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A GENERALIZATION OF THE BOUNDEDNESS OF CERTAIN INTEGRAL OPERATORS IN VARIABLE LEBESGUE SPACES

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Abstract. Let $n \in \mathbb{N}$. Let $A_1,\dots A_m$ be $n\times n$ invertible matrices. Let $0 \leqslant \alpha < n$ and $0 < \alpha_i < n$ such that $\alpha_1 + \dots + \alpha_m = n-\alpha$. We define

$$T_\alpha f(x) = \int \frac{1}{|x-A_1y|^{\alpha_1}\dots|x-A_my|^{\alpha_m}} f(y)dy.$$

In [8] we obtained the boundedness of this operator from $L^{p(\cdot)}(\mathbb{R}^n)$ into $L^{q(\cdot)}(\mathbb{R}^n)$ for $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$, in the case that A_i is a power of certain fixed matrix A and for exponent functions p satisfying log-Holder conditions and $p(Ay) = p(y)$, $y \in \mathbb{R}^n$. We will show now that the hypothesis on p, in certain cases, is necessary for the boundedness of T_α and we also prove the result for more general matrices A_i.

1. Introduction

Let $n \in \mathbb{N}$. Given a measurable function $p(\cdot) : \mathbb{R}^n \to [1,\infty)$, let $L^{p(\cdot)}(\mathbb{R}^n)$ be the Banach space of measurable functions f on $\mathbb{R}^n$ such that for some $\lambda > 0$,

$$\int \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty,$$

with norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leqslant 1\right\}.$$

These spaces are known as *variable exponent spaces* and are a generalization of the classical Lebesgue spaces $L^p(\mathbb{R}^n)$. They have been widely studied lately. See for example [1], [3] and [4]. The first step was to determine sufficient conditions on $p(\cdot)$ for the boundedness on $L^{p(\cdot)}$ of the Hardy Littlewood maximal operator

$$\mathscr{M} f(x) = \sup_B \frac{1}{|B|}\int_B |f(y)|\, dy,$$

where the supremun is taken over all balls B containing x. Let $p_- = ess\inf\ p(x)$ and let $p_+ = ess\sup\ p(x)$. In [3] , D. Cruz Uribe, A. Fiorenza and C. J. Neugebauer proved the following result.

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THEOREM 1. *Let* $p(\cdot):\mathbb{R}^n\to[1,\infty)$ *be such that* $1<p_-\leqslant p_+<\infty$. *Suppose further that* $p(\cdot)$ *satisfies*

$$|p(x)-p(y)|\leqslant\frac{c}{-\log|x-y|},\ \ |x-y|<\frac{1}{2}, \tag{1}$$

and

$$|p(x)-p(y)|\leqslant\frac{c}{\log(e+|x|)},\ \ |y|\geqslant|x|. \tag{2}$$

Then the Hardy Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

We recall that a weight ω is a locally integrable and non negative function. The Muckenhoupt class $\mathscr{A}_p$, $1<p<\infty$, is defined as the class of weights ω such that

$$\sup_Q\left[\left(\frac{1}{|Q|}\int_Q\omega\right)\left(\frac{1}{|Q|}\int_Q\omega^{-\frac{1}{p-1}}\right)^{p-1}\right]<\infty,$$

where Q is a cube in $\mathbb{R}^n$.

For $p=1$, $\mathscr{A}_1$ is the class of weights ω satisfying that there exists $c>0$ such that

$$\mathscr{M}\omega(x)\leqslant c\omega(x)\ a.e.\ x\in\mathbb{R}^n.$$

We denote $[\omega]_{\mathscr{A}_1}$ the infimum of the constant c such that ω satisfies the above inequation.

In [5], B. Muckenhoupt y R.L. Wheeden define $\mathscr{A}(p,q)$, $1<p<\infty$ and $1<q<\infty$, as the class of weights ω such that

$$\sup_Q\left[\left(\frac{1}{|Q|}\int_Q\omega(x)^q dx\right)^{\frac{1}{q}}\left(\frac{1}{|Q|}\int_Q\omega(x)^{-p'}dx\right)^{\frac{1}{p'}}\right]\ <\ \infty.$$

When $p=1$, $\omega\in\mathscr{A}(1,q)$ if only if

$$\sup_Q\left[\|\omega^{-1}\chi_Q\|_\infty\left(\frac{1}{|Q|}\int_Q\omega(x)^q dx\right)^{\frac{1}{q}}\right]\ <\ \infty.$$

Let $0\leqslant\alpha<n$. For $1\leqslant i\leqslant m$, let $0<\alpha_i<n$, be such that

$$\alpha_1+\ldots+\alpha_m=n-\alpha.$$

Let T_α be the positive integral operator given by

$$T_\alpha f(x)=\int k(x,y)\,f(y)\,dy, \tag{3}$$

where

$$k(x,y)=\frac{1}{|x-A_1y|^{\alpha_1}}\cdots\frac{1}{|x-A_my|^{\alpha_m}},$$

and where the matrices A_i are certain invertible matrices such that A_i-A_j is invertible for $i\neq j$, $1\leqslant i,j\leqslant m$.

In the paper [7] the authors studied this kind of integral operators and they obtained weighted (p,q) estimates, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, for weights $w\in A(p,q)$ such that $w(A_i x)\leqslant cw(x)$. In [8] we use extrapolation techniques to obtain $p(\cdot)-q(\cdot)$ and weak type estimates, in the case where $A_i=A^i$, for some invertible matrix A such that $A^N=I$, for some $N\in\mathbb{N}$. This technique allows us to replace the log-Hölder conditions about the exponent $p(\cdot)$ by a more general hypothesis concerning the boundeness of the maximal function $\mathscr{M}$. We obtain the following results.

THEOREM 2. *Let A be an invertible matrix such that $A^N=I$, for some $N\in\mathbb{N}$, let T_α be the integral operator given by (3), where $A_i=A^i$ and such that A_i-A_j is invertible for $i\neq j$, $1\leqslant i,j\leqslant m$. Let $p:\mathbb{R}^n\longrightarrow[1,\infty)$ be such that $1<p_-\leqslant p_+<\frac{n}{\alpha}$ and such that $p(Ax)=p(x)$ a.e. $x\in\mathbb{R}^n$. Let $q(\cdot)$ be defined by $\frac{1}{p(x)}-\frac{1}{q(x)}=\frac{\alpha}{n}$. If the maximal operator $\mathscr{M}$ is bounded on $L^{\left(\frac{n-\alpha p_-}{np_-}q(\cdot)\right)'}$ then T is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ into $L^{q(\cdot)}(\mathbb{R}^n)$.*

THEOREM 3. *Let A be an invertible matrix such that $A^N=I$, for some $N\in\mathbb{N}$, let T_α be the integral operator given by (3), where $A_i=A^i$ and such that A_i-A_j is invertible for $i\neq j$, $1\leqslant i,j\leqslant m$. Let $p:\mathbb{R}^n\longrightarrow[1,\infty)$ be such that $1\leqslant p_-\leqslant p_+<\frac{n}{\alpha}$ and such that $p(Ax)=p(x)$ a.e. $x\in\mathbb{R}^n$. Let $q(\cdot)$ be defined by $\frac{1}{p(x)}-\frac{1}{q(x)}=\frac{\alpha}{n}$. If the maximal operator $\mathscr{M}$ is bounded on $L^{\left(\frac{n-\alpha p_-}{np_-}q(\cdot)\right)'}$ then there exists $c>0$ such that*

$$\left\|t\chi_{\{x:T_\alpha f(x)>t\}}\right\|_{q(\cdot)}\leqslant c\|f\|_{p(\cdot)}.$$

We also showed that this technique applies in the case when each of the matrices A_i is either a power of an orthogonal matrix A or a power of A^{-1}.

In this paper we will prove that these theorems generalize to any invertible matrices $A_1,\ldots,A_m$ such that A_i-A_j is invertible for $i\neq j$, $1\leqslant i,j\leqslant m$. We will also show, in some cases, that the condition $p(A_i x)=p(x)$, $x\in\mathbb{R}^n$ is necessary to obtain $p(.)-q(.)$ boundedness.

2. Necessary conditions on p

Let A be a $n\times n$ invertible matrix and let $0<\alpha<n$. We define

$$T_A f(x)=\int\frac{1}{|x-Ay|^{n-\alpha}}f(y)dy.$$

PROPOSITION 4. *Let A be a $n\times n$ invertible matrix. Let $p:\mathbb{R}^n\to[1,\infty)$ be a measurable function such that p is continuos at y_0 and at Ay_0 for some $y_0\in\mathbb{R}^n$. If $p(Ay_0)>p(y_0)$ then there exists $f\in L^{p(\cdot)}(\mathbb{R}^n)$ such that $T_A f\notin L^{q(\cdot)}(\mathbb{R}^n)$ for $\frac{1}{q(\cdot)}=\frac{1}{p(\cdot)}-\frac{\alpha}{n}$.*

Proof. Since p is continuos at y_0, there exists ball $B=B(y_0,r)$ such that $p(y)\sim p(y_0)$ for $y\in B$. We have that $p(y_0)<p(Ay_0)$. In this case we take

$$f(y)=\frac{\chi_B(y)}{|y-y_0|^{\beta}},$$

for certain $\beta<\frac{n}{p(y_0)}$ that will be chosen later. We will show that, for certain β, $f\in L^{p(\cdot)}(\mathbb{R}^n)$ but $T_Af\notin L^{q(\cdot)}(\mathbb{R}^n)$. Indeed,

$$T_Af(x)=\int\frac{1}{|x-Ay|^{n-\alpha}}f(y)dy=\int_B\frac{1}{|x-Ay|^{n-\alpha}|y-y_0|^{\beta}}dy,$$

so

$$\begin{aligned}\int(T_Af(x))^{q(x)}dx&=\int\left(\int_B\frac{1}{|x-Ay|^{n-\alpha}|y-y_0|^{\beta}}dy\right)^{q(x)}dx\\&\geqslant\int_{B(Ay_0,\varepsilon)}\left(\int_B\frac{1}{|x-Ay|^{n-\alpha}|y-y_0|^{\beta}}dy\right)^{q(x)}dx\\&\geqslant\int_{B(Ay_0,\varepsilon)}\left(\int_{B\cap\{y:|Ay-Ay_0|<|Ay_0-x|\}}\frac{1}{|x-Ay|^{n-\alpha}|y-y_0|^{\beta}}dy\right)^{q(x)}dx.\end{aligned}$$

Now, we denote by $M=\|A\|=\sup\limits_{\|y\|=1}|Ay|$. Now for $\varepsilon<Mr$ and $x\in B(Ay_0,\varepsilon)$, $B(y_0,\frac{1}{M}|Ay_0-x|)\subset B\cap\{y:|Ay-Ay_0|<|Ay_0-x|\}$. Indeed, $|y-y_0|\leqslant\frac{1}{M}|Ay_0-x|\leqslant\frac{1}{M}\varepsilon\leqslant r$ and $|Ay-Ay_0|\leqslant M|y-y_0|\leqslant|Ay_0-x|$, so

$$\geqslant\int_{B(Ay_0,\varepsilon)}\left(\int_{B(y_0,\frac{1}{M}|Ay_0-x|)}\frac{1}{|x-Ay|^{n-\alpha}|y-y_0|^{\beta}}dy\right)^{q(x)}dx,$$

also, for $y\in B(y_0,\frac{1}{M}|Ay_0-x|)$

$$|x-Ay|\leqslant|x-Ay_0|+|Ay_0-Ay|\leqslant|x-Ay_0|+M|y_0-y|\leqslant2|x-Ay_0|,$$

so

$$\begin{aligned}&\geqslant\int_{B(Ay_0,\varepsilon)}\left(\frac{1}{2^{n-\alpha}|x-Ay_0|^{n-\alpha}}\right)^{q(x)}\left(\int_{B(y_0,\frac{1}{M}|Ay_0-x|)}\frac{1}{|y-y_0|^{\beta}}dy\right)^{q(x)}dx\\&=\int_{B(Ay_0,\varepsilon)}\left(\frac{1}{2^{n-\alpha}|x-Ay_0|^{n-\alpha}}\right)^{q(x)}\left(c|Ay_0-x|^{-\beta+n}\right)^{q(x)}dx\\&=\int_{B(Ay_0,\varepsilon)}\left(\frac{c}{2^{n-\alpha}|x-Ay_0|^{\beta-\alpha}}\right)^{q(x)}dx.\end{aligned}$$

Now, since $q(Ay_0) > q(y_0)$, $q(Ay_0) - \gamma > q(y_0)$ for $\gamma = \frac{q(Ay_0)-q(y_0)}{2}$ We observe that if $\frac{1}{q(y_0)} = \frac{1}{p(y_0)} - \frac{\alpha}{n}$, for $\beta_0 = \frac{n}{p(y_0)}$, $(\beta_0 - \alpha)\, q(y_0) = \left(\frac{n}{p(y_0)} - \alpha\right) q(y_0) = n$, so since $q(Ay_0) - \gamma > q(y_0)$, we obtain that $\left(\frac{n}{p(y_0)} - \alpha\right)(q(Ay_0) - \gamma) > n$ and still $(\beta - \alpha)$ $\cdot (q(Ay_0) - \gamma) > n$ for $\beta = \frac{n}{p(y_0)} - \frac{1}{2}\left(\frac{n}{p(y_0)} - \left(\alpha + \frac{n}{q(Ay_0)-\gamma}\right)\right)$. So $\beta = \frac{n}{p(y_0)}(1-\delta)$ for some $\delta > 0$. Since q is continuos, we chose ε so that, for $x \in B(Ay_0, \varepsilon)$, $q(x) > q(Ay_0) - \gamma$ and $\frac{c}{2^{n-\alpha}|x-Ay_0|^{\beta-\alpha}} > 1$ so this last integral is bounded from below by

$$c \int_{B(Ay_0,\varepsilon)} \left(\frac{1}{|x - Ay_0|^{\beta-\alpha}}\right)^{q(Ay_0)-\gamma} dx = \infty.$$

For this β we chose r to obtain that the ball $B = B(y_0, r) \subset \left\{y : p(y) < \frac{p(y_0)}{1-\delta}\right\}$. In this way we obtain that $f \in L^{p(\cdot)}(\mathbb{R}^n)$ but $T_A f \notin L^{q(\cdot)}(\mathbb{R}^n)$. $\square$

COROLLARY 5. *If $A^N = I$ for some $N \in \mathbb{N}$, p is continuos and T_A is bounded from $L^{p(\cdot)}$ into $L^{q(\cdot)}$, then $p(Ay) = p(y)$ for all $y \in \mathbb{R}^n$.*

Proof. We suppose that $p(Ay_0) < p(y_0)$. Since p is continuos in y_0, by the last proposition,

$$p(Ay_0) < p(y_0) = p(A^N y_0) \leqslant p(A^{N-1} y_0) \leqslant \ldots \leqslant p(Ay_0) = p(Ay_0)$$

which is a contradiction. $\square$

3. The main results

Given $0 \leqslant \alpha < n$, we recall that we are studying fractional type integral operators of the form

$$T_\alpha f(x) = \int k(x,y)\, f(y)\, dy, \tag{4}$$

$f \in L_c^\infty(\mathbb{R}^n)$, with a kernel

$$k(x,y) = \frac{1}{|x - A_1 y|^{\alpha_1}} \cdots \frac{1}{|x - A_m y|^{\alpha_m}},$$

$\alpha_1 + \ldots + \alpha_m = n - \alpha$, $0 < \alpha_i < n$.

THEOREM 6. *Let $m \in \mathbb{N}$, let $A_1, \ldots A_m$ be invertible matrices such that $A_i - A_j$ is invertible for $i \neq j$, $1 \leqslant i, j \leqslant m$. Let T_α be the integral operator given by (4), let $p : \mathbb{R}^n \longrightarrow [1, \infty)$ be such that $1 \leqslant p_- \leqslant p_+ < \frac{n}{\alpha}$ and such that $p(A_i x) = p(x)$ a.e. $x \in \mathbb{R}^n$, $1 \leqslant i \leqslant m$. Let $q(\cdot)$ be defined by $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}$. If the maximal operator $\mathscr{M}$ is bounded on $L^{\left(\frac{n-\alpha p_-}{np_-} q(\cdot)\right)'}$ then there exists $c > 0$ such that*

$$\left\| t \chi_{\{x : T_\alpha f(x) > t\}} \right\|_{q(.)} \leqslant c \, \|f\|_{p(\cdot)},$$

$f \in L_c^\infty(\mathbb{R}^n)$.

REMARK 7. *With the hypothesis of Theorem 6, if* $f \in L^{p(\cdot)}(\mathbb{R}^n)$*, the integral in (4) converges a.e.* $x \in \mathbb{R}^n$*, we still call it* $T_\alpha f(x)$ *and we have that there exists* $c > 0$ *such that*

$$\left\| \lambda \chi_{\{x: T_\alpha f(x) > \lambda\}} \right\|_{q(\cdot)} \leqslant c \|f\|_{p(\cdot)}, \ f \in L^{p(\cdot)}(\mathbb{R}^n).$$

Proof. We take $f \geqslant 0$ and a sequence $f_n \in L_c^\infty(\mathbb{R}^n)$ such that $f_n(x) \nearrow f(x)$ *a.e.* $x \in \mathbb{R}^n$. Then $T_\alpha f_n(x) \nearrow T_\alpha f(x)$ *a.e.* $x \in \mathbb{R}^n$ and then

$$\chi_{\{x: T_\alpha f_n(x) > \lambda\}}(x) \to \chi_{\{x: T_\alpha f_n(x) > \lambda\}}(x),$$

and so by Fatou´s Lemma, (see Th. 2.61, p.46 [2])

$$\begin{aligned} \left\| \lambda \chi_{\{x: T_\alpha f(x) > \lambda\}} \right\|_{q(\cdot)} &= \left\| \liminf \lambda \chi_{\{x: T_\alpha f_n(x) > \lambda\}} \right\|_{q(\cdot)} \\ &\leqslant \liminf \left\| \lambda \chi_{\{x: T_\alpha f_n(x) > \lambda\}} \right\|_{q(\cdot)} \leqslant \liminf \|f_n\|_{p(\cdot)} \leqslant \|f\|_{p(\cdot)}. \end{aligned}$$

For general f, as usual, we write $f = f^+ - f^-$. □

THEOREM 8. *Let* $m \in \mathbb{N}$*, let* $A_1, \ldots A_m$ *be invertible matrices such that* $A_i - A_j$ *is invertible for* $i \neq j$*,* $1 \leqslant i, j \leqslant m$*. Let* T_α *be the integral operator given by (4), let* $p : \mathbb{R}^n \longrightarrow [1, \infty)$ *be such that* $1 < p_- \leqslant p_+ < \frac{n}{\alpha}$ *and such that* $p(A_i x) = p(x)$ *a.e.* $x \in \mathbb{R}^n$*,* $1 \leqslant i \leqslant m$*. Let* $q(\cdot)$ *be defined by* $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}$*. If the maximal operator* $\mathscr{M}$ *is bounded on* $L^{\left(\frac{n-\alpha p_-}{np_-} q(\cdot)\right)'}$ *then* T_α *is bounded from* $L^{p(\cdot)}(\mathbb{R}^n)$ *into* $L^{q(\cdot)}(\mathbb{R}^n)$*.*

4. Proofs of the main results

LEMMA 9. *If* $f \in L^1_{loc}(\mathbb{R}^n)$ *and* A *an invertible* $n \times n$ *matrix then*

$$\mathscr{M}(f \circ A)(x) \leqslant c(\mathscr{M}(f) \circ A)(x).$$

Proof. Indeed,

$$\mathscr{M}(f \circ A) = \sup_B \frac{1}{|B|} \int_B |(f \circ A)(y)| \, dy,$$

where the supremun is taken over all balls B containing x. By a change of variable we see that,

$$\frac{1}{|B|} \int_B |(f \circ A)(y)| \, dy = |det(A^{-1})| \frac{1}{|B|} \int_{A(B)} |f(z)| \, dz,$$

where $A(B) = \{Ay : y \in B\}$. Now, if $y \in B = B(x_0, r)$ then $|Ay - Ax_0| \leqslant M|y - x_0| \leqslant Mr$, where $M = \|A\|$. That is $Ay \in \widetilde{B} = B(Ax_0, Mr)$. So

$$\leqslant \frac{M^n |det(A^{-1})|}{|\widetilde{B}|} \int_{\widetilde{B}} f(z) dz \leqslant M^n |det(A^{-1})| \mathscr{M} f(Ax).$$

Therefore we obtain that,

$$\mathcal{M}(f\circ A)\leqslant c(\mathcal{M}(f)\circ A),$$

with $c=M^n|det(A^{-1})|$. $\square$

Proof of Theorem 6. We take $f\in L_c^\infty(\mathbb{R}^n)$. In [7] (See page 459) the authors prove that there exists $c>0$ such that,

$$\sup_{\lambda>0}\lambda\left(\omega^{q_0}\{x:|T_\alpha f(x)|>\lambda\}\right)^{\frac{1}{q_0}}\leqslant\sup_{\lambda>0}\lambda\left(\omega^{q_0}\{x:\sum_{i=1}^m\mathcal{M}_\alpha f(A_i^{-1}x)>c\lambda\}\right)^{\frac{1}{q_0}}$$

for all $\omega\in\mathcal{A}_\infty$ and $f\in L_c^\infty(\mathbb{R}^n)$.

Let $F_\lambda=\lambda^{q_0}\chi_{\{x:|T_\alpha f(x)|>\lambda\}}$ the last inequality implies that,

$$\int_{\mathbb{R}^n}F_\lambda(x)\omega(x)^{q_0}dx\leqslant\sup_{\lambda>0}\int_{\mathbb{R}^n}\lambda^{q_0}\chi(x)_{\{x:\sum_{i=1}^m\mathcal{M}_\alpha f(A_i^{-1}x)>c\lambda\}}\omega(x)^{q_0}dx \qquad (5)$$

for some $c>0$ and for all $\omega\in\mathcal{A}_\infty$. Now by proposition 2.18 in [2], if $\widetilde{q}(\cdot)=\frac{q(\cdot)}{q_0}$,

$$\begin{aligned}\|\lambda\chi_{\{x:|T_\alpha f(x)|>\lambda\}}\|_{q(\cdot)}^{q_0}&=\|\lambda^{q_0}\chi_{\{x:|T_\alpha f(x)|>\lambda\}}\|_{\widetilde{q}(\cdot)}\\&=\|F_\lambda\|_{\widetilde{q}(\cdot)}\leqslant c\sup_{\|h\|_{\widetilde{q}'(\cdot)}=1}\int_{\mathbb{R}^n}F_\lambda(x)h(x)dx.\end{aligned}$$

We define an iteration algorithm on $L^{\widetilde{q}(\cdot)'}$ by

$$\mathcal{R}h(x)=\sum_{k=0}^\infty\frac{\mathcal{M}^kh(x)}{2^k\|\mathcal{M}\|_{\widetilde{q}(\cdot)'}^k}, \qquad (6)$$

where, for $k\geqslant 1$, $\mathcal{M}^k$ denotes k iteration of the maximal operator $\mathcal{M}$ and $\mathcal{M}^0(h)=|h|$. We will check that

a) $|h(x)|\leqslant\mathcal{R}h(x)\ x\in\mathbb{R}^n$,

b) For all $j:1,...,m,\left\|\mathcal{R}h\circ A_j\right\|_{\widetilde{q}(\cdot)'}\leqslant c\|h\|_{\widetilde{q}(\cdot)'}$,

c) For all $j:1,...,m,\mathcal{R}h^{\frac{1}{q_0}}\circ A_j\in\mathcal{A}(p_-,q_0)$

Indeed, *a*) is evident from the definition. To verify *b*),

$$\|\mathcal{R}h\circ A_j\|_{\widetilde{q}(\cdot)'}\leqslant\sum_{k=0}^\infty\frac{\|\mathcal{M}^kh\circ A_j\|_{\widetilde{q}(\cdot)'}}{2^k\|\mathcal{M}\|_{\widetilde{q}(\cdot)'}^k}$$

and

$$\|\mathscr{M}^k h\circ A_j\|_{\widetilde{q}(\cdot)'}=\inf\left\{\lambda>0:\int_{\mathbb{R}^n}\left(\frac{\mathscr{M}^k h(A_jx)}{\lambda}\right)^{\widetilde{q}(x)'}dx\leqslant 1\right\}.$$

But, by a change of variable and using the hypothesis on the exponent,

$$\int_{\mathbb{R}^n}\left(\frac{\mathscr{M}^k h(A_jx)}{\lambda}\right)^{\widetilde{q}(x)'}dx=|det(A_j^{-1})|\int_{\mathbb{R}^n}\left(\frac{\mathscr{M}^k h(y)}{\lambda}\right)^{\widetilde{q}(A_j^{-1}y)'}dy,$$

put D=$max\left\{|det(A_j^{-1})|, j=1...m\right\}$

$$\leqslant D\int_{\mathbb{R}^n}\left(\frac{\mathscr{M}^k h(y)}{\lambda}\right)^{\widetilde{q}'(y)}dy. \tag{7}$$

If $D\leqslant 1$,

$$\|\mathscr{M}^k h\circ A_j\|_{\widetilde{q}(\cdot)'}\leqslant \|\mathscr{M}^k h\|_{\widetilde{q}(\cdot)'}.$$

So,

$$\|\mathscr{R}h\circ A_j\|_{\widetilde{q}(\cdot)'}\leqslant\sum_{k=0}^{\infty}\frac{\|\mathscr{M}^k h(x)\|_{\widetilde{q}(\cdot)'}}{2^k\,\|\mathscr{M}\|^k_{\widetilde{q}(\cdot)'}}\leqslant\|h\|_{\widetilde{q}(\cdot)'}\sum_{k=0}^{\infty}\frac{1}{2^k}=2\|h\|_{\widetilde{q}(\cdot)'}.$$

If $D>1$ then from (7) it is follows that

$$D\int_{\mathbb{R}^n}\left(\frac{\mathscr{M}^k h(y)}{\lambda}\right)^{\widetilde{q}'(y)}dy=\int_{\mathbb{R}^n}\left(\frac{M^k h(y)}{\lambda C^{\frac{1}{\widetilde{q}(y)'}}}\right)^{\widetilde{q}(y)'}dy$$

and $D=\frac{1}{C}$ where $C=min\{|det(A_j)|, j=1...m\}$. So,

$$\leqslant\int_{\mathbb{R}^n}\left(\frac{M^k h(y)}{\lambda C^{\frac{1}{(\widetilde{q}')_-}}}\right)^{\widetilde{q}(y)'}dy.$$

That is,

$$\int_{\mathbb{R}^n}\left(\frac{\mathscr{M}^k h(A_jx)}{\lambda}\right)^{\widetilde{q}(x)'}dx\leqslant\int_{\mathbb{R}^n}\left(\frac{M^k h(x)}{\lambda C^{\frac{1}{(\widetilde{q}')_-}}}\right)^{\widetilde{q}(x)'}dx.$$

From this last inequality it follows that

$$\|\mathscr{M}^k h\circ A_j\|_{\widetilde{q}(\cdot)'}\leqslant D^{\frac{1}{(\widetilde{q}')_-}}\|\mathscr{M}^k h\|_{\widetilde{q}(\cdot)'}$$

and so $b)$ is verified with $c=2D^{\frac{1}{(\widetilde{q}')_-}}$. To see $c)$, by Lemma 9,

$$\mathscr{M}(\mathscr{R}h^{\frac{1}{q_0}}\circ A_j)(x)\leqslant c\mathscr{M}(\mathscr{R}h^{\frac{1}{q_0}})(A_jx)$$

$\mathscr{R}h \in \mathscr{A}_1$ (see [2]) implies that $\mathscr{R}h^{\frac{1}{q_0}} \in \mathscr{A}_1$ and so,

$$\leqslant c\mathscr{R}h^{\frac{1}{q_0}}(A_jx) = c(\mathscr{R}h^{\frac{1}{q_0}} \circ A_j)(x).$$

Then $c)$ follows since a weight $\omega \in \mathscr{A}_1$ implies that $\omega \in \mathscr{A}(p_-, q_0)$.

And so,

$$\begin{aligned} c \sup_{\|h\|_{\tilde{q}'(\cdot)}=1} \int_{\mathbb{R}^n} F_\lambda(x)h(x)dx &\leqslant c \sup_{\|h\|_{\tilde{q}'(\cdot)}=1} \int_{\mathbb{R}^n} F_\lambda(x)\mathscr{R}h(x)dx \\ &= c \sup_{\|h\|_{\tilde{q}'(\cdot)}=1} \int_{\mathbb{R}^n} F_\lambda(x)(\mathscr{R}h^{\frac{1}{q_0}}(x))^{q_0}dx, \end{aligned}$$

and by (5), since $\mathscr{R}h^{\frac{1}{q_0}} \in \mathscr{A}(p_-, q_0)$ and $Rh \in \mathscr{A}_1 \subset \mathscr{A}_\infty$,

$$\leqslant c \sup_{\|h\|_{\tilde{q}'(\cdot)}=1} \sup_{\lambda>0} \int_{\mathbb{R}^n} \lambda^{q_0} \chi_{\{x: \sum_{i=1}^m \mathscr{M}_\alpha f(A_i^{-1}x) > c\lambda\}} (\mathscr{R}h^{\frac{1}{q_0}}(x))^{q_0}dx.$$

Since,

$$\left\{x : \sum_{i=1}^m \mathscr{M}_\alpha f(A_i^{-1}x) > c\lambda\right\} \subseteq \bigcup_{i=1}^m \left\{x : \mathscr{M}_\alpha f(A_i^{-1}x) > \frac{c\lambda}{m}\right\}$$

then,

$$\chi_{\{x: \sum_{i=1}^m \mathscr{M}_\alpha f(A_i^{-1}x) > c\lambda\}} \leqslant \sum_{i=1}^m \chi_{\{x: \mathscr{M}_\alpha f(A_i^{-1}x) > \frac{c\lambda}{m}\}},$$

so

$$\begin{aligned} &\leqslant c \sup_{\|h\|_{\tilde{q}'(\cdot)}=1} \sup_{\lambda>0} \sum_{i=1}^m \int_{\mathbb{R}^n} \lambda^{q_0} \chi(x)_{\{x: \mathscr{M}_\alpha f(A_i^{-1}x) > \frac{c\lambda}{m}\}} (\mathscr{R}h^{\frac{1}{q_0}}(x))^{q_0}dx \\ &= c \sup_{\|h\|_{\tilde{q}'(\cdot)}=1} \sup_{\lambda>0} \sum_{i=1}^m \int_{\{x: \mathscr{M}_\alpha f(A_i^{-1}x) > \frac{c\lambda}{m}\}} \lambda^{q_0} (\mathscr{R}h^{\frac{1}{q_0}}(x))^{q_0}dx \\ &= c \sup_{\|h\|_{\tilde{q}'(\cdot)}=1} \sup_{\lambda>0} \sum_{i=1}^m \lambda^{q_0} |det(A_i)| \int_{A_i^{-1}\left\{x: \mathscr{M}_\alpha f(A_i^{-1}x) > \frac{c\lambda}{m}\right\}} (\mathscr{R}h^{\frac{1}{q_0}}(A_iy))^{q_0}dy, \\ &\leqslant c \sup_{\|h\|_{\tilde{q}'(\cdot)}=1} \sup_{\lambda>0} \sum_{i=1}^m \lambda^{q_0} \int_{\left\{y: \mathscr{M}_\alpha f(y) > \frac{c\lambda}{m}\right\}} (\mathscr{R}h^{\frac{1}{q_0}}(A_iy))^{q_0}dy \\ &\leqslant c \sup_{\|h\|_{\tilde{q}'(\cdot)}=1} \sup_{\lambda>0} \sum_{i=1}^m \left(\int_{\mathbb{R}^n} |f(y)|^{p_-} (\mathscr{R}h^{\frac{p_-}{q_0}}(A_iy))dy\right)^{\frac{q_0}{p_-}} \\ &= c \sup_{\|h\|_{\tilde{q}'(\cdot)}=1} \sum_{i=1}^m \left(\int_{\mathbb{R}^n} |f(y)|^{p_-} (\mathscr{R}h^{\frac{p_-}{q_0}}(A_iy))dy\right)^{\frac{q_0}{p_-}}. \end{aligned}$$

We denote by $\widetilde{p}(\cdot)=\frac{p(\cdot)}{p_-}$. Holder´s inequality, 2) and Proposition 2.18 in [2] and again the hypothesis about A_i and p give

$$\begin{aligned}\|\lambda\chi_{\{x:|T_\alpha f(x)|>\lambda\}}\|_{q(\cdot)}^{q_0} &\leqslant C\|f^{p_-}\|_{\widetilde{p}(\cdot)}^{\frac{q_0}{p_-}}\sup_{\|h\|_{\widetilde{q}(\cdot)'}=1}\sum_{j=1}^{m}\left\|\left(\mathscr{R}h^{\frac{p_-}{q_0}}\right)\circ A_j\right\|_{\widetilde{p}(\cdot)'}^{\frac{q_0}{p_-}} \\ &\leqslant \sup_{\|h\|_{\widetilde{q}(\cdot)'}=1} Cm\|f\|_{p(\cdot)}^{q_0}\|h\|_{\widetilde{q}(\cdot)'}\leqslant C\|f\|_{p(\cdot)}^{q_0}.\end{aligned}$$

Now f is bounded and with compact support, so $T_\alpha f\in L^s(\mathbb{R}^n)$ for $\frac{n}{n-\alpha}<s<\infty$, (see Lemma 2.2 in [7]) thus $\left\|\lambda\chi_{\{x:T_\alpha f(x)>\lambda\}}\right\|_{q(.)}<\infty$. □

Proof of Theorem 7. In the paper [7] the authors obtain an estimate of the form

$$\int (T_\alpha f)^p(x)w(x)dx\leqslant c\sum_{j=1}^{m}\int (\mathscr{M}_\alpha f)^p(x)w(A_jx)dx, \tag{8}$$

for any $w\in\mathscr{A}_\infty$ and $0<p<\infty$ (See the last lines of page 454 in [7]). We denote $\widetilde{q}(\cdot)=\frac{q(\cdot)}{q_0}$, we define an iteration algorithm on $L^{\widetilde{q}(\cdot)'}$ as in the last proof (see (6)). We have $a)$ For all $x\in\mathbb{R}^n, |h(x)|\leqslant\mathscr{R}h(x)$,

$b)$ For all $j:1,...,m, \left\|\mathscr{R}h\circ A_j\right\|_{\widetilde{q}(\cdot)'}\leqslant c\|h\|_{\widetilde{q}(\cdot)'}$,

$c)$ For all $j:1,...,m, \mathscr{R}^{\frac{1}{q_0}}h\circ A_j\in\mathscr{A}(p_-,q_0)$.

We now take a bounded function f with compact support. So as in Theorem 5.24 in [2],

$$\begin{aligned}\|T_\alpha f\|_{q(\cdot)}^{q_0}=\|(T_\alpha f)^{q_0}\|_{\widetilde{q}(\cdot)} &= C\sup_{\|h\|_{\widetilde{q}(\cdot)'}=1}\int (T_\alpha f)^{q_0}(x)h(x)dx \\ &\leqslant C\sup_{\|h\|_{\widetilde{q}(\cdot)'}=1}\int (T_\alpha f)^{q_0}(x)\mathscr{R}h(x)dx \\ &\leqslant C\sup_{\|h\|_{\widetilde{q}(\cdot)'}=1}\sum_{j=1}^{m}\int (\mathscr{M}_\alpha f)^{q_0}(x)\mathscr{R}h(A_jx)dx \\ &\leqslant C\sup_{\|h\|_{\widetilde{q}(\cdot)'}=1}\sum_{j=1}^{m}\left(\int |f(x)|^{p_-}\mathscr{R}h^{\frac{p_-}{q_0}}(A_jx)dx\right)^{\frac{q_0}{p_-}},\end{aligned}$$

where the last inequality follows since $\mathscr{R}h^{\frac{1}{q_0}}\circ A_i$ are weights in $\mathscr{A}(p_-,q_0)$ (by $c)$). Now, following as in the last proof,

$$\leqslant C\|f\|_{p(\cdot)}^{q_0}.$$

Also, as in the last proof, we show that $\|T_\alpha f\|_{q(\cdot)}<\infty$. The theorem follows since bounded functions with compact support are dense in $L^{p(\cdot)}(\mathbb{R}^n)$ (See Corollary 2.73 in [2]). □

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