Correlations in Two-Qubit Systems under Non-Dissipative Decoherence

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Abstract: We built a new set of suitable measures of correlations for bipartite quantum states based upon a recently introduced theoretical framework [Bussandri et al. in Quantum Inf. Proc. 18:57, 2019]. We applied these measures to examine the behavior of correlations in two-qubit states with maximally mixed marginals independently interacting with non-dissipative decohering environments in different dynamical scenarios of physical relevance. In order to get further insight about the physical meaning of the behavior of these correlation measures we compared our results with those obtained by means of well-known correlation measures such as quantum mutual information and quantum discord. On one hand, we found that the behaviors of total and classical correlations, as assessed by means of the measures introduced in this work, are qualitatively in agreement with the behavior displayed by quantum mutual information and the measure of classical correlations typically used to calculate quantum discord. We also found that the optimization of all the measures of classical correlations depends upon a single parameter and the optimal value of this parameter turns out to be the same in all cases. On the other hand, regarding the measures of quantum correlations used in our studies, we found that in general their behavior does not follow the standard quantum discord $\mathcal{D}$. As the quantification by means of standard quantum discord and the measures of quantum correlations introduced in this work depends upon the assumption that total correlations are additive, our results indicate that this property needs a deeper and systematic study in order to gain a further understanding regarding the possibility to obtain reliable quantifiers of quantum correlations within this additive scheme.

Keywords: quantum discord; quantum correlations; non-dissipative decoherence

1. Introduction

The ever-increasing processing power of current classical computers depends upon the corresponding increase in the capability of miniaturization of electronic components. However, according to Moore's law this increase is damned to cease in a few more years [1,2]. Indeed, when the characteristic dimensions of the electronic integrated circuits reach a scale of the order of about 50 nanometers and somewhere below 10 nanometers, the individual properties of the atomic elements are expected to become predominant.

Quantum information processing and quantum computing involve the use of quantum resources to perform tasks of information processing which are either not feasible to be implemented classically or can be performed with classical devices in a way much less efficient.

Thus, a key point is to identify the actual features which make it possible for quantum algorithms to outperform their classical counterparts. At first, entanglement was believed to be the main responsible for this effect. However, at present we know that some separable mixed states can provide for a computational speed-up in some quantum computation models [3,4]. In addition, a number of theoretical and experimental results points to an increment in efficiency due to the existence of correlations of a different nature that entanglement [3–10].

Another aspect to be considered in tasks of Quantum information processing and quantum computing is the fact that quantum correlations are very sensitive to uncontrolled interactions between the system and its environment. This phenomenon is known as decoherence. When decoherence becomes significant, the system becomes less efficient to process quantum information. Thus, a central issue in quantum information processing is to know the time scales along which quantum resources can reliably be preserved and processed.

Quantum discord (QD) is a widely used measure of quantum correlations which is also used to pinpoint a departure from classicality [11–13]. However, it is still not clear the origin of the sources responsible for the quantum speed up. As a result, a number of measures of quantum correlations have been proposed in addition to QD [14–17].

In a previous work, we introduced a generalized framework to define bona fide measures of correlations in bipartite quantum systems [18].

The main aim of this work, is to built an explicit set of new well-behaved measures of correlations for bipartite quantum states, based upon that theoretical framework [18], and apply them to examine the behavior of correlations in two-qubit states with maximally mixed marginals independently interacting with non-dissipative decohering environments for different dynamical scenarios of physical relevance.

This work is organized as follows. In Section 2, we outline the basic theoretical background directly related to our work. In Section 3, we develop our main results, i.e., we introduce a new set of explicit measures of total, classical and quantum correlations. Finally, some conclusions are drawn in Section 4.

2. Theoretical framework

2.1. Quantum Discord

Quantum mutual information $\mathcal{I}(\rho)$ is a widely accepted information-theoretic measure of the total correlations contained in a bipartite quantum state ρ. This quantity is defined as follows:

$$\mathcal{I}(\rho) \doteq S(\rho_A) + S(\rho_B) - S(\rho), \tag{1}$$

where ρ stands for a general bipartite quantum state, $\rho_A = \mathrm{Tr}_B[\rho]$, $\rho_B = \mathrm{Tr}_A[\rho]$ represent the corresponding reduced (marginal) states and $S(\rho)$ is the von Neumann entropy given by

$$S(\rho) \doteq -\mathrm{Tr}[\rho \log_2 \rho], \tag{2}$$

It is worth mentioning that $\mathcal{I}(\rho)$ describes the correlations between the whole subsystems rather than a correlation between just two observables.

Classical correlations present in a quantum state ρ of a bipartite quantum system can be quantified by means of the measure $\mathcal{J}_S(\rho)$ defined as [11,12]

$$\mathcal{J}_S(\rho) \doteq S(\rho_B) - \min_{\mathcal{M}} \sum_j p'_j S(\rho^{\mathcal{M}}_{B|j}), \tag{3}$$

with $\mathcal{M} = \{M_j\}_{j=1}^{m}$ ($m \in \mathbb{N}$) being a von Neumann measurement on subsystem A (i.e., a complete set of rank-1 orthonormal projective measurements on $\mathcal{H}_A$), and

$$\rho_{B|j}^{\mathcal{M}} = \text{Tr}_A \left[(M_j \otimes \mathbb{I})\rho \right] / p_j' \tag{4}$$

$$p_j' = \text{Tr} \left[(M_j \otimes \mathbb{I})\rho \right], \tag{5}$$

the resulting state of the subsystem B after obtaining the result M_j when $\mathcal{M}$ is measured on subsystem A and p_j' being its corresponding probability. States given by Equation (4) are commonly referred to as conditional states.

The difference between total correlations given by $\mathcal{I}(\rho)$ [cf. Equation (1)] and classical correlations as measured by $\mathcal{J}_S(\rho)$ [cf. Equation (3)] provides the measure of quantum correlations known as quantum discord which can be written as [11,12],

$$\mathcal{D}(\rho) \doteq S(\rho_A) - S(\rho) + \min_{\mathcal{M}} \sum_j p_j' \, S(\rho_{B|j}^{\mathcal{M}}). \tag{6}$$

It is worth pointing out that, as the measure $\mathcal{J}_S(\rho)$ is not symmetric under the exchange of subsystems A and B, there exists a directionality over $\mathcal{J}_S(\rho)$ and in consequence over the quantity $\mathcal{D}(\rho)$.

Finally, for future purposes it is worth bearing in mind that the notion of quantum discord to quantify total quantum correlations is implicitly embedded into a theoretical framework where total correlations are additive, i.e., they are given by the addition of classical and quantum correlations. This scheme, as pointed out by Brodutch et al. [19], belong to a class of debatable properties for total correlation measures of quantum states.

2.2. Generalized Measures of Correlations

Following reference [18], we can quantify the total correlations showed by a bipartite quantum state ρ by means of a suitable distance measure $d(\cdot||\cdot)$ between quantum states, satisfying some specific requirements [18], as follows:

$$\mathcal{T}_d(\rho) = d(\rho||\rho_A \otimes \rho_B) \tag{7}$$

As the state $\rho_A \otimes \rho_B$ does not contain correlations of any kind, clearly, Equation (7) measures the amount of total correlation by means of a bonafide measure $d(\cdot||\cdot)$ between an arbitrary bipartite state ρ and the product state $\rho_A \otimes \rho_B$.

In [18] also a measure of classical correlations was defined according with the following idea: once the observable $\mathcal{M}$ is measured on subsystem A it is analyzed how its action conditions the state of the subsystem B. By assuming that $\mathcal{M}$ has a discrete spectrum with eigenvalues labeled with $j \in \mathbb{N}$, it is possible to quantify how different is ρ_B from the states $\rho_{B|j}^{\mathcal{M}}$ which arises after performing the local measurement on A, by means of distance $d(\rho_{B|j}^{\mathcal{M}} || \rho_B)$.

Thus, a measure of classical correlations between the two subsystems is the maximum of the weighted average of the above $m \in \mathbb{N}$ quantities $d(\rho_{B|j}^{\mathcal{M}} || \rho_B)$:

$$\mathcal{J}_d(\rho) := \max_{\{\mathcal{M}\}} \mathcal{J}_d^{\mathcal{M}}(\rho), \tag{8}$$

$$\mathcal{J}_d^{\mathcal{M}}(\rho) := \sum_{j=1}^{m} p_j' \, d\left(\rho_{B|j}^{\mathcal{M}} || \rho_B \right). \tag{9}$$

Having defined the generalized measures of total and classical correlation, a measure for quantum correlations can be advanced, based on an additive approach (as quantum discord), in terms of the difference between total and classical correlations,

$$\mathcal{Q}_d\left(\rho\right) \doteq \mathcal{T}_d\left(\rho\right) - \mathcal{J}_d\left(\rho\right). \tag{10}$$

The necessary properties of $d(\cdot||\cdot)$ in order to generate well-behaved measures of quantum correlations are discussed in [18].

It is important to realize that, when the distance $d(\cdot||\cdot)$ is replaced by the relative entropy

$$S(\rho||\sigma) = \text{Tr}[\rho(\log_2\rho - \log_2\sigma)], \tag{11}$$

the quantities $\mathcal{T}_S$, $\mathcal{J}_S$ and $\mathcal{Q}_S$ do coincide with the quantum mutual information $\mathcal{I}$, the classical correlation measure given by (3) and, quantum discord, respectively.

In the next section, we will analyze the behavior of the correlation measures $\mathcal{T}_d$, $\mathcal{J}_d$ and $\mathcal{Q}_d$ for the following three distance measures between quantum states.

2.2.1. Squared Bures Distance

The squared Bures distance (B) is defined as follows:

$$d_{\text{B}}^2(\rho||\sigma) = 2 - 2\sqrt{F(\rho||\sigma)}, \tag{12}$$

being F the Uhlmann–Jozsa fidelity [20,21]:

$$F(\rho||\sigma) = \left[\text{Tr}\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}} \right) \right]^2. \tag{13}$$

The correlation measures resulting from choosing $d_{\text{B}}(\cdot||\cdot)$ will be denoted as $\mathcal{T}_B$, $\mathcal{J}_B$ and $\mathcal{Q}_B$. See Equations (7), (8) and (10).

2.2.2. Squared Hellinger Distance

Squared Hellinger distance (H) between two arbitrary density operators ρ and σ can be written as follows

$$d_{\text{H}}^2(\rho||\sigma) = 2 - 2\,\text{Tr}\sqrt{\rho}\sqrt{\sigma}. \tag{14}$$

We will denote as $\mathcal{T}_H$, $\mathcal{J}_H$ and $\mathcal{Q}_H$ the correlation measures resulting from choosing the distance $d_{\text{H}}(\cdot||\cdot)$ in Equations (7), (8) and (10), respectively.

It is worth mentioning that we consider the squared versions of the Bures and Hellinger distance because of the convexity property required to give rise to a well-behaved measure of classical correlations [18].

2.2.3. Quantum Jensen–Shannon Divergence

The quantum Jensen–Shannon divergence (QJSD) is a symmetrized version of relative entropy (11) and is defined as follows:

$$D_{JS}(\rho||\sigma) = \frac{1}{2}\left[S\left(\rho||\frac{\rho+\sigma}{2} \right) + S\left(\sigma||\frac{\rho+\sigma}{2} \right) \right]. \tag{15}$$

In this case, the Equations (7), (8) and (10), give rise to the measures $\mathcal{T}_{JS}$, $\mathcal{J}_{JS}$ and $\mathcal{Q}_{JS}$.

2.3. Two-Qubit States with Maximally Mixed Marginals

Bell-diagonal (BD) states are two-qubit states with maximally mixed marginals which can be written as

$$\rho^{BD} = \frac{1}{4} \left(\mathbb{I}_2 \otimes \mathbb{I}_2 + \sum_{i=1}^{3} c_i \sigma_i^A \otimes \sigma_i^B \right), \tag{16}$$

being $\mathbb{I}_2$ the identity matrix of dimension 2 and σ_i^A (σ_i^B) the Pauli operators corresponding to the subsystem A (B).

Any two-qubit state satisfying $\langle \sigma_j^A \rangle = 0 = \langle \sigma_j^B \rangle$, i.e., having maximally mixed marginal density operators $\rho_A = \mathbb{I}_2/2 = \rho_B$, can be brought into a Bell-diagonal form by using local unitary operations upon each qubit. It is worth mentioning that BD states are widely used in the current literature in order to study quantum correlations and also the phenomena of freezing of QD. Since quantum and classical correlations are both invariant under local unitary transformations, for the purpose of this work, it will suffice to consider this kind of quantum states.

For an arbitrary BD state, the eigenvalues are given by

$$\lambda_0 = \frac{1}{4}(1 - c_1 - c_2 - c_3), \tag{17}$$

$$\lambda_1 = \frac{1}{4}(1 - c_1 + c_2 + c_3), \tag{18}$$

$$\lambda_2 = \frac{1}{4}(1 + c_1 - c_2 + c_3), \tag{19}$$

$$\lambda_3 = \frac{1}{4}(1 + c_1 + c_2 - c_3), \tag{20}$$

where the coefficients $\{c_j\}$ are such that $0 \leq \lambda_i \leq 1$ ($i = 0, \dots, 3$), with $\sum_i \lambda_i = 1$.

BD states are a three-parameter set which includes the subsets of separable and classical states [22]. They can be specified by the 3-tuple (c_1, c_2, c_3). Two-qubit states with maximally mixed marginals also includes Werner ($|c_1| = |c_2| = |c_3| = c$) and Bell states ($|c_i| = 1$, $|c_j| = 0$, $|c_k| = 0$, where the triplet of indexes (i, j, k) represents any arbitrary permutation of $(1, 2, 3)$). Thus, the state represented by Equation (16) encompasses a wide set of quantum states. Furthermore, BD states allow us to obtain explicit analytical expressions for the different measures of correlations that will be used in this work (cf. Section 3).

3. Results

In what follows, we shall calculate the corresponding expressions for the total, classical, and quantum correlations measures , taking $d(\cdot||\cdot)$ equal to squared Bures and Hellinger distances, and also considering $d(\cdot||\cdot)$ as the quantum Jensen–Shannon divergence. All these measures fulfill the necessary conditions to provide suitable measures of total, classical, and quantum correlations [18].

The computation of $\mathcal{Q}_d$ [cf. Equation (10)] involves an optimization of the *classical correlations* $\mathcal{J}_d^{\mathcal{M}}$ [cf. Equation (9)] over all possible von Neumann measurements. Let us introduce local measurements for party A,

$$\{E_j = |j\rangle \langle j| \ / \ j \in \{0, 1\}\}, \tag{21}$$

that is, $\{E_j\}$ is a PVM (Projection-Valued Measure) over the subsystem A given in the computational basis $\{|j\rangle\}$. Any other projective measurement will be given by a unitary transformation:

$$\{M_j = V |j\rangle \langle j| V^\dagger \ / \ j \in \{0, 1\}\}, \tag{22}$$

with $V \in U(2)$. A useful parametrization of this unitary operators, up to a constant phase, is

$$V = \mathbf{s} \cdot (\mathbb{I}_2, i\sigma), \tag{23}$$

with $\mathbf{s} \in \Gamma$, and $\Gamma = \{\mathbf{s} \in \mathbb{R}^4 \,/\, s_0^2 + s_1^2 + s_2^2 + s_3^2 = 1\}$.

Once the measurement is parametrized by the vector $\mathbf{s}$, and considering Bell diagonal states (16), the conditional states of the subsystem B [cf. Equation (4)] are given by [23]

$$\rho_{B|0}^{BD}(\mathbf{s}) = \frac{1}{2}\left(\mathbb{I}_2 + \sum_{i=1}^{3} c_i z_i(\mathbf{s})\sigma_i^B\right), \tag{24}$$

$$\rho_{B|1}^{BD}(\mathbf{s}) = \frac{1}{2}\left(\mathbb{I}_2 - \sum_{i=1}^{3} c_i z_i(\mathbf{s})\sigma_i^B\right). \tag{25}$$

In Equations (24) and (25) we defined

$$z_1(\mathbf{s}) = 2(-s_0 s_2 + s_1 s_3), \tag{26}$$
$$z_2(\mathbf{s}) = 2(s_0 s_1 + s_2 s_3), \tag{27}$$
$$z_3(\mathbf{s}) = s_0^2 + s_3^2 - s_1^2 - s_2^2, \tag{28}$$

and the associated conditional probabilities are $p_0(\mathbf{s}) = p_1(\mathbf{s}) = \frac{1}{2}$ for all $\mathbf{s} \in \Gamma$.

By using (26)–(28) it turns out that the quantities $\mathcal{J}_d^{\mathcal{M}}$ [cf. Equation (9)] for the squared Bures and squared Hellinger distances and also for the quantum Jensen–Shannon divergence are non-decreasing functions of the parameter $\theta(\mathbf{s}) := \sqrt{|c_1 z_1(\mathbf{s})|^2 + |c_2 z_2(\mathbf{s})|^2 + |c_3 z_3(\mathbf{s})|^2}$. Therefore, the optimal measurement is common for these three measures and its direction is given by the vector $\mathbf{s}$ such that $\theta(\mathbf{s})$ is maximum.

If we set $c = \max\{|c_1|, |c_2|, |c_3|\}$ it can be verified that $\theta(\mathbf{s}) \leq c$. Thus, the optimal measurement is given by the vector $\mathbf{s}_M$ satisfying $\theta(\mathbf{s}_M) = c$. More specifically, we have the following cases,

1. If $c = |c_1| \Rightarrow |z_1(\mathbf{s}_M)| = 1, z_2(\mathbf{s}_M) = z_3(\mathbf{s}_M) = 0$;
2. If $c = |c_2| \Rightarrow |z_2(\mathbf{s}_M)| = 1, z_1(\mathbf{s}_M) = z_3(\mathbf{s}_M) = 0$;
3. If $c = |c_3| \Rightarrow |z_3(\mathbf{s}_M)| = 1, z_2(\mathbf{s}_M) = z_1(\mathbf{s}_M) = 0$.

Thus, the measures $\mathcal{J}_d$, for the squared Bures and Hellinger distances, the Jensen–Shannon divergence and the relative entropy, are completely equivalent since they are non–decreasing functions of the same parameter $\theta(\mathbf{s}_M)$.

On the other hand, the generalized measures of total correlations can be obtained just diagonalizing the matrix ρ.

For Bell diagonal states, the distances d_B^2 and d_H^2 yield the same analytical expressions for total and classical correlations measures. Thus, we will just consider the the case of squared Bures distance:

$$\mathcal{T}_B(\rho) = 2 - \sum_{i=0}^{3} \sqrt{\lambda_i}, \tag{29}$$

$$\mathcal{J}_B(\rho) = 2 - \sqrt{1 - c} - \sqrt{1 + c}. \tag{30}$$

The quantum Jensen–Shannon divergence quantify the total and classical correlations according to the expressions:

$$\mathcal{T}_{JS}(\rho) = \frac{1}{2}\sum_i \lambda_i \log_2 \lambda_i - \frac{1}{8}\sum_i (1 + 4\lambda_i)\log_2\left(\frac{1 + 4\lambda_i}{8}\right) - 1, \tag{31}$$

$$\mathcal{J}_{JS}(\rho) = \frac{1}{4}\log_2\left[\frac{-4^2(c^2-1)}{(4-c^2)^2}\right] + \frac{1}{4}c\left[\log_2\left(\frac{2-c^2+c}{(2+c)(1-c)}\right)\right]. \tag{32}$$

3.1. Behavior of Correlations under Non-Dissipative Decoherence

Now, we turn to the study of a dynamical scenario where we shall consider two non-interacting qubits A and B under the influence of local and identical non-dissipative decoherence channels. In this case, the evolution of a two-qubit state ρ can be written by means of the Kraus operators formalism, e.g.,

$$\Lambda[\rho] = \sum_{i,j=1}^{4}(E_i^A \otimes E_j^B)\rho(E_i^{A\dagger} \otimes E_j^{B\dagger}), \tag{33}$$

where the Kraus operators are given by:

$$E_k^m = \sqrt{\frac{1-\exp(-\gamma t)}{2}}\sigma_k^m, \tag{34}$$

$$E_4^m = \sqrt{\frac{1+\exp(-\gamma t)}{2}}\mathbb{I}_2, \tag{35}$$

$$E_{i,j\neq k}^m = 0, \tag{36}$$

and $m = A, B$ states for the qubit A or B, $k \in \{1, 2, 3\}$ is in correspondence with {bit flip, bit-phase flip, phase flip} channels, and $\gamma \in \mathbb{R}_{\geq 0}$ represents the decoherence rate. A particular choice of k defines the direction x, y, z of the noise in the Bloch sphere and establishes the decoherence process. From now on, we will consider only the phase-flip case, $k = 3$.

It turns out that, whenever the system $A + B$ is initially in a BD state, its structure remains unchanged for all t [13,24–27]. In this scenario, the coefficients c_i are functions of t and are given by

$$c_1(t) = c_1(0)e^{-2\gamma t}, \tag{37}$$

$$c_2(t) = c_2(0)e^{-2\gamma t}, \tag{38}$$

$$c_3(t) = c_3(0). \tag{39}$$

An interesting behaviour of QD, so-called as *"Freezing phenomenon of quantum discord"*, may occur if certain particular initial conditions are satisfied:

$$c_2(0) = -c_1(0)c_3(0), \tag{40}$$

$$|c_1(0)| > |c_3(0)|, \tag{41}$$

The evolution of the system from the above initial conditions gives rise to a peculiar dynamics. In particular, some measures of quantum correlations [28], remain constant for all $t \in [0, t^*]$ where $t^* = -\frac{1}{2\gamma}\log\frac{|c_3(0)|}{|c_1(0)|}$. However, for $t > t^*$ they start to decay with t.

In what follows, we shall analyze the behavior of correlations under non-dissipative decoherence by means of $\mathcal{T}_d$, $\mathcal{J}_d$ and $\mathcal{Q}_d$, for the squared Bures distance and the quantum Jensen–Shannon divergence and we will compare our results with those obtained by means of $\mathcal{I}$, $\mathcal{J}_S$ and $\mathcal{D}$, respectively. In order to summarize our study, we shall explicitly describe three typical examples [24].

Example 1. *We set $|c_1| \geq |c_2|, |c_3|$ or $|c_2| \geq |c_1|, |c_3|$.*

In this case, the typical behaviors of $\mathcal{T}_B$, $\mathcal{T}_{JS}$ and $\mathcal{I}$ are represented in Figure 1 whereas the behaviors of $\mathcal{J}_B$, $\mathcal{J}_{JS}$ and $\mathcal{J}_S$ are shown in Figure 2. In addition, the behaviors of $\mathcal{Q}_B$, $\mathcal{Q}_{JS}$ and $\mathcal{D}$ are represented in Figure 3. Clearly, on one hand, it can be seen that the behavior of total correlations, as quantified by $\mathcal{T}_B$ and $\mathcal{T}_{JS}$, qualitatively follows the behavior showed by von Newman mutual information $\mathcal{I}$. In addition, it can be seen that the behavior of classical correlations, as quantified by $\mathcal{J}_B$

and $\mathcal{J}_{JS}$, qualitatively follows the behavior displayed by the measure $\mathcal{J}_S$ [cf. Equation (3)]. From this results, we can see that total correlation measures and classical correlation measures impose an equivalent order between the quantum states as Total Mutual Information and the classical correlation measure $\mathcal{J}_S$, respectively [29,30]. On the other hand, regarding the behavior of $\mathcal{Q}_B$, $\mathcal{Q}_{JS}$ and $\mathcal{D}$, in Figure 3 we can see that, while $\mathcal{D}$ is decreasing between 0 and t^*, $\mathcal{Q}_{JS}$ and $\mathcal{Q}_B$ are increasing instead in this time interval. However, later on for $t > t^*$, all three quantum correlation measures do decrease monotonically. In this case, for $t < t^*$, the order imposed by the measures of quantum correlations $\mathcal{Q}_B$ and $\mathcal{Q}_{JS}$ turns out to be different from the order imposed by standard quantum discord $\mathcal{D}$.
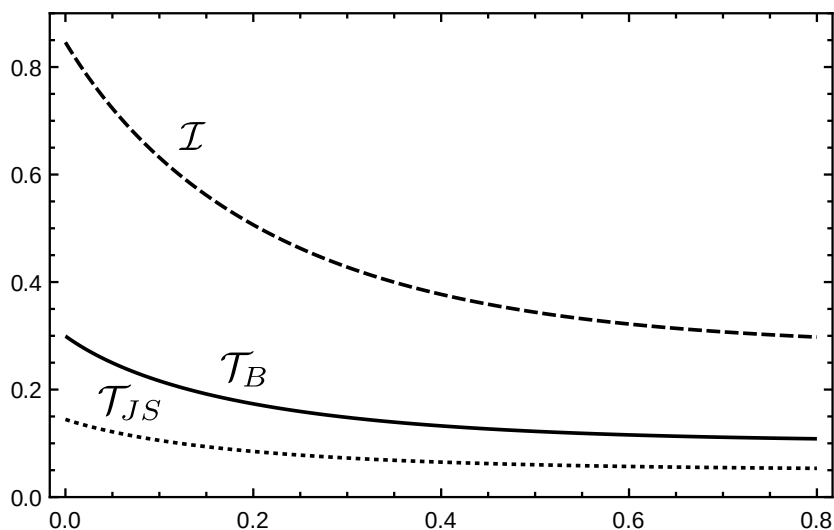
Figure 1. Dynamics of the generalized total correlation measures as a function of γt for $c_1(0) = 0.8$, $c_2(0) = -c_3(0)$, $c_3(0) = 0.6$.
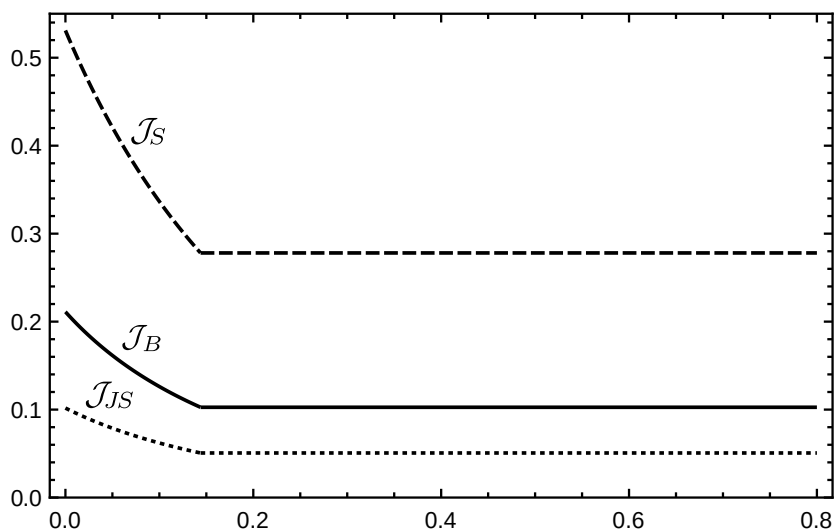

Figure 2. Dynamics of the generalized classical correlation measures as a function of γt for $c_1(0) = 0.8$, $c_2(0) = -c_3(0)$, $c_3(0) = 0.6$.
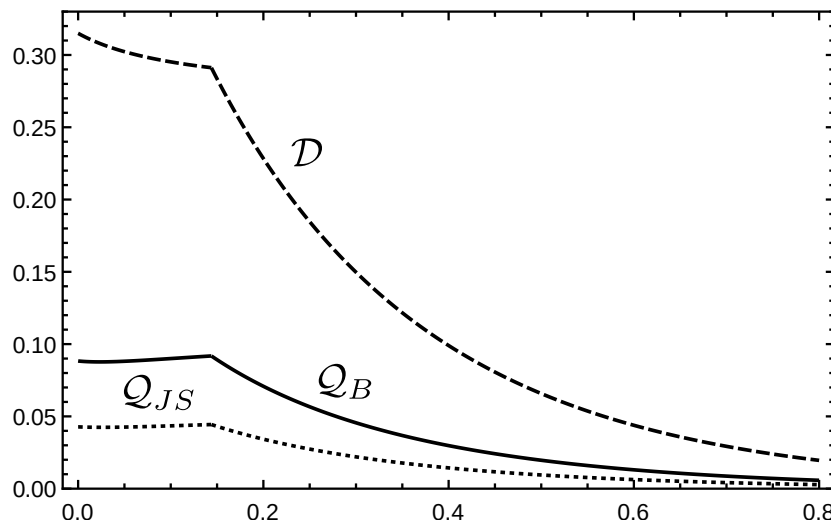
Figure 3. Dynamics of the generalized quantum correlation measures as a function of γt for $c_1(0) = 0.8$, $c_2(0) = -c_3(0)$, $c_3(0) = 0.6$.

In addition, within the same kind of initial conditions studied in this case, in Figures 4 and 5 we present the results corresponding to a dynamical scenario where the "freezing phenomena of quantum discord" takes place. We can see that, while quantum discord $\mathcal{D}$ remains constant for $t < t^*$, the quantities $\mathcal{Q}_{JS}$ and $\mathcal{Q}_B$ are increasing instead. Besides, in this case quantum discord turns out to be equal to the measure of classical correlations $\mathcal{J}_S$ for t^* (see the inset in Figures 4 and 5), i.e., the time interval in which the "sudden change" occurs [24]. This peculiar behavior is also not observed in the cases of the measures $\mathcal{Q}_B$ and $\mathcal{Q}_{JS}$.
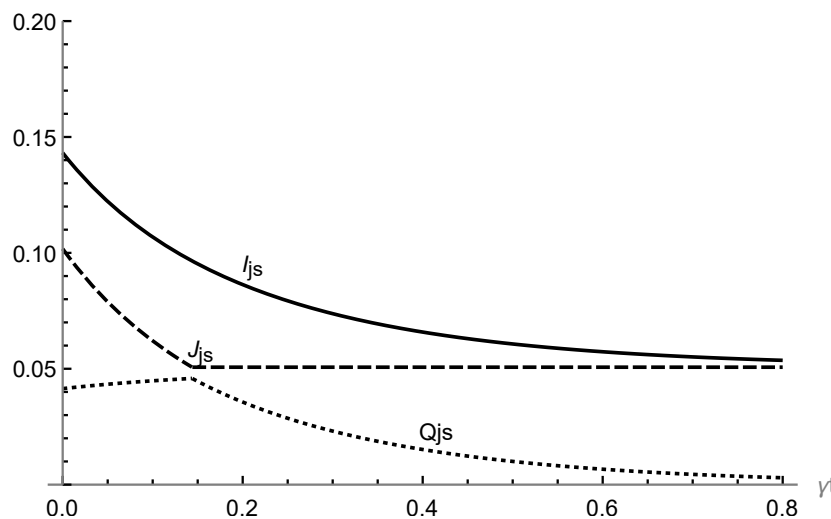

Figure 4. Dynamics of the generalized quantum correlation measures for the quantum Jensen–Shannon divergence, as a function of γt, considering the initial conditions of the freezing phenomenon of quantum discord: $c_1(0) = 0.8$, $c_2(0) = -c_1(0)c_3(0)$, $c_3(0) = 0.6$. $t^* \approx 0.144$.

On the other hand, it should be noted that the initial conditions that give rise to the freezing phenomenon of quantum discord are included in Example 1. In this particular case, as we see in Figures 4 and 5, we have that while $\mathcal{D}$ remains constant, the quantities $\mathcal{Q}_{JS}$ and $\mathcal{Q}_B$ are increasing. Besides, the quantum discord is equal to the measure of classical correlations $\mathcal{J}_S$ for t^* in which the "sudden change" occurs [24]. This peculiar equality is also not seen in the cases of the measures $\mathcal{Q}_B$ and $\mathcal{Q}_{JS}$.
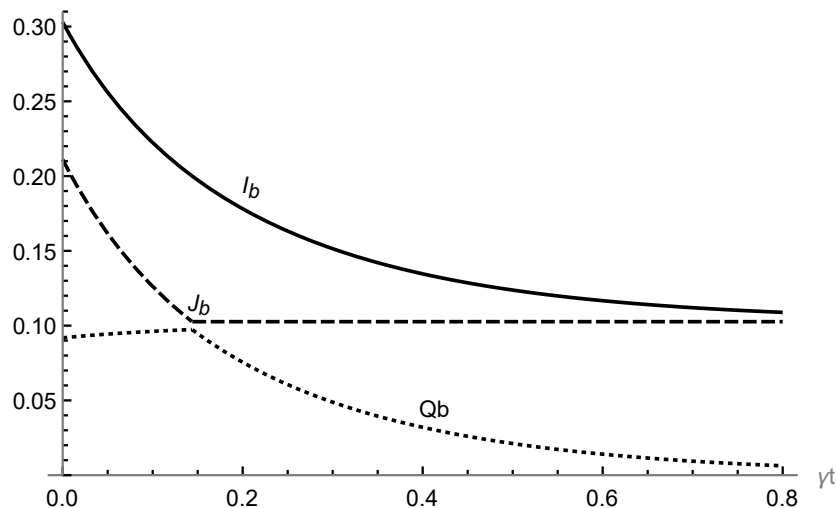
Figure 5. Dynamics of the generalized quantum correlation measures for the squared Bures distance case, as a function of γt, considering the initial conditions of the freezing phenomenon of quantum discord: $c_1(0) = 0.8$, $c_2(0) = -c_1(0)c_3(0)$, $c_3(0) = 0.6$. $t^* \approx 0.144$.

Example 2. *In this case we set* $|c_3| \geq |c_1|, |c_2|$.

In this case we find that all three quantities $\mathcal{J}_B$, $\mathcal{J}_{JS}$ and $\mathcal{J}_S$ remain constant. Therefore, the correlation measures $\mathcal{Q}_d$ and $\mathcal{T}_d$, for all the distance measures considered here, show the same behavior up to an additive constant (i.e., $\mathcal{J}_d$).

Example 3. *Now, we set* $|c_3| = 0$.

When we consider these initial conditions, it is straightforward to see that all correlation measures give rise to the same ordering of the states since they decay monotonically with t.

4. Concluding Remarks

In this work, we built new measures of correlations in bipartite quantum states based upon a recently introduced theoretical framework. All measures satisfy a set of requirements, as discussed in ref. [18], in order to be bona fide measures of correlations. In particular, we used squared Bures and squared Hellinger distances, in addition to quantum Jensen–Shannon divergence, to define measures of total, classical and quantum correlations. We applied these measures to analyze the behavior of correlations under non-dissipative decoherence in two-qubit states with maximally mixed marginals in different dynamical scenarios of physical relevance.

On one hand, we found that the behaviors of the measures of total and classical correlations introduced in this work, are qualitatively in agreement with the behavior displayed by quantum mutual information and the measure $\mathcal{J}_S$ of classical correlations typically used to calculate quantum discord [cf. Equation (3)]. On the other hand, regarding the measures of quantum correlations used in our studies, we found that in general their behavior does not follow the standard quantum discord $\mathcal{D}$. It is worth mentioning that, quantum discord, as well as the measures of quantum correlations used in this work, relies upon the assumption that total correlations are additive. However, this assumption, as pointed out by Brodutch et al., is indeed a debatable property required for total correlations. In the light of our findings, perhaps, this property needs a deeper and systematic study in order to gain more insight regarding the possibility to obtain reliable quantifiers of quantum correlations within this additive scheme.

Regarding the optimization of all the measures of classical correlations $\mathcal{J}_d$ introduced in this work, we found that all of these quantities are non-decreasing functions of a single parameter θ. Thus,

the optimization of each one of these measures relies on our ability to find the maximum value of the parameter θ (i.e., the value θ_{max}). As an interesting result, we found that the value θ_{max} which optimize our measures of classical correlations is the same which optimizes the quantity $\mathcal{J}_S$ and therefore, is the same value used to obtain the value of the quantum discord $\mathcal{D}$ [cf. Eqs. (3) and (6)]. Furthermore, one can observe a sudden change in the dynamics of the classical correlations for a time t^*. In addition, it is worth mentioning that at t^* there is a change in the direction of the measurement that optimizes all the measures of classical correlations.

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