - v)^2\frac{1}{\sigma}\,dx\,d\sigma\,d\tau \\ &+ \int_0^1\int_0^1\int_{|\nabla v|<|\nabla u^\sigma|} B_s(x,u^{\sigma(1-\tau)},\nabla u^{\sigma(1-\tau)})(u^\sigma - v)^2\frac{1}{\sigma}\,dx\,d\sigma\,d\tau \\ &+ \int_0^1\int_0^1\int_{|\nabla v|\ge|\nabla u^\sigma|} \nabla_\eta B(x,u^{\sigma\tau},\nabla u^{\sigma\tau})\cdot\nabla(u^\sigma - v)(u^\sigma - v)\frac{1}{\sigma}\,dx\,d\sigma\,d\tau \\ &+ \int_0^1\int_0^1\int_{|\nabla v|<|\nabla u^\sigma|} \nabla_\eta B(x,u^{\sigma(1-\tau)},\nabla u^{\sigma(1-\tau)})\cdot\nabla(u^\sigma - v)(u^\sigma - v)\frac{1}{\sigma}\,dx\,d\sigma\,d\tau. \end{aligned} \tag{3.21}$$

Finally, using that $A_s(x,s,\eta)=\nabla_\eta B(x,s,\eta)$, the assumption (3.14) and estimates (3.17), (3.18), (3.20) and (3.21), we get (3.16). □

We now prove a comparison principle for equation (3.1), which holds under assumption (3.14).

Proposition 3.4. *Let p, A and B be as in Section 2. Assume moreover that condition (3.14) holds. Let $u, v \in W^{1,p(\cdot)}(\Omega)$ be such that*

$$\begin{gathered}\operatorname{div}A(x,u,\nabla u)\geq B(x,u,\nabla u)\quad in\ \Omega,\\ \operatorname{div}A(x,v,\nabla v)\leq B(x,v,\nabla v)\quad in\ \Omega,\\ u\leq v\quad on\ \partial\Omega.\end{gathered}\tag{3.22}$$

Then,

$$u\leq v\quad in\ \Omega.\tag{3.23}$$

Proof. We will use arguments similar to those in Proposition 3.3. In fact, for $R>0$ we consider the nonnegative function $w_R\in W_0^{1,p(\cdot)}(\Omega)\cap L^\infty(\Omega)$ given by

$$w_R=\begin{cases}0 & \text{if } u-v\leq 0,\\ u-v & \text{if } 0<u-v<R,\\ R & \text{if } u-v\geq R,\end{cases}\tag{3.24}$$

and by (3.22) we have

$$0\geq\int_\Omega\big(A(x,u,\nabla u)-A(x,v,\nabla v)\big)\cdot\nabla w_R\,dx+\int_\Omega\big(B(x,u,\nabla u)-B(x,v,\nabla v)\big)w_R\,dx=I+II.\tag{3.25}$$

Then, denoting $\Omega_R=\Omega\cap\{0<u-v<R\}$ and, for $0\leq\tau\leq 1$, $u^\tau=v+\tau(u-v)$, we get

$$\begin{aligned}I&=\int_0^1\int_{\Omega_R\cap\{|\nabla v|\geq|\nabla u|\}}A_s(x,u^\tau,\nabla u^\tau)\cdot\nabla(u-v)(u-v)\,dx\,d\tau\\&+\int_0^1\int_{\Omega_R\cap\{|\nabla v|<|\nabla u|\}}A_s(x,u^{(1-\tau)},\nabla u^{(1-\tau)})\cdot\nabla(u-v)(u-v)\,dx\,d\tau\\&+\int_0^1\int_{\Omega_R\cap\{|\nabla v|\geq|\nabla u|\}}\sum_{i,j}\frac{\partial A_i}{\partial\eta_j}(x,u^\tau,\nabla u^\tau)(u-v)_{x_i}(u-v)_{x_j}\,dx\,d\tau\\&+\int_0^1\int_{\Omega_R\cap\{|\nabla v|<|\nabla u|\}}\sum_{i,j}\frac{\partial A_i}{\partial\eta_j}(x,u^{(1-\tau)},\nabla u^{(1-\tau)})(u-v)_{x_i}(u-v)_{x_j}\,dx\,d\tau\\&=I_1+I_2+I_3+I_4.\end{aligned}\tag{3.26}$$

Now, proceeding as in Proposition 3.3, we obtain

$$\begin{aligned}I_3+I_4&\geq\int_0^1\int_{\Omega_R\cap\{|\nabla v|\geq|\nabla u|\}}\lambda_0|\nabla u^\tau|^{p(x)-2}|\nabla(u-v)|^2\,dx\,d\tau\\&+\int_0^1\int_{\Omega_R\cap\{|\nabla v|<|\nabla u|\}}\lambda_0|\nabla u^{(1-\tau)}|^{p(x)-2}|\nabla(u-v)|^2\,dx\,d\tau\\&\geq\tilde\alpha\lambda_0\Big(\int_{\Omega_R\cap\{p(x)\geq 2\}}|\nabla u-\nabla v|^{p(x)}\,dx+\int_{\Omega_R\cap\{p(x)<2\}}\Big(|\nabla u|+|\nabla v|\Big)^{p(x)-2}|\nabla u-\nabla v|^2\,dx\Big),\end{aligned}\tag{3.27}$$

where $\tilde\alpha=\tilde\alpha(p_{\min},p_{\max})$ and λ_0 is as in (2.2).

On the other hand, we observe that the evaluation of (3.14) in $\xi = 0$ implies that $B_s(x, s, \eta) \geq 0$. Then, we get

$$
\begin{aligned}
II \geq & \int_0^1 \int_{\Omega_R \cap \{|\nabla v| \geq |\nabla u|\}} B_s(x, u^\tau, \nabla u^\tau)(u-v)^2 \, dx \, d\tau \\
& + \int_0^1 \int_{\Omega_R \cap \{|\nabla v| < |\nabla u|\}} B_s(x, u^{(1-\tau)}, \nabla u^{(1-\tau)})(u-v)^2 \, dx \, d\tau \\
& + \int_0^1 \int_{\Omega_R \cap \{|\nabla v| \geq |\nabla u|\}} \nabla_\eta B(x, u^\tau, \nabla u^\tau) \cdot \nabla(u-v)(u-v) \, dx \, d\tau \\
& + \int_0^1 \int_{\Omega_R \cap \{|\nabla v| < |\nabla u|\}} \nabla_\eta B(x, u^{(1-\tau)}, \nabla u^{(1-\tau)}) \cdot \nabla(u-v)(u-v) \, dx \, d\tau \\
& + \int_0^1 \int_{\{u-v>R\} \cap \{|\nabla v| \geq |\nabla u|\}} B_s(x, u^\tau, \nabla u^\tau) w_R^2 \, dx \, d\tau \\
& + \int_0^1 \int_{\{u-v>R\} \cap \{|\nabla v| < |\nabla u|\}} B_s(x, u^{(1-\tau)}, \nabla u^{(1-\tau)}) w_R^2 \, dx \, d\tau \\
& + \int_0^1 \int_{\{u-v>R\} \cap \{|\nabla v| \geq |\nabla u|\}} \nabla_\eta B(x, u^\tau, \nabla u^\tau) \cdot \nabla(u-v) w_R \, dx \, d\tau \\
& + \int_0^1 \int_{\{u-v>R\} \cap \{|\nabla v| < |\nabla u|\}} \nabla_\eta B(x, u^{(1-\tau)}, \nabla u^{(1-\tau)}) \cdot \nabla(u-v) w_R \, dx \, d\tau.
\end{aligned} \tag{3.28}
$$

Now, using that $A_s(x, s, \eta) = \nabla_\eta B(x, s, \eta)$, (2.3), (3.19), assumption (3.14) and estimates (3.25), (3.26), (3.27) and (3.28), we get

$$
\begin{aligned}
0 \geq & \frac{1}{2} \tilde{\alpha} \lambda_0 \Big(\int_{\Omega_R \cap \{p(x) \geq 2\}} |\nabla u - \nabla v|^{p(x)} \, dx + \int_{\Omega_R \cap \{p(x) < 2\}} \Big(|\nabla u| + |\nabla v|\Big)^{p(x)-2} |\nabla u - \nabla v|^2 \, dx \Big) \\
& - \hat{\alpha} \Lambda_0 \Big(\int_{\{u-v>R\} \cap \{p(x) \geq 2\}} \Big(|\nabla u| + |\nabla v|\Big)^{p(x)} dx + \int_{\{u-v>R\} \cap \{p(x) < 2\}} |\nabla u - \nabla v|^{p(x)} \, dx \Big),
\end{aligned} \tag{3.29}
$$

where $\hat{\alpha} = \hat{\alpha}(p_{\min}, p_{\max})$ and Λ_0 is as in (2.3). Since $R > 0$ is arbirtrary, we can use that $u, v \in W^{1,p(\cdot)}(\Omega)$ and let $R \to \infty$ and we obtain

$$
0 \geq \frac{1}{2} \tilde{\alpha} \lambda_0 \Big(\int_{\Omega \cap \{p(x) \geq 2\}} |\nabla (u-v)^+|^{p(x)} \, dx + \int_{\Omega \cap \{p(x) < 2\}} \Big(|\nabla u| + |\nabla v|\Big)^{p(x)-2} |\nabla (u-v)^+|^2 \, dx \Big), \tag{3.30}
$$

which implies that $\nabla(u-v)^+ = 0$ in Ω. Since $(u-v)^+ \in W_0^{1,p(\cdot)}(\Omega)$, Poincare's inequality (Theorem A.4) gives $(u-v)^+ = 0$ in Ω. That is, (3.23) holds. □

As a corollary of Propostion 3.4 we obtain the following uniqueness result

Corollary 3.2. *Let p, A and B be as in Section 2. Assume moreover that condition* (3.14) *holds. Let $\varphi \in W^{1,p(\cdot)}(\Omega)$ and let $u_1, u_2 \in W^{1,p(\cdot)}(\Omega)$ be such that*

$$
\begin{cases} \operatorname{div} A(x, u_i, \nabla u_i) = B(x, u_i, \nabla u_i) & \text{in } \Omega, \\ u_i = \varphi & \text{on } \partial\Omega, \end{cases} \tag{3.31}
$$

for $i = 1, 2$. Then, $u_1 = u_2$ in Ω.

We next prove that solutions to (3.1) with bounded boundary data are bounded, under the assumptions of Proposition 3.1.

Proposition 3.5. *Let p, A and B be as in Section 2 and let $\Omega' \subset \Omega$ be a C^1 domain. Assume moreover, that conditions* (3.3), (3.4) *and* (3.14) *hold in Ω' for some $p^+ - p^- < \delta < p_{\min}$ where $p^+ = \sup_{\Omega'} p$ and $p^- = \inf_{\Omega'} p$ and with τ satisfying* (2.7)*. Let us also assume that there exists a positive constant Λ_0 such that the following condition holds:*

$$|B(x,s,\eta)| \leq \Lambda_0(1 + |s|^{p(x)-1} + |\eta|^{p(x)-1}), \tag{3.32}$$

for every $(x,s,\eta) \in \overline{\Omega'} \times \mathbb{R} \times \mathbb{R}^N$. Let $u \in W^{1,p(\cdot)}(\Omega')$ be such that

$$\begin{cases} \mathrm{div} A(x,u,\nabla u) = B(x,u,\nabla u) & \text{in } \Omega', \\ |u| \leq M & \text{on } \partial\Omega', \end{cases} \tag{3.33}$$

for some positive constant M. Then, there exists C such that $|u| \leq C$ in Ω', where C depends only on M, $|\Omega'|$, $\mathrm{diam}(\Omega')$, $N, \lambda_0, \Lambda_0, L, p^-, p^+$, δ, $\|g\|_{L^1(\Omega')}$, $||\tau||_{L^\infty(\Omega')}$, ν and c_1.

Proof. Let v^+ be the solution to (3.1) with boundary data M. Then, from the proof of Proposition 3.1 it follows that $||v^+||_{W^{1,p(\cdot)}(\Omega')}$ depends only on the constants in the structural conditions, on $|\Omega'|$, $\mathrm{diam}(\Omega')$ and M. Since (recall Remark 2.1) we are under the assumptions of Theorem 4.1 in [11], then $v^+ \in L^\infty(\Omega')$ with bounds depending only on the constants in the structural conditions, on $|\Omega'|$, $\mathrm{diam}(\Omega')$ and M. Now, the comparison principle (Proposition 3.4) implies that $u \leq v^+$ in Ω' and the upper bound follows. Proceeding in an analogous way with v^- the solution to (3.1) with boundary data $-M$, we obtain the lower bound, thus concluding the proof. □

As a corollary of Propositions 3.1 and 3.5 we get

Corollary 3.3. *Let p, F, A and B as in Section 2 and let $\Omega' \subset \Omega$ be a C^1 domain. Assume, moreover that F satisfies* (3.8) *and* (3.9) *with τ satisfying* (2.7) *and A and B satisfy* (3.14) *and* (3.32) *in Ω' for some $p^+ - p^- < \delta < p_{\min}$ where $p^+ = \sup_{\Omega'} p$ and $p^- = \inf_{\Omega'} p$.*

Let $u \in W^{1,p(\cdot)}(\Omega') \cap L^\infty(\Omega')$. Then, there exists $v \in u + W_0^{1,p(\cdot)}(\Omega')$ a solution to

$$\mathrm{div} A(x,v,\nabla v) = B(x,v,\nabla v) \quad \text{in} \quad \Omega'.$$

Moreover, $v \in L^\infty(\Omega')$ and $\|v\|_{L^\infty(\Omega')}$ is bounded by a constant C that depends only on $\|u\|_{L^\infty(\Omega')}$, $|\Omega'|$, $\mathrm{diam}(\Omega')$, $N, \lambda_0, \Lambda_0, L, p^-, p^+$, δ, $||\tau||_{L^\infty(\Omega')}$, ν and c_1.

We also prove the following maximum principle

Proposition 3.6. *Let p, A and B be as in Section 2. Assume moreover that condition* (3.14) *holds. We also assume that $B(x,0,0) \equiv 0$ for every $x \in \overline{\Omega}$. Let $u \in W^{1,p(\cdot)}(\Omega)$ be such that*

$$\begin{cases} \mathrm{div} A(x,u,\nabla u) = B(x,u,\nabla u) & \text{in } \Omega, \\ -M_1 \leq u \leq M_2 & \text{on } \partial\Omega, \end{cases} \tag{3.34}$$

for some nonnegative constants M_1, M_2. Then, $-M_1 \leq u \leq M_2$ in Ω.

Proof. Since condition (3.14) implies that $B_s(x,s,\eta) \geq 0$ in $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \setminus \{0\}$, we have $B(x,M_2,0) \geq 0$ and also $B(x,-M_1,0) \leq 0$, for every $x \in \Omega$. Recalling (2.1), we take $v^+ \equiv M_2$ and $v^- \equiv -M_1$ and observe that $\mathrm{div} A(x,v^+,\nabla v^+) \leq B(x,v^+,\nabla v^+)$ and $\mathrm{div} A(x,v^-,\nabla v^-) \geq B(x,v^-,\nabla v^-)$ in Ω. Then, we can apply the comparison principle (Proposition 3.4) and obtain $-M_1 \equiv v^- \leq u \leq v^+ \equiv M_2$ in Ω and the conclusion follows. □

As a corollary of Propositions 3.1 and 3.6 we get

Corollary 3.4. *Let p, F, A and B as in Section 2 and let $\Omega' \subset \Omega$ be a C^1 domain. Assume, moreover that F satisfies* (3.8) *and* (3.9) *with τ satisfying* (2.7) *and A and B satisfy* (3.14) *in Ω' for some $p^+ - p^- < \delta < p_{\min}$ where $p^+ = \sup_{\Omega'} p$ and $p^- = \inf_{\Omega'} p$. We also assume that $B(x,0,0) \equiv 0$ for every $x \in \overline{\Omega'}$.*

Let $u \in W^{1,p(\cdot)}(\Omega') \cap L^\infty(\Omega')$. Then, there exists $v \in u + W_0^{1,p(\cdot)}(\Omega')$ a solution to

$$\mathrm{div} A(x, v, \nabla v) = B(x, v, \nabla v) \quad \text{in} \quad \Omega'.$$

Moreover, $v \in L^\infty(\Omega')$ and $\|v\|_{L^\infty(\Omega')} \leq \|u\|_{L^\infty(\Omega')}$.

We also have the following existence result of a bounded solution

Proposition 3.7. *Let p as in Section 2. Assume that $F(x,\cdot,\cdot)$ is locally Lipschitz in $\mathbb{R} \times \mathbb{R}^N$ for almost every $x \in \Omega$ and that $F(x,s,\cdot) \in C^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{0\})$ for $s \in \mathbb{R}$ and almost every $x \in \Omega$. Let $A = \nabla_\eta F$, $B = F_s$. Assume that A satisfies* (2.2) *and* (2.5),

$$|A(x,s,\eta)|,\ |B(x,s,\eta)| \leq \Lambda_0 (1 + |s|^{\tau(x)} + |\eta|^{p(x)}) \quad \text{a.e. in } \ \Omega \times \mathbb{R} \times \mathbb{R}^N,$$

and F satisfies (3.3) *and* (3.4), *where τ satisfies* (2.7). *Assume moreover that*

$$F(x,s,\eta) = G(x,s,\eta) + f(x,s) \text{ with } G,\ f \text{ measurable functions} \tag{3.35}$$

and,

$$G \geq 0 \text{ in } \Omega \times \mathbb{R} \times \mathbb{R}^N, \quad G(x,s,\eta) = 0 \iff \eta = 0, \tag{3.36}$$

$$f(x,\cdot) \text{ monotone decreasing in } (-\infty, 0] \text{ and monotone increasing in } [0, +\infty). \tag{3.37}$$

Then, for every $\Omega' \subset \Omega$ of class C^1 there holds that, if $p^+ - p^- < \delta < p_{\min}$ where $p^+ = \sup_{\Omega'} p$ and $p^- = \inf_{\Omega'} p$ for δ in (3.3), *given $u \in W^{1,p(\cdot)}(\Omega')$ such that $0 \leq u \leq M$ in Ω' there exists v that minimizes the functional $\mathcal{J}_{\Omega'}(v)$ in $u + W_0^{1,p(\cdot)}(\Omega')$. Moreover, $0 \leq v \leq M$ in Ω'.*

In addition, if there exists $\varepsilon_0 > 0$ such that for almost every $x \in \Omega$, $F(x,\cdot,\cdot) \in C^1((-\varepsilon_0, M + \varepsilon_0) \times \mathbb{R}^N)$, then there holds that v is a solution to

$$\begin{cases} \mathrm{div} A(x,v,\nabla v) = B(x,v,\nabla v) & \text{in } \Omega', \\ v = u & \text{on } \partial\Omega'. \end{cases} \tag{3.38}$$

Proof. To begin with, the existence of a minimizer v follows proceeding as in Proposition 3.1. Let us prove that a minimizer satisfies $0 \leq v \leq M$. In fact, both $w_1 = v - (v-M)^+$ and $w_2 = v + v^-$ are admissible functions. So that on the one hand,

$$\begin{aligned} 0 &\leq \int_{\Omega'} F(x,w_1,\nabla w_1) - F(x,v,\nabla v) = \int_{v>M} F(x,M,0) - F(x,v,\nabla v) \\ &= \int_{v>M} f(x,M) - f(x,v) - \int_{v>M} G(x,v,\nabla v) \\ &\leq -\int_{v>M} G(x,v,\nabla v) \leq 0. \end{aligned}$$

Hence, $G(x,v,\nabla v) = 0$ in $\{v > M\}$. So that, $\nabla(v-M)^+ = 0$ in Ω'. As $(v-M)^+ = 0$ on $\partial\Omega'$, we deduce that $v \leq M$ in Ω'.

On the other hand, proceeding in a similar way with w_2,

$$\begin{aligned}
0 &\le \int_{\Omega'} F(x,w_2,\nabla w_2) - F(x,v,\nabla v) = \int_{v<0} F(x,0,0) - F(x,v,\nabla v)\\
&= \int_{v<0} f(x,0) - f(x,v) - \int_{v<0} G(x,v,\nabla v)\\
&\le -\int_{v<0} G(x,v,\nabla v) \le 0,
\end{aligned}$$

and we deduce as before that $v^- = 0$. This is, $v \ge 0$ in Ω'.

Now, in order to proceed with the proof we assume further regularity of F for $-\varepsilon_0 \le s \le M+\varepsilon_0$. Let $0 \le \varphi \in C_0^\infty(\Omega')$ and $0 < \varepsilon < \varepsilon_0/\|\varphi\|_{L^\infty}$. Then, $w = v + \varepsilon\varphi$ is an admissible function, $-\varepsilon_0 < w < M + \varepsilon_0$ and we deduce that

$$\text{div} A(x,v,\nabla v) \le B(x,v,\nabla v) \quad \text{in } \Omega'.$$

Replacing φ by $-\varphi$ we reverse the inequality. So that, v is a solution to (3.38). □

4. Energy minimizers of energy functional (1.1)

In this section we prove properties of nonnegative local minimizers of the energy functional (1.1). We prove that nonnegative local minimizers are locally Hölder continuous (Theorem 4.3) and are solutions to

$$\text{div} A(x,u,\nabla u) = B(x,u,\nabla u) \quad \text{in } \{u>0\},$$

where $A(x,s,\eta) = \nabla_\eta F(x,s,\eta)$ and $B(x,s,\eta) = F_s(x,s,\eta)$. In particular we prove our main result which is the local Lipschitz continuity on nonnegative local minimizers (Theorem 4.5).

We start with a definition, some related remarks and an existence result of a minimizer. We also prove nonnegativity and boundedness, under suitable assumptions.

Definition 4.1. Let p, F and λ be as in Section 2. Assume that F satisfies (3.3) and (3.4) with τ satisfying (2.7). We say that $u \in W^{1,p(\cdot)}(\Omega)$ is a local minimizer in Ω of

$$J(v) = J_\Omega(v) = \int_\Omega \big(F(x,v,\nabla v) + \lambda(x)\chi_{\{v>0\}}\big)\,dx$$

if for every $\Omega' \subset\subset \Omega$ and for every $v \in W^{1,p(\cdot)}(\Omega)$ such that $v = u$ in $\Omega \setminus \Omega'$ there holds that $J(v) \ge J(u)$.

We point out that the energy J is well defined in $W^{1,p(\cdot)}(\Omega)$ since, under the conditions in (2.7), the embedding theorem (see Theorem A.5) applies.

Remark 4.1. Let u be as in Definition 4.1. Let $\Omega' \subset\subset \Omega$ and $w - u \in W_0^{1,p(\cdot)}(\Omega')$. If we define

$$\bar w = \begin{cases} w & \text{in } \Omega',\\ u & \text{in } \Omega\setminus\Omega',\end{cases}$$

then $\bar w \in W^{1,p(\cdot)}(\Omega)$ and therefore $J(\bar w) \ge J(u)$. If we now let

$$J_{\Omega'}(v) = \int_{\Omega'} \big(F(x,v,\nabla v) + \lambda(x)\chi_{\{v>0\}}\big)\,dx$$

it follows that $J_{\Omega'}(w) \ge J_{\Omega'}(u)$.

Remark 4.2. Let J be as in Definition 4.1. If $u \in W^{1,p(\cdot)}(\Omega)$ is a minimizer of J among the functions $v \in u + W_0^{1,p(\cdot)}(\Omega)$, then u is a local minimizer of J in Ω.

We start with an existence result of a minimizer to (1.1).

Theorem 4.1. *Let p, F, A, B and λ be as in Section 2. Let $\phi \in W^{1,p(\cdot)}(\Omega)$ and assume moreover that F satisfies* (3.10) *and* (3.11) *with τ satisfying* (2.7).

Then, there exists a minimizer $u \in \phi + W_0^{1,p(\cdot)}(\Omega)$ to (1.1) *and there holds that $\|u\|_{W^{1,p(\cdot)}(\Omega)} \leq C$, for a constant C depending only on $\|\phi\|_{W^{1,p(\cdot)}(\Omega)}$ $\|g\|_{L^1(\Omega)}$, $\lambda_{\max}$, $|\Omega|$,* diam(Ω), *N, $p_{\min}$, $p_{\max}$, q, L, ν, c_1, $||\tau||_{L^\infty(\Omega)}$ and the C^1 norm of $\partial\Omega$.*

Proof. The proof is immediate from the computations in the proof of Proposition 3.2. □

We also have,

Theorem 4.2. *Let p and λ be as in Section 2. Let F, A and B be as in Propostion 3.7, except for the fact that we require that F satisfies* (3.10) *and* (3.11) *with τ satisfying* (2.7), *instead of* (3.3) *and* (3.4), *and with no oscillation assumption on p. Let $\phi \in W^{1,p(\cdot)}(\Omega)$ such that $0 \leq \phi \leq M$, for some $M > 0$.*

Then, there exists a minimizer $u \in \phi + W_0^{1,p(\cdot)}(\Omega)$ to (1.1) *and $0 \leq u \leq M$ in Ω.*

Proof. Proceeding as in the proof of Proposition 3.2 we obtain that there exists a minimizer $u \in \phi + W_0^{1,p(\cdot)}(\Omega)$ to (1.1). The proof that $0 \leq u \leq M$ is similar to that of Proposition 3.7. We only have to observe that

$$\{u - (u - M)^+ > 0\} = \{u > 0\} \qquad \text{and} \qquad \{u + u^- > 0\} = \{u > 0\}.$$

□

For local minimizers of (1.1) we first have

Lemma 4.1. *Let p, F, A, B and λ be as in Section 2. Assume that F satisfies* (3.3) *and* (3.4) *with τ satisfying* (2.7). *Let $u \in W^{1,p(\cdot)}(\Omega)$ be a local minimizer of*

$$J(v) = \int_\Omega \big(F(x, v, \nabla v) + \lambda(x)\chi_{\{v>0\}}\big)\, dx.$$

Then

$$\text{div} A(x, u, \nabla u) \geq B(x, u, \nabla u) \quad \text{in } \Omega, \tag{4.1}$$

where $A(x, s, \eta) = \nabla_\eta F(x, s, \eta)$ and $B(x, s, \eta) = F_s(x, s, \eta)$.

Proof. In fact, let $t > 0$ and $0 \leq \xi \in C_0^\infty(\Omega)$. Using the minimality of u we have

$$0 \leq \frac{1}{t}(J(u - t\xi) - J(u)) \leq \frac{1}{t}\int_\Omega \big(F(x, u - t\xi, \nabla u - t\nabla\xi) - F(x, u, \nabla u)\big)\, dx$$

and if we take $t \to 0$, we obtain

$$0 \leq -\int_\Omega \nabla_\eta F(x, u, \nabla u) \cdot \nabla\xi\, dx - \int_\Omega F_s(x, u, \nabla u)\xi\, dx, \tag{4.2}$$

which gives (4.1). □

From now on we will deal with nonnegative, bounded, local minimizers of (1.1). Next we will prove that they are locally Lipschitz continuous.

We first prove that nonnegative, bounded, local minimizers are locally Hölder continuous.

Theorem 4.3. *Let p, F, A, B and λ be as in Section 2. Assume that F satisfies* (3.3) *and* (3.4) *with τ satisfying* (2.7). *Let $x_0 \in \Omega$, $\hat{r}_0 > 0$ such that $B_{\hat{r}_0}(x_0) \subset\subset \Omega$. Assume that A, B satisfy condition* (3.14) *in $B_{\hat{r}_0}(x_0)$ and either $B(x, 0, 0) \equiv 0$ for $x \in B_{\hat{r}_0}(x_0)$ or B satisfies* (3.32) *for $x \in B_{\hat{r}_0}(x_0)$. Let $u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ be a nonnegative local minimizer of* (1.1). *Then, there exist $0 < \gamma < 1$, $\gamma = \gamma(N, p_{\min})$ and $0 < \hat{\rho}_0 < \hat{r}_0$, such that $u \in C^\gamma(\overline{B_{\hat{\rho}_0}(x_0)})$. Moreover, $\|u\|_{C^\gamma(\overline{B_{\hat{\rho}_0}(x_0)})} \le C$ with $\hat{\rho}_0$ and C depending only on $\beta, p_{\max}, p_{\min}, N, L, \hat{r}_0, \lambda_0, \Lambda_0$, $\|g\|_{L^1(B_{\hat{r}_0}(x_0))}$, ν, c_1, $\lambda_{\max}$, $\|u\|_{L^\infty(B_{\hat{r}_0}(x_0))}$, $||\tau||_{L^\infty(B_{\hat{r}_0}(x_0))}$ and δ.*

Proof. We will prove that there exist $0 < \gamma < 1$ and $0 < \rho_0 < r_0 < \hat{r}_0$ such that, if $B_{r_0}(y) \subset B_{\hat{r}_0}(x_0)$ and $\rho \le \rho_0$, then

$$\Big(\fint_{B_\rho(y)} |\nabla u|^{p_-}\,dx\Big)^{1/p_-} \le C\rho^{\gamma-1}, \tag{4.3}$$

where $p_- = \inf\{p(x), x \in B_{r_0}(y)\}$. Without loss of generality we will assume that $y = 0$.

In fact, let $0 < r_0 \le \min\{\frac{\hat{r}_0}{2}, 1\}$, $0 < r \le r_0$ and v the solution of

$$\operatorname{div} A(x, v, \nabla v) = B(x, v, \nabla v) \quad \text{in } B_r, \qquad v - u \in W_0^{1,p(\cdot)}(B_r). \tag{4.4}$$

Observe that, under our assumptions we can apply either Proposition 3.1 and Proposition 3.5 or Proposition 3.6 and deduce that such a solution exists and it is bounded in $\overline{B}_r$ if r_0 is small enough depending on δ and $L = \|\nabla p\|_{L^\infty(\Omega)}$. Hence, by Proposition 3.3, we have

$$\begin{aligned}&\int_{B_r} \big(F(x, u, \nabla u) - F(x, v, \nabla v)\big)\,dx \ge \\ &\quad \frac{1}{2}\alpha\lambda_0\Big(\int_{B_r\cap\{p(x)\ge 2\}} |\nabla u - \nabla v|^{p(x)}\,dx + \int_{B_r\cap\{p(x)<2\}} \Big(|\nabla u| + |\nabla v|\Big)^{p(x)-2}|\nabla u - \nabla v|^2\,dx\Big),\end{aligned} \tag{4.5}$$

where $\alpha = \alpha(p_{\min}, p_{\max})$ and λ_0 is as in (2.2).

By the minimality of u, we have (if $A_1 = B_r \cap \{p(x) < 2\}$ and $A_2 = B_r \cap \{p(x) \ge 2\}$)

$$\int_{A_2} |\nabla u - \nabla v|^{p(x)}\,dx \le Cr^N, \tag{4.6}$$

$$\int_{A_1} |\nabla u - \nabla v|^2(|\nabla u| + |\nabla v|)^{p(x)-2}\,dx \le Cr^N, \tag{4.7}$$

where $C = C(p_{\min}, p_{\max}, N, \lambda_{\max}, \lambda_0)$.

Let $\varepsilon > 0$. Take $\rho = r^{1+\varepsilon}$ and suppose that $r^\varepsilon \le 1/2$. Take $0 < \eta < 1$ to be chosen later. Then, by Young's inequality, the definition of A_1 and (4.7), we obtain

$$\begin{aligned}\int_{A_1\cap B_\rho} |\nabla u - \nabla v|^{p(x)}\,dx \le& \frac{C}{\eta^{2/p_{\min}}}\int_{A_1\cap B_r} (|\nabla u| + |\nabla v|)^{p(x)-2}|\nabla u - \nabla v|^2\,dx \\ &+ C\eta\int_{B_\rho\cap A_1} (|\nabla u| + |\nabla v|)^{p(x)}\,dx \\ \le& \frac{C}{\eta^{2/p_{\min}}} r^N + C\eta\int_{B_\rho\cap A_1} (|\nabla u| + |\nabla v|)^{p(x)}\,dx.\end{aligned} \tag{4.8}$$

Therefore, by (4.6) and (4.8), we get

$$\int_{B_\rho} |\nabla u - \nabla v|^{p(x)}\,dx \le \frac{C}{\eta^{2/p_{\min}}} r^N + C\eta\int_{B_\rho\cap A_1} (|\nabla u| + |\nabla v|)^{p(x)}\,dx, \tag{4.9}$$

where $C = C(p_{\min}, p_{\max}, N, \lambda_{\max}, \lambda_0)$.

Since, $|\nabla u|^q \le C(|\nabla u - \nabla v|^q + |\nabla v|)^q)$, for any $q > 1$, with $C = C(q)$, we have, by (4.9), choosing η small, that

$$\int_{B_\rho} |\nabla u|^{p(x)}\,dx \le Cr^N + C\int_{B_\rho} |\nabla v|^{p(x)}\,dx, \tag{4.10}$$

where $C = C(p_{\min}, p_{\max}, N, \lambda_{\max}, \lambda_0)$.

Now let $M \ge 1$ such that $||v||_{L^\infty(B_r)} \le M$ and define

$$w(x) = \frac{v(rx)}{M} \quad \text{in} \quad B_1.$$

Observe that M depends only on $\|u\|_{L^\infty(B_{\hat r_0}(x_0))}$ if $B(x,0,0) \equiv 0$ or it depends also on the structural conditions on F, A and B, on $\hat r_0$ and on the bound L of $\|\nabla p\|_{L^\infty}$ if not.

There holds that,

$$\mathrm{div}\bar A(x, w, \nabla w) = \bar B(x, w, \nabla w) \quad \text{in} \quad B_1$$

where

$$\bar A(x, s, \eta) = A(rx, Ms, \tfrac{M}{r}\eta), \qquad \bar B(x, s, \eta) = rB(rx, Ms, \tfrac{M}{r}\eta).$$

Now, let

$$\widetilde A(x, s, \eta) = \Big(\frac{r}{M}\Big)^{p_r^- - 1} \bar A(x, s, \eta), \qquad \widetilde B(x, s, \eta) = \Big(\frac{r}{M}\Big)^{p_r^- - 1} \bar B(x, s, \eta).$$

Observe that $w \in W^{1,\bar p(\cdot)}(B_1) \cap L^\infty(B_1)$ satisfies

$$\mathrm{div}\widetilde A(x, w, \nabla w) = \widetilde B(x, w, \nabla w) \quad \text{in} \quad B_1, \tag{4.11}$$

where $\bar p(x) = p(rx)$.

Let us see that (4.11) is under the conditions of Theorem 1.1 in [10].

First, we clearly have $\widetilde A(x, s, 0) = 0$. Moreover, as $1 \le r^{p_r^- - p_r^+} \le C_L < \infty$ if $r \le 1$ and we have assumed that $M \ge 1$,

$$\begin{aligned} \sum_{ij} \frac{\partial \widetilde A_i}{\partial \eta_j}(x, s, \eta)\xi_i\xi_j &= \Big(\frac{r}{M}\Big)^{p_r^- - 1}\Big(\frac{M}{r}\Big) \sum_{ij} \frac{\partial A_i}{\partial \eta_j}(rx, Ms, \tfrac{M}{r}\eta)\xi_i\xi_j \\ &\ge \lambda_0 \Big(\frac{r}{M}\Big)^{p_r^- - 1}\Big(\frac{M}{r}\Big)^{p(rx)-1} |\eta|^{\bar p(x)-2}|\xi|^2 \ge \lambda_0|\eta|^{\bar p(x)-2}|\xi|^2. \end{aligned} \tag{4.12}$$

On the other hand,

$$\sum_{ij} \Big|\frac{\partial \widetilde A_i}{\partial \eta_j}(x, s, \eta)\Big| \le \Lambda_0 \Big(\frac{r}{M}\Big)^{p_r^- - 1}\Big(\frac{M}{r}\Big)^{p(rx)-1} |\eta|^{\bar p(x)-2} \le \Lambda_0 C_L M^{p_{\max} - p_{\min}} |\eta|^{\bar p(x)-2}. \tag{4.13}$$

Then, assuming without loss of generality that $p(rx_1) \ge p(rx_2)$,

$$\begin{aligned} |\widetilde A(x_1, s, \eta) - \widetilde A(x_2, s, \eta)| &\le \Big(\frac{r}{M}\Big)^{p_r^- - 1} \big|A(rx_1, Ms, \tfrac{M}{r}\eta) - A(rx_2, Ms, \tfrac{M}{r}\eta)\big| \\ &\le \Big(\frac{r}{M}\Big)^{p_r^- - 1} \Lambda_0 \Big(\Big(\frac{M}{r}\Big)^{p(rx_1)-1} |\eta|^{p(rx_1)-1} + \Big(\frac{M}{r}\Big)^{p(rx_2)-1} |\eta|^{p(rx_2)-1}\Big) \\ &\qquad \big(1 + \big|\log\big(\tfrac{M}{r}|\eta|\big)\big|\big) r^\beta |x_1 - x_2|^\beta \\ &\le \Lambda_0 C_L M^{p_{\max} - p_{\min}} \big(|\eta|^{\bar p(x_1)-1} + |\eta|^{\bar p(x_2)-1}\big)\big(1 + \big|\log|\eta|\big|\big)|x_1 - x_2|^\beta \end{aligned} \tag{4.14}$$

if $r \le r_{M,\beta}$.

Similarly,

$$|\widetilde{A}(x,s_1,\eta)-\widetilde{A}(x,s_2,\eta)|\le \Lambda_4|s_1-s_2||\eta|^{\bar p(x)-1}$$

with $\Lambda_4=\Lambda_0C_LM^{p_{\max}-p_{\min}+1}$.

On the other hand, denoting $\bar\tau(x)=\tau(rx)$,

$$\begin{aligned}|\widetilde{B}(x,s,\eta)|&\le \Lambda_0 r\Big(\frac{r}{M}\Big)^{p_r^- -1}+\Lambda_0C_LM^{p_{\max}-p_{\min}+1}|\eta|^{\bar p(x)}+\Lambda_0 r\Big(\frac{r}{M}\Big)^{p_r^- -1}|Ms|^{\bar\tau(x)}\\ &\le \Lambda_5\big(1+|\eta|^{\bar p(x)}+|s|^{\bar\tau(x)}\big)\end{aligned}$$

with Λ_5 depending on Λ_0, L, $p_{\max}$, $p_{\min}$, M and $||\tau||_{L^\infty(B_{\hat r_0}(x_0))}$.

Since $|w|\le 1$, we may assume that

$$\widetilde{B}(x,s,\eta)\le \Lambda_6\big(1+|\eta|^{\bar p(x)}\big),$$

with Λ_6 depending on Λ_0, L, $p_{\min}$, $p_{\max}$, M and $||\tau||_{L^\infty(B_{\hat r_0}(x_0))}$.

From Theorem 1.1 in [10], it follows that $w\in C^{1,\alpha}_{\rm loc}(B_1)$ for some $0<\alpha<1$ and that

$$\sup_{B_{1/2}}|\nabla w|\le C,$$

with C depending only on $\beta, p_{\max}, p_{\min}, N, L, \lambda_0, \Lambda_0$, M and $||\tau||_{L^\infty(B_{\hat r_0}(x_0))}$, which implies

$$\sup_{B_{r/2}}|\nabla v|\le \frac{CM}{r}. \tag{4.15}$$

Therefore, from (4.10) and (4.15), we deduce that if r is small depending on M and β,

$$\int_{B_\rho}|\nabla u|^{p(x)}\,dx\le Cr^N+C\rho^N r^{-p_+}, \tag{4.16}$$

with $p_+=\sup\{p(x), x\in B_{r_0}\}$ and C depending on $\beta, p_{\max}, p_{\min}, N, L, \lambda_0, \Lambda_0$, $\lambda_{\max}$, M and $||\tau||_{L^\infty(B_{\hat r_0}(x_0))}$.

Then, if we take $\varepsilon\le \frac{p_{\min}}{N}$, we have by (4.16) and by our election of ρ, that

$$\begin{aligned}\fint_{B_\rho}|\nabla u|^{p_-}\,dx&\le \fint_{B_\rho}|\nabla u|^{p(x)}\,dx+\frac{1}{|B_\rho|}\int_{B_\rho\cap\{|\nabla u|<1\}}|\nabla u|^{p_-}\,dx\\ &\le \fint_{B_\rho}|\nabla u|^{p(x)}\,dx+1\\ &\le 1+C\Big(\frac{r}{\rho}\Big)^N+Cr^{-p_+}\\ &\le 1+Cr^{-\varepsilon N}+Cr^{-p_+}\\ &\le Cr^{-p_+}=C\rho^{-\frac{p_+}{(1+\varepsilon)}}.\end{aligned}$$

Now let $r_0\le r_0(\varepsilon,p_{\min},L)$ so that

$$\frac{p_+}{p_-}=\frac{p_+(B_{r_0})}{p_-(B_{r_0})}\le 1+\frac{\varepsilon}{2},$$

and small enough so that, in addition, $r_0^\varepsilon \leq 1/2$. Then, if $\rho \leq \rho_0 = r_0^{1+\varepsilon}$ and moreover, r_0 is small depending on M and β,

$$\fint_{B_\rho} |\nabla u|^{p_-}\, dx \leq C\rho^{-\frac{(1+\frac{\varepsilon}{2})}{(1+\varepsilon)}p_-} = C\rho^{-(1-\gamma)p_-},$$

where $\gamma = \frac{\frac{\varepsilon}{2}}{(1+\varepsilon)} = \gamma(N, p_{\min})$. That is, if $\rho \leq \rho_0 = r_0^{1+\varepsilon}$

$$\Big(\fint_{B_\rho} |\nabla u|^{p_-}\, dx\Big)^{1/p_-} \leq C\rho^{\gamma-1}.$$

Thus (4.3) holds, with C depending only on $\beta, p_{\max}, p_{\min}, N, L, \hat{r}_0, \lambda_0, \Lambda_0$, $\|g\|_{L^1(B_{\hat{r}_0}(x_0))}$, ν, c_1, $\lambda_{\max}$, $\|u\|_{L^\infty(B_{\hat{r}_0}(x_0))}$, $||\tau||_{L^\infty(B_{\hat{r}_0}(x_0))}$ and δ.

Applying Morrey's Theorem, see e.g. [18], Theorem 1.53, we conclude that $u \in C^\gamma(B_{\rho_0}(x_0))$ and $\|u\|_{C^\gamma(\overline{B_{\rho_0/2}(x_0)})} \leq C$ for C depending only on $\beta, p_{\max}, p_{\min}, N, L, \hat{r}_0, \lambda_0, \Lambda_0$, $\|g\|_{L^1(B_{\hat{r}_0}(x_0))}$, ν, c_1, $\lambda_{\max}$, $\|u\|_{L^\infty(B_{\hat{r}_0}(x_0))}$, $||\tau||_{L^\infty(B_{\hat{r}_0}(x_0))}$ and δ. □

As a corollary we obtain

Corollary 4.1. *Let p, F, A, B and λ be as in Section 2. Assume that F satisfies* (3.8) *and* (3.9) *with τ satisfying* (2.7). *Assume that A, B satisfy condition* (3.14) *and either $B(x, 0, 0) \equiv 0$ for $x \in \Omega$ or B satisfies* (3.32) *for $x \in \Omega$. Let $u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ be a nonnegative local minimizer of* (1.1). *Then, there exists $0 < \gamma < 1$, $\gamma = \gamma(N, p_{\min})$ such that $u \in C^\gamma(\Omega)$. Moreover, if $\Omega' \subset\subset \Omega$, then $\|u\|_{C^\gamma(\overline{\Omega'})} \leq C$ with C depending only on* $\operatorname{dist}(\Omega', \partial\Omega)$, *$\beta$, N, $p_{\min}$, $p_{\max}$, L, $\lambda_{\max}$, λ_0, Λ_0, ν, c_1, $\|u\|_{L^\infty(\Omega)}$, $||\tau||_{L^\infty(\Omega)}$ and δ.*

Then, under the assumptions of the previous corollary we have that u is continuous in Ω and therefore, $\{u > 0\}$ is open. We can now prove the following property for nonnegative local minimizers of (1.1)

Lemma 4.2. *Let p, F, A, B and λ be as in Corollary 4.1. If $u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ is a nonnegative local minimizer of*

$$J(v) = \int_\Omega \big(F(x, v, \nabla v) + \lambda(x)\chi_{\{v>0\}}\big)\, dx,$$

there holds that,

$$\operatorname{div} A(x, u, \nabla u) = B(x, u, \nabla u) \quad \text{in } \{u > 0\}, \tag{4.17}$$

where $A(x, s, \eta) = \nabla_\eta F(x, s, \eta)$ and $B(x, s, \eta) = F_s(x, s, \eta)$.

Proof. From Lemma 4.1 we already know that (4.1) holds. In order to obtain the opposite inequality in $\{u > 0\}$, we let $0 \leq \xi \in C_0^\infty(\{u > 0\})$ and consider $u - t\xi$, for $t < 0$, with $|t|$ small.

Using the minimality of u we have

$$0 \geq \frac{1}{t}(J(u - t\xi) - J(u)) = \frac{1}{t}\int_\Omega \big(F(x, u - t\xi, \nabla u - t\nabla\xi) - F(x, u, \nabla u)\big)\, dx$$

and if we take $t \to 0$, we obtain

$$0 \geq -\int_\Omega \nabla_\eta F(x, u, \nabla u) \cdot \nabla\xi\, dx - \int_\Omega F_s(x, u, \nabla u)\xi\, dx,$$

which gives the desired inequality, so (4.17) follows. □

We will next prove the Lipschitz continuity of nonnegative local minimizers of (1.1).

Before getting the Lipschitz continuity we prove the following result

Theorem 4.4. *Let p, F, A, B, λ and u be as in Corollary 4.1. Let $\Omega' \subset\subset \Omega$. There exist constants $C > 0$, $r_0 > 0$ such that if $x_0 \in \Omega' \cap \partial\{u > 0\}$ and $r \leq r_0$ then*

$$\sup_{B_r(x_0)} u \leq Cr.$$

The constants depend only on $\operatorname{dist}(\Omega', \partial\Omega)$, β, N, $p_{\min}$, $p_{\max}$, L, $\lambda_{\max}$, λ_0, Λ_0, ν, c_1, $\|u\|_{L^\infty(\Omega)}$, $||\tau||_{L^\infty(\Omega)}$ *and* δ.

Proof. Let us suppose by contradiction that there exist a sequence of nonnegative local minimizers u_k corresponding to functionals J_k given by

$$J_k(v) = \int_\Omega \big(F_k(x, v, \nabla v) + \lambda_k(x)\chi_{\{v>0\}}\big)\, dx,$$

with $u_k \in W^{1,p_k(\cdot)}(\Omega) \cap L^\infty(\Omega)$, $p_{\min} \leq p_k(x) \leq p_{\max}$, $\|\nabla p_k\|_{L^\infty} \leq L$, $0 \leq \lambda_k(x) \leq \lambda_{\max}$, $||u_k||_{L^\infty(\Omega)} \leq M$, for some $M \geq 1$, and points $\bar{x}_k \in \Omega' \cap \partial\{u_k > 0\}$, such that

$$\sup_{B_{r_k/4}(\bar{x}_k)} u_k \geq kr_k \quad \text{and} \quad r_k \leq \frac{1}{k}.$$

We denote $A_k(x, s, \eta) = \nabla_\eta F_k(x, s, \eta)$ and $B_k(x, s, \eta) = (F_k)_s(x, s, \eta)$ and we also suppose that p_k, F_k, A_k, B_k and τ_k satisfy the assumptions in Section 2 with constants λ_0, Λ_0 and β, we assume that A_k, B_k satisfy condition (3.14) and F_k satisfy (3.8) and (3.9) with τ_k satisfying (2.7) and either and $B_k(x, 0, 0) \equiv 0$ for $x \in \Omega$ or B_k satisfy (3.32) for $x \in \Omega$. All these conditions with exponent p_k and constants independent of k and with $||\tau_k||_{L^\infty(\Omega)} \leq \tau_0$, for some $\tau_0 > 0$.

Without loss of generality we will assume that $\bar{x}_k = 0$.

Let us define in B_1, for k large, $\bar{u}_k(x) = \frac{1}{r_k} u_k(r_k x)$, $\bar{p}_k(x) = p_k(r_k x)$ and $\bar{\lambda}_k(x) = \lambda_k(r_k x)$. Then $p_{\min} \leq \bar{p}_k(x) \leq p_{\max}$, $\|\nabla \bar{p}_k\|_{L^\infty(B_1)} \leq Lr_k$ and $0 \leq \bar{\lambda}_k(x) \leq \lambda_{\max}$. Moreover, $\bar{u}_k$ is a nonnegative minimizer in $\bar{u}_k + W_0^{1,\bar{p}_k(\cdot)}(B_1)$ of the functional

$$\bar{J}_k(v) = \int_{B_1} \Big(\bar{F}_k(x, v, \nabla v) + \bar{\lambda}_k(x)\chi_{\{v>0\}}\Big)\, dx, \tag{4.18}$$

where

$$\bar{F}_k(x, s, \eta) = F_k(r_k x, r_k s, \eta),$$

with

$$\bar{u}_k(0) = 0 \qquad \text{and} \qquad \max_{\overline{B}_{1/4}} \bar{u}_k(x) > k.$$

Let $d_k(x) = \operatorname{dist}(x, \{\bar{u}_k = 0\})$ and $\mathcal{O}_k = \Big\{x \in B_1 : d_k(x) \leq \dfrac{1 - |x|}{3}\Big\}$. Since $\bar{u}_k(0) = 0$ then $\overline{B}_{1/4} \subset \mathcal{O}_k$, therefore

$$m_k := \sup_{\mathcal{O}_k}(1 - |x|)\bar{u}_k(x) \geq \max_{\overline{B}_{1/4}}(1 - |x|)\bar{u}_k(x) \geq \frac{3}{4} \max_{\overline{B}_{1/4}} \bar{u}_k(x) > \frac{3}{4}k.$$

For each fixed k, $\bar{u}_k$ is bounded, then $(1 - |x|)\bar{u}_k(x) \to 0$ when $|x| \to 1$ which means that there exists $x_k \in \mathcal{O}_k$ such that $(1 - |x_k|)\bar{u}_k(x_k) = \sup_{\mathcal{O}_k}(1 - |x|)\bar{u}_k(x)$, and then

$$\bar{u}_k(x_k) = \frac{m_k}{1 - |x_k|} \geq m_k > \frac{3}{4}k \tag{4.19}$$

as $x_k \in \mathcal{O}_k$. Observe that $\delta_k := d_k(x_k) \leq \frac{1-|x_k|}{3}$. Let $y_k \in \partial\{\bar{u}_k > 0\} \cap B_1$ such that $|y_k - x_k| = \delta_k$. Then,

(1) $B_{2\delta_k}(y_k) \subset B_1$,

since if $y \in B_{2\delta_k}(y_k) \Rightarrow |y| < 3\delta_k + |x_k| \leq 1$,

(2) $B_{\frac{\delta_k}{2}}(y_k) \subset \mathcal{O}_k$,

since if $y \in B_{\frac{\delta_k}{2}}(y_k) \Rightarrow |y| \leq \dfrac{3}{2}\delta_k + |x_k| \leq 1 - \dfrac{3}{2}\delta_k \Rightarrow d_k(y) \leq \dfrac{\delta_k}{2} \leq \dfrac{1-|y|}{3}$ and

(3) if $z \in B_{\frac{\delta_k}{2}}(y_k) \Rightarrow 1 - |z| \geq 1 - |x_k| - |x_k - z| \geq 1 - |x_k| - \dfrac{3}{2}\delta_k \geq \dfrac{1-|x_k|}{2}$.

By (2) we have

$$\max_{\mathcal{O}_k}(1-|x|)\bar{u}_k(x) \geq \max_{B_{\frac{\delta_k}{2}}(y_k)}(1-|x|)\bar{u}_k(x) \geq \max_{B_{\frac{\delta_k}{2}}(y_k)} \frac{(1-|x_k|)}{2}\bar{u}_k(x),$$

where in the last inequality we are using (3). Then,

$$2\bar{u}_k(x_k) \geq \max_{B_{\frac{\delta_k}{2}}(y_k)} \bar{u}_k(x). \tag{4.20}$$

As $B_{\delta_k}(x_k) \subset \{\bar{u}_k > 0\}$, then $B_{r_k\delta_k}(r_kx_k) \subset \{u_k > 0\}$. Hence, $\text{div}A_k(x, u_k, \nabla u_k) = B_k(x, u_k, \nabla u_k)$ in $B_{r_k\delta_k}(r_kx_k)$. Recalling that $||u_k||_{L^\infty(B_{r_k\delta_k}(r_kx_k))} \leq M$, we can replace $|s|^{\tau_k(x)}$ in (2.6) for B_k by M^{τ_0}. Then we can apply Harnack's inequality (Theorem 3.2 in [23]) and we thus have

$$\max_{B_{\frac{3}{4}r_k\delta_k}(r_kx_k)} u_k(x) \leq C\big[\min_{B_{\frac{3}{4}r_k\delta_k}(r_kx_k)} u_k(x) + r_k\delta_k\big], \tag{4.21}$$

with C a positive constant depending only on $N, p_{\min}, p_{\max}, L$, M, λ_0, Λ_0 and τ_0.

It follows that

$$\max_{B_{\frac{3}{4}\delta_k}(x_k)} \bar{u}_k(x) \leq C\big[\min_{B_{\frac{3}{4}\delta_k}(x_k)} \bar{u}_k(x) + \delta_k\big]. \tag{4.22}$$

Recalling (4.19), we get from (4.22), for k large,

$$\min_{B_{\frac{3}{4}\delta_k}(x_k)} \bar{u}_k(x) \geq c\bar{u}_k(x_k), \tag{4.23}$$

with c a positive constant depending only on $N, p_{\min}, p_{\max}, L$ M, λ_0, Λ_0 and τ_0. As $\overline{B_{\frac{3}{4}\delta_k}}(x_k) \cap \overline{B_{\frac{\delta_k}{4}}}(y_k) \neq \emptyset$ we have by (4.23)

$$\max_{B_{\frac{\delta_k}{4}}(y_k)} \bar{u}_k(x) \geq c\bar{u}_k(x_k). \tag{4.24}$$

Let $w_k(x) = \dfrac{\bar{u}_k(y_k + \frac{\delta_k}{2}x)}{\bar{u}_k(x_k)}$. Then, $w_k(0) = 0$ and, by (4.20) and (4.24), we have

$$\max_{B_1} w_k \leq 2 \qquad \max_{B_{1/2}} w_k \geq c > 0. \tag{4.25}$$

Now, recalling that $\bar u_k$ is a nonnegative minimizer in $\bar u_k+W_0^{1,\bar p_k(\cdot)}(B_1)$ of the functional $\bar J_k$ in (4.18) and that $B_{\frac{\delta_k}{2}}(y_k)\subset B_1$, we see that w_k is a nonnegative minimizer of $\hat J_k$ in $w_k+W_0^{1,\bar p_k(y_k+\frac{\delta_k}{2}x)}(B_1)$, where

$$\hat J_k(v)=\int_{B_1}\Big(\hat F_k(x,v,\nabla v)+\hat\lambda_k(x)\chi_{\{v>0\}}\Big)\,dx,$$

$$\hat F_k(x,s,\eta)=\bar F_k\big(y_k+\frac{\delta_k}{2}x,\bar u_k(x_k)s,\frac{2\bar u_k(x_k)}{\delta_k}\eta\big)\qquad\text{and}\qquad \hat\lambda_k(x)=\bar\lambda_k(y_k+\frac{\delta_k}{2}x).$$

We let $c_k=\frac{2\bar u_k(x_k)}{\delta_k}$ and we notice that $c_k\to\infty$. So we define $\tilde p_k(x)=\bar p_k(y_k+\frac{\delta_k}{2}x)$ and divide the functional $\hat J_k$ by $c_k^{\tilde p_k^-}$, with $\tilde p_k^-=\inf_{B_1}\tilde p_k$. Then, it follows that w_k is a nonnegative minimizer of $\tilde J_k$ in $w_k+W_0^{1,\tilde p_k(\cdot)}(B_1)$, where

$$\tilde J_k(v)=\int_{B_1}\Big(\tilde F_k(x,v,\nabla v)+\tilde\lambda_k(x)\chi_{\{v>0\}}\Big)\,dx,$$

$$\tilde F_k(x,s,\eta)=c_k^{-\tilde p_k^-}\hat F_k(x,s,\eta)\qquad\text{and}\qquad\tilde\lambda_k(x)=c_k^{-\tilde p_k^-}\hat\lambda_k(x).$$

We claim that

$$\tilde\lambda_k\to0\quad\text{uniformly in }B_1,\tag{4.26}$$

$$c_k^{\tilde p_k(x)-\tilde p_k^-}\to1\ \text{ uniformly},\quad 1\le c_k^{\tilde p_k(x)-\tilde p_k^-}\le M_1\quad\text{in }B_1,\tag{4.27}$$

$$\tilde p_k\to p_0\ \text{ uniformly}\quad\text{and}\quad p_{\min}\le p_0\le p_{\max}\quad\text{in }B_1,\tag{4.28}$$

up to a subsequence, for some constants M_1 and p_0, where $M_1=M_1(M,L)$.

On the one hand, $0<\tilde\lambda_k(x)\le\lambda_{\max}c_k^{-1}\to0$ gives (4.26).

In addition, in B_1 there holds, for k large, that $1\le c_k^{\tilde p_k(x)-\tilde p_k^-}\le e^{2\|\nabla\tilde p_k\|_{L^\infty}\log c_k}$. But we have $\|\nabla\tilde p_k\|_{L^\infty}\log c_k\le Lr_k\frac{\delta_k}{2}\log\big(\frac{2M}{r_k\delta_k}\big)\to0$, which implies (4.27).

To see (4.28) we observe that $p_{\min}\le p_k(x)\le p_{\max}$ and $\|\nabla p_k\|_{L^\infty(\Omega)}\le L$ and then, for a subsequence, $p_k\to p$ uniformly on compacts of Ω, so $\tilde p_k(x)=p_k(r_k(y_k+\frac{\delta_k}{2}x))\to p_0=p(0)$ uniformly in B_1.

We define $\tilde A_k=\nabla_\eta\tilde F_k$ and $\tilde B_k=(\tilde F_k)_s$ and we observe that

$$\tilde p_k(x)=p_k(r_k(y_k+\frac{\delta_k}{2}x)),\qquad\tilde\tau_k(x)=\tau_k(r_k(y_k+\frac{\delta_k}{2}x)),$$

$$\begin{aligned}\tilde F_k(x,s,\eta)&=c_k^{-\tilde p_k^-}\hat F_k(x,s,\eta)=c_k^{-\tilde p_k^-}\bar F_k\big(y_k+\frac{\delta_k}{2}x,\bar u_k(x_k)s,\frac{2\bar u_k(x_k)}{\delta_k}\eta\big)\\&=c_k^{-\tilde p_k^-}F_k\big(r_k(y_k+\frac{\delta_k}{2}x),r_k\bar u_k(x_k)s,c_k\eta\big),\\ \tilde A_k(x,s,\eta)&=c_k^{-\tilde p_k^-}c_kA_k\big(r_k(y_k+\frac{\delta_k}{2}x),r_k\bar u_k(x_k)s,c_k\eta\big),\\ \tilde B_k(x,s,\eta)&=c_k^{-\tilde p_k^-}r_k\bar u_k(x_k)B_k\big(r_k(y_k+\frac{\delta_k}{2}x),r_k\bar u_k(x_k)s,c_k\eta\big).\end{aligned}$$

There holds that $\tilde p_k$, $\tilde F_k$, $\tilde A_k$, $\tilde B_k$ and $\tilde\tau_k$ are under the assumptions of Section 2, with constants independent of k. In fact, recalling (4.27), we get for k large

$$p_{\min}\le\tilde p_k(x)\le p_{\max},\quad\|\nabla\tilde p_k\|_{L^\infty(\Omega)}\le L,\quad\tilde p_k(x)\le\tilde\tau_k(x)\le\tau_0,$$

$$\tilde{A}_k(x,s,0)=0,$$

$$\begin{aligned}\sum_{i,j}\frac{\partial(\tilde{A}_k)_i}{\partial\eta_j}(x,s,\eta)\xi_i\xi_j =&c_k^{-\tilde{p}_k^-}c_k^2\sum_{i,j}\frac{\partial(A_k)_i}{\partial\eta_j}\big(r_k(y_k+\frac{\delta_k}{2}x),r_k\bar{u}_k(x_k)s,c_k\eta\big)\xi_i\xi_j\\ \geq&\lambda_0 c_k^{\tilde{p}_k(x)-\tilde{p}_k^-}|\eta|^{\tilde{p}_k(x)-2}|\xi|^2\geq\lambda_0|\eta|^{\tilde{p}_k(x)-2}|\xi|^2,\end{aligned}\tag{4.29}$$

$$\begin{aligned}\sum_{i,j}\Big|\frac{\partial(\tilde{A}_k)_i}{\partial\eta_j}(x,s,\eta)\Big| =&c_k^{-\tilde{p}_k^-}c_k^2\sum_{i,j}\Big|\frac{\partial(A_k)_i}{\partial\eta_j}\big(r_k(y_k+\frac{\delta_k}{2}x),r_k\bar{u}_k(x_k)s,c_k\eta\big)\Big|\\ \leq&\Lambda_0 c_k^{\tilde{p}_k(x)-\tilde{p}_k^-}|\eta|^{\tilde{p}_k(x)-2}\leq\Lambda_0 M_1|\eta|^{\tilde{p}_k(x)-2}.\end{aligned}\tag{4.30}$$

Assuming, without loss of generality, that $\tilde{p}_k(x_1)\geq\tilde{p}_k(x_2)$ and using that $(r_k\frac{\delta_k}{2})^{\beta_1}\log c_k\leq(r_k\frac{\delta_k}{2})^{\beta_1}\log\big(\frac{2M}{r_k\delta_k}\big)\to 0$, we get

$$\begin{aligned}\big|\tilde{A}_k(x_1,s,\eta)-\tilde{A}_k(x_2,s,\eta)\big|\leq& c_k^{-\tilde{p}_k^-}c_k\Lambda_0(r_k\frac{\delta_k}{2})^{\beta_1}|x_1-x_2|^{\beta}\big(|c_k\eta|^{\tilde{p}_k(x_1)-1}+|c_k\eta|^{\tilde{p}_k(x_2)-1}\big)\\ &\big(1+\big|\log|c_k\eta|\big|\big)\leq 2M_1\Lambda_0|x_1-x_2|^{\beta}\big(|\eta|^{\tilde{p}_k(x_1)-1}+|\eta|^{\tilde{p}_k(x_2)-1}\big)\big(1+\big|\log|\eta|\big|\big).\end{aligned}\tag{4.31}$$

Finally, recalling that $r_k\bar{u}_k(x_k)\leq M$, we obtain

$$\begin{aligned}\big|\tilde{A}_k(x,s_1,\eta)-\tilde{A}_k(x,s_2,\eta)\big|\leq& c_k^{-\tilde{p}_k^-}c_k\Lambda_0 r_k\bar{u}_k(x_k)|s_1-s_2||c_k\eta|^{\tilde{p}_k(x)-1}\\ \leq&\Lambda_0 M_1 M|s_1-s_2||\eta|^{\tilde{p}_k(x)-1},\end{aligned}\tag{4.32}$$

$$\begin{aligned}\big|\tilde{B}_k(x,s,\eta)\big|\leq& c_k^{-\tilde{p}_k^-}r_k\bar{u}_k(x_k)\Lambda_0\Big(1+|c_k\eta|^{\tilde{p}_k(x)}+|r_k\bar{u}_k(x_k)s|^{\tilde{\tau}_k(x)}\Big)\\ \leq& M\Lambda_0\big(c_k^{-\tilde{p}_k^-}+M_1|\eta|^{\tilde{p}_k(x)}+c_k^{-\tilde{p}_k^-}|Ms|^{\tilde{\tau}_k(x)}\big)\leq M_1M\Lambda_0\big(1+|\eta|^{\tilde{p}_k(x)}+M^{\tau_0}|s|^{\tilde{\tau}_k(x)}\big).\end{aligned}\tag{4.33}$$

On the other hand, $\tilde{A}_k$ and $\tilde{B}_k$ satisfy condition (3.14). In fact, since A_k and B_k satisfy condition (3.14),

$$\begin{aligned}&\frac{1}{2}\sum_{i,j}\frac{\partial(\tilde{A}_k)_i}{\partial\eta_j}(x,s,\eta)\xi_i\xi_j+(\tilde{B}_k)_s(x,s,\eta)w^2\\ &\quad=\frac{1}{2}c_k^{-\tilde{p}_k^-}c_k^2\sum_{i,j}\frac{\partial(A_k)_i}{\partial\eta_j}\big(r_k(y_k+\frac{\delta_k}{2}x),r_k\bar{u}_k(x_k)s,c_k\eta\big)\xi_i\xi_j\\ &\quad+c_k^{-\tilde{p}_k^-}(r_k\bar{u}_k(x_k))^2(B_k)_s\big(r_k(y_k+\frac{\delta_k}{2}x),r_k\bar{u}_k(x_k)s,c_k\eta\big)w^2\\ &\quad\geq c_k^{-\tilde{p}_k^-}2\big|(A_k)_s\big(r_k(y_k+\frac{\delta_k}{2}x),r_k\bar{u}_k(x_k)s,c_k\eta\big)\cdot(c_k\xi)(r_k\bar{u}_k(x_k)w)\big|\\ &\quad=2\big|(\tilde{A}_k)_s(x,s,\eta)\cdot\xi w\big|.\end{aligned}$$

Also, since F_k satisfy (3.8) and (3.9) with τ_k satisfying (2.7), with exponent p_k and constants independent of k, then $\tilde{F}_k$ satisfy (3.8) and (3.9) with $\tilde{\tau}_k$ satisfying (2.7), with exponent $\tilde{p}_k$ and constants independent of k. In fact,

$$\begin{aligned}\tilde{F}_k(x,s,\eta)\geq& c_k^{-\tilde{p}_k^-}\nu|c_k\eta|^{\tilde{p}_k(x)}-c_k^{-\tilde{p}_k^-}c_1\big(|r_k\bar{u}_k(x_k)s|^{\tilde{p}_k(x)-\delta}+1\big)\\ \geq&\nu|\eta|^{\tilde{p}_k(x)}-c_1M^{p_{\max}}\big(|s|^{\tilde{p}_k(x)-\delta}+1\big).\end{aligned}\tag{4.34}$$

Analogously,

$$\tilde{F}_k(x,s,\eta) \le c_k^{-\tilde{p}_k^-}\nu^{-1}|c_k\eta|^{\tilde{p}_k(x)} + c_k^{-\tilde{p}_k^-}c_1\big(|r_k\bar{u}_k(x_k)s|^{\tilde{\tau}_k(x)}+1\big) \\ \le M_1\nu^{-1}|\eta|^{\tilde{p}_k(x)} + c_1M^{\tau_0}\big(|s|^{\tilde{\tau}_k(x)}+1\big). \tag{4.35}$$

If $B_k(x,0,0)\equiv 0$ for $x\in\Omega$, then $\tilde{B}_k(x,0,0)\equiv 0$ for $x\in B_1$.

On the other hand, if B_k satisfy (3.32) for $x\in\Omega$ with exponent p_k and constant independent of k, then $\tilde{B}_k$ satisfy (3.32) for $x\in B_1$ with exponent $\tilde{p}_k$ and constant independent of k. In fact,

$$\begin{aligned} |\tilde{B}_k(x,s,\eta)| =& c_k^{-\tilde{p}_k^-}r_k\bar{u}_k(x_k)|B_k\big(r_k(y_k+\frac{\delta_k}{2}x), r_k\bar{u}_k(x_k)s, c_k\eta\big)| \\ \le& c_k^{-\tilde{p}_k^-}r_k\bar{u}_k(x_k)\Lambda_0(1+|r_k\bar{u}_k(x_k)s|^{\tilde{p}_k(x)-1}+|c_k\eta|^{\tilde{p}_k(x)-1}) \\ \le& c_k^{-1}M_1M^{p_{\max}}\Lambda_0(1+|s|^{\tilde{p}_k(x)-1}+|\eta|^{\tilde{p}_k(x)-1}) \\ \le& M_1M^{p_{\max}}\Lambda_0(1+|s|^{\tilde{p}_k(x)-1}+|\eta|^{\tilde{p}_k(x)-1}). \end{aligned} \tag{4.36}$$

We now take v_k the solution of

$$\mathrm{div}\tilde{A}_k(x,v_k,\nabla v_k) = \tilde{B}_k(x,v_k,\nabla v_k) \quad \text{in } B_{3/4}, \qquad v_k - w_k \in W_0^{1,\tilde{p}_k(\cdot)}(B_{3/4}). \tag{4.37}$$

In fact, from Corollaries 3.3, 3.4 and 3.2 and the upper bound in (4.25), it follows that if k is large enough

$$||v_k||_{L^\infty(B_{3/4})} \le \bar{C}, \tag{4.38}$$

where $\bar{C}$ depends only on N, $p_{\min}$, $p_{\max}$, L, λ_0, Λ_0, ν, c_1, δ, M and τ_0. Here we have used that $\sup_{B_{3/4}}\tilde{p}_k - \inf_{B_{3/4}}\tilde{p}_k \le \|\nabla\tilde{p}_k\|_{L^\infty}\frac{3}{2} \le 3Lr_k\frac{\delta_k}{4} < \delta$ in (3.8), for k large.

Then, by (4.38), we can replace $|s|^{\tilde{\tau}_k(x)}$ in (4.33) by $1+\bar{C}^{\tau_0}$ and applying Theorem 1.1 in [10] we obtain that, for k large,

$$||v_k||_{C^{1,\alpha}(\overline{B_{1/2}})} \le \hat{C} \quad \text{with} \quad 0<\alpha<1 \tag{4.39}$$

where $\hat{C}$ depends only on β, N, $p_{\min}$, $p_{\max}$, L, λ_0, Λ_0, ν, c_1, δ, M and τ_0. Therefore, there is a function $v_0\in C^{1,\alpha}(\overline{B_{1/2}})$ such that, for a subsequence,

$$v_k\to v_0 \quad \text{and} \quad \nabla v_k\to\nabla v_0 \quad \text{uniformly in } \overline{B_{1/2}}. \tag{4.40}$$

Let us now show that

$$w_k - v_k \to 0 \quad \text{in } L^{p_{\min}}(B_{3/4}). \tag{4.41}$$

From the minimality of w_k we have

$$\int_{B_{3/4}} \tilde{F}_k(x,w_k\nabla w_k) - \tilde{F}_k(x,v_k\nabla v_k) \le C(N)\|\tilde{\lambda}_k\|_{L^\infty(B_{3/4})}, \tag{4.42}$$

which together with Proposition 3.3 gives

$$\int_{A_2^k}|\nabla w_k-\nabla v_k|^{\tilde{p}_k(x)}\,dx \le C\|\tilde{\lambda}_k\|_{L^\infty(B_{3/4})}, \tag{4.43}$$

$$\int_{A_1^k}|\nabla w_k-\nabla v_k|^2(|\nabla w_k|+|\nabla v_k|)^{\tilde{p}_k(x)-2}\,dx \le C\|\tilde{\lambda}_k\|_{L^\infty(B_{3/4})}, \tag{4.44}$$

where $A_1^k = B_{3/4}\cap\{\tilde{p}_k(x)<2\}$, $A_2^k = B_{3/4}\cap\{\tilde{p}_k(x)\ge 2\}$ and $C = C(p_{\min},p_{\max},N,\lambda_0)$.

Applying Hölder's inequality (Theorem A.3) with exponents $\frac{2}{\tilde{p}_k(x)}$ and $\frac{2}{2-\tilde{p}_k(x)}$, we get

$$\int_{A_1^k} |\nabla w_k - \nabla v_k|^{\tilde{p}_k(x)}\,dx \le 2\,\|G_k^a\|_{L^{2/\tilde{p}_k(\cdot)}(A_1^k)}\|G_k^b\|_{L^{2/(2-\tilde{p}_k(\cdot))}(A_1^k)}, \tag{4.45}$$

where

$$G_k^a = |\nabla w_k - \nabla v_k|^{\tilde{p}_k}(|\nabla w_k| + |\nabla v_k|)^{(\tilde{p}_k-2)\tilde{p}_k/2}$$
$$G_k^b = (|\nabla w_k| + |\nabla v_k|)^{(2-\tilde{p}_k)\tilde{p}_k/2}.$$

Since

$$\int_{A_1^k} |G_k^a|^{2/\tilde{p}_k(x)}\,dx = \int_{A_1^k} |\nabla w_k - \nabla v_k|^2(|\nabla w_k| + |\nabla v_k|)^{\tilde{p}_k(x)-2}\,dx,$$

then, from (4.44), (4.26) and Proposition A.1, we get, for k large,

$$\|G_k^a\|_{L^{2/\tilde{p}_k(\cdot)}(A_1^k)} \le C\|\tilde{\lambda}_k\|_{L^\infty(B_{3/4})}^{p_{\min}/2}, \tag{4.46}$$

$C = C(p_{\min}, p_{\max}, N, \lambda_0)$. On the other hand, (4.37) and the bounds (4.34), (4.35) and (4.38) give

$$\begin{aligned} C_1 \int_{B_{3/4}} |\nabla v_k|^{\tilde{p}_k(x)} &\le \int_{B_{3/4}} \tilde{F}_k(x, v_k \nabla v_k) + C_2 \\ &\le \int_{B_{3/4}} \tilde{F}_k(x, w_k \nabla w_k) + C_2 \\ &\le C\big(1 + \int_{B_{3/4}} |\nabla w_k|^{\tilde{p}_k(x)}\big). \end{aligned}$$

This implies

$$\int_{A_1^k} |G_k^b|^{2/(2-\tilde{p}_k(x))}\,dx \le C\int_{B_{3/4}} (|\nabla w_k|^{\tilde{p}_k(x)} + |\nabla v_k|^{\tilde{p}_k(x)})\,dx \le \tilde{C}\big(1 + \int_{B_{3/4}} |\nabla w_k|^{\tilde{p}_k(x)}\big), \tag{4.47}$$

for some $\tilde{C} \ge 1$, depending only on $p_{\min}$, $p_{\max}$ and the uniform constants and functions in (4.34), (4.35) and (4.38). Now (4.47) and Proposition A.1 give

$$\|G_k^b\|_{L^{2/(2-\tilde{p}_k(\cdot))}(A_1^k)} \le \tilde{C}\big(1 + \int_{B_{3/4}} |\nabla w_k|^{\tilde{p}_k(x)}\big). \tag{4.48}$$

Let us show that the right hand side in (4.48) can be bounded independently of k.

In fact, let $\tilde{v}_k$ be the solution of

$$\mathrm{div}\tilde{A}_k(x, \tilde{v}_k, \nabla \tilde{v}_k) = \tilde{B}_k(x, \tilde{v}_k, \nabla \tilde{v}_k) \quad \text{in } B_{7/8}, \qquad \tilde{v}_k - w_k \in W_0^{1,\tilde{p}_k(\cdot)}(B_{7/8}). \tag{4.49}$$

Then, similar arguments to those leading to (4.38) and (4.39), give, for k large enough,

$$||\tilde{v}_k||_{L^\infty(B_{7/8})} \le \bar{C}, \tag{4.50}$$

and

$$||\tilde{v}_k||_{C^{1,\alpha}(\overline{B_{3/4}})} \le \hat{C} \quad \text{with} \quad 0 < \alpha < 1, \tag{4.51}$$

where $\bar{C}$ and $\hat{C}$ depend only β, N, $p_{\min}$, $p_{\max}$, L, λ_0, Λ_0, ν, c_1, δ, M and τ_0.

Since w_k is a nonnegative minimizer of $\tilde{J}_k$ in B_1, then we can argue as in the proof of Theorem 4.3 and get estimate (4.10) for $u = w_k$, $v = \tilde{v}_k$, $p(x) = \tilde{p}_k(x)$, $\lambda(x) = \tilde{\lambda}_k(x)$, $r = 7/8$ and $\rho = 3/4$. That is,

$$\int_{B_{3/4}} |\nabla w_k|^{\tilde{p}_k(x)}\, dx \le C + C \int_{B_{3/4}} |\nabla \tilde{v}_k|^{\tilde{p}_k(x)}\, dx, \tag{4.52}$$

where $C = C(p_{\min}, p_{\max}, N, \lambda_{\max}, \lambda_0)$. Therefore (4.52) and (4.51) give, for k large, a uniform bound for the right hand side in (4.48). That is,

$$\|G_k^b\|_{L^{2/(2-\tilde{p}_k(\cdot))}(A_1^k)} \le \bar{C}, \tag{4.53}$$

with $\bar{C}$ a constant depending only on β, N, $p_{\min}$, $p_{\max}$, L, λ_0, Λ_0, ν, c_1, δ, M and τ_0.

Now, putting together (4.43), (4.45), (4.46), (4.53) and (4.26), we obtain

$$\int_{B_{3/4}} |\nabla w_k - \nabla v_k|^{\tilde{p}_k(x)} \to 0. \tag{4.54}$$

Thus, using Poincare's inequality (Theorem A.4) and Theorem A.2, we get (4.41).

In order to conclude the proof, we now observe that, since $\tilde{p}_k$, $\tilde{F}_k$, $\tilde{A}_k$, $\tilde{B}_k$, $\tilde{\tau}_k$, $\tilde{\lambda}_k$ and w_k fall (uniformly) under the assumption of Corollary 4.1 in B_1, there exists $0 < \gamma < 1$, $\gamma = \gamma(N, p_{\min})$, such that

$$\|w_k\|_{C^\gamma(\overline{B_{1/2}})} \le C$$

with C depending only on β, N, $p_{\min}$, $p_{\max}$, L, $\lambda_{\max}$, λ_0, Λ_0, ν, c_1, τ_0 and δ (recall that $\|w_k\|_{L^\infty(B_1)} \le 2$).

Therefore, there is a function $w_0 \in C^\gamma(\overline{B_{1/2}})$ such that, for a subsequence,

$$w_k \to w_0 \quad \text{uniformly in } \overline{B_{1/2}}. \tag{4.55}$$

In addition, recalling (4.40) and (4.41), we get $v_0 = w_0$ in $\overline{B_{1/2}}$.

We then observe that, since there holds that $w_k \ge 0$, $w_k(0) = 0$ and (4.25), then (4.55) implies

$$w_0 \ge 0, \quad w_0(0) = 0, \quad \max_{\overline{B_{1/2}}} w_0 \ge c > 0.$$

That is,

$$v_0 \ge 0, \quad v_0(0) = 0, \quad \max_{\overline{B_{1/2}}} v_0 \ge c > 0. \tag{4.56}$$

Let us show that (4.56) gives a contradiction. We will divide the proof in two cases.

Case I. Assume that $\tilde{B}_k(x, 0, 0) \equiv 0$ for $x \in B_1$.

We first observe that, since $w_k \ge 0$, from Proposition 3.6 we deduce that $v_k \ge 0$.

Recalling (4.39), we choose $M_0 > 0$ such that, for every k,

$$||v_k||_{L^\infty(B_{1/2})} \le M_0, \quad ||\nabla v_k||_{L^\infty(B_{1/2})} \le M_0,$$

and define

$$\begin{aligned}
\tilde{\tilde{A}}_k(x, s, \eta) &= a(s, \eta)\tilde{A}_k(x, s, \eta) + (1 - a(s, \eta))|\eta|^{p_0 - 2}\eta,\\
\tilde{\tilde{B}}_k(x, s, \eta) &= a(s, \eta)\tilde{B}_k(x, s, \eta),
\end{aligned}$$

where

$$a(s, \eta) = \chi_{\{|s| \le M_0, |\eta| \le M_0\}}.$$

Then,

$$\operatorname{div}\tilde{\tilde{A}}_k(x,v_k,\nabla v_k)=\tilde{\tilde{B}}_k(x,v_k,\nabla v_k)\quad \text{in } B_{1/2}.$$

From (4.29) and (4.30) (recall Remark 2.1) we deduce

$$\begin{aligned}|\tilde{A}_k(x,s,\eta)|&\le \tilde{\Lambda}_0|\eta|^{\tilde{p}_k(x)-1},\\ \tilde{A}_k(x,s,\eta)\cdot\eta&\ge \tilde{\Lambda}_0^{-1}|\eta|^{\tilde{p}_k(x)},\end{aligned}\tag{4.57}$$

for some constant $\tilde{\Lambda}_0>0$ independent of k.

Let us now fix $\varepsilon>0$. Then, if $k\ge k_0(\varepsilon)$, (4.57), (4.33) and (4.28) give, for large k,

$$\begin{aligned}|\tilde{\tilde{A}}_k(x,s,\eta)|&\le \tilde{\tilde{\Lambda}}_0|\eta|^{p_0-1}+c\varepsilon,\\ \tilde{\tilde{A}}_k(x,s,\eta)\cdot\eta&\ge \tilde{\tilde{\Lambda}}_0^{-1}|\eta|^{p_0}-c\varepsilon,\\ |\tilde{\tilde{B}}_k(x,s,\eta)|&\le \tilde{\tilde{\Lambda}}_0|\eta|^{p_0-1}+c\varepsilon,\end{aligned}$$

for some positive constants $\tilde{\tilde{\Lambda}}_0$ and c (independent of ε and k).

Applying Harnack's inequality (see [22], Theorems 5 and 6 and Section 5), we get for any $0<r<1$

$$\max_{B_{r/2}} v_k\le C_r\big(\min_{B_{r/2}} v_k+\varepsilon^{\frac{1}{p_0}}\big),$$

with C_r a positive constant.

Now, letting $k\to\infty$ first, and then $\varepsilon\to 0$, we get

$$\max_{B_{r/2}} v_0\le C_r\min_{B_{r/2}} v_0,\tag{4.58}$$

with

$$v_0\ge 0,\qquad v_0(0)=0.\tag{4.59}$$

Since $0<r<1$ is arbitrary, we get $v_0\equiv 0$ in $B_{1/2}$. This is in contradiction with (4.56) and concludes the proof of *Case I.*

Case II. Assume that $\tilde{B}_k$ satisfy (3.32) for $x\in B_1$ with exponent $\tilde{p}_k$ and constant independent of k. Then, (4.30), (4.31), (4.32) and (4.36) imply that, for a subsequence,

$$\begin{aligned}\tilde{A}_k\to\tilde{A}&\quad\text{uniformly on compacts of } B_{1/2}\times\mathbb{R}\times\mathbb{R}^N\setminus\{0\}\text{ and pointwise on } B_{1/2}\times\mathbb{R}\times\mathbb{R}^N,\\ \tilde{B}_k\to 0&\quad\text{uniformly on compacts of } B_{1/2}\times\mathbb{R}\times\mathbb{R}^N,\end{aligned}$$

and from (4.29) and (4.30) (recall Remark 2.1) we deduce

$$\begin{aligned}|\tilde{A}(x,s,\eta)|&\le\tilde{\Lambda}_0|\eta|^{p_0-1},\\ \tilde{A}(x,s,\eta)\cdot\eta&\ge\tilde{\Lambda}_0^{-1}|\eta|^{p_0},\end{aligned}$$

for some constant $\tilde{\Lambda}_0>0$. Then, (4.37) and (4.40) imply that

$$\operatorname{div}\tilde{A}(x,v_0,\nabla v_0)=0\quad\text{in } B_{1/2}.$$

Applying Harnack's inequality (see [22], Theorems 5 and 6 and Section 5), we get again, that (4.58) and (4.59) holds for any $0<r<1$. This contradicts once more (4.56) and concludes the proof. □

We can now prove the Lipschitz continuity of nonnegative local minimizers of (1.1)

Theorem 4.5. *Let p, F, A, B, λ and u be as in Corollary 4.1. Then u is locally Lipschitz continuous in Ω. Moreover, for any $\Omega' \subset\subset \Omega$ the Lipschitz constant of u in Ω' can be estimated by a constant C depending only on $\operatorname{dist}(\Omega', \partial\Omega)$, β, N, $p_{\min}$, $p_{\max}$, L, $\lambda_{\max}$, λ_0, Λ_0, ν, c_1, $\|u\|_{L^\infty(\Omega)}$, $||\tau||_{L^\infty(\Omega)}$ and δ.*

Proof. The result is a consequence of Corollary 4.1, Lemma 4.2 and Theorem 4.4 above, and Proposition 2.1 in [16]. We point out that, although the proof of Proposition 2.1 in [16] is written for the particular case in which $A(x, s, \eta) = |\eta|^{p(x)-2}\eta$ and $B(x, s, \eta) = f(x)$, this same proof is valid for general A and B under the present assumptions, without changes. □

5. Examples

In this section we present some examples of application of our results.

Theorem 5.1. *Let $f(x, s)$ be a measurable function such that $f(x, \cdot) \in C^2(\mathbb{R})$ for every $x \in \Omega$. Let $a(x, s)$ be a Hölder continuous function with exponent α, $a(x, \cdot) \in C^2(\mathbb{R})$ for every $x \in \Omega$. Let p, τ and λ as in Section 2 and $0 < \delta < p_{\min}$. Assume that there exist positive constants a_0, a_1, a_2, c_1 and Λ_0 such that*

- f1 $-c_1(1 + |s|^{p(x)-\delta}) \le f(x, s) \le c_1(1 + |s|^{\tau(x)})$ *in* $\Omega \times \mathbb{R}$.
- f2 $f_s(x, 0) \equiv 0$ *in* Ω.
- f3 $f_{ss}(x, s) \ge 0$ *in* $\Omega \times \mathbb{R}$.
- f4 $|f_s(x, s)| \le \Lambda_0(1 + |s|^{\tau(x)})$ *in* $\Omega \times \mathbb{R}$.

And

- a1 $0 < a_0 \le a(x, s) \le a_1 < \infty$ *in* $\Omega \times \mathbb{R}$.
- a2 $|a_s(x, s)| \le a_2$ *in* $\Omega \times \mathbb{R}$.
- a3 $\big(a(x, s)^{1-\gamma(x)}\big)_{ss} \le 0$ *in* $\Omega \times \mathbb{R}$ *with* $\gamma(x) = \frac{2p(x)}{\min\{1, p(x)-1\}} > 1$.

Let

$$F(x, s, \eta) = a(x, s)\frac{|\eta|^{p(x)}}{p(x)} + f(x, s)$$

and let $u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ a nonnegative, local minimizer of (1.1). Then, u is locally Lipschitz continuous in Ω.

Proof. We only have to see that F, A, B satisfy the hypotheses of Theorem 4.5.

There holds that

$$A(x, s, \eta) = a(x, s)|\eta|^{p(x)-2}\eta, \qquad B(x, s, \eta) = a_s(x, s)\frac{|\eta|^{p(x)}}{p(x)} + f_s(x, s).$$

And

$$\frac{a_0}{p_{\max}}|\eta|^{p(x)} - c_1(1 + |s|^{p(x)-\delta}) \le F(x, s, \eta) \le \frac{a_1}{p_{\min}}|\eta|^{p(x)} + c_1(1 + |s|^{\tau(x)}).$$

Moreover,

(1) $A(x, s, 0) = 0$.
(2) $\sum_{i,j} \frac{\partial A_i}{\partial \eta_j}(x, s, \eta)\xi_i\xi_j \ge \lambda_0|\eta|^{p(x)-2}|\xi|^2$. In fact,

$$\sum_{i,j} \frac{\partial A_i}{\partial \eta_j}(x, s, \eta)\xi_i\xi_j = a(x, s)\Big[(p(x) - 2)|\eta|^{p(x)-4}\langle\eta, \xi\rangle^2 + |\eta|^{p(x)-2}|\xi|^2\Big] \tag{5.1}$$

$$\ge a(x, s)\min\{1, p(x) - 1\}|\eta|^{p(x)-2}|\xi|^2 \ge \lambda_0|\eta|^{p(x)-2}|\xi|^2$$

with $\lambda_0 = a_0 \min\{1, p_{\min} - 1\}$.

(3) $\sum_{i,j}\left|\frac{\partial A_i}{\partial \eta_j}(x,s,\eta)\right| \le \Lambda_0|\eta|^{p(x)-2}$ if $\Lambda_0 \ge a_1 N(p_{\max}+3)$.

(4) $|A(x_1,s,\eta)-A(x_2,s,\eta)| \le \Lambda_0|x_1-x_2|^\alpha\big(|\eta|^{p(x_1)-1}+|\eta|^{p(x_2)-1}\big)\big|\big(1+\big|\log|\eta|\big|\big)$ for a big enough constant Λ_0. In fact, without loss of generality we may assume that $p(x_1)\ge p(x_2)$. There holds,

$$|A(x_1,s,\eta)-A(x_2,s,\eta)| \le a(x_1,s)\big||\eta|^{p(x_1)-1}-|\eta|^{p(x_2)-1}\big| + \big|a(x_1,s)-a(x_2,s)\big||\eta|^{p(x_2)-1}.$$

Now, if $|\eta|\ge 1$,

$$\big||\eta|^{p(x_1)-1}-|\eta|^{p(x_2)-1}\big| \le L|x_1-x_2||\eta|^{p(x_1)-1}\big|\log|\eta|\big| \le L|x_1-x_2|\Big(|\eta|^{p(x_1)-1}+|\eta|^{p(x_2)-1}\Big)\big|\log|\eta|\big|.$$

A similar inequality holds if $|\eta|\le 1$. So that,

$$|A(x_1,s,\eta)-A(x_2,s,\eta)| \le a_1L|x_1-x_2|\Big(|\eta|^{p(x_1)-1}+|\eta|^{p(x_2)-1}\Big)\big|\log|\eta|\big| + C_a|x_1-x_2|^\alpha|\eta|^{p(x_2)-1},$$

where C_a is the Holder constant of the function a. And the result follows if $\Lambda_0 \ge a_1Ld(\Omega)^{1-\alpha}+C_a$ with $d(\Omega)$ the diameter of Ω.

(5) $|A(x,s_1,\eta)-A(x,s_2,\eta)| \le a_2|\eta|^{p(x)-1}|s_1-s_2|$.

We clearly have,

(1) $|B(x,s,\eta)| \le \Lambda_0(1+|\eta|^{p(x)}+|s|^{\tau(x)})$ (as we may assume, without loss of generality that $\Lambda_0 \ge \frac{a_2}{p_{\min}}$).

(2) $B(x,0,0)=0$.

Finally, let us see that

$$2|A_s(x,s,\eta)\cdot\xi\, w| \le \frac{1}{2}\sum_{i,j}\frac{\partial A_i}{\partial\eta_j}(x,s,\eta)\xi_i\xi_j + B_s(x,s,\eta)\,w^2.$$

In fact, let

$$\ell(x)=\frac{p(x)-2}{2(p(x)-1)} \qquad \varepsilon(x,s)=a(x,s)\min\{1,p(x)-1\}.$$

Then,

$$\begin{aligned}
2|A_s(x,s,\eta)\cdot\xi\, w| &\le \Big(\sqrt{\varepsilon(x,s)}|\eta|^{\ell(x)(p(x)-1)}|\xi|\Big)\Big(\frac{2}{\sqrt{\varepsilon(x,s)}}|a_s(x,s)||\eta|^{(1-\ell(x))(p(x)-1)}\,|w|\Big)\\
&\le \frac{\varepsilon(x,s)}{2}|\eta|^{p(x)-2}|\xi|^2 + \frac{2}{\varepsilon(x,s)}a_s(x,s)^2|\eta|^{p(x)}w^2\\
&= \frac{1}{2}a(x,s)\min\{1,p(x)-1\}|\eta|^{p(x)-2}|\xi|^2 + \frac{2a_s(x,s)^2}{a(x,s)\min\{1,p(x)-1\}}|\eta|^{p(x)}w^2.
\end{aligned}$$

By (5.1), we only have to check that

$$B_s(x,s,\eta) \ge \frac{2a_s(x,s)^2}{a(x,s)\min\{1,p(x)-1\}}|\eta|^{p(x)}.$$

Since $f_{ss}(x,s)\ge 0$ it is enough to check that

$$a_{ss}(x,s) \ge \gamma(x)\frac{a_s(x,s)^2}{a(x,s)} \quad \text{with} \quad \gamma(x)=\frac{2p(x)}{\min\{1,p(x)-1\}}>1. \tag{5.2}$$

And, (5.2) holds by hypothesis a3. □

If $a(x,s)$ is smooth in $-M_1 < s < M_2$ with $M_1, M_2 > 0$, condition a3 only holds in $0 \le s \le M < M_2$ and the local minimizer u satisfies that $0 \le u \le M$, we can still apply the results in this paper and get that u is locally Lipschitz continuous.

Theorem 5.2. *Let $f(x,s)$ be a measurable function such that $f(x,\cdot) \in C^2(\mathbb{R})$ for every $x \in \Omega$. Let $a(x,s)$ be a Hölder continuous function with exponent α, $a(x,\cdot) \in C^2(-M_1, M_2) \cap Lip(\mathbb{R})$ for almost every $x \in \Omega$ with $M_1, M_2 > 0$. Let p, τ and λ as in Section 2 and $0 < \delta < p_{\min}$. Assume that there exist positive constants $a_0, a_1, a_2, c_1, \Lambda_0$ and $0 < M < M_2$ such that*

f1 $-c_1(1 + |s|^{p(x)-\delta}) \le f(x,s) \le c_1(1 + |s|^{\tau(x)})$ *in* $\Omega \times \mathbb{R}$.
f2 $f_s(x,0) \equiv 0$ *in* Ω.
f3 $f_{ss}(x,s) \ge 0$ *in* $\Omega \times \mathbb{R}$.
f4 $|f_s(x,s)| \le \Lambda_0(1 + |s|^{\tau(x)})$ *in* $\Omega \times \mathbb{R}$.

And

a1 $0 < a_0 \le a(x,s) \le a_1 < \infty$ *in* $\Omega \times \mathbb{R}$.
a2 $|a_s(x,s)| \le a_2$ *a.e. in* $\Omega \times \mathbb{R}$.
a3' $\left(a(x,s)^{1-\gamma(x)}\right)_{ss} \le 0$ *in* $\Omega \times [0,M]$ *with* $\gamma(x) = \frac{2p(x)}{\min\{1, p(x)-1\}} > 1$.

Let

$$F(x,s,\eta) = a(x,s)\frac{|\eta|^{p(x)}}{p(x)} + f(x,s)$$

and let $u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ be a local minimizer of (1.1) *such that $0 \le u \le M$. Then, u is locally Lipschitz continuous in Ω.*

Proof. By Proposition 3.7 for such a function f and with a satisfying a1 and a2, for every ball $B_r(x_0) \subset \Omega$ with r small enough there exists a solution $v \in u + W_0^{1,p(\cdot)}(B_r(x_0))$ of (1.2) such that $0 \le v \le \|u\|_{L^\infty(B_r(x_0))}$. And this result also holds for all the rescaled equations and functions that appear in the proofs of Section 4. Hence, condition (3.14) is only needed for $s \in (0,M)$ and this is a consequence of a3'. □

Example 5.1. A possible example of functions a and f satisfying the assumptions of Theorem 5.2 is

$$a(x,s) = \begin{cases} (1+s)^{-q(x)} & \text{if } -1/2 \le s \le M_2, \\ 2^{q(x)} & \text{if } s \le -1/2, \\ (1+M_2)^{-q(x)} & \text{if } s \ge M_2, \end{cases}$$

with $M_2 > 0$ and $q \in L^\infty(\Omega)$ a Hölder continuous function such that $0 < q(x) < \frac{1}{\gamma(x)-1}$ and

$$f(x,s) = b(x)|s|^{\tau(x)}$$

with $0 \le b \in L^\infty(\Omega)$ and $\tau(x) \ge 2$ in Ω satisfying (2.7).

Another possible choice of f is

$$f(x,s) = b(x)\widetilde{f}(x,s) \tag{5.3}$$

with $0 \le b \in L^\infty(\Omega)$ and

$$\widetilde{f}(x,s) = \begin{cases} s^2 & \text{if } |s| \le 1, \\ \widetilde{a}(x)|s|^{\tau(x)} + \widetilde{b}(x)|s| + \widetilde{c}(x) & \text{if } |s| \ge 1, \end{cases}$$

where $\tau(x)$ satisfies (2.7) and the functions $\widetilde{a}, \widetilde{b}, \widetilde{c} \in L^\infty(\Omega)$ are chosen in such a way that $\widetilde{f}(x,\cdot) \in C^2(\mathbb{R})$ for every $x \in \Omega$.

With this choice of a and f, for every $0 < M < M_2$ there holds that any local minimizer u such that $0 \le u \le M$ is locally Lipschitz continuous in Ω.

Observe that, by Theorem 4.2, if $\phi \in W^{1,p(\cdot)}(\Omega)$ is such that $0 \le \phi \le M < M_2$, such a minimizers always exists.

We have another example.

Theorem 5.3. *Let $f(x,s)$ be a measurable function such that $f(x,\cdot) \in C^2(\mathbb{R})$ for every $x \in \Omega$. Let $G(x,\eta)$ be a measurable function such that $G(x,\cdot) \in C^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{0\})$ for every $x \in \Omega$. Let p and λ as in Section 2 and assume that either f satisfies conditions* f1, $\cdots$,f4 *in Theorem 5.1 or f satisfies* f1, f3 *in Theorem 5.1 and*

f4' $|f_s(x,s)| \le \Lambda_0\big(1 + |s|^{p(x)-1}\big)$.

On the other hand, G satisfies

G1 $\nu\big(|\eta|^{p(x)} - 1\big) \le G(x,\eta) \le \nu^{-1}\big(|\eta|^{p(x)} + 1\big)$ *with* $\nu > 0$.

G2 $\nabla_\eta G(x,0) \equiv 0$ *in* Ω.

G3 $\sum_{i,j} \frac{\partial^2 G}{\partial\eta_i \partial\eta_j}\xi_i\xi_j \ge \lambda_0|\eta|^{p(x)-2}|\xi|^2$.

G4 $\sum_{i,j} \left|\frac{\partial^2 G}{\partial\eta_i \partial\eta_j}\right| \le \Lambda_0|\eta|^{p(x)-2}$.

G5 $|\nabla_\eta G(x_1,\eta) - \nabla_\eta G(x_2,\eta)| \le \Lambda_0|x_1 - x_2|^\beta\big(|\eta|^{p(x_1)-1} + |\eta|^{p(x_2)-1}\big)\big(1 + \big|\log|\eta|\big|\big)$ *for some* $0 < \beta \le 1$.

Let

$$F(x,s,\eta) = G(x,\eta) + f(x,s)$$

and let $u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ be a nonnegative, local minimizer of (1.1). *Then, u is locally Lipschitz continuous in Ω.*

Proof. There holds that

$$A(x,s,\eta) = \nabla_\eta G(x,\eta), \qquad B(x,s,\eta) = f_s(x,s).$$

And it is clear that F, A and B satisfy the assumptions in Theorem 4.5. □

Example 5.2. A possible example of function G satisfying the assumptions of Theorem 5.3 is

$$G(x,\eta) = a(x)\widetilde{G}\big(|\eta|^{p(x)}\big),$$

with $p(x)$ as in Section 2, $a(x)$ a Hölder continuous function such that $a_0 \le a(x) \le a_1$, with a_0, a_1 positive constants and $\widetilde{G} \in C^2\big([0,\infty)\big)$ a function satisfying:

$$c_0 \le \widetilde{G}'(t) \le C_0,$$

$$0 \le \widetilde{G}''(t) \le \frac{C_0}{1+t} \qquad c_0, C_0 \text{ positive constants.}$$

In fact, since $c_0 \le \widetilde{G}'(t) \le C_0$, condition G1 in Theorem 5.3 holds. We have $\nabla_\eta G(x,\eta) = a(x)\widetilde{G}'\big(|\eta|^{p(x)}\big)p(x)|\eta|^{p(x)-2}\eta$, so we get condition G2. We obtain condition G3 by reasoning as in (5.1), using that in the present case we have $\widetilde{G}''(t) \ge 0$ and $\widetilde{G}'(t) \ge c_0$.

We get condition G4 by using in our computations that $\widetilde{G}'(t) \le C_0$ and $\widetilde{G}''(t)t \le C_0$.

Finally, applying again that $\widetilde{G}''(t)t \le C_0$, we can obtain the estimate

$$|\widetilde{G}'\big(|\eta|^{p(x_1)}\big) - \widetilde{G}'\big(|\eta|^{p(x_2)}\big)| \le C_0|p(x_1) - p(x_2)||\log|\eta||,$$

which combined with computations similar as those in (4) in Theorem 5.1 leads to condition G5.

A possible example of function f satisfying the assumptions of Theorem 5.3 is

$$f(x,s)=g(x)s, \qquad \text{with} \quad g\in L^\infty(\Omega).$$

In fact, it is immediate that f satisfies conditions f1, f3 and f4'.

On the other hand, $f(x,s)=b(x)|s|^{\tau(x)}$ with b and τ as in Example 5.1 and $f(x,s)$ as in (5.3) are other possible choices.

Let us present another example

Example 5.3. Another possible example of function G satisfying the assumptions of Theorem 5.3 is

$$G(x,\eta)=\widetilde{A}(x)\eta\cdot\eta|\eta|^{p(x)-2},$$

with $p(x)$ as in Section 2 and $\widetilde{A}(x)\in\mathbb{R}^{N\times N}$, symmetric, Hölder continuous in Ω and such that

$$\lambda(x)I\le\widetilde{A}(x)\le\Lambda(x)I.$$

Here $\lambda_0\le\lambda(x)\le\Lambda(x)\le\Lambda_0$ with λ_0,Λ_0 positive constants and $\Lambda(x)-\lambda(x)\le c_0$, with c_0 a suitable positive constant depending only on N, $p_{\min}$, $p_{\max}$ and λ_0.

In fact, conditions G1 and G2 in Theorem 5.3 are easy to verify. The computations leading to G4 and G5 are similar to the computations in Theorem 5.1.

In order to verify G3, we observe that, denoting $a(x)$ the smaller eigenvalue of $\widetilde{A}(x)$, there holds that

$$\widetilde{A}(x)=a(x)I+\widetilde{B}(x), \qquad \text{with} \quad ||\widetilde{B}(x)||_{L^\infty(\Omega)}\le||\Lambda(x)-\lambda(x)||_{L^\infty(\Omega)}.$$

Then we can write

$$\begin{aligned}G(x,\eta)=&a(x)|\eta|^{p(x)}+\widetilde{B}(x)\eta\cdot\eta|\eta|^{p(x)-2}\\=&G_1(x,\eta)+G_2(x,\eta).\end{aligned}$$

Now, proceeding as in Theorem 5.1, we get

$$\sum_{i,j}\frac{\partial^2G_1}{\partial\eta_i\partial\eta_j}\xi_i\xi_j\ge c_{p_{\min}}\lambda_0|\eta|^{p(x)-2}|\xi|^2. \tag{5.4}$$

It is not hard to see that

$$\sum_{i,j}\Big|\frac{\partial^2G_2}{\partial\eta_i\partial\eta_j}\Big|\le C||\Lambda(x)-\lambda(x)||_{L^\infty(\Omega)}|\eta|^{p(x)-2}, \tag{5.5}$$

with C depending only on N, $p_{\min}$ and $p_{\max}$. Then, combining (5.4) and (5.5) we deduce that $G(x,\eta)$ satisfies condition G3, if we take $||\Lambda(x)-\lambda(x)||_{L^\infty(\Omega)}\le c_0$, with c_0 depending only on λ_0, N, $p_{\min}$ and $p_{\max}$.

For choices of suitable functions $f(x,s)$ for this $G(x,\eta)$ we refer to Example 5.2.

Remark 5.1. We can present further examples of functions satisfying our assumptions. Let p and λ be as in Section 2. Let F_1 and F_2 satisfy the assumptions on Theorem 4.5, with $B_i=\partial_sF_i$ satisfying $B_i(x,0,0)\equiv0$ for $x\in\Omega$, $i=1,2$. Then Theorem 4.5 also applies to the function

$$F(x,s,\eta)=a_1(x)F_1(x,s,\eta)+a_2(x)F_2(x,s,\eta),$$

for any choice of Hölder continuous functions $a_1(x),a_2(x)$, which are bounded from above and below by positive constants.

The same result holds if F_1 and F_2 satisfy the assumptions on Theorem 4.5, with $B_i=\partial_sF_i$ satisfying (3.32) for $x\in\Omega$, $i=1,2$.

Similar consideration applies to functions F_1 and F_2 under the assumptions of Theorem 5.2.

Appendix A

In Section 1 we included some preliminaries on Lebesgue and Sobolev spaces with variable exponent. For the sake of completeness we collect here some additional results on these spaces.

Proposition A.1. *There holds*

$$\min\Big\{\Big(\int_\Omega |u|^{p(x)}\,dx\Big)^{1/p_{\min}},\Big(\int_\Omega |u|^{p(x)}\,dx\Big)^{1/p_{\max}}\Big\} \le \|u\|_{L^{p(\cdot)}(\Omega)}$$
$$\le \max\Big\{\Big(\int_\Omega |u|^{p(x)}\,dx\Big)^{1/p_{\min}},\Big(\int_\Omega |u|^{p(x)}\,dx\Big)^{1/p_{\max}}\Big\}.$$

Some important results for these spaces are

Theorem A.1. *Let $p'(x)$ such that*

$$\frac{1}{p(x)}+\frac{1}{p'(x)}=1.$$

Then $L^{p'(\cdot)}(\Omega)$ is the dual of $L^{p(\cdot)}(\Omega)$. Moreover, if $p_{\min}>1$, $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are reflexive.

Theorem A.2. *Let $q(x)\le p(x)$. If Ω has finite measure, then $L^{p(\cdot)}(\Omega)\hookrightarrow L^{q(\cdot)}(\Omega)$ continuously.*

We also have the following Hölder's inequality

Theorem A.3. *Let $p'(x)$ be as in Theorem A.1. Then there holds*

$$\int_\Omega |f||g|\,dx \le 2\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)},$$

for all $f\in L^{p(\cdot)}(\Omega)$ and $g\in L^{p'(\cdot)}(\Omega)$.

The following version of Poincare's inequality holds

Theorem A.4. *Let Ω be bounded. Assume that $p(x)$ is log-Hölder continuous in Ω (that is, p has a modulus of continuity $\omega(r)=C(\log\frac{1}{r})^{-1}$). For every $u\in W_0^{1,p(\cdot)}(\Omega)$, the inequality*

$$\|u\|_{L^{p(\cdot)}(\Omega)}\le C\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

holds with a constant C depending only on N, $\operatorname{diam}(\Omega)$ and the log-Hölder modulus of continuity of $p(x)$.

The following Sobolev embedding holds. We assume for simplicity that the domain is C^1, but the result holds with weaker assumptions on the smoothness of the boundary.

Theorem A.5. *Let Ω be a C^1 bounded domain. Assume that $p(x)$ is log-Hölder continuous in Ω and $1<p_{\min}\le p(x)\le p_{\max}<\infty$. Let τ be such that $\tau(x)\ge p(x)$ and $\tau\in C(\overline{\Omega})$. Assume moreover that $\tau(x)\le p^*(x)=\frac{Np(x)}{N-p(x)}$ if $p_{\max}<N$, $\tau(x)$ is arbitrary if $p_{\min}>N$, $\tau(x)=p(x)$ if $p_{\min}\le N\le p_{\max}$.*

Then, $W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{\tau(\cdot)}(\Omega)$ continuously. The embedding constant depends only on N, $|\Omega|$, the log-Hölder modulus of continuity of $p(x)$, $p_{\min}$, $p_{\max}$, $||\tau||_{L^\infty}$ and the C^1 norm of $\partial\Omega$.

For the proof of these results and more about these spaces, see [9], [14], [20], [13] and the references therein.

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