A GENERALIZED GELFAND PAIR ATTACHED TO A 3-STEP NILPOTENT LIE GROUP

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May 26, 2020

Abstract. Let N be a nilpotent Lie group and K a compact subgroup of the automorphism group $Aut(N)$ of N. It is well-known that if $(K \ltimes N, K)$ is a Gelfand pair then N is at most 2-step nilpotent Lie group.

The notion of Gelfand pair was generalized when K is a non-compact group. In this work, we give an example of a 3-step nilpotent Lie group and a non-compact subgroup K of $Aut(N)$ such that $(K \ltimes N, N)$ is a generalized Gelfand pair.

1. Introduction

Let G be a Lie group and K a compact subgroup of G. We denote by $\mathcal{D}(G/K)$ the space of C^∞-functions on G/K with compact support and by $\mathcal{D}_K(G)$ the subspace of $\mathcal{D}(G)$ of functions on G which are right K-invariant. Both spaces are identified by mapping $f \in \mathcal{D}(G/K)$ to $f_0 := f \circ \mathcal{P}$, where $\mathcal{P} : G \to G/K$ is the natural projection.

It follows from the Schwartz's kernel Theorem, that every linear operator which maps continously $\mathcal{D}(G/K)$ in $\mathcal{D}'(G/K)$ with respect to the standard topologies and commuting with the action of G is a convolution operator with a K-bi-invariant distribution in $\mathcal{D}'(G)$.

In particular, we consider the subalgebra of convolution operators which kernels are K-bi-invariant integrable functions on G. When this algebra is commutative, we can expect a kind of simultaneous "diagonalization" of all these operators. This motivated, in part, the study of Gelfand pairs and the corresponding spherical analysis. In this sense, we begin by introducing the concept of Gelfand pair. The following statements are equivalent:

i) The convolution algebra of K-bi-invariant integrable functions on G is commutative.

ii) For any irreducible unitary representation $(\pi, \mathcal{H})$ of G, the subspace $\mathcal{H}_K$ of vectors fixed by K is at most one dimensional.

When any of the above holds, we say that (G, K) is a Gelfand pair.

Very well studied examples of Gelfand pairs are provided by symmetric pairs of compact or non-compact types. More recent works have put attention on Gelfand pairs of the form $(K \ltimes N, K)$ where N is a nilpotent Lie group and K is a subgroup of the automorphism group $Aut(N)$ of N (see [1],[2],[3],[4],[7],[8],[12], among others).

2010 *Mathematics Subject Classification.* 43A80, 22E25.

Key words and phrases. generalizd Gelfand pairs, nilpotent Lie group.

The authors are partially supported by CONICET and SECYT-UNC.

One of the first results, proved in [2], stated that if $(K \ltimes N, K)$ (in short (K, N)) is a Gelfand pair then N is abelian or a 2-step nilpotent group.

The notion of Gelfand pair was extended to the case when K is *non-compact*. Observe that, in this case, the space of K-invariant integrable functions on G/K is trivial. Thus, one attempt is to generalize (ii). Seminal papers are due to J. Faraut [6] and E.G. Thomas [11], and there is a nice survey in [13]. First of all, we assume that G and K are *unimodular* groups.

Let $(\pi, \mathcal{H})$ be a unitary representation of G, and denote by $\mathcal{H}^\infty$ the space of $\mathcal{C}^\infty$-vectors, that is, $\mathcal{H}^\infty = \{v \in \mathcal{H} : g \mapsto \pi(g)v \in \mathcal{C}^\infty(G)\}$. $\mathcal{H}^\infty$ is a Fréchet space equipped with a natural Sobolev topology. Let $\mathcal{H}^{-\infty}$ be the antidual of $\mathcal{H}^\infty$, with the strong topology (uniform convergence on bounded sets of $\mathcal{H}^\infty$). This yields natural imbeddings

$$\mathcal{H}^\infty \subset \mathcal{H} \subset \mathcal{H}^{-\infty}.$$

We denote by π_∞ the restriction of π to $\mathcal{H}^\infty$, and for $g \in G$ define $\pi_{-\infty}(g)$ on $\mathcal{H}^{-\infty}$ by duality: for $\phi \in \mathcal{H}^{-\infty}, v \in \mathcal{H}^\infty$

$$\langle \pi_{-\infty}(g)\phi, v\rangle := \langle \phi, \pi_\infty(g)v\rangle.$$

The elements of $\mathcal{H}^{-\infty}$ are called *distribution vectors*.

We say that (G, K) is a *generalized Gelfand pair if for any irreducible unitary representation* $(\pi, \mathcal{H})$ of G *the space* $\mathcal{H}_K^{-\infty}$ *of distribution vectors fixed by* K is at most one dimensional.

In this work, we give an example of a 3-step nilpotent Lie group and a non-compact subgroup K of $Aut(N)$ such that (K, N) is a generalized Gelfand pair. This is stated in Theorem 3.2.

Acknowledgements: We are grateful to G. Ratcliff who let us know the article [10].

2. Preliminaries

We begin this section by recalling some known results about generalized Gelfand pairs.

When (G, K) is a Gelfand pair, there is a one-to-one correspondence between K-bi-invariant functions on G of positive type and equivalent classes of unitary representations having a cyclic vector fixed by K. Moreover, for a K-bi-invariant function ψ on G of positive type it holds a Bochner-Godement Theorem

$$\psi = \int_\Sigma \varphi \, d\mu\,(\varphi)$$

where Σ denotes the set of *extremal* K-bi-invariant functions on G of positive type (or *spherical functions of positive type*) and $d\mu$ is a Radon measure on Σ. This allows to see that any *spherical* representation of G decomposes multiplicity free.

There is an analogous result for a generalized Gelfand pair: Let $(\pi, \mathcal{H})$ be a unitary representation of G having a nonn zero distribution vector $\phi \in \mathcal{H}_K^{-\infty}$. Then for

$f \in \mathcal{D}(G)$, it is easy to see that $\pi_\infty(f)\phi \in \mathcal{H}^\infty$ and T_ϕ defined by

$$T_\phi(f) = \langle \phi, \pi_\infty(f)\phi \rangle,$$

is a K-bi-invariant distribution of positive type.

Conversely, let T be a K-bi-invariant distribution of positive type. For $f \in \mathcal{D}(G/K)$, let $f_0 = f \circ \mathcal{P}$. On $\mathcal{D}(G/K)$ let us consider the scalar product $\langle f, g \rangle = T_\phi(f_0^* * g_0)$, where $f_0^*(x) = \overline{f_0(x^{-1})}$, and denote by $\mathcal{N}$ the subspace of vectors of lenght zero. The Hilbert subspace $\mathcal{H}$ of $\mathcal{D}'(G/K)$ associated to T is the completion of $\mathcal{D}(G/K)/\mathcal{N}$, and an easy computation (using that a Hilbert space $\mathcal{H}$ is identified with its dual) shows that if $J^* : \mathcal{D}(G/K) \to \mathcal{D}(G/K)/\mathcal{N}$ is the natural projection and $J : \mathcal{H} \to \mathcal{D}(G/K)'$ is the dual map, then for $f \in \mathcal{D}(G/K)$, $(J \circ J^*)f = f_0 * T$.

T is called the *reproducing kernel* of $\mathcal{H}$.

Thus, we have the following result (for a detailed proof see [13]).

Theorem (A)**.** *There is a one-to-one correspondence between unitary representations of G having a cyclic distribution vector fixed by K and K-bi-invariant distributions of positive type in $\mathcal{D}'(G)$ (the corresponding representation is realized as an invariant Hilbert subspace of $\mathcal{D}'(G/K)$).*

Also the following results hold:

- Bochner-Godement's theorem: For every K-bi-invariant distribution T of positive type there exists a Radon measure on the set Σ of extremal K-bi-invariant distributions of positive type, such that $T = \int_\Sigma T_s d\mu(s)$;
- Every G-invariant Hilbert subspace of $\mathcal{D}'(G/K)$ decomposes multiplicity free.

Now, let us consider a unimodular Lie group H such that for any $(\gamma, \mathcal{V}) \in \widehat{H}$, $\gamma(f)$ is a trace class operator for all $f \in \mathcal{D}(H)$ (this property holds for a wide class of Lie groups such as nilpotent or semisimple Lie groups).

Let us consider the pair (G, K) where $G = H \times H$ and $K = diag(H \times H)$ which is naturally identified with H. Also G/K can be identified with H. Let us denote by $(\gamma^*, \mathcal{V}^*)$ the contragradient representation of $(\gamma, \mathcal{V}_\gamma)$. On the first hand, $\mathcal{V}^* \otimes \mathcal{V}$ is canonically isomorphic to the Hilbert subspace $\mathcal{H}_\gamma$ of $\mathcal{D}'(H)$ of distributions of the form $f * \chi_\gamma, f \in \mathcal{D}(H)$. On the other hand, $\gamma^* \otimes \gamma$ corresponds to the representation of $H \times H$ on $\mathcal{D}'(H)$ given by $(h_1, h_2) \mapsto L(h_1)R(h_2)$. Thus, χ_γ is the reproducing kernel of $\mathcal{H}_\gamma$ and clearly χ_γ is a distribution vector in $\mathcal{H}_\gamma^{-\infty}$ fixed by H.

The complete result, due to Mokni and Thomas in [9] yields an analogous of a Carcano criterion for Gelfand pair.

Theorem (B)**.** *Let $(\omega, \mathcal{W}), (\gamma, \mathcal{V})$ be unitary representations of H such that γ is irreducible. Then γ appears in the decomposition of ω into irreducible components if and only if $\gamma^* \otimes \omega$ has a distribution vector fixed by H as $(H \times H)$-module.*

Let N be a nilpotent Lie group and denote by $\widehat{N}$ the set of equivalent class of irreducible unitary representation of N. We describe $\widehat{N}$ according to Kirillov's theory. Let $\mathfrak{n}$ be the Lie algebra of N. The group N acts on $\mathfrak{n}$ by the adjoint action Ad, and N acts on $\mathfrak{n}^*$, the dual space of $\mathfrak{n}$, by the dual representation $Ad^*(n)\Lambda = \Lambda \circ Ad(n^{-1})$. Fixed a non trivial $\Lambda \in \mathfrak{n}^*$, let $O_\Lambda := \{Ad^*(n)\Lambda : n \in N\}$ be its coadjoint orbit.

From Kirillov's theory it follows that there is a correspondence between $\widehat{N}$ and the set of coadjoint orbits in $\mathfrak{n}^*$. Indeed, let

$$B_\Lambda(u,v) := \Lambda([u,v]),\ u,v \in \mathfrak{n}. \tag{2.1}$$

Let $\mathfrak{m}$ be a maximal isotropic subspace of $\mathfrak{n}$, and set $M = \exp(\mathfrak{m})$. Defining on M the character $\chi_\Lambda(\exp u) = e^{i\Lambda(u)}$, the irreducible representation of N corresponding to O_Λ is the induced representation $\rho_\Lambda := Ind_M^N(\chi_\Lambda)$.

Let K be a subgroup of $Aut(N)$. Given $k \in K$, $\Lambda \in \mathfrak{n}^*$ there is a new representation of N defined by $\rho_\Lambda^k(n) := \rho_\Lambda(k \cdot n)$. The stabilizer of ρ_Λ is $K_\Lambda := \{k \mid \rho_\Lambda \sim \rho_\Lambda^k\}$. For each $k \in K_\Lambda$, one can choose an intertwining operator $\omega_\Lambda(k)$ such that $\rho_\Lambda^k(n) = \omega_\Lambda(k)\rho_\Lambda(n)\omega_\Lambda(k^{-1})$ for all $n \in N$. The map $k \mapsto \omega_\Lambda(k)$ is a projective representation of K_Λ, i.e, $\omega_\Lambda(k_1k_2) = \sigma_\Lambda(k_1,k_2)\omega_\Lambda(k_1)\omega_\Lambda(k_2)$, with $|\sigma_\Lambda(k_1,k_2)| = 1$ for all $k_1, k_2 \in K_\Lambda$. The map ω_Λ is called the intertwining representation of ρ_Λ or metaplectic representation and σ_Λ the multiplier for the projective representation ω_Λ.

Here we shall consider a 3-step nilpotent Lie group introduced in [10] and a certain subgroup K of $Aut(N)$, such that

(i) for all $\Lambda \in \mathfrak{n}^*, K_\Lambda = K$,
(ii) ω_Λ is a true representation of K.

In this situation, Mackey theory asserts that for $\sigma \in \widehat{K}$,

$$\rho_{\sigma,\Lambda}(k,n) = \sigma(k) \otimes \omega_\Lambda(k)\rho_\Lambda(n)$$

is an irreducible representation of $K \ltimes N$ and by varying $\sigma \in \widehat{K}$ and $\rho_\Lambda \in \widehat{N}$ this construction exhausts $\widehat{K \ltimes N}$.

Notice that the representation $\rho_{\sigma,\Lambda|_K}$ coincides with $\sigma \otimes \omega_\Lambda$. Thus, the Theorem B implies that the irreducible representation $\sigma \otimes \omega_\Lambda\rho_\Lambda$ has a distribution vector fixed by K if and only if the dual representation σ^* of K appears in the decomposition into irreducible components of ω_Λ (see [9], Theorem 2.1).

As a consequence, we have the following result.

Theorem (C)**.** (K,N) *is a generalized Gelfand pair if and only if* ω_Λ *is multiplicity free.*

3. Example

The group N to be considered is the simplest case of the family introduced by G. Ratcliff in [10]. Let H_1 be the 3-dimensional Heisenberg group with Lie algebra $\mathfrak{h}_1$ whose coordinates are $(x,y,t) \in \mathbb{R}^3$ and Lie bracket defined by

$$[(x,y,t),(x',y',t')] = (0,0,xy' - yx').$$

Let S be the subgroup of $Sp(1) \subseteq Aut(H_1)$ consinting of the matrices

$$\mathbf{s} = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix},\ s \in \mathbb{R}.$$

Let us consider the action of S on H_1 given by

$$\mathbf{s} \cdot (x,y,t) = (x, sx + y, t).$$

This action gives rise to a semidirect product $N = S \ltimes H_1$ such that

$$(s,x,y,t)(s',x',y',t') = (s+s', x+x', sx'+y+y', t+t'+\frac{y'x-x'y}{2}). \tag{3.1}$$

Let $\mathfrak{s}$ be the Lie algebra of S. The Lie algebra $\mathfrak{n}$ associated to N is a 3-step nilpotent Lie algebra with coordinates (s,x,y,t), where $s \in \mathfrak{s}$, $x,y,t \in \mathbb{R}$, and product

$$[(s,x,y,t),(s',x',y',t')] = (0,0,sx'-s'x, xy'-x'y); \tag{3.2}$$

its one-dimensional center is $\mathfrak{c} = \{(0,0,0,t) \mid t \in \mathbb{R}\}$.

We denote by $Aut_0(N)$ the group of automorphisms of N acting on $\mathfrak{c}$ by the identity. Since the exponential map is the identity

$$Aut_0(N) = \{k \in GL(4,\mathbb{R}) : k([u,v]) = [k(u),k(v)] \text{ for all } u,v \in \mathfrak{n}, k\mid_{\mathfrak{c}} = I\}. \tag{3.3}$$

Let $\Phi \in Aut_0(N)$, and $\mathcal{B} = \{e_j\}_{j=1}^4$ be the cannonical basis of $\mathbb{R}^4$. According to (3.2) and (3.3), Φ must satisfy the following relationships

$$\Phi([e_1,e_2]) = \Phi(e_3); \quad \Phi([e_2,e_3]) = \Phi(e_4);$$

$$\Phi([e_1,e_3]) = \Phi([e_1,e_4]) = \Phi([e_2,e_4]) = \Phi([e_3,e_4]) = \Phi(0) = 0.$$

Thus, we obtain that

$$Aut_0(N) = \left\{ \begin{pmatrix} r & a & 0 & 0 \\ 0 & r^{-\frac{1}{2}} & 0 & 0 \\ d & b & r^{\frac{1}{2}} & 0 \\ e & c & -dr^{-\frac{1}{2}} & 1 \end{pmatrix} \ : \ r,a,b,c,d,e \in \mathbb{R} \text{ and } r \neq 0 \right\}.$$

We define

$$A := \left\{ \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & r^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & r^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ : \ r \in \mathbb{R} \right\} \text{ and}$$

$$M := \left\{ \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ d & b & 1 & 0 \\ e & c & -d & 1 \end{pmatrix} \ : \ a,b,c,d,e \in \mathbb{R} \right\}.$$

Then we have that A is acting by conjugation over M and $Aut_0(N) = A \ltimes M$. Writing the elements in M as 5-uplas (a,b,c,d,e), we have that the product is given by

$$(a,b,c,d,e)(a',b',c',d',e') = (a+a', b+b'+da', c+c'+ea'-db', d+d', e+e'-dd') \tag{3.4}$$

Moreover, M is isomorphic to $H \ltimes \mathbb{R}^3$, where

$$H := \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ d & 0 & 1 & 0 \\ e & 0 & -d & 1 \end{pmatrix} \ : \ d,e \in \mathbb{R} \right\}, \text{and}$$

$$M_0 := \left\{ \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & 1 & 0 \\ 0 & c & 0 & 1 \end{pmatrix} \ : \ a,b,c \in \mathbb{R} \right\} \simeq \mathbb{R}^3.$$

Indeed, the product on H is

$$(0,0,0,d,e)(0,0,0,d',e') = (0,0,0,d+d',e+e'-dd'),$$

and thus we can indentify H with the subgroup $\{(d,-d,e) \ : \ d,e \in \mathbb{R}\}$ of H_1. By considering the action of H over $\mathbb{R}^3$ given by

$$(d,e)\cdot(a',b',c') = (a',b'+da',c'+ea'-db'),$$

and using (3.4) we obtain that $M = H \ltimes M_0$.

The subgroup K of $Aut_0(N)$ that we will consider is

$$K = \left\{ k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & k_1 & 1 & 0 \\ 0 & k_2 & 0 & 1 \end{pmatrix} \ : \ k_1,k_2 \in \mathbb{R} \right\} \simeq \mathbb{R}^2. \tag{3.5}$$

Let K_1 (resp. K_2) be the subgroup of K whose elements have matrix entry $k_2 = 0$ (resp. $k_1 = 0$).

We denote by (α,μ,ν,λ) the elemnts of $\mathfrak{n}^*$. The pairing between $\mathfrak{n}$ and $\mathfrak{n}^*$ is given by

$$(\alpha,\mu,\nu,\lambda)[(s,x,y,t)] = \alpha s + \mu x + \nu y + \lambda t.$$

For $\Lambda \in \mathfrak{n}^*$, it is easy to see that $K_\Lambda = \{k \in K \mid k\cdot\Lambda \in O_\Lambda\}$ where O_Λ is the coadjoint orbit of Λ. Let $X_\Lambda \in \mathfrak{n}$ such that $\Lambda(Y) = \langle Y, X_\Lambda\rangle$ for all $Y \in \mathfrak{n}$, hence it follows that $k\cdot\Lambda(Y) = \langle Y, k^{-1^t}X_\Lambda\rangle$. So

$$K_\Lambda := \{k \in K \mid k^t \cdot X_\Lambda \in O_\Lambda\}, \tag{3.6}$$

where k^t denotes the transposed of k.

The *generic orbits* are those which correspond to the representations with non-zero Plancherel measure and were computed in [10]. They are parametrized by $\Lambda = (\alpha,0,0,\lambda)$ with $\lambda \neq 0$, and if $O_{\alpha,\lambda}$ is the coadjoint orbit of Λ, then

$$O_{\alpha,\lambda} = \{(\alpha - \frac{1}{2\lambda}\nu^2,\mu,\nu,\lambda) \mid \mu,\nu \in \mathbb{R}\}.$$

In the case of the non-generic orbits with $\Lambda = (\alpha,\mu,\nu,0)$, by the well-known equality $Ad\circ exp = exp\circ ad$, we obtain that $O_\Lambda = \{(\beta,\eta,\nu,0) \mid \beta,\eta \in \mathbb{R}\}$. Let $(0,0,\nu,0)$ be a representative of $O_{(\alpha,\mu,\nu,0)}$, and we denote $O_{(\alpha,\mu,\nu,0)}$ by O_ν.

We now compute explicitly the representation ρ_Λ corresponding to the orbits O_Λ for all $\Lambda \in \mathfrak{n}^*$. We denote ρ_Λ by $\rho_{\alpha,\lambda}$ in the case $\Lambda = (\alpha,0,0,\lambda)$ and by ρ_ν in the case $\Lambda = (0,0,\nu,0)$.

Fixed $\Lambda = (\alpha,0,0,\lambda)$ with $\lambda \neq 0$, the non-degenerate skew-symetric form associated is

$$\begin{aligned} B_\Lambda((s,x,y,t),(s',x',y',t')) &= (\alpha,0,0,\lambda)([(s,x,y,t),(s',x',y',t')]) \\ &= (\alpha,0,0,\lambda)(0,0,sx'-s'x,xy'-x'y) \end{aligned}$$

$$= \lambda(xy' - x'y).$$

Thus, a maximal isotropic subspace associated to Λ is given by

$$\mathfrak{M}_\Lambda = \{(s,0,y,t) \in \mathfrak{n} \mid s,y,t \in \mathbb{R}\}.$$

The character χ_Λ defined on $M_\Lambda = \exp(\mathfrak{M}_\Lambda)$ is $\chi_\Lambda(s,0,y,t) = e^{\Lambda(s,0,y,t)} = e^{i(\alpha s + \lambda t)}$, and

$$\rho_{\alpha,\lambda} = Ind_{M_\Lambda}^N(\chi_\Lambda).$$

We recall that the induced representation is the pair $(\rho_{\alpha,\lambda}, H_{\alpha,\lambda})$ where $H_{\alpha,\lambda}$ is the completion of

$$\{f \in C_c(N) \mid f(nm) = \chi_\Lambda(m^{-1})f(n) \text{ for all } m \in M_\Lambda, n \in N\},$$

with respect to the inner product $\langle f,g\rangle = \int_{N/M_\Lambda} f(u)\overline{g(u)}du$, and the action is given by the regular left translation, that is $(\rho_{\alpha,\lambda}(n)f)(n') = f(n^{-1}n')$, $n,n' \in N$. Notice that setting

$$(s,x,y,t) = (0,x,0,0)(s,0,y,t - \frac{xy}{2}),$$

we can identify $H_{\alpha,\lambda}$ with $L^2(\mathbb{R})$ via the map $(0,u,0,0) \mapsto u$.

Since

$$(s,x,y,t)^{-1} = (-s,-x,sx-y,-t),$$

for $f \in H_{\alpha,\lambda}$ we obtain

$$\begin{aligned}
[\rho_{\alpha,\lambda}(s,0,0,0)f](u) &= f((s,0,0,0)^{-1}(0,u,0,0)) \\
&= f((-s,0,0,0)(0,u,0,0)) \\
&= f(-s,u,-su,0) \\
&= f((0,u,0,0)(-s,0,-su,\frac{su^2}{2}) \\
&= \chi_\Lambda(s,0,su,-\frac{su^2}{2})f(0,u,0,0) \\
&= e^{s\alpha - \lambda\frac{su^2}{2}}f(u).
\end{aligned}$$

Analogously, we have

$$[\rho_{\alpha,\lambda}(s,0,0,0)f](u) = e^{is(\alpha - \frac{\lambda u^2}{2})}f(u),$$
$$[\rho_{\alpha,\lambda}(0,x,0,0)f](u) = f(u-x),$$
$$[\rho_{\alpha,\lambda}(0,0,y,0)f](u) = e^{-i\lambda uy}f(u),$$
$$[\rho_{\alpha,\lambda}(0,0,0,t)f](u) = e^{i\lambda t}f(u).$$

We observe that the representations $\rho_{\alpha,\lambda}$ with $\lambda \neq 0$, are extensions of irreducible representations of H_1.

We now describe the representations corresponding to non-generic orbits O_ν with $\nu \neq 0$. In this case, $B_\Lambda((s,x,y,t),(s',x',y',t')) = \nu(sx' - s'x)$, and a maximal isotropic subspace is again $\mathfrak{M}_\Lambda = \{(s,0,y,t) : s,y,t \in \mathbb{R}\}$. The character associated is $\chi_\Lambda(s,0,y,t) = e^{i\nu y}$. With similar computations to the above case, we obtain

$$[\rho_\nu(s,0,0,0)f](u) = e^{i\nu su}f(u),$$

$$[\rho_\nu(0,x,0,0)f](u) = f(u-x),$$
$$[\rho_\nu(0,0,y,0)f](u) = e^{i\nu y}f(u),$$
$$[\rho_\nu(0,0,0,t)f](u) = f(u).$$

By (3.6) we can easily see that $K_\Lambda = K$ for all $\Lambda \in \mathfrak{n}^*$. Thus, the metaplectic representation ω_Λ must satisfy

$$\rho^k_\Lambda(n)\omega_\Lambda(k) = \omega_\Lambda(k)\rho_\Lambda(n) \quad \text{for all } k \in K \text{ and } n \in N.$$

The subgroups K_1 and K_2 fix the elements $(s,0,0,0)$, $(0,0,y,0)$ and $(0,0,0,t)$ for all $s,y,t \in \mathbb{R}$. Then, we have to find an unitary operator ω_Λ on $L^2(\mathbb{R})$ such that

$$\rho^k_\Lambda(0,x,0,0)\omega_\Lambda(k) = \omega_\Lambda(k)\rho_\Lambda(0,x,0,0), \quad \forall x \in \mathbb{R}, k \in K \tag{3.7}$$

and

$$\rho_\Lambda(n)\omega_\Lambda(k) = \omega_\Lambda(k)\rho_\Lambda(n) \tag{3.8}$$

for $n = (s,0,0,0)$, $n = (0,0,y,0)$ and $n = (0,0,0,t)$, $k \in K$.

Proposition 3.1. *For all $\Lambda \in \mathfrak{n}^*$ the representation ω_Λ of K is multiplicity free.*

Proof. We denote by $\omega_{\alpha,\lambda}$ (resp. ω_ν) the metaplectic representation corresponding to $\Lambda = (\alpha,0,0,\lambda)$ (resp. $\Lambda = (0,0,\nu,0)$).

It is easy to see that given $k_1 \in K_1$, we have

$$k_1 \cdot (0,x,0,0) = (0,x,k_1x,0).$$

By writing $(0,x,k_1x,0) = (0,x,0,0)(0,0,k_1x,0)(0,0,-\frac{k_1x^2}{2})$, we get that

$$[\rho_{\alpha,\lambda}(0,x,k_1x,0)f](u) = e^{-i\lambda k_1 xu + i\lambda k_1 \frac{x^2}{2}} f(u-x),$$

and hence we define

$$[\omega_{\alpha,\lambda}(k_1,0)f](u) = e^{-i\lambda\frac{u^2}{2}k_1}f(u).$$

Also, given $k_2 \in K_2$,

$$k_2 \cdot (0,x,0,0) = (0,x,0,k_2x).$$

Then

$$[\rho_{\alpha,\lambda}(0,x,0,k_2x)f](u) = e^{i\lambda k_2 x}f(u-x),$$

and we set

$$[\omega_{\alpha,\lambda}(0,k_2)f](u) = e^{i\lambda k_2 u}f(u).$$

That is,

$$[\omega_{\alpha,\lambda}(k_1,k_2)f](u) = e^{-i\lambda\frac{u^2}{2}k_1 + i\lambda k_2 u}f(u).$$

The analysis to the non generic orbit is similar, and we obtain

$$[\rho_\nu(0,x,k_1x,0)f](u) = e^{i\nu k_1 x}f(u-x),$$
$$[\rho_\nu(0,x,0,k_2x)f](u) = f(u-x).$$

Then,

$$[\omega_\nu(k_1,0)f](u) = e^{i\nu k_1 u} f(u),$$
$$[\omega_\nu(0,k_2)f](u) = f(u).$$

That is,

$$[\omega_\nu(k_1,k_2)f](u) = e^{i\nu k_1 x} f(u).$$

It follows straightforward that (3.8) holds for all $\Lambda \in \mathfrak{n}^*$.

Thus we can conclude that the decomposition of $\omega_{\alpha,\lambda}$ on $L^2(\mathbb{R})$ is

$$L^2(\mathbb{R}) = \int_{\mathbb{R}} \chi_{-\lambda\frac{u^2}{2},\lambda u}\, du \tag{3.9}$$

where $\chi_{-\lambda\frac{u^2}{2},\lambda u}(k_1,k_2) = e^{i(-k_1\lambda\frac{u^2}{2}+\lambda u k_2)}$.

Analogously, the decomposition of ω_ν on $L^2(\mathbb{R})$ is

$$L^2(\mathbb{R}) = \int_{\mathbb{R}} \chi_{\nu u,0}\, du \tag{3.10}$$

where $\chi_{\nu u,0}(k_1,k_2) = e^{ik_1\nu u}$.

In the last case, $\Lambda \equiv 0$ thus $\mathfrak{m}_\Lambda = \mathfrak{n}$ and $\chi_\Lambda \equiv 1$. Hence ρ_Λ is the trivial representation. This case concludes the analysis of the metaplectic representation obtaining that ω_Λ is multiplicity free for all $\Lambda \in \mathfrak{n}^*$. □

Then, we obtain the following result.

Theorem 3.2. *Let $N = S \ltimes H_1$ and $K \subseteq Aut_0(N)$ defined in* (3.5)*. Then, the pair (K,N) is a generalized Gelfand pair.*

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