The Steklov eigenvalue problem in a cuspidal domain

María G. Armentano and Ariel L. Lombardi

Abstract In this paper we analyze the approximation, by piecewise linear finite elements, of a Steklov eigenvalue problem in a plane domain with an external cusp. This problem is not covered by the literature and its analysis requires a special treatment. Indeed, we develop new trace theorems and we also obtain regularity results for the source counterpart. Moreover, under appropriate assumptions on the meshes, we present interpolation error estimates for functions in fractional Sobolev spaces. These estimates allow us to obtain appropriate convergence results of the source counterpart which, in the context of the theory of compact operator, are a fundamental tool in order to prove the convergence of the eigenpairs. At the end, we prove the convergence of the eigenpairs by using graded meshes and present some numerical tests.

Keywords Steklov eigenvalue problem · finite elements · Cuspidal domains · Graded meshes

Mathematics Subject Classification (2000) 65N25, 65N30

1 Introduction

Problems with an eigenvalue parameter on the boundary, as the Steklov eigenvalue problem, arise in the study of many interesting physical situations. We refer to the survey article [16] where several applications and features of the Steklov spectrum are discussed. One of them is its essential role for the implementation of electrical impedance tomography in medical and geophysical imaging, and the analysis of photonic crystal. If Ω is two-dimensional, the Steklov eigenvalues can be thought as the squares of the natural frequencies of a vibrating free membrane with its mass concentrated along its boundary. This problem was also considered in [19], for modeling the sloshing problem which describes the natural frequencies and the corresponding modes of a free wave motion in an infinitely long canal having an uniform cross section. The Steklov eigenvalue problem under consideration arises, for example, in the study of the normal modes of oscillation of a mass of liquid in a finite basin. In this case the eigenvalue equation is satisfied only on the top of the boundary (the free surface), while on the bottom the vanishing of the normal derivative is required (see, for example, [14]). There are features of the Steklov eigenvalues which are, in comparison with the Laplacian eigenvalue problem with Dirichlet or Neumann conditions, highly

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sensible to the regularity of the boundary, as is explained in [16]. Indeed, even for Lipschitz domains, one–term spectral asymptotics have not been proved yet (see [16] and references therein).

Numerical approximations of eigenvalues and eigenfunctions, by the finite element method, have been widely analyzed for several kinds of spectral problems, and, in particular, for Steklov eigenvalue problems (see, for example, [2,6,7,9] and the reference therein). However, as far as we know, all of them only deal with polygonal or smooth domains.

The goal of this paper is the analysis of a piecewise linear finite element approximation of a Steklov eigenvalue problem in a rough plane domain, precisely a domain with an external cusp. Such a domain is curved and non-Lipschitz, and therefore the classical spectral theory (see [10,12,23]) cannot be directly applied and special treatment is required. In order to be able to use part of the theory developed in [2–4] and with the aim to add new tools to it, we consider the model problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda u & \text{on } \Gamma_3, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1, \\ \quad u = 0 & \text{on } \Gamma_2, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is the domain defined by

$$\Omega = \{(x, y) : 0 < x < 1, \, 0 < y < x^\alpha\},$$

α is a parameter satisfying $1 < \alpha < 2$, ν denotes the unitary outward normal to Ω (see Figure 1). With $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ we denote the boundary of Ω, where

$$\Gamma_1 = \{0 \le x \le 1, y = 0\}, \quad \Gamma_2 = \{x = 1, 0 \le y \le 1\} \quad \text{and} \quad \Gamma_3 = \{0 \le x \le 1, y = x^\alpha\}.$$

In view of the boundary conditions in the Steklov eigenvalue problem, a different analysis than the one developed in [2] for the Laplacian eigenvalue problem in this kind of domain, is required. Therefore, the results of the previous works in domains with an external cusp (see, [1–4] and the references therein) need to be extended or generalized. In particular, in this work we prove that functions in $H^1(\Omega)$ have traces in fractional Sobolev spaces, extending the trace theorems of [3]. We also analyze the existence of solutions for the source counterpart, to cover the case in which the boundary datum is less regular, extending the regularity results given in [3]. Moreover, we need piecewise linear interpolation error estimates for functions in fractional Sobolev spaces strictly between H^1 and H^2, which are obtained by using the theory of interpolation of operators. Since the Lagrange interpolation, used in [2,4], is not defined for functions in H^1, we introduce a Scott–Zhang type quasi–interpolation and obtain stability and error estimates under appropriate conditions of the family of anisotropic meshes. It is important to emphasize that, due to the characteristics of the domain under consideration, these interpolation error estimates are not covered by the literature and provide, in particular, a new contribution to the interpolation theory in cuspidal domains. All these results allow us to obtain suitable convergence results for the source counterpart and prove, by using the abstract approximation theory stated in [10,12,23], the convergence of the eigenfunctions and the eigenvalues when suitable graded meshes are used.

Let $V = \{v \in H^1(\Omega) : v|_{\Gamma_2} = 0\}$. The weak form of the problem is: Find $\lambda \in \mathbb{R}$ and $u \in V$, $u \ne 0$ satisfying

$$\begin{cases} a(u, v) = \lambda b(u, v) & \forall \, v \in V, \\ \|u\|_{L^2(\Gamma_3)} = 1, \end{cases} \tag{1}$$

where $a(u, v) = \int_\Omega \nabla u \cdot \nabla v$, which is continuous and coercive on V, and $b(u, v) = \int_{\Gamma_3} uv$.

To approximate the solution of (1) we replace Ω with a polygonal domain Ω_h and use the standard linear finite element method. We will construct Ω_h in such a way that $\Omega \subset \Omega_h$ and the nodes on Γ_h, the boundary of Ω_h, are on Γ.
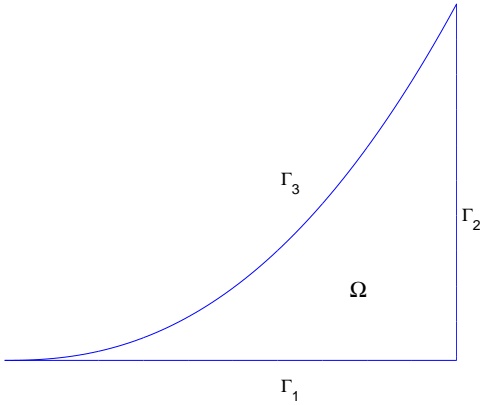
Fig. 1 A cuspidal domain

Let $\{\mathcal{T}_h\}$ be a family of triangulations of Ω_h. Associated with $\{\mathcal{T}_h\}$ we have the finite element space

$$V_h = \{v \in H^1(\Omega_h) : v|_{\Gamma_2} = 0 \text{ and } v|_K \in \mathcal{P}_1 \quad \forall\, K \in \mathcal{T}_h\},$$

where $\mathcal{P}_1$ denotes the space of linear polynomials.

Then, we propose the following discrete problem: find $\lambda_h \in \mathbb{R}$ and $u_h \in V_h$, $u_h \neq 0$ such that

$$\begin{cases} a_h(u_h, v) = \lambda_h b(u_h, v) & \forall v \in V_h, \\ \|u_h\|_{L^2(\Gamma_3)} = 1, \end{cases} \tag{2}$$

where $a_h(u_h, v) = \int_{\Omega_h} \nabla u_h \cdot \nabla v$. We note that, although our numerical scheme (2) conserves the continuous bilinear form b of (1) on the right hand side, it is non conforming, since the integral in the definition of a_h is on Ω_h.

Let $L^2_\beta(\Gamma_3)$ be the space defined as $L^2_\beta(\Gamma_3) = \{v : \|vx^\beta\|_{L^2(\Gamma_3)} < \infty\}$, with $\beta \in \mathbb{R}$ such that $\beta = 0$ if $\alpha \leq 2$ (see, [3,22]) and $\beta > \frac{\alpha}{2} - 1$ if $\alpha > 2$. We consider the following problem associated with the Steklov eigenvalues of the Laplacian operator: Given $g \in L^2_{-\beta}(\Gamma_3)$, find $u \in V$ such that,

$$a(u, v) = \int_{\Gamma_3} gv \quad \forall\, v \in V. \tag{3}$$

From the trace theorems obtained in [3] we can affirm that, this problem has a unique solution $u \in V$.

In order to analyze the convergence properties of (2) we use the abstract approximation theory stated in [10,12,23]. Therefore we need to introduce appropriate compact operators T and T_h associated to (1) and (2) respectively defined in the same functional space, a requirement not easy to fulfill, and such that $T_h \to T$ in norm. Due to the Neumann boundary conditions in the Steklov eigenvalue problem, we need to develop new trace theorems which in particular allow us to prove the compactness of the involved operators. We also show the regularity of the solution of the source counterpart in fractional Sobolev spaces. Then, in order to prove the convergence of T_h to T, we obtain interpolation error estimates in fractional Sobolev spaces taking into account that our meshes necessarily include anisotropic elements, such as those with a vertex at the tip of the cusp, and therefore these estimates are not standard and require a special analysis.

The rest of the paper is organized as follows. In Section 2 we state the spectral problem, so we introduce linear operators T and T_h associated to the eigenvalue problem and its approximation. We present a trace theorem, which is an important tool in order to prove the compactness of the operators, and we also obtain some regularity results for the solution of the associated source problems. In Section 3 we present the interpolation error estimates in fractional Sobolev spaces for some anisotropic graded meshes. In section 4 we obtain L^2 error estimates on Γ_3. In Section 5, we prove the convergence of the

eigenfunctions and eigenvalues. Finally, in section 6 we analyze numerical aspects concerning the solution of the discrete generalized eigenvalue problem (2) and we present some numerical examples.

2 The spectral problems

In this section we introduce appropriate linear operators T and T_h with spectra related to those of problems (1) and (2) respectively. Despite of α is a parameter satisfying $1 < \alpha < 2$, some results, as specified in the statements, are also valid for $\alpha = 2$.

Let $B : L^2(\Gamma_3) \to H^1(\Omega)$ be the solution operator of problem (3) defined by $Bg = u$. In view of Theorem 2.2 of [3] and, for the case $\alpha = 2$, Lemma 2.1 of [22], we can define the operator $T : L^2(\Gamma_3) \to L^2(\Gamma_3)$ in such a way that for $g \in L^2(\Gamma_3)$, Tg is the restriction of Bg on Γ_3: $Tg = Bg|_{\Gamma_3}$. In this case we have that

$$\|Tg\|_{L^2(\Gamma_3)} \leq C\|u\|_{H^1(\Omega)} \leq C\|g\|_{L^2(\Gamma_3)}.$$

This inequality guarantee that the operator T is bounded in $L^2(\Gamma_3)$.

Hence, we want to obtain conditions under which the operator T is compact. In the case of a Lipschitz domain, the compactness of this kind of operator follows by using well known results on restrictions of $H^1(\Omega)$ to the boundary (see, for example, [6]) but, since our domain is not Lipschitz, those arguments can not be straightforward applied.

For any subset $D \subset \mathbb{R}^2$ and $0 < s < 1$, we recall that the space $H^s(D)$ is defined as

$$H^s(D) = \left\{ v \in L^2(D) : \|v\|_{H^s(D)} < \infty \right\},$$

with

$$\|v\|_{H^s(D)} = \|v\|_{L^2(D)} + |v|_{H^s(D)}$$

and

$$|v|_{H^s(D)} = \left(\int_D \int_D \frac{|v(\mathbf{x}) - v(\mathbf{y}))|^2}{\|\mathbf{x} - \mathbf{y}\|^{1+2s}} \, d\mathbf{x} \, d\mathbf{y} \right)^{\frac{1}{2}}.$$

Then, the following example shows that although the restriction of $H^1(\Omega)$ functions is in $L^2(\Gamma)$ for $\alpha \leq 2$, it is not necessarily in $H^s(\Gamma)$ for $s > 1 - \frac{\alpha}{2}$, and so, the trace may be not in $H^{\frac{1}{2}}(\Gamma)$.

Example 1 Let v be the function $v(x, y) = x^\gamma$ with $\gamma < 0$. Then, an easy computation shows that $v \in H^1(\Omega)$ iff $\gamma > \frac{1-\alpha}{2}$ and $v \in L^2(\Gamma)$ iff $\gamma > -\frac{1}{2}$. So, for $\alpha < 2$ and $\gamma > \frac{1-\alpha}{2}$ this function is in $H^1(\Omega)$ and in $L^2(\Gamma)$. However, we show that v is not in $H^s(\Gamma)$ if $\gamma \leq s - \frac{1}{2}$ and therefore, for $\alpha < 2$ there exist functions which are in $H^1(\Omega)$ and $L^2(\Gamma)$ but not in $H^s(\Gamma)$ for s with $1 > s > 1 - \frac{\alpha}{2}$ (observe that $1 - \frac{\alpha}{2} \nearrow \frac{1}{2}$ when $\alpha \searrow 1$, and $1 - \frac{\alpha}{2} \searrow 0$ when $\alpha \nearrow 2$).

In fact,

$$\int_{\Gamma_1} \int_{\Gamma_1} \frac{|v(\mathbf{x}) - v(\mathbf{y}))|^2}{\|\mathbf{x} - \mathbf{y}\|^{1+2s}} \, d\mathbf{x} \, d\mathbf{y} = \int_0^1 \int_0^y \frac{|y^\gamma - x^\gamma|^2}{(y-x)^{1+2s}} \, dx \, dy + \int_0^1 \int_y^1 \frac{|x^\gamma - y^\gamma|^2}{(x-y)^{1+2s}} \, dx \, dy$$
$$= I + II.$$

Now, since $s < 1$ and by introducing $\theta_{x,y}$ with $x < \theta_{x,y} < y$ and defined by $x^\gamma - y^\gamma = \gamma \theta_{x,y}^{\gamma-1}(x-y)$, we have

$$I = \int_0^1 \int_0^y \frac{\gamma^2 \theta_{x,y}^{2\gamma-2}(y-x)^2}{(y-x)^{1+2s}} \, dx \, dy \geq \int_0^1 \int_0^y \frac{\gamma^2 y^{2\gamma-2}}{(y-x)^{2s-1}} \, dx \, dy$$
$$= \int_0^1 \frac{\gamma^2}{2-2s} y^{2\gamma-2} y^{2-2s} \, dy = \frac{\gamma^2}{2-2s} \int_0^1 y^{2\gamma-2s} \, dy.$$

Then, taking into account that $II \geq 0$, we get

$$|v|_{H^s(\Gamma_1)} = I + II \geq \frac{\gamma^2}{2-2s} \int_0^1 y^{2\gamma-2s} \, dy.$$

Therefore, if $\gamma \leq s - \frac{1}{2}$, the function $v \notin H^s(\Gamma_1)$ (and the same result holds on Γ_3 since $|(1, \alpha x^{\alpha-1})|$ is bounded). Moreover, since $\|v\|_{H^s(\Gamma_i)} < \|v\|_{H^s(\Gamma)}$, $i = 1, 2, 3$, we can conclude that $v \notin H^s(\Gamma)$.

Next theorem allows us to prove that, for $1 < \alpha \leq 2$, the restriction of $H^1(\Omega)$ functions are in $H^\sigma(\Gamma)$, with $\sigma = 1 - \frac{\alpha}{2}$.

Theorem 1 *Let $1 < \alpha < 2$. There exists a constant C, depending only on α, such that for any $u \in H^1(\Omega)$*

$$\|u\|_{H^\sigma(\Gamma_i)} \leq C\|u\|_{H^1(\Omega)},$$

with $\sigma = 1 - \frac{\alpha}{2}$ and $i = 1, 2, 3$.

Proof. Let $\mathcal{D} = \{(x, y) : -x < y < x^\alpha, \, 0 < x < 1\}$ and $\tilde{u} : \mathcal{D} \to \mathbb{R}$ be the extension of u to $\mathcal{D}$ defined in [4]. By examining the proof of Lemma 3.1 of [4] we know that $\tilde{u} \in H^1(\mathcal{D}, r^{\alpha-1})$ and

$$\|\tilde{u}\|_{H^1(\mathcal{D}, r^{\alpha-1})} \leq 2\|u\|_{H^1(\Omega)},$$

where $r = \sqrt{x^2 + y^2}$ and, for any function $\rho \geq 0$ and $s \in \mathbb{R}$, we define

$$\|v\|_{H^1(\mathcal{D}, \rho^s)}^2 := \|v\rho^{\frac{s}{2}}\|_{L^2(\mathcal{D})}^2 + \|\nabla v\rho^{\frac{s}{2}}\|_{L^2(\mathcal{D})}^2.$$

We observe that, if $d(x, y)$ denotes the distance between (x, y) and $\partial\mathcal{D}$ then, $d(x, y) \leq r(x, y)$ for any $(x, y) \in \mathcal{D}$. Thus,

$$\tilde{u} \in H^1(\mathcal{D}, d^{\alpha-1}), \qquad \|\tilde{u}\|_{H^1(\mathcal{D}, d^{\alpha-1})} \leq C\|u\|_{H^1(\Omega)}.$$

Now, since $\mathcal{D}$ is Lipschitz we can apply [18, Theorem 2.13] (taking $p = 2, d = 2, k = 1, \theta = \alpha + 1$ and $\sigma = 1 - \frac{\alpha}{2}$), or [13, Theorem 3.2] (with $p = 2$ and $a = \frac{\alpha-1}{2}$), to conclude that for a constant C, depending only on α, we have

$$\|\tilde{u}\|_{H^\sigma(\partial\mathcal{D})} \leq C\|\tilde{u}\|_{H^1(\mathcal{D}, d^{\alpha-1})} \leq C\|u\|_{H^1(\Omega)}.$$

In particular, for $i = 2, 3$ we have that

$$\|u\|_{H^\sigma(\Gamma_i)} \leq \|\tilde{u}\|_{H^\sigma(\Gamma)} \leq C\|u\|_{H^1(\Omega)}.$$

The estimate for $i = 1$ follows by similar arguments.   □

Corollary 1 *There exists a constant C, depending only on α, such that for any $u \in H^1(\Omega)$ we have*

$$\|u\|_{H^\sigma(\Gamma)} \leq C\|u\|_{H^1(\Omega)}, \tag{4}$$

with $\sigma = 1 - \frac{\alpha}{2}$, $1 < \alpha < 2$.

Proof. From Theorem 1 we know that, for $i = 1, 2, 3$, $\gamma_i = u|_{\Gamma_i}$ is in $H^\sigma(\Gamma_i)$ with $\sigma = 1 - \frac{\alpha}{2} < \frac{1}{2}$. Since $\sigma < \frac{1}{2}$, if we consider $\tilde{\gamma}_i$ the extension by zero of γ_i to Γ we can affirm that $\tilde{\gamma}_i$ is in $H^\sigma(\Gamma)$ [20, Theorem 11.4]. Then

$$\|\tilde{\gamma}_i\|_{H^\sigma(\Gamma)} = C\|\tilde{\gamma}_i\|_{H^\sigma(\Gamma_i)} \leq C\|u\|_{H^1(\Omega)}.$$

Hence, $u|_\Gamma = \tilde{\gamma}_1 + \tilde{\gamma}_2 + \tilde{\gamma}_3$, and

$$\|u\|_{H^\sigma(\Gamma)} = \|\tilde{\gamma}_1 + \tilde{\gamma}_2 + \tilde{\gamma}_3\|_{H^\sigma(\Gamma)} \leq \|\tilde{\gamma}_1\|_{H^\sigma(\Gamma)} + \|\tilde{\gamma}_2\|_{H^\sigma(\Gamma)} + \|\tilde{\gamma}_3\|_{H^\sigma(\Gamma)} \leq C\|u\|_{H^1(\Omega)},$$

which concludes the proof.   □

Next theorem (which is a consequence of Theorem 3.1 of [3]) guarantees the regularity of our source problem under appropriate conditions on the function g.

Proposition 1 *Let $1 < \alpha \leq 2$ and g such that, if $h(t) = g(t, t^\alpha)$, h is in $H^1(0,1)$ and $h(0) = 0$. Then, the solution u of the source problem (3) is in $H^2(\Omega)$ and there exists a constant C such that*

$$\|u\|_{H^2(\Omega)} \leq C\|h'\|_{L^2(0,1)}.$$

Proof. We note that if $\varphi(t) = t^\alpha$ then

1. $\|h(t)\varphi^{-\frac{1}{2}}\|_{L^2(0,1)} = \|h(t)t^{-\frac{\alpha}{2}}\|_{L^2(0,1)} \leq C\|h'\|_{L^2(0,1)}$, using the classical Hardy inequality and taking into account that $1 < \alpha \leq 2$,
2. $(\varphi'')^{-\frac{1}{2}} \in L^\infty([0,1])$,
3. $\frac{\varphi''\varphi}{(\varphi')^2}$ is constant.

So the desired result follows by using Theorem 3.1 of [3] with $f = 0$.   □

Given a Sobolev space $W(\Gamma_3)$, we denote by $\overset{o}{W}(\Gamma_3)$ the space defined as

$$\overset{o}{W}(\Gamma_3) = \overline{\{\phi \in C^\infty(\Gamma_3) : \phi(0,0) = 0\}}^{\|\cdot\|_{W(\Gamma_3)}}.$$

From [3, Theorem 2.2] and Proposition 1 we know that, if $g \in L^2(\Gamma_3) = \overset{o}{L^2}(\Gamma_3)$ the solution of (3) is in $H^1(\Omega)$ and if $g \in \overset{o}{H^1}(\Gamma_3)$ then the solution of (3) is in $H^2(\Omega)$. Let $0 < s < \frac{1}{2}$. Following the theory developed in [20, Section 11.5], we have

$$\left[L^2(\Gamma_3), \overset{o}{H^1}(\Gamma_3)\right]_s = \overset{o}{H^s}(\Gamma_3) = H^s(\Gamma_3),$$

by using, for example, Corollary 1.4.4.5 of [17], for the last equality. Then if $g \in H^s(\Gamma_3)$, it follows from the theory of Interpolation of Operators (see for example [20, Theorem 5.1]) that the solution u verifies

$$u \in [H^1(\Omega), H^2(\Omega)]_s = H^{1+s}(\Omega)$$

and we have the following theorem.

Theorem 2 *Let $g \in H^s(\Gamma_3)$ with $0 \leq s < \frac{1}{2}$. Then, the solution u of (3) is in $H^{1+s}(\Omega)$ and there exists a constant C, independent of s but might depend on α, such that*

$$\|u\|_{H^{1+s}(\Omega)} \leq C\|g\|_{H^s(\Gamma_3)}.$$

Proposition 2 *Assume $1 < \alpha < 2$. Then the operator $T : L^2(\Gamma_3) \to L^2(\Gamma_3)$ is compact and self–adjoint.*

Proof. Using Theorem 1 with $\sigma = 1 - \frac{\alpha}{2} > 0$, we have that

$$\|Tg\|_{H^\sigma(\Gamma_3)} = \|u|_{\Gamma_3}\|_{H^\sigma(\Gamma_3)} \leq C\|u\|_{H^1(\Omega)} \leq C\|g\|_{L^2(\Gamma_3)},$$

for all $g \in L^2(\Gamma_3)$. Therefore, the compactness of T follows from the fact that for $\sigma > 0$, $H^\sigma(\Gamma_3)$ is compactly embedded in $L^2(\Gamma_3)$. Now, for $g_1, g_2 \in L^2(\Gamma_3)$, consider $u_1 = Bg_1$ and $u_2 = Bg_2$. Then, recalling the definition of B given at the beginning of this Section (i.e., the solution operator of problem (3)) we have

$$(Tg_1, g_2)_{\Gamma_3} = (u_1, g_2)_{\Gamma_3} = a(u_2, u_1) = a(u_1, u_2) = (g_1, u_2)_{\Gamma_3} = (g_1, Tg_2)_{\Gamma_3},$$

which proves that T is self–adjoint.   □

Now, if we denote by μ a nonzero, real and simple eigenvalue of T and by $z \in L^2(\Gamma_3)$ the associated eigenfunction, i.e., $Tz = \mu z$, we get

$$a(Bz, v) = \int_{\Gamma_3} zv = \frac{1}{\mu} \int_{\Gamma_3} Tz \, v = \frac{1}{\mu} \int_{\Gamma_3} Bz \, v$$

and so $\frac{1}{\mu}$ is an eigenvalue of (1) with eigenfunction Bz.

Through this correspondence, properties of the eigenvalue problem (1) can be derived from the spectral theory for compact operators (see, for example, [6,7,10,26]). Thus, we may infer that the problem (1) has a sequence of eigenpairs (λ_j, u_j), with positive eigenvalues λ_j diverging to $+\infty$. We assume the eigenvalues to be increasingly ordered: $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots$.

Now, we want to introduce the analogous discrete operator T_h. Hence, we can consider the discrete problem: Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = b(g, v_h) \qquad \forall v_h \in V_h. \tag{5}$$

We can define the solution operator $B_h : L^2(\Gamma_3) \to H^1(\Omega_h)$ by $B_h g = u_h$. Then, we take $T_h : L^2(\Gamma_3) \to L^2(\Gamma_3)$ as the restriction on Γ_3 of applying B_h: $T_h g = B_h g|_{\Gamma_3}$ which satisfies

$$\|T_h g\|_{L^2(\Gamma_3)} \leq C \|u_h\|_{H^1(\Omega_h)} \leq C \|g\|_{L^2(\Gamma_3)}.$$

We observe that, our formulation of the discrete problem, allows us to define T_h in the same functional space as T. Now, if we denote by μ_h a nonzero and real eigenvalue of T_h and by $z_h \in L^2(\Gamma_3)$ the associated eigenfunction, i.e., $T_h z_h = \mu_h z_h$, we get

$$a_h(B_h z_h, v_h) = \int_{\Gamma_3} z_h v_h = \frac{1}{\mu_h} \int_{\Gamma_3} T_h z_h \, v_h = \frac{1}{\mu_h} \int_{\Gamma_3} B_h z_h \, v_h,$$

as a consequence $\frac{1}{\mu_h}$ is an eigenvalue of (2) with eigenfunction $B_h z_h$.

In order to prove the convergence of the eigenvalues we want to use the classical spectral theory. Our first goal is to prove the convergence of T_h to T in norm.

3 Graded meshes and interpolation error estimates

We will assume that the family of meshes $\{\mathcal{T}_h\}$ satisfies the same properties considered in [1,2,4]. Indeed, let $\mathcal{T}_h$ be a triangulation of Ω_h, where Ω_h is a polygonal approximation of Ω with all its vertices belonging to Γ, and $h > 0$ be a parameter that goes to 0. Let be $\gamma = (\alpha - 1)/2$ with $1 < \alpha < 2$. For each $K \in \mathcal{T}_h$ we denote by h_K its diameter and by θ_K its maximum angle, we assume that there exist positive constants τ and $\theta_M < \pi$, independent of h, such that

M.1 $\theta_K < \theta_M, \forall K \in \mathcal{T}_h$ *(the maximum angle condition)*.
M.2 $h_K \sim \tau \, h^{\frac{1}{1-\gamma}}$, if $(0,0) \in K$.
M.3 $h_K \leq \tau \, h \inf_K x^\gamma$, if $(0,0) \notin K$.

In order to simplify the proofs we also assume

M.4 Every triangle in $\mathcal{T}_h$ has a vertical (parallel to the y-axis) side.

In what follows we use the letter C to denote a generic constant which is independent of h.

We say that two elements K and K' are neighbors if $\overline{K} \cap \overline{K'} \neq \emptyset$. We need the next assumption concerning neighboring elements:
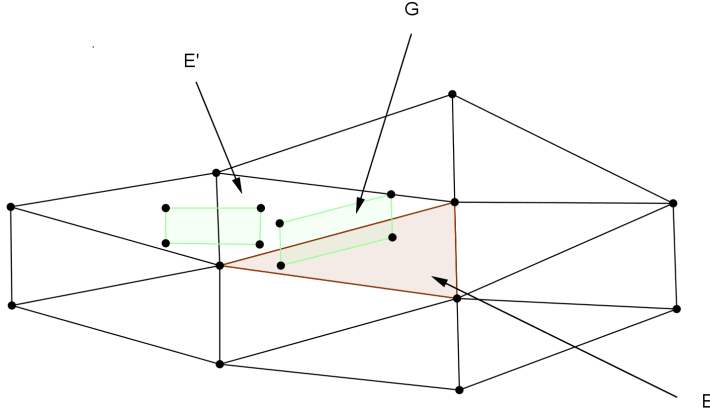
Fig. 2

M.5 Let K and K' be neighbor elements, and suppose that ℓ_1, ℓ_2 are two different edges of K and ℓ'_1, ℓ'_2 are different edges of K', being ℓ_1 and ℓ'_1 vertical edges. Then we have

$$|\ell_1| \leq C|\ell'_1|, \qquad |\ell_2| \leq C|\ell'_2|,$$

with the constant C independent of K, K' and h.

Assumption M.5 together with the maximum angle condition M.1 implies

$$|K| \leq C|K'|, \qquad \forall K, K' \in \mathcal{T}_h \text{ such that } \overline{K} \cap \overline{K}' \neq \emptyset. \tag{6}$$

We finally assume

M.6 For any element K of $\mathcal{T}_h$ with vertices $\mathbf{n}_1, \mathbf{n}_2$ and $\mathbf{n}_3$, we denote by $\overline{\mathbf{n}_i \mathbf{n}_j}$ the edge with end points $\mathbf{n}_i$ and $\mathbf{n}_j$, $i,j = 1,2,3$. Suppose that $\overline{\mathbf{n}_2 \mathbf{n}_3}$ is the vertical edge of K. Then $|\overline{\mathbf{n}_2 \mathbf{n}_3}| \leq C|\overline{\mathbf{n}_1 \mathbf{n}_2}|$ and $|\overline{\mathbf{n}_2 \mathbf{n}_3}| \leq C|\overline{\mathbf{n}_1 \mathbf{n}_3}|$, with the constant C independent of K and h. Moreover, in the case that $\overline{K}$ intersects Γ_2 we also assume that $|\overline{\mathbf{n}_1 \mathbf{n}_2}| \leq C|\overline{\mathbf{n}_2 \mathbf{n}_3}|$ and $|\overline{\mathbf{n}_1 \mathbf{n}_3}| \leq C|\overline{\mathbf{n}_2 \mathbf{n}_3}|$.

We denote by $\mathcal{N}_h$ and $\mathcal{E}_h$ the sets of nodes and edges (element sides) of $\mathcal{T}_h$. Let $\mathcal{T}_h^{\Omega}$ be the partition of Ω defined by $\mathcal{T}_h$, that is

$$\mathcal{T}_h^{\Omega} = \{E = K \cap \Omega : K \in \mathcal{T}_h\}.$$

Note that an element in $\mathcal{T}_h^{\Omega}$ is either an element of $\mathcal{T}_h$ or a curved element with the curved edge on Γ_3. From now on, the word element refers to curved or non–curved elements. For each node $\mathbf{n}$, let $\omega_{\mathbf{n}} = \cup\{E \in \mathcal{T}_h^{\Omega} : \mathbf{n} \in \overline{E}\}$, and for an edge ℓ let $\omega_{\ell} = \omega_{\mathbf{n}} \cup \omega_{\mathbf{n}'}$ if $\mathbf{n}$ and $\mathbf{n}'$ are the end points of ℓ. We put $\omega_E = \cup\{\omega_{\ell} : \ell \text{ edge of } E\}$ and finally ω'_E is the subset of ω_E obtained by excluding curved elements.

The goal of this section is to obtain interpolation error estimates in fractional Sobolev spaces. First, we present the following estimate that will be useful later on.

Proposition 3 *Let E be a (non curved) element of $\mathcal{T}_h^{\Omega}$. Then there exists a constant C, depending only on the maximum angle of $\mathcal{T}_h$, and on the number of neighbours of E such that for all $v \in H^1(\omega_E)$ we have*

$$\|v - c_E\|_{L^2(\omega'_E)} \leq C \left(h_{E,1} \|\partial_x v\|_{L^2(\omega'_E)} + h_{E,2} \|\partial_y v\|_{L^2(\omega'_E)} \right),$$

where $h_{E,2}$ is the length of the vertical side of E, $h_{E,1}$ is the length of E in the x-direction and c_E is the mean value of v on E.

Proof. Although the proof follows the arguments of [15, Theorem 7.1], we include it in detail with the purpose of showing that the interpolation constant depends only on the maximum angle of the mesh. Let E' be an element in ω'_E sharing an edge with E. We first prove the inequality in $E \cup E'$. Let G be a parallelogram with

$$|G \cap E| \sim |E|, \qquad |G \cap E'| \sim |E|, \tag{7}$$

see Figure 2. The existence of such a G follows from the assumptions on the mesh $\mathcal{T}_h$. Given $v \in H^1(\omega_E)$ we denote by c_S the mean of v on the set S. We know that

$$\|v - c_E\|_{L^2(E)} \leq C \left(h_{E,1} \|\partial_x v\|_{L^2(E)} + h_{E,2} \|\partial_y v\|_{L^2(E)} \right), \tag{8}$$

$$\|v - c_{E'}\|_{L^2(E')} \leq C \left(h_{E,1} \|\partial_x v\|_{L^2(E')} + h_{E,2} \|\partial_y v\|_{L^2(E')} \right), \tag{9}$$

$$\|v - c_G\|_{L^2(G)} \leq C \left(h_{E,1} \|\partial_x v\|_{L^2(G)} + h_{E,2} \|\partial_y v\|_{L^2(G)} \right), \tag{10}$$

where we used that the diameters of E, E' and G in the directions x and y are similar. Then

$$\|v - c_E\|_{L^2(E \cup G)} \leq \|v - c_E\|_{L^2(E)} + \|v - c_E\|_{L^2(G)}$$
$$\leq \|v - c_E\|_{L^2(E)} + \|v - c_G\|_{L^2(G)} + \|c_G - c_E\|_{L^2(G)}.$$

From the equivalence (7) of measures and with c_G and c_E being constants we obtain

$$\|v - c_E\|_{L^2(E \cup G)} \leq \|v - c_E\|_{L^2(E)} + \|v - c_G\|_{L^2(G)} + C\|c_G - c_E\|_{L^2(E \cap G)}$$
$$\leq (C + 1) \left(\|v - c_E\|_{L^2(E)} + \|v - c_G\|_{L^2(G)} \right) \tag{11}$$
$$\leq C \left(h_{E,1} \|\partial_x v\|_{L^2(E \cup G)} + h_{E,2} \|\partial_y v\|_{L^2(E \cup G)} \right),$$

where in the last line we used (8) and (10). Now, applying the same technique we have

$$\|v - c_E\|_{L^2(E \cup E')} \leq \|v - c_E\|_{L^2(E \cup G)} + \|v - c_E\|_{L^2(E')}$$
$$\leq \|v - c_E\|_{L^2(E \cup G)} + \|v - c_{E'}\|_{L^2(E')} + \|c_E - c_{E'}\|_{L^2(E')}$$
$$\leq \|v - c_E\|_{L^2(E \cup G)} + \|v - c_{E'}\|_{L^2(E')} + C\|c_{E'} - c_E\|_{L^2(E' \cap G)}$$
$$\leq (C + 1) \left(\|v - c_E\|_{L^2(E \cup G)} + \|v - c_{E'}\|_{L^2(E')} \right)$$
$$\leq C \left(h_{E,1} \|\partial_x v\|_{L^2(E \cup E')} + h_{E,2} \|\partial_y v\|_{L^2(E \cup E')} \right)$$

where we used (11) and (9) in the last line. Repeating this argument for all the elements in ω'_E we obtain the result. □

Remark 1 Let E_0 be the element of $\mathcal{T}_h$ with $(0,0)$ as a vertex. Then, analogously to the proof of the last Proposition, we can obtain that there exist a constant c_{E_0} such that

$$\|v - c_{E_0}\|_{L^2(\omega''_{E_0})} \leq C \left(h_{E_0,1} \|\partial_x v\|_{L^2(\omega''_{E_0})} + h_{E_0,2} \|\partial_y v\|_{L^2(\omega''_{E_0})} \right),$$

where $\omega''_{E_0} = E_0 \cup \omega'_{E_0}$.

Now we introduce a quasi-interpolation operator of Scott-Zhang type [25] $Q_h : H^1(\Omega) \to V_h$ associated with the graded meshes $\mathcal{T}_h$. For each node $\mathbf{n} \in \mathcal{N}_h$ we choose an edge $\ell_{\mathbf{n}} \in \mathcal{E}_h$ following the next rules:

Q.1 $\mathbf{n}$ is an endpoint of $\ell_{\mathbf{n}}$.
Q.2 $\ell_{\mathbf{n}} \subset \overline{\Omega}$ for every node $\mathbf{n}$.
Q.3 If $\mathbf{n} \in \overline{\Omega} \setminus \Gamma_2$ then $\ell_{\mathbf{n}}$ is a non-vertical edge.
Q.4 If $\mathbf{n}_1$ and $\mathbf{n}_2$ are nodes along the same vertical line, then the projections of $\ell_{\mathbf{n}_1}$ and $\ell_{\mathbf{n}_2}$ on the x-axis coincide.
Q.5 If $\mathbf{n} \in \Gamma_2$ then $\ell_{\mathbf{n}} \subset \Gamma_2$.

For each edge $\ell \in \mathcal{E}_h$ let $\Pi_\ell : L^2(\ell) \to P_1(\ell)$ be the $L^2(\ell)$ projection on the space of linear polynomials on ℓ. If $\mathbf{n} \in \mathcal{N}_h$, $\phi_\mathbf{n}$ denotes the Lagrangian basis associated with $\mathbf{n}$, that is, $\phi_\mathbf{n}$ is a piecewise linear function on $\mathcal{T}_h$ with $\phi_\mathbf{n}(\mathbf{n}') = \delta_{\mathbf{n},\mathbf{n}'}$ for every $\mathbf{n}' \in \mathcal{N}_h$.

We are ready to introduce the operator Q_h which, for a function $v \in H^1(\Omega)$, is defined by

$$Q_h v = \sum_{\mathbf{n} \in \mathcal{N}_h} \Pi_{\ell_\mathbf{n}} v(\mathbf{n}) \phi_\mathbf{n}.$$

Following [25, Section 2], for each node $\mathbf{n}$ there exists a unique linear polynomial $\psi_\mathbf{n} \in P_1(\ell_\mathbf{n})$, such that

$$p(\mathbf{n}) = \int_{\ell_\mathbf{n}} p \psi_\mathbf{n}, \qquad \forall p \in P_1(\ell_\mathbf{n}).$$

Indeed, let $\hat{\ell} = [0,1]$, and $\hat{\psi}$ be the linear function such that

$$\int_{\hat{\ell}} \hat{\psi} \hat{p} = \hat{p}(0), \qquad \forall \hat{p} \in P_1([0,1]).$$

Explicitly, $\hat{\psi}(t) = -6t + 4, t \in [0,1]$. Then, if $\ell_\mathbf{n} = \overline{\mathbf{n}\mathbf{n}'}$ it follows that

$$\psi_\mathbf{n}(x) = |\overline{\mathbf{n}\mathbf{n}'}|^{-1} \hat{\psi}(t), \qquad \text{if } x = (1-t)\mathbf{n} + t\mathbf{n}', t \in [0,1]. \tag{12}$$

Moreover, it holds (see also [25, Lemma 3.1])

$$\|\psi_\mathbf{n}\|_{L^\infty(\ell_\mathbf{n})} \leq 4|\ell_\mathbf{n}|^{-1}. \tag{13}$$

Now, taking $p = \Pi_{\ell_\mathbf{n}} v$ and using the definition of $\Pi_{\ell_\mathbf{n}}$, we have

$$\Pi_{\ell_\mathbf{n}} v(\mathbf{n}) = \int_{\ell_\mathbf{n}} \left(\Pi_{\ell_\mathbf{n}} v\right) \psi_\mathbf{n} = \int_{\ell_\mathbf{n}} v \psi_\mathbf{n},$$

from where we can write ([25, equation (2.13)])

$$Q_h v = \sum_{\mathbf{n} \in \mathcal{N}_h} \left(\int_{\ell_\mathbf{n}} v \psi_\mathbf{n} \right) \phi_\mathbf{n}.$$

Proposition 4 *There exists a constant C, depending only on the maximum angle of $\mathcal{T}_h$, such that for all $v \in H^1(\Omega)$*

$$\|\nabla Q_h v\|_{L^2(\Omega)} \leq C \|\nabla v\|_{L^2(\Omega)}. \tag{14}$$

Proof. Let $E \in \mathcal{T}_h^\Omega$ be an element with vertices $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$, and suppose that $\overline{\mathbf{n}_2\mathbf{n}_3}$ is the vertical edge of E. Then, for a constant C, which depends only on the maximum angle of E, we have

$$\|\nabla Q_h v\|_{L^2(E)} \leq C \left(\|\partial_y Q_h v\|_{L^2(E)} + \|\partial_{\overline{\mathbf{n}_1\mathbf{n}_2}} Q_h v\|_{L^2(E)} \right),$$

where $\partial_{\overline{\mathbf{n}_1\mathbf{n}_2}}(\cdot) = \nabla(\cdot) \cdot \frac{\overline{\mathbf{n}_1\mathbf{n}_2}}{|\overline{\mathbf{n}_1\mathbf{n}_2}|}$. Let start with the estimate for $\|\partial_y Q_h v\|_{L^2(E)}$. We first assume that E has no edge on Γ_2. We have

$$\partial_y Q_h v = |\overline{\mathbf{n}_2\mathbf{n}_3}|^{-1} \left(Q_h v(\mathbf{n}_3) - Q_h v(\mathbf{n}_2)\right) = |\overline{\mathbf{n}_2\mathbf{n}_3}|^{-1} \left(\int_{\ell_{\mathbf{n}_3}} v \psi_{\mathbf{n}_3} - \int_{\ell_{\mathbf{n}_2}} v \psi_{\mathbf{n}_2} \right).$$

Suppose that $\ell_{\mathbf{n}_2} = \overline{\mathbf{n}_2\mathbf{n}_2'}$ and $\ell_{\mathbf{n}_3} = \overline{\mathbf{n}_3\mathbf{n}_3'}$. Then

$$\int_{\ell_{\mathbf{n}_2}} v \psi_{\mathbf{n}_2} = \int_0^1 v((1-t)\mathbf{n}_2 + t\mathbf{n}_2')\psi_{\mathbf{n}_2}((1-t)\mathbf{n}_2 + t\mathbf{n}_2')|\overline{\mathbf{n}_2\mathbf{n}_2'}|\, dt$$

and

$$\int_{\ell_{\mathbf{n}_3}} v\psi_{\mathbf{n}_3} = \int_0^1 v((1-t)\mathbf{n}_3 + t\mathbf{n}_3')\psi_{\mathbf{n}_3}((1-t)\mathbf{n}_3 + t\mathbf{n}_3')|\overline{\mathbf{n}_3\mathbf{n}_3'}|\, dt.$$

Taking into account assumptions Q.4 and Q.3, and property (12), we have

$$\psi_{\mathbf{n}_2}((1-t)\mathbf{n}_2 + t\mathbf{n}_2')|\overline{\mathbf{n}_2\mathbf{n}_2'}| = \psi_{\mathbf{n}_3}((1-t)\mathbf{n}_3 + t\mathbf{n}_3')|\overline{\mathbf{n}_3\mathbf{n}_3'}|, \qquad \forall t \in [0,1]$$

and then we can write

$$\partial_y Q_h v = |\overline{\mathbf{n}_2\mathbf{n}_3}|^{-1} \int_0^1 \left[v((1-t)\mathbf{n}_3 + t\mathbf{n}_3') - v((1-t)\mathbf{n}_2 + t\mathbf{n}_2')\right]\psi_{\mathbf{n}_3}((1-t)\mathbf{n}_3 + t\mathbf{n}_3')|\overline{\mathbf{n}_3\mathbf{n}_3'}|\, dt.$$

Using (13) we obtain

$$|\partial_y Q_h v| \le 4|\overline{\mathbf{n}_2\mathbf{n}_3}|^{-1} \int_0^1 |v((1-t)\mathbf{n}_3 + t\mathbf{n}_3') - v(1-t)\mathbf{n}_2 + t\mathbf{n}_2'|\, dt.$$

From Q.4 it also follows that the first component of the points $(1-t)\mathbf{n}_2 + t\mathbf{n}_2'$ and $(1-t)\mathbf{n}_3 + t\mathbf{n}_3'$ coincide for all $t \in [0,1]$ and therefore, if D_E^2 denotes the subset of Ω limited by $\overline{\mathbf{n}_2\mathbf{n}_3}$, $\ell_{\mathbf{n}_2}$, $\ell_{\mathbf{n}_3}$ and a vertical line (or eventually just a point), we have

$$|\partial_y Q_h v| \le 4|\overline{\mathbf{n}_2\mathbf{n}_3}|^{-1}|(\mathbf{n}_3' - \mathbf{n}_3)_1|^{-1} \iint_{D_E^2} |\partial_y v|.$$

We remark that, $(\mathbf{n}_3'-\mathbf{n}_3)_1$ denotes the first component of $\mathbf{n}_3'-\mathbf{n}_3$. Hence, using Cauchy–Schwarz inequality we have

$$\begin{aligned}
\|\partial_y Q_h v\|_{L^2(E)} &\le 4|E|^{\frac{1}{2}}|\overline{\mathbf{n}_2\mathbf{n}_3}|^{-1}|(\mathbf{n}_3' - \mathbf{n}_3)_1|^{-1} \iint_{D_E^2} |\partial_y v| \\
&\le C|E||\overline{\mathbf{n}_2\mathbf{n}_3}|^{-1}|(\mathbf{n}_3' - \mathbf{n}_3)_1|^{-1}\|\partial_y v\|_{L^2(D_E^2)} \\
&\le C\|\partial_y v\|_{L^2(\omega_E)},
\end{aligned} \tag{15}$$

where we used (6).

Now we deal with $\partial_{\overline{\mathbf{n}_1\mathbf{n}_2}} Q_h v$. We have

$$\partial_{\overline{\mathbf{n}_1\mathbf{n}_2}} Q_h v = |\overline{\mathbf{n}_1\mathbf{n}_2}|^{-1}\left[Q_h v(\mathbf{n}_2) - Q_h v(\mathbf{n}_1)\right] = |\overline{\mathbf{n}_1\mathbf{n}_2}|^{-1}\left[\Pi_{\ell_{\mathbf{n}_2}} v(\mathbf{n}_2) - \Pi_{\ell_{\mathbf{n}_1}} v(\mathbf{n}_1)\right].$$

Since for all edge ℓ, $\Pi_\ell p = p$ for $P_1(\ell)$–functions p, and in particular, for constant functions, we can write

$$\partial_{\overline{\mathbf{n}_1\mathbf{n}_2}} Q_h v = |\overline{\mathbf{n}_1\mathbf{n}_2}|^{-1}\left[\Pi_{\sigma_{\mathbf{n}_2}}(v - c_E)(\mathbf{n}_2) - \Pi_{\ell_{\mathbf{n}_1}}(v - c_E)(\mathbf{n}_1)\right],$$

where c_E is a constant to be chosen later. We consider now the case when $\ell_{\mathbf{n}_2} \not\subset \Gamma_2$. We note that $\mathbf{n}_2 \ne (0,0)$ and so, from Q.4 and Q.3, $\omega_{\ell_{\mathbf{n}_2}}$ always contains a parallelogram with sides $\ell_{\mathbf{n}_2}$ and a vertical one of length $c|\overline{\mathbf{n}_2\mathbf{n}_3}|$ with c independent of E and h. Furthermore we can also assume that that parallelogram is contained in ω_E'. Then we can use a trace inequality to obtain

$$\|v - c_E\|_{L^2(\ell_2)} \le C\left(|\overline{\mathbf{n}_2\mathbf{n}_3}|^{-\frac{1}{2}}\|v - c_E\|_{L^2(\omega_E')} + |\overline{\mathbf{n}_2\mathbf{n}_3}|^{\frac{1}{2}}\|\partial_y v\|_{L^2(\omega_E')}\right).$$

It follows from (6) that $|\ell_{\mathbf{n}_1}|, |\ell_{\mathbf{n}_2}| \sim |\overline{\mathbf{n}_1\mathbf{n}_2}|$. Then, by (13), we have

$$\begin{aligned}
\left|\Pi_{\ell_{\mathbf{n}_2}}(v - c_E)(\mathbf{n}_2)\right| = \left|\int_{\ell_{\mathbf{n}_2}} (v - c_E)\psi_{\ell_{\mathbf{n}_2}}\right| &\le C|\overline{\mathbf{n}_1\mathbf{n}_2}|^{-1}\int_{\ell_{\mathbf{n}_2}} |v - c_E| \\
&\le C|\overline{\mathbf{n}_1\mathbf{n}_2}|^{-1}|\ell_{\mathbf{n}_2}|^{\frac{1}{2}}\|v - c_E\|_{L^2(\ell_{\mathbf{n}_2})} \\
&\le C|\overline{\mathbf{n}_1\mathbf{n}_2}|^{-\frac{1}{2}}\left(|\overline{\mathbf{n}_2\mathbf{n}_3}|^{-\frac{1}{2}}\|v - c_E\|_{L^2(\omega_E')} + |\overline{\mathbf{n}_2\mathbf{n}_3}|^{\frac{1}{2}}\|\partial_y v\|_{L^2(\omega_E')}\right).
\end{aligned}$$

If $\mathbf{n}_1 \neq (0,0)$ a similar estimate can be obtained for $\left|\Pi_{\ell_{\mathbf{n}_1}}(v - c_E)(\mathbf{n}_1)\right|$ and therefore

$$|\partial_{\overline{\mathbf{n}_1\mathbf{n}_2}} Q_h v| \leq C|\overline{\mathbf{n}_1\mathbf{n}_2}|^{-\frac{3}{2}} \left(|\overline{\mathbf{n}_2\mathbf{n}_3}|^{-\frac{1}{2}} \|v - c_E\|_{L^2(\omega'_E)} + |\overline{\mathbf{n}_2\mathbf{n}_3}|^{\frac{1}{2}} \|\partial_y v\|_{L^2(\omega'_E)} \right), \tag{16}$$

and so

$$\|\partial_{\overline{\mathbf{n}_1\mathbf{n}_2}} Q_h v\|_{L^2(E)} \leq C|E|^{\frac{1}{2}} |\overline{\mathbf{n}_1\mathbf{n}_2}|^{-\frac{3}{2}} \left(|\overline{\mathbf{n}_2\mathbf{n}_3}|^{-\frac{1}{2}} \|v - c_E\|_{L^2(\omega'_E)} + |\overline{\mathbf{n}_2\mathbf{n}_3}|^{\frac{1}{2}} \|\partial_y v\|_{L^2(\omega'_E)} \right).$$

Now we can take c_E following Proposition 3 to have

$$\|v - c_E\|_{L^2(\omega'_E)} \leq C \left(|\overline{\mathbf{n}_1\mathbf{n}_2}| \|\partial_x v\|_{L^2(\omega'_E)} + |\overline{\mathbf{n}_2\mathbf{n}_3}| \|\partial_y v\|_{L^2(\omega'_E)} \right).$$

Inserting this inequality in (16) and taking into account that $|E| \sim |\overline{\mathbf{n}_1\mathbf{n}_2}||\overline{\mathbf{n}_2\mathbf{n}_3}|$ we obtain

$$\|\partial_{\overline{\mathbf{n}_1\mathbf{n}_2}} Q_h v\|_{L^2(E)} \leq C \left(\|\partial_x v\|_{L^2(\omega'_E)} + \frac{|\overline{\mathbf{n}_2\mathbf{n}_3}|}{|\overline{\mathbf{n}_1\mathbf{n}_2}|} \|\partial_y v\|_{L^2(\omega'_E)} \right)$$

$$\leq C \left(\|\partial_x v\|_{L^2(\omega_E)} + \|\partial_y v\|_{L^2(\omega_E)} \right),$$

where we used that $|\overline{\mathbf{n}_2\mathbf{n}_3}| \leq C|\overline{\mathbf{n}_1\mathbf{n}_2}|$ thanks to assumption M.6.

Now we consider the case in which E has the edge $\overline{\mathbf{n}_2\mathbf{n}_3}$ on Γ_2. In this case, $\ell_{\mathbf{n}_2}$ and $\ell_{\mathbf{n}_3}$ are both contained in Γ_2 and, on the other hand we know from condition M.6 since $\bar{E} \cap \Gamma_2 \neq \emptyset$, that $|\overline{\mathbf{n}_2\mathbf{n}_3}| \sim |\ell_{\mathbf{n}_2}|$, $|\overline{\mathbf{n}_2\mathbf{n}_3}| \sim |\ell_{\mathbf{n}_3}|$, $|\overline{\mathbf{n}_2\mathbf{n}_3}| \sim |\overline{\mathbf{n}_1\mathbf{n}_2}|$ and $\overline{\mathbf{n}_2\mathbf{n}_3} \sim h_E$. We have, again for a constant c_E to be chosen later,

$$\partial_y Q_h v = |\overline{\mathbf{n}_2\mathbf{n}_3}|^{-1} \left(\int_{\ell_{\mathbf{n}_3}} (v - c_E)\psi_{\mathbf{n}_3} - \int_{\ell_{\mathbf{n}_2}} (v - c_E)\psi_{\mathbf{n}_2} \right),$$

and using the estimates for $|\psi_{\mathbf{n}_2}|$ and $|\psi_{\mathbf{n}_3}|$ and trace inequalities we obtain

$$|\partial_y Q_h v| \leq C h_E^{-2} \left(\int_{\sigma_{\mathbf{n}_2}} |v - c_E| + \int_{\sigma_{\mathbf{n}_3}} |v - c_E| \right)$$

$$\leq C \left(h_E^{-2} \|v - c_E\|_{L^2(\omega_E)} + h_E^{-1} \|\nabla v\|_{L^2(\omega_E)} \right),$$

and now, taking c_E as the mean value of v on an element contained in ω_E, by applying Poincaré inequality we have

$$|\partial_y Q_h v| \leq C h_E^{-1} \|\nabla v\|_{L^2(\omega_E)},$$

from where we finally obtain

$$\|\partial_y Q_h v\|_{L^2(E)} \leq C \|\nabla v\|_{L^2(\omega_E)}.$$

Clearly for this case, with $\overline{\mathbf{n}_2\mathbf{n}_3} \subset \Gamma_2$, exploiting the shape regularity of E (which follows from condition M.6) we similarly obtain

$$\|\partial_x Q_h v\|_{L^2(E)} \leq C \|\nabla v\|_{L^2(\omega_E)},$$

from where we have

$$\|\nabla Q_h v\|_{L^2(E)} \leq C \|\nabla v\|_{L^2(\omega_E)}.$$

It just remains to consider the case in which E has $(0,0)$ as a vertex. We have yet dealt with $\|\partial_y Q_h v\|_{L^2(E)}$ obtaining inequality (15). So, it remains to estimate $\|\partial_x Q_h v\|_{L^2(E)}$. We put $\mathbf{n}_1 = (0,0)$ and $\mathbf{n}_2$ and $\mathbf{n}_3$ the other two vertices, with $(\mathbf{n}_2)_2 = 0$. Again we have (now $\partial_{\overline{\mathbf{n}_1\mathbf{n}_2}} = \partial_x$)

$$\partial_x Q_h v = |\overline{\mathbf{n}_1\mathbf{n}_2}|^{-1} \left[\Pi_{\ell_{\mathbf{n}_2}}(v - c_E)(\mathbf{n}_2) - \Pi_{\ell_{\mathbf{n}_1}}(v - c_E)(\mathbf{n}_1) \right], \tag{17}$$

for a constant c_E. We choose c_E as the mean value of v on a triangular element contained in ω'_E as in Proposition 3. The first term inside the square brackets can be treated as before, using Poincaré and trace inequalities,

$$\left| \Pi_{\ell_{\mathbf{n}_2}}(v - c_E)(\mathbf{n}_2) \right| \leq C \left[\left(\frac{|\overline{\mathbf{n}_1\mathbf{n}_2}|}{|\overline{\mathbf{n}_2\mathbf{n}_3}|} \right)^{\frac{1}{2}} \|\partial_x v\|_{L^2(\omega_E)} + \left(\frac{|\overline{\mathbf{n}_2\mathbf{n}_3}|}{|\overline{\mathbf{n}_1\mathbf{n}_2}|} \right)^{\frac{1}{2}} \|\partial_y v\|_{L^2(\omega_E)} \right]. \tag{18}$$

On the other hand, by using the trace inequality [3, Ineq. (2.4)] we have

$$
\left| \Pi_{\ell_{\mathbf{n}_1}}(v - c_E)(\mathbf{n}_1) \right| = \left| \int_{\ell_{\mathbf{n}_1}} (v - c_E)\psi_{\mathbf{n}_1} \right|
$$
$$
\leq C|\overline{\mathbf{n}_1\mathbf{n}_2}|^{-1} \|v - c_E\|_{L^1(\ell_{\mathbf{n}_1})}
$$
$$
\leq C|\overline{\mathbf{n}_1\mathbf{n}_2}|^{-1} \left(\|(v - c_E)x^{-\alpha}\|_{L^1(E)} + \|\nabla v\|_{L^1(E)} \right)
$$
$$
\leq C|\overline{\mathbf{n}_1\mathbf{n}_2}|^{-1} \left(|\overline{\mathbf{n}_1\mathbf{n}_2}|^{-\frac{\alpha+1}{p}+1}\|v - c_E\|_{L^p(E)} + |\overline{\mathbf{n}_1\mathbf{n}_2}|^{\frac{\alpha+1}{2}}\|\nabla v\|_{L^2(E)} \right),
$$

where we used Hölder p, q inequality with $p > 1 + \alpha$, $\|x^{-\alpha}\|_{L^q(E)} = |\overline{\mathbf{n}_1\mathbf{n}_2}|^{-\frac{\alpha+1}{p}+1}$ and $|E| \sim |\overline{\mathbf{n}_1\mathbf{n}_2}|^{\alpha+1}$. Now, in E we have the scaled embedding inequality, which follows from the Sobolev embedding theorem in cusps [5, Theorem 5.35] by standard rescaling arguments

$$
\|w\|_{L^p(E)} \leq C|\overline{\mathbf{n}_1\mathbf{n}_2}|^{(\alpha+1)\frac{2-p}{2p}} \left(\|w\|_{L^2(E)} + |\overline{\mathbf{n}_1\mathbf{n}_2}|\|\partial_x w\|_{L^2(E)} + |\overline{\mathbf{n}_1\mathbf{n}_2}|^{\alpha}\|\partial_y w\|_{L^2(E)} \right),
$$

valid for all $w \in H^1(E)$ and $p < 6$. Then we obtain

$$
\left| \Pi_{\ell_{\mathbf{n}_1}}(v - c_E)(\mathbf{n}_1) \right| \leq
$$
$$
|\overline{\mathbf{n}_1\mathbf{n}_2}|^{-\frac{\alpha+1}{2}} \left(|\overline{\mathbf{n}_1\mathbf{n}_2}|\|\partial_x v\|_{L^2(E)} + |\overline{\mathbf{n}_1\mathbf{n}_2}|^{\alpha}\|\partial_y v\|_{L^2(E)} \right) + |\overline{\mathbf{n}_1\mathbf{n}_2}|^{\frac{\alpha-1}{2}}\|\nabla v\|_{L^2(E)}, \quad (19)
$$

where we have used the Poincaré inequality as stated in Remark 1. From (17)–(19), and since $|E| \sim |\overline{\mathbf{n}_1\mathbf{n}_2}|^{\alpha+1}$ and $|\overline{\mathbf{n}_2\mathbf{n}_3}| = |\overline{\mathbf{n}_1\mathbf{n}_2}|^{\alpha}$ we have

$$
\|\partial_x Q_h v\|_{L^2(E)} \leq C|\overline{\mathbf{n}_1\mathbf{n}_2}|^{\frac{\alpha+1}{2}-1} \left(|\overline{\mathbf{n}_1\mathbf{n}_2}|^{\frac{1-\alpha}{2}}\|\partial_x v\|_{L^2(\omega_E)} + |\overline{\mathbf{n}_1\mathbf{n}_2}|^{\frac{\alpha-1}{2}}\|\partial_y v\|_{L^2(\omega_E)} \right)
$$
$$
+ C \left(\|\partial_x v\|_{L^2(E)} + |\overline{\mathbf{n}_1\mathbf{n}_2}|^{\alpha-1}\|\nabla v\|_{L^2(E)} \right)
$$
$$
\leq C \left(\|\partial_x v\|_{L^2(\omega_E)} + |\overline{\mathbf{n}_1\mathbf{n}_2}|^{\alpha-1}\|\partial_y v\|_{L^2(\omega_E)} \right).
$$

The proof conclude by adding the inequalities obtained for each element $E \in \mathcal{T}_h^{\Omega}$ and taking into account that
$$
\# \left\{ E' \in \mathcal{T}_h^{\Omega} : E \cap E' \neq \emptyset \right\} \leq C,
$$
for all $E \in \mathcal{T}_h^{\Omega}$. $\qquad\square$

Corollary 2 *There exists a constant C, depending only on the maximum angle of $\mathcal{T}_h$, such that for all $v \in H^1(\Omega)$ we have*
$$
\|\nabla Q_h v\|_{L^2(\Omega_h)} \leq C\|\nabla v\|_{L^2(\Omega)}. \tag{20}
$$

Proof. We note that, for a constant C which depends only on α, if $K \in \mathcal{T}_h$ and $E = K \cap \Omega$ then
$$
|K| \leq C|E|.
$$

Hence, taking into account that $\nabla Q_h v|_K \in P_0(K)^2$, we have
$$
\|\nabla Q_h v\|_{L^2(K)} = |K|^{\frac{1}{2}} |\nabla Q_h v|_K| \leq C|E|^{\frac{1}{2}} |\nabla Q_h v|_K| = C\|\nabla Q_h v\|_{L^2(E)}.
$$

Therefore
$$
\|\nabla Q_h v\|_{L^2(\Omega_h \setminus \Omega)} \leq C\|\nabla Q_h v\|_{L^2(\Omega)},
$$
which together with (14) gives (20). $\qquad\square$

Proposition 5 *There exists a constant C, independent of h, such that the following interpolation error estimates hold:*
$$
\|\nabla(v - Q_h v)\|_{L^2(\Omega)} \leq C|v|_{H^1(\Omega)}, \qquad \forall v \in H^1(\Omega) \tag{21}
$$
$$
\|\nabla(v - Q_h v)\|_{L^2(\Omega)} \leq Ch\|v\|_{H^2(\Omega)}, \qquad \forall v \in H^2(\Omega). \tag{22}
$$

Proof. Using the triangle inequality and (14) we have, for all $v \in H^1(\Omega)$,

$$\|\nabla(v - Q_h v)\|_{L^2(\Omega)} \leq \|\nabla v\|_{L^2(\Omega)} + \|\nabla Q_h v\|_{L^2(\Omega)} \leq C\|\nabla v\|_{L^2(\Omega)},$$

which gives (21).

Denote by $\mathcal{L}_h : H^2(\Omega) \to V_h$ the Lagrange interpolation operator defined on the graded mesh $\mathcal{T}_h$. It follows from [1, Theorem 2.2 and equation (1.5)] that

$$\|\nabla(v - \mathcal{L}_h v)\|_{L^2(\Omega)} \leq Ch\|v\|_{H^2(\Omega)}, \qquad \forall v \in H^2(\Omega). \tag{23}$$

Taking into account that for any function w in V_h and $\mathbf{n} \in \mathcal{N}_h$, $\Pi_\ell w(\mathbf{n}) = w(\mathbf{n})$, we have that $Q_h \mathcal{L}_h v = \mathcal{L}_h v$. Then, using (14) and (23) we obtain

$$\begin{aligned}
\|\nabla(v - Q_h v)\|_{L^2(\Omega)} &\leq \|\nabla(v - \mathcal{L}_h v)\|_{L^2(\Omega)} + \|\nabla(\mathcal{L}_h v - Q_h v)\|_{L^2(\Omega)} \\
&= \|\nabla(v - \mathcal{L}_h v)\|_{L^2(\Omega)} + \|\nabla Q_h(v - \mathcal{L}_h v)\|_{L^2(\Omega)} \\
&\leq \|\nabla(v - \mathcal{L}_h v)\|_{L^2(\Omega)} + C\|\nabla(v - \mathcal{L}_h v)\|_{L^2(\Omega)} \\
&\leq Ch\|v\|_{H^2(\Omega)},
\end{aligned}$$

which proves (22). $\qquad\qquad\square$

From the operator interpolation theory we obtain the next result.

Corollary 3 *Let $s \in [0,1]$. There exists a constant C, independent of h, such that*

$$\|\nabla(v - Q_h v)\|_{L^2(\Omega)} \leq Ch^s\|v\|_{H^{1+s}(\Omega)}, \qquad \forall v \in H^{1+s}(\Omega).$$

Let $v \in H^2(\Omega)$. We know from [21, Section 5.4.2], with $l = p = n = 2$ and $\varphi(x) = x^\alpha$, that there exists an extension $\tilde{v} \in H^2_\alpha(\mathbb{R}^2)$ such that

$$\|\tilde{v}\|_{H^2_\alpha(\mathbb{R}^2)} \leq C\|v\|_{H^2(\Omega)}.$$

where the weighted Sobolev space $H^2_\alpha(\mathcal{D})$ on a domain $\mathcal{D} \subset \mathbb{R}^2$ is defined as

$$H^2_\alpha(\mathcal{D}) = \left\{ v : r^{\frac{\alpha-1}{2}} D^\delta v \in L^2(\mathcal{D}) \ \ \forall \delta, |\delta| \leq 2 \right\},$$

with $r = \sqrt{x^2 + y^2}$, and

$$\|v\|^2_{H^2_\alpha(\mathcal{D})} = \sum_{|\delta| \leq 2} \|r^{\frac{\alpha-1}{2}} D^\delta v\|^2_{L^2(\mathcal{D})}.$$

Now, we show the following result for functions in $H^2_\alpha(\Omega_h)$ which will be useful later on.
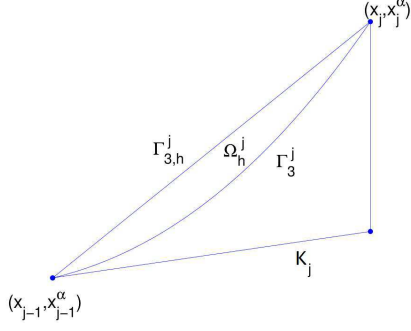
Proposition 6 *There exists a constant C, depending only on α, such that*

$$\|\nabla(v - Q_h v)\|_{L^2(\Omega_h)} \leq Ch\|v\|_{H^2_\alpha(\Omega_h)}, \qquad \forall v \in H^2_\alpha(\Omega_h).$$

Proof. Using Corollary 2 we have

$$\begin{aligned}
\|\nabla(v - Q_h v)\|_{L^2(\Omega_h)} &\leq \|\nabla(v - \mathcal{L}_h v)\|_{L^2(\Omega_h)} + \|\nabla(Q_h(v - \mathcal{L}_h v))\|_{L^2(\Omega_h)} \\
&\leq \|\nabla(v - \mathcal{L}_h v)\|_{L^2(\Omega_h)} + C\|\nabla(v - \mathcal{L}_h v)\|_{L^2(\Omega)} \\
&\leq C\|\nabla(v - \mathcal{L}_h v)\|_{L^2(\Omega_h)}.
\end{aligned}$$

Now we use Theorem 2.2 of [1] and conclude the proof. $\qquad\qquad\square$

Fig. 3 The region Ω_h^j and the corresponding triangle K_j satisfying (Ha).

4 Finite element approximations

For each $j = 1, \ldots, n$ we denote by $\Gamma_{3,h}^j$ the edge on the boundary of Ω_h with endpoints $P_{j-1} = (x_{j-1}, x_{j-1}^\alpha)$ and $P_j = (x_j, x_j^\alpha)$, being $x_0 = 0$ and $x_n = 1$, and by Γ_3^j the part of Γ_3 with the same endpoints (see Figure 3). By Ω_h^j we denote the region bounded by Γ_3^j and $\Gamma_{3,h}^j$.

In addition to the assumptions M.1-M.6 we also assume the following hypothesis on the meshes:

(Ha) For $1 \le j \le n$ the region Ω_h^j is strictly contained in only one triangle denoted by K_j. We denote the diameter of K_j by h_j.

We will show in Section 6 how meshes verifying conditions M1-M6 and (Ha) can indeed be constructed. We know, for these graded meshes, that the following properties hold (see [1, Lemma 2.1 and Lemma 2.3]):

P.1 For $2 \le j \le n$, $x_{j-1} \le x_j \le Cx_{j-1}$ with C depending only on α and τ.
P.2 $|\Omega_h \setminus \Omega| \le Ch^2$.
P.3 Let $K \in \mathcal{T}_h$ be such that $K \cap \Gamma_3 \neq \emptyset$, and put $E = K \cap \Omega$. Then $|E| \le |K| \le C|E|$ with C independent of K and h.

Proposition 7 *Let $s \in [0,1]$. There exists a constant C, independent of s and h but might depend on α, such that we have*

$$\|\nabla Q_h \phi\|_{L^2(\Omega_h \setminus \Omega)} \le C \left(h \sqrt{\log \frac{1}{h}} \right)^s \|\phi\|_{H^{1+s}(\Omega)}, \qquad \forall \phi \in H^{1+s}(\Omega). \tag{24}$$

Proof. From Corollary 2 we have

$$\|\nabla Q_h v\|_{L^2(\Omega_h \setminus \Omega)} \le C\|v\|_{H^1(\Omega)}, \qquad \forall v \in H^1(\Omega). \tag{25}$$

Now, let $v \in H^2(\Omega)$. We know from [21, Section 5.4.2] that there exists an extension $\tilde{v} \in H_\alpha^2(\mathbb{R}^2)$ such that

$$\|\tilde{v}\|_{H_\alpha^2(\mathbb{R}^2)} \le C\|v\|_{H^2(\Omega)}.$$

From Proposition 6, and since $Q_h v = Q_h \tilde{v}$ we have

$$\|\nabla(\tilde{v} - Q_h v)\|_{L^2(\Omega_h)} \le Ch\|\tilde{v}\|_{H_\alpha^2(\Omega_h)} \le Ch\|v\|_{H^2(\Omega)}.$$

Then

$$\|\nabla Q_h v\|_{L^2(\Omega_h \setminus \Omega)} \le \|\nabla(\tilde{v} - Q_h v)\|_{L^2(\Omega_h \setminus \Omega)} + \|\nabla \tilde{v}\|_{L^2(\Omega_h \setminus \Omega)}$$
$$\le Ch\|v\|_{H^2(\Omega)} + \|\nabla \tilde{v}\|_{L^2(\Omega_h \setminus \Omega)}.$$

From [4, Lemma 4.1] we have

$$\|\nabla \tilde{v}\|_{L^2(\Omega_h \setminus \Omega)} \le Ch\sqrt{\log\frac{1}{h}}\|v\|_{H^2(\Omega)},$$

from where we obtain

$$\|\nabla Q_h v\|_{L^2(\Omega_h \setminus \Omega)} \le Ch\sqrt{\log\frac{1}{h}}\|v\|_{H^2(\Omega)}.$$

So we proved

$$\|\nabla Q_h v\|_{L^2(\Omega_h \setminus \Omega)} \le Ch\sqrt{\log\frac{1}{h}}\|v\|_{H^2(\Omega)}, \qquad \forall v \in H^2(\Omega). \tag{26}$$

Inequality (24) follows from (25) and (26) by the operator interpolation theory. □

Now, we will obtain error estimates for the solutions of the source problems (3) and (5) in $L^2(\Gamma_3)$–norm.

Theorem 3 *Assume $1 < \alpha < 2$ and $\sigma = 1 - \frac{\alpha}{2}$. Let u and u_h be the solutions of problems (3) and (5) respectively. If $g \in H^s(\Gamma_3)$, with $0 \le s < \frac{1}{2}$, then there exists a positive constant C, dependent on α but independent of s and h, such that*

$$\|u - u_h\|_{L^2(\Gamma_3)} \le C\left(h\sqrt{\log(\frac{1}{h})}\right)^{\frac{\sigma}{2}+s}\|g\|_{H^s(\Gamma_3)}.$$

Proof. Let $e = u - u_h$. In order to obtain error estimates in $L^2(\Gamma_3)$ we consider the following auxiliary problem:

$$\begin{aligned}
-\triangle \phi &= 0 && \text{in } \Omega, \\
\frac{\partial \phi}{\partial \nu} &= 0 && \text{on } \Gamma_1, \\
\phi &= 0 && \text{on } \Gamma_2, \\
\frac{\partial \phi}{\partial \nu} &= e && \text{on } \Gamma_3.
\end{aligned}$$

Then,

$$\|e\|_{L^2(\Gamma_3)}^2 = \int_{\Gamma_3} e^2 = \int_{\Gamma_3} \frac{\partial \phi}{\partial \nu} e = \int_{\Omega} \nabla\phi\nabla e$$
$$= \int_{\Omega} \nabla(\phi - Q_h\phi)\nabla e + \int_{\Omega} \nabla(Q_h\phi)\nabla e.$$

Now, from the error equation we know that for any $v \in V_h$ we have

$$\int_{\Omega} \nabla e \cdot \nabla v = \int_{\Gamma_3} gv - \int_{\Omega} \nabla u_h \cdot \nabla v$$
$$= \int_{\Gamma_3} gv - \int_{\Omega_h} \nabla u_h \nabla v + \int_{\Omega_h \setminus \Omega} \nabla u_h \nabla v$$
$$= \int_{\Omega_h \setminus \Omega} \nabla u_h \nabla v.$$

Therefore,

$$\|e\|_{L^2(\Gamma_3)}^2 \le \|\nabla e\|_{L^2(\Omega)}\|\nabla(\phi - Q_h\phi)\|_{L^2(\Omega)} + \|\nabla u_h\|_{L^2(\Omega_h\setminus\Omega)}\|\nabla Q_h\phi\|_{L^2(\Omega_h\setminus\Omega)}.$$

Since $e \in H^1(\Omega)$ from (4) we know that $e|_{\Gamma_3} \in H^\sigma(\Gamma_3)$. In view of Theorem 2, $\phi \in H^{1+\sigma}(\Omega)$, and in particular, $\phi \in H^{1+\frac{\sigma}{2}}(\Omega)$. Using the interpolation error estimates obtained in Corollary 3 we get

$$\|e\|_{L^2(\Gamma_3)}^2 \le Ch^{\frac{\sigma}{2}}\|\phi\|_{H^{1+\frac{\sigma}{2}}(\Omega)}\|\nabla e\|_{L^2(\Omega)} + \|\nabla u_h\|_{L^2(\Omega_h\setminus\Omega)}\|\nabla Q_h\phi\|_{L^2(\Omega_h\setminus\Omega)}.$$

Using Proposition 7 we obtain

$$\|e\|_{L^2(\Gamma_3)}^2 \le Ch^{\frac{\sigma}{2}}\|\phi\|_{H^{1+\frac{\sigma}{2}}(\Omega)}\left[\|\nabla e\|_{L^2(\Omega)} + \left(\sqrt{\log\frac{1}{h}}\right)^{\frac{\sigma}{2}}\|\nabla u_h\|_{L^2(\Omega_h\setminus\Omega)}\right].$$

So

$$\|e\|_{L^2(\Gamma_3)}^2 \le C\left(h\sqrt{\log\frac{1}{h}}\right)^{\frac{\sigma}{2}}\|\phi\|_{H^{1+\frac{\sigma}{2}}(\Omega)}\left(\|\nabla e\|_{L^2(\Omega)} + \|\nabla u_h\|_{L^2(\Omega_h)}\right), \qquad (27)$$

and for $1 \le \alpha < 2$ we have from the definitions of u and u_h and a trace inequality

$$\|e\|_{L^2(\Gamma_3)}^2 \le C\left(h\sqrt{\log\frac{1}{h}}\right)^{\frac{\sigma}{2}}\|\phi\|_{H^{1+\frac{\sigma}{2}}(\Omega)}\|g\|_{L^2(\Gamma_3)}.$$

Now, we can apply similar arguments to those given in the proof of Proposition 4.4 of [11]. Indeed, from Theorem 2 we know that $\phi \in H^{1+\sigma}(\Omega)$, with $\sigma = 1 - \frac{\alpha}{2}$ and $\|\phi\|_{H^{1+\sigma}(\Omega)} \le C\|e\|_{H^\sigma(\Gamma_3)}$. On the other hand, the problem (3) is well defined for any $e \in H^{-\sigma}(\Gamma_3)$ and $\|\phi\|_{H^1(\Omega)} \le C\|e\|_{H^{-\sigma}(\Gamma_3)}$. Therefore, by interpolation of Sobolev spaces, we can conclude that

$$\|\phi\|_{H^{1+\frac{\sigma}{2}}(\Omega)} \le C\|e\|_{L^2(\Gamma_3)}.$$

Then

$$\|e\|_{L^2(\Gamma_3)}^2 \le C\left(h\sqrt{\log\frac{1}{h}}\right)^{\frac{\sigma}{2}}\|e\|_{L^2(\Gamma_3)}\|g\|_{L^2(\Gamma_3)},$$

which proves the assertion for $s = 0$.

It was proved in [4, Theorem 4.2] (and its proof) that if $g \in H^1(\Gamma_3)$ with $g(0,0) = 0$ then

$$\|\nabla e\|_{L^2(\Omega)} \le \|\nabla e\|_{L^2(\Omega_h)} \le Ch\sqrt{\log\frac{1}{h}}\|g\|_{H^1(\Gamma_3)}$$

and using this and also [4, Lemma 4.1] we have (always for $\alpha < 2$)

$$\|\nabla u_h\|_{L^2(\Omega_h\setminus\Omega)} \le \|\nabla(u_h - \tilde{u})\|_{L^2(\Omega_h\setminus\Omega)} + \|\nabla\tilde{u}\|_{L^2(\Omega_h\setminus\Omega)}$$

$$\le Ch\sqrt{\log\frac{1}{h}}\|g\|_{H^1(\Gamma_3)}.$$

On the other hand we also have (again for $\alpha < 2$)

$$\|\nabla e\|_{L^2(\Omega)} \le C\|g\|_{L^2(\Gamma_3)}$$

and

$$\|\nabla u_h\|_{L^2(\Omega_h\setminus\Omega)} \le C\|\nabla u_h\|_{L^2(\Omega)} \le C\|g\|_{L^2(\Gamma_3)}.$$

Recalling that

$$0 \le s < \frac{1}{2} \qquad \Longrightarrow \qquad H^s(\Gamma_3) = \left[L^2(\Gamma_3), \overset{o}{H}{}^1(\Gamma_3)\right]_s,$$

by interpolation we obtain

$$\|\nabla e\|_{L^2(\Omega)} \le C \left(h\sqrt{\log \frac{1}{h}} \right)^s \|g\|_{H^s(\Gamma_3)} \tag{28}$$

and

$$\|\nabla u_h\|_{L^2(\Omega_h \setminus \Omega)} \le C \left(h\sqrt{\log \frac{1}{h}} \right)^s \|g\|_{H^s(\Gamma_3)}. \tag{29}$$

Inserting these inequalities in (27) we arrive at

$$\|e\|^2_{L^2(\Gamma_3)} \le C \left(h\sqrt{\log \frac{1}{h}} \right)^{\frac{\sigma}{2}+s} \|\phi\|_{H^{1+\frac{\sigma}{2}}(\Omega)} \|g\|_{H^s(\Gamma_3)}$$

$$\le C \left(h\sqrt{\log \frac{1}{h}} \right)^{\frac{\sigma}{2}+s} \|e\|_{L^2(\Gamma_3)} \|g\|_{H^s(\Gamma_3)},$$

which proves the assertion for $s > 0$. $\square$

5 Spectral approximation

In this section we prove the convergence of the eigenfunctions and eigenvalues. In what follows we assume that $1 < \alpha < 2$ and $\sigma = 1 - \frac{\alpha}{2}$.

Next Lemma, which is a direct consequence of Theorem 3, shows that T_h converge to T in operator norm when h goes to zero.

Lemma 1 *There exists a positive constant C such that for any $g \in L^2(\Gamma_3)$*

$$\|Tg - T_h g\|_{L^2(\Gamma_3)} \le C \left(h\sqrt{\log \frac{1}{h}} \right)^{\frac{\sigma}{2}} \|g\|_{L^2(\Gamma_3)}. \tag{30}$$

Despite of the poor order given in (30) this result allows us to state the convergence for eigenvalues. In the following theorem we study the regularity of the eigenfunctions of problem (1). In Particular, it shows that, for any $\alpha \in (1, 2)$ the eigenfunctions are always in $H^{\frac{1}{2}}(\Gamma_3)$.

Theorem 4 *For all $\epsilon > 0$, the restrictions to Γ_3 of eigenfunctions of problem (1) are in $H^{\frac{1}{2}+\sigma-\epsilon}(\Gamma_3)$.*

Proof. Let γ be the trace operator on Γ_3. From Theorem 1 we know that

$$\gamma : H^1(\Omega) \to H^\sigma(\Gamma_3)$$

and from which one can deduce that also

$$\gamma : H^2(\Omega) \to H^{1+\sigma}(\Gamma_3).$$

Therefore, by interpolation we have

$$\gamma : H^{1+s}(\Omega) \to H^{\sigma+s}(\Gamma_3), \qquad s \in [0, 1]. \tag{31}$$

On the other hand, denoting $\overset{o}{H}{}^1(\Gamma_3) = \{g \in H^1(\Gamma_3) : g(0,0) = 0\}$, we know from [4] that the operator B, introduced at the beginning of Section 2, verifies

$$B : \overset{o}{H}{}^1(\Gamma_3) \to H^2(\Omega),$$

and from Theorem 2 that

$$B : H^\sigma(\Gamma_3) \to H^{1+\sigma}(\Omega).$$

Then we have

$$B : H^s(\Gamma_3) \to H^{1+s}(\Omega), \qquad \sigma \le s < \frac{1}{2}. \tag{32}$$

Let $u \in H^1(\Omega)$ be an eigenfunction of (1). Then from Theorem 1 we know that $u|_{\Gamma_3} \in H^\sigma(\Gamma_3)$. It follows from Theorem 2 that $u \in H^{1+\sigma}(\Omega)$. Then from (31) we have $u|_{\Gamma_3} \in H^{2\sigma}(\Gamma_3)$.

If $2\sigma > \frac{1}{2}$ then we have

$$u|_{\Gamma_3} \in H^{\frac{1}{2}-\epsilon}(\Gamma_3) = \left[L^2(\Gamma_3), \overset{o}{H}^1(\Gamma_3) \right]_{\frac{1}{2}-\epsilon}.$$

Now, from (32) we have $u \in H^{\frac{3}{2}-\epsilon}(\Omega)$. Using again (31) we obtain that $u \in H^{\frac{1}{2}+\sigma-\epsilon}(\Gamma_3)$.

If $2\sigma < \frac{1}{2}$ we obtain from (32) that $u \in H^{1+2\sigma}(\Omega)$, and so, from (31) $u|_{\Gamma_3} \in H^{3\sigma}(\Gamma_3)$. Now we conclude as before if $3\sigma > \frac{1}{2}$ or we repeat the argument. $\qquad\square$

Thus, we can apply Theorem 3 with g being an eigenfunction of the operator T, taking into account that from Theorem 4 we know that, at least, $g \in H^{\frac{1}{2}-\epsilon}(\Gamma_3)$ for all $\epsilon > 0$ (indeed, g is probably more regular but we observe that we can only take $s < \frac{1}{2}$ in Theorem 3). Then we have the next result.

Corollary 4 *There exists a positive constant C, depending on α, such that for any eigenfunction g of T*

$$\|Tg - T_h g\|_{L^2(\Gamma_3)} \le C \left(h\sqrt{\log\frac{1}{h}} \right)^{\frac{\sigma}{2}+\frac{1}{2}-\epsilon} \|g\|_{H^{\frac{1}{2}-\epsilon}(\Gamma_3)}.$$

Now, recalling that since $1 < \alpha < 2$, T is compact (see Proposition 2) we are able to apply the spectral approximation theory for compact operators given in [10]. We remark that since the linear operator T on the Hilbert space $L^2(\Gamma_3)$ is self–adjoint, for each μ eigenvalue of T the geometric and algebraic multiplicities of μ are equal.

Let μ be a fixed eigenvalue of T with multiplicity m and denote by $\mathcal{E}$ its eigenspace. From Lemma 1 we know that T_h converge to T in norm and so, there exist exactly m eigenvalues of T_h, $\mu_h^1, \ldots, \mu_h^m$ repeated according to their algebraic multiplicities converging to μ. Let $\mathcal{E}_h$ be the direct sum of the associated eigenspaces. Define the gap $\bar{\delta}$ between two subspaces, $\mathcal{X}$ and $\mathcal{Y}$ of $L^2(\Gamma_3)$

$$\bar{\delta}(\mathcal{X}, \mathcal{Y}) := \max\left\{ \delta(\mathcal{X}, \mathcal{Y}), \delta(\mathcal{Y}, \mathcal{X}) \right\},$$

with

$$\delta(\mathcal{X}, \mathcal{Y}) = \sup_{v \in \mathcal{X}, \|v\|_{L^2(\Gamma_3)}=1} \left(\inf_{w \in \mathcal{Y}} \|v - w\|_{L^2(\Gamma_3)} \right).$$

In view of Corollary 4, and [23, Theorem 1] we have the following result.

Lemma 2 *There exists a constant C, dependent on α and the eigenvalue μ but independent on h, such that*

$$\bar{\delta}(\mathcal{E}, \mathcal{E}_h) \le C \left(h\sqrt{\log\frac{1}{h}} \right)^{\frac{\sigma}{2}+\frac{1}{2}-\epsilon}.$$

Furthermore, from Corollary 4 and [10, Theorem 7.4] we also have the next Corollary, which for simplicity we state just for simple eigenvalues (see, for example, [10] for the general case).

Corollary 5 *Let μ be a simple eigenvalue of T and let μ_h be the associated eigenvalue of T_h. Then, the corresponding eigenfunctions u and u_h can be chosen such that $\|u\|_{L^2(\Gamma_3)} = 1$, $\|u_h\|_{L^2(\Gamma_3)} = 1$ and*

$$\|u - u_h\|_{L^2(\Gamma_3)} \le C \left(h\sqrt{\log \frac{1}{h}} \right)^{\frac{\sigma}{2} + \frac{1}{2} - \epsilon},$$

with C a strictly positive constant, dependent on α and the eigenvalue μ but independent on h.

Proof. Let $\mathcal{E}$ be set of (generalized) eigenfunctions for μ. Then $\mathcal{E}$ is a one–dimensional subspace. Let $g \in \mathcal{E}$ be an eigenfunction with $\|g\|_{L^2(\Gamma_3)} = 1$. Then

$$\|(T - T_h)|_{\mathcal{E}}\|_{\mathcal{L}(L^2(\Gamma_3))} = \|Tg - T_h g\|_{L^2(\Gamma_3)}$$

and from Corollary 4 we obtain

$$\|(T - T_h)|_{\mathcal{E}}\|_{\mathcal{L}(L^2(\Gamma_3))} \le C \left(h\sqrt{\log \frac{1}{h}} \right)^{\frac{\sigma}{2} + \frac{1}{2} - \epsilon} \|g\|_{H^{\frac{1}{2} - \epsilon}(\Gamma_3)}.$$

Now, from Lemma 4 (and $\alpha < 2$) we know that g is at least a function in $H^{\frac{1}{2}}(\Gamma_3)$, so we can write the last inequality as

$$\|(T - T_h)|_{\mathcal{E}}\|_{\mathcal{L}(L^2(\Gamma_3))} \le C \left(h\sqrt{\log \frac{1}{h}} \right)^{\frac{\sigma}{2} + \frac{1}{2} - \epsilon},$$

where the fractional norm of g was incorporated to C, the constant therefore depends only on the eigenvalue μ and is independent of ϵ.

It is stated in [10, Theorem 7.4] that the eigenfunctions u and u_h in the statement of the Corollary can be chosen such that

$$\|u - u_h\|_{L^2(\Gamma_3)} \le C \|(T - T_h)|_{\mathcal{E}}\|_{\mathcal{L}(L^2(\Gamma_3))},$$

from where we obtain the result.   □

Hence, with [10, Theorem 7.3] in mind, the convergence order for the approximate eigenvalues can be improved by using the following lemma.

Lemma 3 *For all $\epsilon > 0$ there exists a positive constant C such that for any $f, g \in \mathcal{E}$*

$$\left| \int_{\Gamma_3} (T - T_h) fg \right| \le C \left(h\sqrt{\log \frac{1}{h}} \right)^{1 - \epsilon} \|f\|_{H^{\frac{1}{2} - \epsilon}(\Gamma_3)} \|g\|_{H^{\frac{1}{2} - \epsilon}(\Gamma_3)}.$$

Proof. The proof follows the same ideas given in [2, Lemma 4.5], we include it for the sake of completeness. Let $f, g \in \mathcal{E}$ and let

$$u = Bf, \quad u_h = B_h f,$$
$$v = Bg, \quad v_h = B_h g.$$

Then, we have that

$$\int_{\Gamma_3} (T - T_h) fg = \int_{\Gamma_3} (u - u_h) g = \int_{\Gamma_3} ug - \int_{\Gamma_3} u_h g.$$

On the other hand, from (3) and (5), we have that

$$\int_{\Omega} \nabla u \nabla w = \int_{\Gamma_3} fw \quad \forall w \in V,$$
$$\int_{\Omega} \nabla v \nabla w = \int_{\Gamma_3} gw \quad \forall w \in V$$

and

$$\int_{\Omega_h} \nabla u_h \nabla w_h = \int_{\Gamma_3} f w_h \quad \forall w_h \in V_h,$$

$$\int_{\Omega_h} \nabla v_h \nabla w_h = \int_{\Gamma_3} g w_h \quad \forall w_h \in V_h.$$

In particular,

$$\int_{\Gamma_3} u g = \int_{\Omega} \nabla u \nabla v,$$

$$\int_{\Gamma_3} u_h g = \int_{\Omega_h} \nabla v_h \nabla u_h$$

and, since functions in V_h restricted to Ω belong to V, we also have that

$$\int_{\Omega_h} \nabla u_h \nabla v_h = \int_{\Gamma_3} f v_h = \int_{\Omega} \nabla u \nabla v_h$$

and

$$\int_{\Omega_h} \nabla v_h \nabla u_h = \int_{\Gamma_3} g u_h = \int_{\Omega} \nabla v \nabla u_h.$$

As a consequence,

$$\int_{\Gamma_3} u g - \int_{\Gamma_3} u_h g = \int_{\Omega} \nabla v \nabla u - \int_{\Omega} \nabla v \nabla u_h + \int_{\Omega_h} \nabla v_h \nabla u_h - \int_{\Omega} \nabla v_h \nabla u$$

$$= \int_{\Omega} \nabla (u - u_h) \nabla (v - v_h) + \int_{\Omega_h \setminus \Omega} \nabla u_h \nabla v_h.$$

Now, since f and g are eigenfunctions, $f, g \in H^{\frac{1}{2} - \epsilon}(\Gamma_3)$, and the solutions u and v are in $H^{\frac{3}{2} - \epsilon}(\Omega)$. Then, from (28) we can affirm that $\|u - u_h\|_{H^1(\Omega)} \leq C \left(h \sqrt{\log \frac{1}{h}} \right)^{\frac{1}{2} - \epsilon} \|f\|_{H^{\frac{1}{2} - \epsilon}(\Gamma_3)}$, and from (29) that $\|\nabla u_h\|_{L^2(\Omega_h \setminus \Omega)} \leq C \left(h \sqrt{\log \frac{1}{h}} \right)^{\frac{1}{2} - \epsilon} \|f\|_{H^{\frac{1}{2} - \epsilon}(\Gamma_3)}$ (analogously for the error $\|v - v_h\|_{H^1(\Omega)}$ and $\|\nabla v_h\|_{L^2(\Omega_h \setminus \Omega)}$) and the proof concludes. $\square$

Theorem 5 *Let μ be a simple eigenvalue of T and μ_h the associated eigenvalue of T_h. Then, for any $\epsilon > 0$ there exists a constant C such that*

$$|\mu - \mu_h| \leq C \left(h \sqrt{\log \frac{1}{h}} \right)^{1 - \epsilon}.$$

Proof. It follows from the previous lemma, Corollary 4, and [10, Theorem 7.3]. $\square$

We recall from Section 2 that problem (1) has a divergence sequence of positive eigenvalues with λ being an eigenvalue of (1) if and only if $\lambda = \frac{1}{\mu}$ for an eigenvalue μ of T. Analogously, λ_h is an eigenvalue of (2) if and only if $\lambda_h = \frac{1}{\mu_h}$ with μ_h being an eigenvalue of T_h. The boundedness of eigenvalues of T and T_h gives us the next Corollary.

Corollary 6 *Let λ be a simple eigenvalue of (1). Then, for any h small enough, there exists an eigenvalue λ_h of (2) such that $\lambda_h \to \lambda$ when $h \to 0$ and we have, for any $\epsilon > 0$,*

$$|\lambda - \lambda_h| \leq C \left(h \sqrt{\log \frac{1}{h}} \right)^{1 - \epsilon},$$

with the constant C independent of h (but may depend of ϵ).

The next lemma gives a relationship between the eigenvalue λ of problem (1) and its approximation λ_h of problem (2) and it is, in particular, a tool to obtain error estimates for eigenfunctions in H^1 norm.

Lemma 4 *Let (λ, u) and (λ_h, u_h) be eigenpairs solutions of problems (1) and (2) respectively, with $\|u\|_{L^2(\Omega)} = 1$ and $\|u_h\|_{L^2(\Omega)} = 1$. Then we have that*

$$\lambda_h - \lambda = \|\nabla(u_h - u)\|_{L^2(\Omega)}^2 - \lambda\|u_h - u\|_{L^2(\Gamma_3)}^2 + \|\nabla u_h\|_{L^2(\Omega_h \setminus \Omega)}^2. \tag{33}$$

Proof. The proof follows the same ideas of, for example, [10, Lemma 9.1], we include it for the sake of completeness. We have

$$\begin{aligned}
\lambda + \lambda_h &= \lambda\|u\|_{L^2(\Gamma_3)}^2 + \lambda_h\|u_h\|_{L^2(\Gamma_3)}^2 = a(u, u) + a_h(u_h, u_h) \\
&= a(u - u_h, u - u_h) + 2a(u, u_h) + \|\nabla u_h\|_{L^2(\Omega_h \setminus \Omega)}^2 \\
&= \|\nabla(u - u_h)\|_{L^2(\Omega)}^2 + 2\lambda \int_{\Gamma_3} u u_h + \|\nabla u_h\|_{L^2(\Omega_h \setminus \Omega)}^2 \\
&= \|\nabla(u - u_h)\|_{L^2(\Omega)}^2 + 2\lambda - \lambda\|u - u_h\|_{L^2(\Gamma_3)}^2 + \|\nabla u_h\|_{L^2(\Omega_h \setminus \Omega)}^2,
\end{aligned}$$

then (33) holds. $\square$

Theorem 6 *Let λ be a (simple) eigenvalue of problem (1) and let λ_h be the associated eigenvalue solution of (2). Then the corresponding eigenfunctions u and u_h can be chosen such that $\|u\|_{L^2(\Gamma_3)} = \|u_h\|_{L^2(\Gamma_3)} = 1$ and*

$$\|u - u_h\|_{H^1(\Omega)} \leq C\left(h\sqrt{\log \frac{1}{h}}\right)^{\frac{1}{2} - \epsilon},$$

with C independent of h.

Proof. From (33) we have

$$\|\nabla(u - u_h)\|_{L^2(\Omega)}^2 + \|\nabla u_h\|_{L^2(\Omega_h \setminus \Omega)}^2 = \lambda_h - \lambda + \lambda\|u - u_h\|_{L^2(\Gamma_3)}^2$$

and therefore

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \leq |\lambda_h - \lambda| + \lambda\|u - u_h\|_{L^2(\Gamma_3)}^2.$$

Then Corollary 5 and Corollary 6 give us

$$\|\nabla(u - u_h)\|_{L^2(\Omega)}^2 \leq C\left\{\left(h\sqrt{\log \frac{1}{h}}\right)^{1-\epsilon} + |\lambda|\left(h\sqrt{\log \frac{1}{h}}\right)^{1+\sigma-\epsilon}\right\} \leq C\left(h\sqrt{\log \frac{1}{h}}\right)^{1-\epsilon},$$

as we wanted, with the constant C depending on λ and α, but independent of h. $\square$

6 Numerical aspects and numerical experiments

In this section we analyze numerical aspects concerning the solution of the discrete problem (2), we present numerical approximations of the first eigenvalue and the numerical orders of convergence.
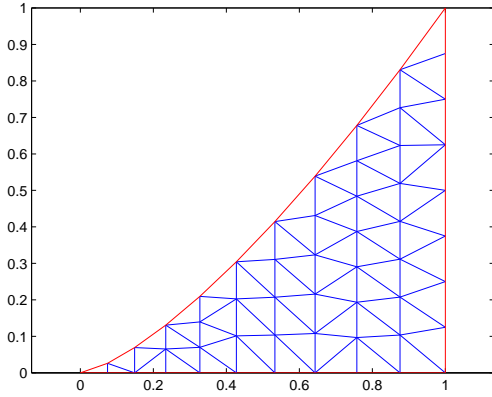
Fig. 4 Graded mesh with $\alpha = 1.4$ and $n = 10$.

6.1 Numerical aspects

First, we show that meshes satisfying the hypotheses M.1 - M.6 and (Ha) can be constructed. In fact, to define the mesh $\mathcal{T}_h$, with $h = 1/n$ we recall the method given in [17, page 393] and [24] which has been also used in [1, 2, 4].

1. Introduce the partition of the interval $(0, 1)$ given by

$$x_j = \left(\frac{j}{n}\right)^{\frac{2}{3-\alpha}} \qquad 0 \leq j \leq n. \tag{34}$$

2. Take the points $(x_j, 0)$ in Γ_1, (x_j, x_j^α) in Γ_3, and for $j > 1$, divide each of the vertical lines $\{(x_j, y) : 0 \leq y \leq x_j^\alpha\}$ uniformly into subintervals such that each has length $\sim x_j - x_{j-1}$.

Figure 4 shows an example of one of these meshes.

Now, we analyze some aspects about the resolution of the generalized eigenvalue problem (2).

Let $\mathcal{N}_{h,ND}$ be the set of vertices of the mesh excluding the ones on the Dirichlet boundary Γ_2, and split it as $\mathcal{N}_{h,ND} = \mathcal{N}_{h,o} \cup \mathcal{N}_{h,3}$ with $\mathcal{N}_{h,3}$ being the subset of $\mathcal{N}_{h,ND}$ consisting of the vertices which belong to a triangle having two vertices on Γ_3, and $\mathcal{N}_{h,o} = \mathcal{N}_{h,ND} \setminus \mathcal{N}_{h,3}$.

We introduce the matrices

$$\boldsymbol{A}_{rs} := \left(\int_\Omega \nabla \beta_i \cdot \nabla \beta_j\right)_{P_i \in \mathcal{N}_{h,r}, P_j \in \mathcal{N}_{h,s}}, \qquad \boldsymbol{B}_{rs} := \left(\int_{\Gamma_3} \beta_i \beta_j\right)_{P_i \in \mathcal{N}_{h,r}, P_j \in \mathcal{N}_{h,s}} \qquad r, s = o, 3,$$

with $\{\beta_i\}_{P_i \in \mathcal{N}_{h,ND}}$ being the nodal basis, that is, $\beta_i \in V_h$ with $\beta_i(P_j) = \delta_{ij}$. Clearly we have

$$\boldsymbol{B}_{oo} = \boldsymbol{0}, \qquad \boldsymbol{B}_{o3} = \boldsymbol{B}_{3o}^t = \boldsymbol{0}.$$

A function $v \in V_h$ can be written as

$$v = \sum_{P_i \in \mathcal{N}_{h,o}} v(P_i)\beta_i + \sum_{P_j \in \mathcal{N}_{h,3}} v(P_j)\beta_j.$$

Introducing the vectors

$$\boldsymbol{u}_o := \left(u_h(P_i)\right)_{P_i \in \mathcal{N}_{h,o}} \in \mathbb{R}^{N_o} \qquad \text{and} \qquad \boldsymbol{u}_3 := \left(u_h(P_i)\right)_{P_i \in \mathcal{N}_{h,3}} \in \mathbb{R}^{\mathbf{n}_3},$$

with $n_i = \#\mathcal{N}_{h,i}$, $i = o, 3$, we have that u_h is defined by the system

$$\begin{pmatrix} \boldsymbol{A}_{oo} & \boldsymbol{A}_{o3} \\ \boldsymbol{A}'_{o3} & \boldsymbol{A}_{33} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_o \\ \boldsymbol{u}_3 \end{pmatrix} = \lambda_h \begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B}_{33} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_o \\ \boldsymbol{u}_3 \end{pmatrix}. \tag{35}$$

This system can be transformed into a standard symmetric generalized eigenvalue problem. We observe that the matrix on the left hand side of (35) is symmetric and positive definite. Then, since the submatrix $\boldsymbol{A}_{oo}$ is invertible (indeed, symmetric and positive definite, too), we can write

$$\boldsymbol{u}_o = -\boldsymbol{A}_{oo}^{-1} \boldsymbol{A}_{o3} \boldsymbol{u}_3.$$

Now, by eliminating $\boldsymbol{u}_o$ from (35) we arrive at

$$\boldsymbol{C} \boldsymbol{u}_3 = \lambda_h \boldsymbol{B}_{33} \boldsymbol{u}_3,$$

with $\boldsymbol{C} := \boldsymbol{A}_{33} - \boldsymbol{A}_{o3}^t \boldsymbol{A}_{oo}^{-1} \boldsymbol{A}_{o3}$. This problem is equivalent to (35). Notice that although the matrix $\boldsymbol{C}$ is not sparse, its dimension is $\mathtt{n}_3 \times \mathtt{n}_3$ and, hence significantly smaller than the size of system (35). Moreover, in actual computations, the matrix $\boldsymbol{A}_{oo}^{-1}$ is not explicitly computed, in fact, the columns of matrix $\boldsymbol{A}_{oo}^{-1} \boldsymbol{A}_{o3}$ are obtained as the solution of linear systems with the same matrix $\boldsymbol{A}_{oo} \in \mathbb{R}^{N_o \times N_o}$, which is sparse, symmetric and positive definite. Then, this is a well posed (and very small) problem that can be efficiently solved by any standard eigensolver.

6.2 Numerical experiments

In Table 1 we present numerical approximations of the first six eigenvalues of the Steklov problem (2) for $\alpha = 1.4$. Then, we solve the problem for different values of α with the purpose of try to estimate the order of the approximation.

h	# nodes	$\lambda_{h,1}$	$\lambda_{h,2}$	$\lambda_{h,3}$	$\lambda_{h,4}$	$\lambda_{h,5}$	$\lambda_{h,6}$
2^{-3}	53	0.8041273434	2.9242421362	5.2402919893	7.9437328154	11.1409802166	15.0438670074
2^{-4}	162	0.7946744205	2.7816190225	4.8508461523	7.0132049915	9.3718201883	11.9190288842
2^{-5}	571	0.7916544790	2.7286690230	4.7131025238	6.7138702509	8.8188973607	10.9909317572
2^{-6}	2157	0.7907200884	2.7095889679	4.6605475331	6.6028687328	8.6288870966	10.6936643446
2^{-7}	8382	0.7904487012	2.7035314255	4.6428771968	6.5641325567	8.5619389967	10.5898209679
2^{-8}	33102	0.7903710292	2.7016789808	4.6371925732	6.5511662493	8.5388907520	10.5534888493

Table 1 Numerical approximation of the first six eigenvalues for $\alpha = 1.4$ with graded meshes.

Assuming that the error $|\lambda - \lambda_h|$ behaves as $C(h \log h)^R$ (with h being the parameter defining the graded meshes, which behaves as $\frac{1}{\sqrt{N}}$, where N is the number of nodes), Table 2 presents the experimental order of convergence (e.o.c), $R = R(\alpha)$, which, for different values of α is computed by using a least-squares fitting with the last four results of an iterative process, in which the last iteration is performed with $h = \frac{1}{256}$ for $\alpha \leq 1.5$, with $h = \frac{1}{128}$ for $\alpha = 1.6$ and with $h = \frac{1}{32}$ for $\alpha \geq 1.7$ (we notice that, from (34), for a given value of n, $x_1 \sim \frac{1}{n}$ when α is next to 1 while $x_1 \sim \frac{1}{n^2}$ for α next to 2 and therefore the number of nodes increases considerably as α increases).

We observe that the e.o.c. increase when α approach 1, while when α approach 2 the order predicted by the theory seems to be obtained. We recall that, in view of Theorem 3, our estimates are restricted to the worst case in which the eigenfunctions are in $H^{\frac{1}{2} - \epsilon}$ for all $\epsilon > 0$ (although they are probably more regular).

α	$\lambda_{h,1}$	$\lambda_{h,2}$	$\lambda_{h,3}$	$\lambda_{h,4}$	$\lambda_{h,5}$	$\lambda_{h,6}$
1.1	1.96	1.93	1.96	1.97	1.97	1.97
1.2	1.92	1.88	1.91	1.93	1.94	1.96
1.3	1.86	1.78	1.78	1.78	1.79	1.81
1.4	1.79	1.78	1.60	1.54	1.52	1.51
1.5	1.76	1.55	1.40	1.31	1.27	1.26
1.6	1.62	1.31	1.04	0.97	0.94	0.99
1.7	1.44	1.09	0.83	0.86	0.96	1.04
1.8	1.33	1.06	0.90	1.11	1.47	1.92
1.9	1.03	0.97	0.85	0.83	1.00	1.27

Table 2 For the first six eigenvalues, the table shows the e.o.c with respect to the parameter h for different values of α using graded meshes.

In all the cases computed we observe that the discrete sequence monotonically decreases with h. The reason for this behaviour could be that, from Lemma 4, we have

$$\lambda_h - \lambda = \frac{(\|\nabla(u - u_h)\|^2_{L^2(\Omega)} + \|\nabla u_h\|^2_{L^2(\Omega_h \setminus \Omega)}) - \lambda_h \|u - u_h\|^2_{L^2(\Gamma_3)}}{1 - \|u - u_h\|^2_{L^2(\Gamma_3)}},$$

and for h small the sign of $\lambda_h - \lambda$ depends on the difference of the two terms given in the numerator. Since $1 < \alpha < 2$ (and therefore $\sigma > 0$), taking into account the results obtained in Corollary 5, Theorem 6 and inequality (29), we would expect that the second one to be of higher order than the first one, we can conjecture that $\lambda_h - \lambda > 0$ for h small enough.

On the other hand, assuming $\lambda_h - \lambda > 0$ and considering λ_H and λ_h, eigenvalues of problem (2) with $H > h$, we can write $\lambda_H - \lambda_h = \lambda_H - \lambda - (\lambda_h - \lambda)$. Then, Theorem 5 would imply the monotonicity of the discrete sequence of eigenvalues.

We end this paper by highlighting that, the trace results introduced in Section 1 and the interpolation theorems obtained in Section 4 for cuspidal domains (joint with the theory developed in [1–4]) could be a fundamental tool to approximate another kind of boundary eigenvalue problems as, for example, the fluid-solid vibrations problems considered in [8].

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