Comment on "Boosted Kerr black holes in general relativity"

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We discuss a recently presented boosted Kerr black hole solution which had already been used by other authors. This boosted metric is based on wrong assumptions regarding asymptotic inertial observers, and moreover the performed boost is not a proper Lorentz transformation. This Comment aims to clarify some of the issues that arise when boosting black holes and the care required to interpret them.

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I. INTRODUCTION

Recently, an algorithm to construct boosted Kerr black hole solutions was presented in the peer-reviewed Refs. [1,2]. In the first work [1], the author presented a simplified analysis, where the Kerr black hole is boosted along the z axis. The subsequent article [2] covered the general boost in arbitrary directions. In both, the author claimed that these solutions represent boosted Kerr metrics as "seen" by an asymptotic inertial observer. The proposed mechanism seems to be simple, making it favorable for studying the physical effects of moving rotating black holes. Indeed, follow-up work by other authors [3,4] using these metrics seems to validate it.

Boosted black holes are relevant in gravitational physics. For example, the final black hole remnant of a binary black hole merger is in general boosted with respect to the rest frame of the two initial black holes. This property has an important bearing for gravitational-wave physics as it gives rise to an additional observable in gravitational-wave astronomy, the gravitational-wave memory [5–9], which is the permanent displacement of test masses after the passage of a gravitational wave. This memory effect can be decomposed into a two parts: an ordinary or linear memory effect related to a boost [10,11] and a null memory effect related to the loss of energy of the radiating system by massless particles (see, e.g., Refs. [8,12–14]). In particular, the extraction of physical observables like the gravitational-wave memory or the "classical" observables like gravitational radiation [15,16] and linear and angular momentum [17–19] at null infinity needs to be done in a generalization of an inertial frame.

Given this relevance, we analyze the metric presented in Refs. [1,2] in greater detail and clarify some of the issues arising from a misunderstanding of the meaning of an asymptotic inertial observer. We further show that for the metrics presented in Refs. [1,2] one cannot deduce that it is the coordinate representation of a boosted Kerr metric of an asymptotic Lorentzian observer. It turns out that the discussed metrics contain an incomplete piece of a Lorentz transformation in a certain sense. More precisely, the coordinate representation of the "boosted Kerr metrics" in Refs. [1,2] only makes use of an angular coordinate transformation of the original Kerr metric that could be thought of as associated to an asymptotic Lorentzian observer. However, the additional transformations of the timelike and radial coordinates are still missing. Therefore, the chosen coordinates do not represent adapted coordinates with respect to an inertial observer. Consequently, care must be taken in the interpretation of the "boosted" Kerr metrics of Refs. [1,2], which can give rise to misleading results with respect to the physics related to moving black holes as measured by such observers.

II. FAULTY POINTS IN THE BOOSTED SOLUTION

With respect to the coordinates $\tilde{x}^\alpha = (\tilde{u}, \tilde{r}, \tilde{\theta}, \tilde{\phi})$, the outgoing Eddington-Finkelstein form of the Kerr metric is given by [20][1]

[1]It was pointed out in Ref. [21] that Kerr's original paper [20] should be corrected by exchanging $u \to -u$ and $a \to -a$.

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$$ds^2 = (\tilde{r}^2 + a^2\cos^2\tilde{\theta})(d\tilde{\theta}^2 + \sin^2\tilde{\theta}d\tilde{\phi}^2) - 2(d\tilde{u} + a\sin^2\tilde{\theta}d\tilde{\phi})(d\tilde{r} - a\sin^2\tilde{\theta}d\tilde{\phi}) - \left(1 - \frac{2m\tilde{r}}{\tilde{r}^2 + a^2\cos^2\tilde{\theta}}\right)(d\tilde{u} + a\sin^2\tilde{\theta}d\tilde{\phi})^2, \quad (1)$$

where m is the mass and a is the specific angular momentum. In Ref. [2], the most general "boosted" version of this metric with respect to the coordinates (u, r, θ, ϕ) was presented as [Eq. (27) in Ref. [2]][2]

$$ds^2 = \frac{r^2+\Sigma^2}{K^2}(d\theta^2 + \sin^2\theta d\phi^2) + \left(\frac{r^2 - 2mr + \Sigma^2}{r^2+\Sigma^2}\right)\left[du - 2L\cot\left(\frac{\theta}{2}\right)d\phi\right]^2 - 2\left[du - 2L\cot\left(\frac{\theta}{2}\right)d\phi\right]\left\{dr + \frac{a}{K^2}[-n_1\sin^2\theta + (n_2\cos\phi + n_3\sin\phi)\sin\theta\cos\theta]d\phi + \frac{a}{K^2}(n_2\sin\phi - n_3\cos\phi)d\theta\right\}, \quad (2)$$

where

$$K = A + B(\hat{x}^i n_i), \qquad A^2 - B^2 = 1, \quad (3)$$

$$\Sigma = a\frac{B + A(\hat{x}^i n_i)}{A + B(\hat{x}^i n_i)}, \quad (4)$$

$$L = \left(\frac{1-\cos\theta}{\sin\theta}\right)\left(\frac{a}{2B^2} - \int\frac{\Sigma}{K}\sin\theta d\theta\right), \quad (5)$$

where the general direction of the boost $n_i = (n_1, n_2, n_3)$ is subject to $\delta_{ij}n^i n^j = 1$, and the rapidity ξ determines $A = \cosh\xi$ and $B = \sinh\xi$, and $\hat{x}^i = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)$. In Ref. [2], it was claimed that "For $n_2 = 0 = n_3$ and $B = 0$ the metric (27)[3] is the original Kerr metric in retarded Bondi-Sachs-type coordinates." Therein, one also finds "The derivation and interpretation of this solution will be framed in the Bondi-Sachs (BS) characteristic formulation of gravitational-wave emission in general relativity, where we have a clear and complete derivation of physical quantities and its conservation laws...". Both statements are not true. Regarding the former, an expression for the Kerr metric in an explicit closed form in Bondi-Sachs-type coordinates is not known. Concerning the latter, a retarded Bondi coordinate system is characterized by a surface-forming null coordinate $\hat{u}$ such that null hypersurfaces $\hat{u} = \text{const}$ are generated by a null geodesic congruence $\ell_\mu = (d\hat{u})_\mu$ reaching future null infinity $\mathcal{J}^+$. Consequently, the necessary condition $g^{\hat{u}\hat{u}} = 0$ must be satisfied by the coordinates. It is easy to see that this is not the case for the coordinates (2). An equivalent statement is that the metric has to obey the conditions $g_{rr} = g_{r\theta} = g_{r\phi} = 0$ [16], which are violated in Eq. (2) by the presence of the term $g_{r\phi}$.

If the parameter $a = 0$, the metric (1) reduces to the Schwarzschild solution in outgoing-null polar coordinates:

$$ds^2 = -\left(1 - \frac{2m}{\tilde{r}}\right)d\tilde{u}^2 - 2d\tilde{u}d\tilde{r} + \tilde{r}^2(d\tilde{\theta}^2 + \sin^2\tilde{\theta}d\tilde{\phi}^2). \quad (6)$$

Let us concentrate first on the boosted solution of Ref. [1]. For large values of $\tilde{r}$ on hypersurfaces $\tilde{u} = \text{const}$, Eq. (1) takes the form

$$ds^2 = (\tilde{r}^2 + a^2\cos^2\tilde{\theta})(d\tilde{\theta}^2 + \sin^2\tilde{\theta}d\tilde{\phi}^2) - 2(d\tilde{u} + a\sin^2\tilde{\theta}d\tilde{\phi})(d\tilde{r} - a\sin^2\tilde{\theta}d\tilde{\phi}) - (d\tilde{u} + a\sin^2\tilde{\theta}d\tilde{\phi})^2 + \mathcal{O}\left(\frac{m}{\tilde{r}}\right), \quad (7)$$

which is a flat metric, as can be shown by calculating the (vanishing) components of the Riemann tensor at leading order.

Given the standard Minkowski metric $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$ in Cartesian coordinates $\tilde{x}^\mu = (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$, its coordinate representation for an inertial observer in outgoing null coordinates $\{\tilde{u}, \tilde{r}, \tilde{\theta}, \tilde{\phi}\}$ in a rest frame follows from the coordinate transformation $\tilde{t} = \tilde{u} + \tilde{r}$,

$$\tilde{x} = \tilde{r}\sin\tilde{\theta}\cos\tilde{\phi}, \qquad \tilde{y} = \tilde{r}\sin\tilde{\theta}\sin\tilde{\phi}, \qquad \tilde{z} = \tilde{r}\cos\tilde{\theta},$$

and has the form

$$ds^2 = -d\tilde{u}^2 - 2d\tilde{u}d\tilde{r} + \tilde{r}^2(d\tilde{\theta}^2 + \sin^2\tilde{\theta}d\tilde{\phi}^2) \quad (8)$$

(see, e.g., Refs. [22,23] for a recent discussion regarding boosted black holes and inertial frames). The metric (8) is the inertial metric $\eta_{\mu\nu}$ in outgoing polar null coordinates. If a general metric in outgoing null coordinates approaches the particular form of Eq. (8) at large distances from the source, the asymptotic frame is called the (inertial) Bondi frame [15,16,24].

The leading-order term of Eq. (7) is not such a Bondi frame for $a \neq 0$. That is, if $a \neq 0$, the coordinates used in

[2]Note that some slight change in notation is required to match the standard notation for the Kerr metric: to obtain Eq. (2) in Ref. [2], one must make the substitutions $a \to \omega$, $A \to a$, and $B \to b$.

[3]Our Eq. (2).

Eq. (7) do not correspond to those of an inertial observer. However, if $a = 0$, Eq. (7) corresponds to a Bondi frame.

We proceed with the boosted metric of Ref. [1] while assuming $a = 0$ and show that even in this case the resulting boosted Schwarzschild metric is not properly boosted with respect to an asymptotic observer in the associated inertial Bondi coordinates.

The "boosted" Schwarzschild metric of Ref. [1] is

$$ds^2 = \frac{r^2(d\theta^2 + \sin^2\theta d\phi^2)}{(A + B\cos\theta)^2} - 2dudr - \left(1 - \frac{2m}{r}\right)du^2. \quad (9)$$

This metric is easily obtained by a simple change of *only one* of the angular coordinates in Eq. (6).

Setting $a = 0$ in Eq. (1) and performing the coordinate transformation

$$\tilde{u} = u, \tilde{r} = r, \qquad \cos\tilde{\theta} = \frac{B + A\cos(\theta)}{A + B\cos(\theta)}, \qquad \tilde{\phi} = \phi,$$
$$\text{with} \quad A^2 - B^2 = 1 \quad (10)$$

implies Eq. (9). According to Ref. [1], the functions A and B are related to the boost velocity β as $\beta = B/A$. Moreover, it was never mentioned in Ref. [1] that their "boosted" Kerr metric in their Eq. (23) can be easily obtained by applying the *same* transformation (10) to the Kerr metric (1), which is reproduced here for completeness:

$$ds^2 = \frac{r^2 + \Sigma^2}{(A + B\cos\theta)^2}(d\theta^2 + \sin^2\theta d\phi^2)$$
$$- 2\left[du + \frac{a\sin^2\theta}{(A + B\cos\theta)^2}d\phi\right]\left[dr - \frac{a\sin^2\theta d\phi}{(A + B\cos\theta)^2}\right]$$
$$- \left(1 - \frac{2mr}{r^2 + \Sigma^2}\right)\left(du + \frac{a\sin^2\theta d\phi}{(A + B\cos\theta)^2}\right)^2, \quad (11)$$

where $\Sigma = a(B + A\cos\theta)(A + B\cos\theta)^{-1}$.

Despite the claim in Ref. [1] that the "boosted" Kerr metric (11) is obtained as an *exact stationary analytic solution*, we remark that it is the original Kerr metric in different angular coordinates. We demonstrate below that Eq. (10) is not a proper asymptotic Lorentz transformation, because it must map one asymptotic inertial metric $\eta_{\mu\nu}(\tilde{x}^\alpha)$ to another $\eta_{\mu\nu}(x^\alpha)$. For large values of r, any asymptotically flat metric in Bondi coordinates $\{u, r, \theta, \phi\}$ transforms to another set of Bondi coordinates $\{\tilde{u}, \tilde{r}, \tilde{\theta}, \tilde{\phi}\}$ like

$$-d\tilde{u}^2 + 2d\tilde{u}d\tilde{r} + \tilde{r}^2(d\tilde{\theta}^2 + \sin^2\tilde{\theta}d\tilde{\phi}^2) + \mathcal{O}(1/\tilde{r}),$$
$$= -du^2 - 2dvdr + r^2(d\theta^2 + \sin^2\tilde{\theta}d\phi^2) + \mathcal{O}(1/r). \quad (12)$$

Consider a boost in the z direction at large distances in flat space, where the boosted coordinates are (t, x, y, z) and the corresponding null coordinates $x^\mu = (u, r, \theta, \phi)$ are defined similar to their (tilded) unboosted counterparts. Taking $\tilde{t}^\mu\partial_\mu = \partial_{\tilde{t}}$ as tangent to the world lines of the unboosted observers, the corresponding boosted observers are tangent to $v^\mu = \Gamma(1, \beta^i)$, with $\Gamma = -v^\mu\tilde{t}_\mu = (1 - \delta_{ij}\beta^i\beta^j)^{-1/2}$. The Lorentz transformations for the coordinates $x^\mu \to \tilde{x}^a$ and $r \to \tilde{r}$ are given by [25]

$$r^2 \to \tilde{r}^2 = x^\alpha x_\alpha + (v^\alpha x_\alpha)^2, \quad (13)$$

$$x^\mu \to \tilde{x}^\mu = x^\mu + \frac{[t^\nu x_\nu - (2\Gamma + 1)v^\nu x_\nu]t^\mu}{1 + \Gamma}$$
$$+ \frac{[t^\nu x_\nu + v^\nu x_\nu]v^\mu}{1 + \Gamma}. \quad (14)$$

For a (inverse) boost in the z direction with $\beta^x = \beta^y = 0$ and $\beta^z = \beta$, we find the relations

$$\tilde{u} = \Gamma[u + r(1 + \beta\cos\theta)]$$
$$- r\sqrt{1 + \Gamma^2\left(\frac{u}{r} + 1 + \beta\cos\theta\right)^2 - \left(\frac{u}{r} + 1\right)^2}, \quad (15)$$

$$\tilde{r} = r\sqrt{1 + \Gamma^2\left(\frac{u}{r} + 1 + \beta\cos\theta\right)^2 - \left(\frac{u}{r} + 1\right)^2}, \quad (16)$$

$$\cos(\tilde{\theta}) = \frac{\tilde{z}}{\tilde{r}}$$
$$= \frac{\Gamma[\cos\theta + \beta(\frac{u}{r} + 1)]}{\sqrt{1 + \Gamma^2(\frac{u}{r} + 1 + \beta\cos\theta)^2 - (\frac{u}{r} + 1)^2}}, \quad (17)$$

$$\tilde{\phi} = \arctan\left(\frac{\tilde{y}}{\tilde{x}}\right) = \arctan\left(\frac{y}{x}\right) = \phi \quad (18)$$

between the unboosted and boosted null coordinates. For large distances (keeping u, θ, and ϕ fixed), Eqs. (15)–(18) reduce to

$$\tilde{u} = \frac{u}{\mathcal{K}(\theta)} + \mathcal{O}\left(\frac{1}{r}\right), \qquad \tilde{r} = \mathcal{K}(\theta)r + \mathcal{O}(r^0), \quad (19)$$

$$\cos(\tilde{\theta}) = \frac{\beta + \cos(\theta)}{1 + \beta\cos(\theta)} + \mathcal{O}\left(\frac{1}{r}\right), \qquad \tilde{\phi} = \phi, \quad (20)$$

with $\mathcal{K}(\theta) = \Gamma(1 + \beta\cos\theta)$.

The first part of Eq. (20) is the relativistic aberration formula. Equations (19) and (20) are the asymptotic Lorentz transformation for a boost along the z axis, which are a subset of the larger Bondi-Metzner-Sachs (BMS) group [26].

Applying Eqs. (19)–(20) to Eq. (6) maps the metric of an asymptotic inertial observer in coordinates $\tilde{x}^\mu$ to the metric of another in coordinates x^μ, as required by Eq. (12).[4] We

[4]In fact, expressions for the $\mathcal{O}(r^{-1}, r^0)$ terms in Eqs. (19) and (20) are needed from the expansions of Eqs. (15)–(17).

stress here that the main point is that to make a Lorentz boost, a transformation in the $\tilde{u}$ and $\tilde{r}$ coordinates is needed.

This transformation is *not* contained in Eq. (10).

Therefore, despite the claims of Ref. [1], the presented metric is not a properly boosted Kerr metric with respect to the adapted coordinates of an asymptotic inertial frame, since the transformation is not even completely carried out in the Schwarzschild limit. More generally, discarding supertranslations, BMS transformations in a neighborhood of null infinity can be written in terms of stereographic coordinates [related to spherical coordinates via $\zeta = e^{i\phi}\cot(\frac{\theta}{2})$] as

$$\tilde{u} = \frac{u}{\mathcal{K}(\zeta,\bar{\zeta})} + \mathcal{O}\left(\frac{1}{r}\right), \qquad \tilde{r} = \mathcal{K}(\zeta,\bar{\zeta})r + \mathcal{O}(r^0), \tag{21}$$

$$\tilde{\zeta} = \frac{\hat{a}\zeta + \hat{b}}{\hat{c}\zeta + \hat{d}} + \mathcal{O}\left(\frac{1}{r}\right), \tag{22}$$

where $\{\hat{a},\hat{b},\hat{c},\hat{d}\}$ are four complex parameters subject to the constraint $\hat{a}\,\hat{d} - \hat{b}\,\hat{c} = 1$, and $\mathcal{K}(\zeta,\bar{\zeta})$ is given by [27]

$$\mathcal{K}(\zeta,\bar{\zeta}) = \frac{(\hat{a}\zeta + \hat{c})(\bar{\hat{a}}\,\bar{\zeta} + \bar{\hat{c}}) + (\hat{b}\zeta + \hat{d})(\bar{\hat{b}}\,\bar{\zeta} + \bar{\hat{d}})}{1 + \zeta\bar{\zeta}}. \tag{23}$$

The unit sphere metric in stereographic coordinates reads $ds^2 = \frac{d\zeta d\bar{\zeta}}{P_0^2}$, where $P_0 = \frac{1}{2}(1 + \zeta\bar{\zeta})$.

Indeed, the "generally boosted" Kerr metric presented in Ref. [2] can *also* be obtained from the Kerr metric (1) via a particular angular transformation (22) associated to a general boost. In fact, in Ref. [2], to obtain the "generally boosted" metric a system of differential equations [given by Eqs. (5)–(7) in Ref. [2]] was solved in a hierarchical way: first for P, then for Σ, and finally for L. In particular, L was expressed in Ref. [2] in terms of a set of involved integrals. However, all of these functions simply follow from the leading-order behavior of the angular transformation (22), i.e.,

$$\tilde{\zeta} = f(\zeta) = \frac{\hat{a}\zeta + \hat{b}}{\hat{c}\zeta + \hat{d}}, \tag{24}$$

while keeping $\tilde{u} = u$ and $\tilde{r} = r$. As shown in Ref. [28] (page 442), under these transformations

$$\tilde{P}^2 = f'\bar{f}'P^2, \tag{25}$$

$$\tilde{\Sigma} = \Sigma, \tag{26}$$

$$\tilde{L} = f'^{-1}L, \tag{27}$$

$$\partial_{\tilde{\zeta}} = f'^{-1}\partial_{\zeta}, \tag{28}$$

where $f' = \partial_{\zeta}f$.

Starting with the unboosted Kerr value of $\tilde{P}(\tilde{\zeta},\bar{\tilde{\zeta}}) = \frac{1}{2}(1 + \tilde{\zeta}\bar{\tilde{\zeta}})$ under Eq. (24), it transforms to $P = |f'(\zeta)|^{-1}\tilde{P}(f(\zeta),\bar{f}(\bar{\zeta})) = P_0\mathcal{K}$ which was obtained in Ref. [2] by solving its differential equation (5). Similarly, the expression for Σ [obtained in Ref. [2] by solving its Eq. (12)] follows directly from Eq. (26) using the unboosted Kerr expression $\tilde{\Sigma} = a(\tilde{\zeta}\bar{\tilde{\zeta}} - 1)(\tilde{\zeta}\bar{\tilde{\zeta}} + 1)^{-1}$. Finally, the differential equation for L [given by Eq. (7) in Ref. [2]] preserves its form; that is, if $\tilde{P}, \tilde{\Sigma}$, and $\tilde{L}$ satisfy Eq. (7) of Ref. [2], the transformed L automatically satisfies the same untilded version of this equation. Hence, instead of solving this equation for L as in Ref. [2] (expressed in terms of integrals or involved trigonometric functions), it is easier to obtain L from the Kerr expression $\tilde{L} = -\frac{ia\bar{\tilde{\zeta}}}{2\tilde{P}^2}$ using Eq. (27).

However, as previously mentioned, extra transformations are necessary because for a Bondi system u must be a surface-forming null coordinate, i.e., $u = \text{const}$ should be surfaces generated by null vector fields reaching $\mathcal{J}^+$. This is not the case for the u coordinate employed in Refs. [1,2].

Even though the $u = \text{const}$ hypersurfaces are indeed null surfaces reaching null infinity in the Schwarzschild case (9), the coordinates do not realize a Bondi coordinate system either. In fact, Eq. (9) is expressed in Newman-Unti coordinates (NU) [29].

In terms of stereographic coordinates, it is a particular metric of the family of Robinson-Trautman geometries given by [30]

$$ds^2 = r^2\frac{d\zeta d\bar{\zeta}}{(P_0V)^2} - 2dudr - \left(1 - \frac{2m}{r} + \frac{V_{,u}}{V}r\right)du^2, \tag{29}$$

with $V = V(u,\zeta,\bar{\zeta})$ and $m = m(u)$. Regarding Eq. (9), we have $m_{,u} = 0$ and

$$V = A + B\cos\theta = A + B\frac{\zeta\bar{\zeta} - 1}{1 + \zeta\bar{\zeta}}, \tag{30}$$

showing that also $V_{,u} = 0$. The coordinates $\{u,r,\zeta,\bar{\zeta}\}$ correspond to a Bondi system if and only if $V = 1$. Note that we are not saying that the metric (29) could not be interpreted as a boosted black hole; what we are saying is that if these NU coordinates are used, they must be related to a Bondi system to extract physical quantities. For example, as discussed in Ref. [17], the total linear momentum P^α for the metric (29) can be computed in a non-Bondi system from the formula

$$P^\alpha = \int \frac{m}{V^3}\hat{\ell}^a dS^2, \tag{31}$$

where dS^2 is the surface element of a unit sphere and

$$\hat{\ell}^a = \left(1, \frac{\zeta + \bar{\zeta}}{1 + \zeta\bar{\zeta}}, \frac{\zeta - \bar{\zeta}}{i(1 + \zeta\bar{\zeta})}, \frac{\zeta\bar{\zeta} - 1}{1 + \zeta\bar{\zeta}}\right). \tag{32}$$

Note that this expression was also used in Ref. [2] to compute the four-momentum of its metrics.

Nevertheless, some of the further analysis carried out on the metrics [1,2] is misleading. For example, the location of the horizon for the "boosted" metric (11) is measured to coincide with the same coordinate values as in the Kerr metric. This was interpreted as a consequence of the fact that a boost does not change null surfaces. It is true that boosts do not distort null surfaces, but its coordinate representation for an asymptotic boosted inertial observer would be in general different. The coordinate location of the horizon for the "boosted" Kerr metric (11) takes the same value as in the Kerr metric because the radial coordinate was not changed by the coordinate transformation [cf. Eq. (10)]. Notwithstanding, it is well known that the shapes of the boosted and unboosted horizons are coordinate dependent [31,32]. We note that if we were to attempt a similar procedure as in Refs. [1,2] for the location of a photon sphere $S_{\rm ph}$ in the boosted Schwarzschild metric (9), we would find it at the same radial coordinate $r = 3m$ as in the unboosted black hole, albeit the surface $S_{\rm ph}$ is not a null hypersurface. Again, this only happens because of the improper transformation of the radial and timelike coordinates.

Additionally, it was claimed in Ref. [1] that "The boosted Kerr geometry also presents an ergosphere"; this is not surprising at all because the metric in Ref. [1] is the Kerr metric after the coordinate transformation (10). The coordinate expression for the ergosphere of Refs. [1,2] exhibits a most complex dependence on the angular coordinates. Again, the relevant expression is analyzed by using the unboosted (Kerr) radial coordinate and the "boosted" angular coordinates. That is, there is again no proper use of the associated "boosted" radial coordinate. In any case, the geometrical definition of the ergosphere of the Kerr black hole is given by the set of points where the (global) timelike Killing vector $\frac{\partial}{\partial u}$ becomes a null vector. This is a geometrical (coordinate-independent) definition. However, for the analysis of the ergosphere of a boosted Kerr black hole by an asymptotic observer, the associated inertial coordinates $\{u, r, \theta, \phi\}$ should be used instead of the mixed set of coordinates $\{\tilde{u}, \tilde{r}, \theta, \phi\}$ like in Refs. [1,2].[5]

[5]It is worth emphasizing that for an asymptotic observer there exists another notion of an (observer-dependent) ergosphere based on the asymptotic Killing vector aligned with the asymptotic observer, which again should be expressed in the adapted coordinates of this observer (see, e.g., Ref. [33] where an analysis of these "resulting ergospheres" of a boosted Schwarzschild black hole can be found). Let us note that for the metric (9) these kind of ergospheres of Ref. [33] cannot be obtained from the procedure followed in Refs. [1,2].

We also stress the well-known fact that Kerr's original metric did not approach the Minkowski metric of an inertial observer for large radii [also seen in Eq. (7)]. But, the Boyer-Lindquist form as well as the Kerr-Schild form of the Kerr metric have this property. On top of that, a Kerr-Schild metric has the (defining) property that it is written as $g_{\mu\nu} = \eta_{\mu\nu} + H k_\mu k_\nu$, where H is a scalar function and k_μ is a null vector with respect to $\eta_{\mu\nu}$ and $g_{\mu\nu}$. It had recently been pointed out that the spacetimes of the Schwarzschild and Kerr black holes in Kerr-Schild form have not only one inertial frame serving as a background spacetime to define a boost, but two such distinct frames [22,23]. These two backgrounds are tied to the ingoing and outgoing principal null directions of the respective Kerr-Schild metric. The inertial coordinates of the two transform between each other via a *nonlinear* coordinate transformation. Indeed, it was shown in Refs. [11,22,23] that for the correct value of the boost memory at *future* null infinity, the discussion of the boost must be done in the Minkowski background of the *ingoing* formulation.

For a Schwarzschild/Kerr black hole that is initially at rest and then ejected with mass m and velocity β along the z axis, the boost memory at null infinity is [8,11,22,23]

$$\Delta\sigma = \frac{4\Gamma m \beta^2 \sin^2\theta}{1 - \beta\cos\theta}. \tag{33}$$

The supertranslation α relating the retarded time cuts $u = \infty$ and $u = -\infty$ at null infinity is [11]

$$\alpha = 4m\Gamma(1 - \beta\cos\theta)\ln(1 - \beta\cos\theta). \tag{34}$$

Equations (33) and (34) can by no means be reproduced from Eq. (11).

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