

Augmented Lagrangians quadratic growth and second-order sufficient optimality conditions

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ABSTRACT

It is well-known that the primal quadratic growth condition of the classical augmented Lagrangian around a local minimizer can be obtained under the second-order sufficient optimality condition. In this paper we show that those conditions are indeed equivalent. Moreover, we prove that the primal quadratic growth condition of the sharp augmented Lagrangian around a local minimizer is in fact equivalent to the weak second-order sufficient optimality condition. In addition, we present some secondary results involving the sharp augmented Lagrangian.

KEYWORDS

Augmented Lagrangian; sharp augmented Lagrangian; second-order sufficient optimality condition; weak second-order sufficient optimality condition

AMS CLASSIFICATION

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1. Introduction

The study of the augmented Lagrangian method is an important topic in the optimization community. Computational implementations such as LANCELOT or ALGENCAN are examples of the state-of-the-art software to solve large nonlinear constrained optimization problems. Theoretical aspects such as its global convergence [1] or its local analysis that does not depend on constraints qualifications [2], puts this method in a preferable position. Most of the existing literature deal with the classical augmented Lagrangian, also known as the Powell-Hestenes-Rockafellar augmented Lagrangian, or proximal Lagrangian [3]. In several convergence results using this Lagrangian, the penalty parameter is driving to infinity to achieve a suitable rate of convergence [4–6]. This behaviour may be related to the quadratic nature of the penalization in the construction of this Lagrangian. The same quadratic nature allows the existence of a duality gap for some problems. It is known that the duality gap can be avoided by using a nondifferentiable penalization, that produces the so-called sharp Lagrangian [3].

In this direction, we believe that by changing the classical by the sharp augmented Lagrangian the penalty parameter will remain bounded for a wide class of problems. Unfortunately, for the primal iteration we should solve a nonconvex nondifferentiable minimization problem. We can try to solve those problems inexactly, as in the standard case. But in order to guarantee convergence some stability result is needed.

Following the analysis of the local behaviour in [2] for the classical augmented Lagrangian, it can be seen that the primal quadratic growth condition of the augmented Lagrangian is the cornerstone of such analysis. In this case, quadratic growth follows from the second-order sufficient optimality condition [7].

The first question that arises is about the existence of a weaker hypothesis to guarantee the primal quadratic growth of the classical augmented Lagrangian. We show that the former is the weakest. Moreover, mimicking this reasoning we obtain its counterpart for the primal quadratic growth of the sharp augmented Lagrangian. In this case, it is equivalent to the weak second-order sufficient optimality condition.

In the sequel we shall use classical relations between weak second-order optimality condition and quadratic growth condition of the objective function [8]. Also, some calculus involving second order tangent sets [9] are used.

We stress that no constraint qualifications are needed for the equivalences between second-order optimality conditions and primal quadratic growth of augmented Lagrangians.

This paper is organized as follows: Section 2 introduces the problem, definitions and notation. Section 3 introduces an auxiliary problem to take advantage of the structure of the augmented Lagrangians. Section 4 links second-order conditions of the original and auxiliary problems. Section 5 presents the main results. Section 6 shows secondary results from the previous theory. Section 7 closes with some final remarks.

2. Framework and notations

Given $f : \mathbb{R}^n \mapsto \mathbb{R}$ and $g : \mathbb{R}^n \mapsto \mathbb{R}^m$ twice continuously differentiable, consider the following nonlinear program

$$\begin{aligned} & \text{minimize} && f(x) \\ & \text{subject to} && g(x) \in Q^\circ, \\ & && x \in \Omega, \end{aligned} \tag{1}$$

where Ω is a polyhedral set and $Q^\circ = \{\xi \in \mathbb{R}^m \mid \xi_i = 0, i = 1, \dots, l; \xi_i \leq 0, i = l + 1, \dots, m\}$. Note that Q° is the (negative) polar cone of the closed convex cone $Q = \mathbb{R}^l \times \mathbb{R}_+^{m-l}$.

As can be seen in [10, (1.2)] or [11, (3.16)], stationary points of problem (1) and the associated Lagrange multipliers (x, μ) , are characterized by the Karush–Kuhn–Tucker (KKT) system:

$$\begin{aligned} 0 & \in \frac{\partial L}{\partial x}(x, \mu) + \mathcal{N}_\Omega(x), \\ 0 & \in -g(x) + \mathcal{N}_Q(\mu), \end{aligned} \tag{2}$$

where $\mathcal{N}_\Omega(x)$ is the standard normal cone of Ω at x and $L : \Omega \times \mathbb{R}^m \mapsto \mathbb{R}$ is the Lagrangian function of problem (1) for nonlinear constraints, i.e.,

$$L(x, \mu) = f(x) + \langle \mu, g(x) \rangle,$$

where we use $\langle \cdot, \cdot \rangle$ to denote the Euclidean inner product. Let us denote by $\mathcal{M}(\bar{x})$ the set of Lagrange multipliers for problem (1) associated with \bar{x} , that is, $\mu \in \mathcal{M}(\bar{x})$ if and only if (\bar{x}, μ) satisfies (2). Clearly, $\mathcal{M}(\bar{x}) \neq \emptyset$ implies that $g(\bar{x}) \in Q^\circ$ and $\bar{x} \in \Omega$, i.e., \bar{x} is feasible for problem (1). According to [10, Definition 2.1] or [11, (3.20)], the critical cone associated to problem (1) at a stationary point \bar{x} is given by

$$\mathcal{C}(\bar{x}) = \{u \in \mathcal{T}_\Omega(\bar{x}) \mid \langle f'(\bar{x}), u \rangle \leq 0, g'(\bar{x})u \in \mathcal{T}_{Q^\circ}(g(\bar{x}))\}, \quad (3)$$

where $\mathcal{T}_\Omega(x)$ is the standard tangent cone of Ω at x . If $\mathcal{M}(\bar{x}) \neq \emptyset$ we can guarantee that \bar{x} is a local solution of (1) under some second order condition.

We say that the second order sufficient optimality condition (SOSC) holds at \bar{x} if

$$\exists \bar{\mu} \in \mathcal{M}(\bar{x}) \text{ s.t. } \forall u \in \mathcal{C}(\bar{x}) \setminus \{0\}, \quad \left\langle \frac{\partial^2 L}{\partial x^2}(\bar{x}, \bar{\mu})u, u \right\rangle > 0. \quad (4)$$

We say that the weak second order sufficient optimality condition (WSOSC) holds at \bar{x} if

$$\forall u \in \mathcal{C}(\bar{x}) \setminus \{0\}, \exists \bar{\mu} \in \mathcal{M}(\bar{x}) \text{ s.t. } \left\langle \frac{\partial^2 L}{\partial x^2}(\bar{x}, \bar{\mu})u, u \right\rangle > 0. \quad (5)$$

From [12, Theorem 7], [11, Theorem 3.63] or [3, Example 13.25] we know that the fulfillment of one of the previous second order optimality condition guarantee the quadratic growth condition for f at \bar{x} , i.e., $f(x) \geq f(\bar{x}) + \beta \|x - \bar{x}\|^2$ for some $\beta > 0$ and all feasible x close enough to \bar{x} . Clearly, the latter ensures that \bar{x} is an isolated local minimizer. Also, the quadratic growth condition is useless for an isolated feasible \bar{x} . Therefore, we are interested in the behaviour at a nonisolated feasible \bar{x} .

On the other hand, among the augmented Lagrangians associated with problem (1) existing in the literature, we will focused our analysis in two of them.

The first, that can be seen in [3, Example 11.57], is the classical augmented Lagrangian $\bar{L}_2 : \Omega \times \mathbb{R}^m \times (0, \infty) \mapsto \mathbb{R}$ such that

$$\bar{L}_2(x; \mu, r) = \inf_{p \in \mathbb{R}^m} \left\{ f(x) + I_{Q^\circ}(g(x) - p) + \langle \mu, p \rangle + \frac{r}{2} \|p\|^2 \right\},$$

where $\|\cdot\|$ is the Euclidean norm and I_{Q° is the indicator function for the set Q° . An equivalent expression, known as the Powell-Hestenes-Rockafellar augmented Lagrangian is

$$\bar{L}_2(x; \mu, r) = f(x) + \frac{1}{2r} \left(\|\Pi_Q(\mu + rg(x))\|^2 - \|\mu\|^2 \right) \quad (6)$$

$$= f(x) + \sum_{i=1}^l \mu_i g_i(x) + \frac{r}{2} g_i(x)^2 + \frac{1}{2r} \sum_{i=l+1}^m \max\{0, \mu_i + rg_i(x)\}^2 - \mu_i^2, \quad (7)$$

where Π_Q is the orthogonal projection onto Q . Since the function $x \mapsto \max\{0, x\}^2$ is continuously differentiable but not twice differentiable at $x = 0$, it can be seen that the function $x \mapsto \bar{L}_2(x; \mu, r)$ is continuously differentiable, but in presence of inequality constraints it may not be twice differentiable, as noticed in [7, Proposition 7.2].

The second, following [3, Example 11.58], is the sharp augmented Lagrangian $\bar{L}_1 : \Omega \times \mathbb{R}^m \times (0, \infty) \mapsto \mathbb{R}$ such that

$$\bar{L}_1(x; \mu, r) = \inf_{p \in \mathbb{R}^m} \{f(x) + I_{Q^\circ}(g(x) - p) + \langle \mu, p \rangle + r\|p\|\}. \quad (8)$$

An equivalent expression for equality constrained problems (i.e., $Q = \mathbb{R}^m$) is

$$\bar{L}_1(x; \mu, r) = f(x) + \langle \mu, g(x) \rangle + r\|g(x)\|.$$

When $\mu = 0$, the sharp augmented Lagrangian is the well-know nondifferentiable merit function

$$\bar{L}_1(x; 0, r) = f(x) + r\|\Pi_Q(g(x))\|.$$

Note that in general the function $x \mapsto \bar{L}_1(x; \mu, r)$ may not be differentiable.

For a unified analysis, let us define

$$\bar{L}_s(x; \mu, r) = \inf_{p \in \mathbb{R}^m} \left\{ f(x) + I_{Q^\circ}(g(x) - p) + \langle \mu, p \rangle + \frac{1}{s}r\|p\|^s \right\},$$

for $s \in \{1, 2\}$. We stress that, in this work, only two augmented Lagrangians are studied. The behaviour for any other parameter $s \in [1, 2]$ is not our concern.

3. Auxiliary Problem

In order to use second order derivatives to study the quadratic growth of the augmented Lagrangians we shall manage the nondifferentiability in a suitable way.

To this end, let $\mu \in \mathbb{R}^m$, $r \in (0, \infty)$, $s \in \{1, 2\}$ and consider the auxiliary problem

$$\begin{aligned} & \text{minimize} && \tilde{f}(z) \\ & \text{subject to} && \tilde{g}(z) \in K, \end{aligned} \quad (9)$$

where $z = (x, p, t)$,

$$\tilde{f}(x, p, t) = f(x) + \langle \mu, p \rangle + \frac{1}{s}rt^s, \quad \tilde{g}(x, p, t) = (g(x) - p, x, p, t),$$

and $K = Q^\circ \times \Omega \times S$ with $S = \{(p, t) \mid \|p\| \leq t\}$.

We will denote by \tilde{L} , $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{C}}$ the Lagrangian function, Lagrange multipliers set and critical cone associated to problem (9), respectively. Clearly, problem (9) depends on the parameter $(\mu, r, s) \in \mathbb{R}^m \times (0, \infty) \times \{1, 2\}$ but we omit it to have a clean notation.

This auxiliary problem appears naturally from the definition of \bar{L}_s , carrying the nondifferentiability into a second order cone constraint. The smoothness of the objective and constraint functions are the same as those of the original problem. Also, the study of the quadratic growth of the (possible nonsmooth) augmented Lagrangian \bar{L}_s is equivalent to the study of the quadratic growth of the objective function \tilde{f} . Another advantage of this auxiliary problem is the fulfillment of the Robinson constraint qualification, which is sufficient for the existence of Lagrange multipliers as shown in [13].

The existence of Lagrange multipliers under the Robinson constraint qualification can also be found in [11, Theorem 3.9].

Proposition 3.1. *The Robinson constraint qualification holds at any feasible point of problem (9).*

Proof. Let $\tilde{g}(z) \in K$. By [11, Corollary 2.98], it is enough to show that

$$\text{Ker}(\tilde{g}'(z)^\top) \cap \mathcal{N}_K(\tilde{g}(z)) = \{0\}.$$

Consider an arbitrary vector λ in this intersection. Since

$$\tilde{g}'(x, p, t) = \begin{bmatrix} g'(x) & -\mathbf{I} & 0 \\ \mathbf{I} & 0 & 0 \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \text{Ker}(\tilde{g}'(z)^\top)$ if and only if

$$0 = g'(x)^\top \lambda_1 + \lambda_2, \quad \lambda_1 = \lambda_3, \quad \lambda_4 = 0. \quad (10)$$

On the other hand, $\mathcal{N}_K(\tilde{g}(z)) = \mathcal{N}_{Q^\circ}(g(x) - p) \times \mathcal{N}_\Omega(x) \times \mathcal{N}_S(p, t)$ and $(\lambda_3, \lambda_4) \in \mathcal{N}_S(p, t)$ if and only if

$$0 = \langle p, \lambda_3 \rangle + t\lambda_4, \quad (\lambda_3, \lambda_4) \in S^\circ = -S.$$

Thus, by (10), $(\lambda_3, 0) \in S^\circ$, which implies that $\lambda_3 = 0$ and hence $\lambda_1 = 0$, obtaining from the first equation in (10) that $\lambda_2 = 0$. Concluding that $\lambda = 0$. \square

From [9, Theorem 4.2], we know that under the Robinson constraint qualification the quadratic growth condition of the objective function implies the WSOSC (5). This result is valid if Ω and Q° are polyhedral sets. For nonpolyhedral sets a support function is involved, as stated in the cited reference. Then, quadratic growth condition of the objective function is equivalent to WSOSC, under the Robinson constraint qualification. This equivalence can also be seen in [11, Theorem 3.86]. In the sequel, we will use the structure of the Lagrange multipliers set of the auxiliary problem that is shown below.

Proposition 3.2. *If $(\bar{x}, \bar{p}, \bar{t})$ is a stationary point of problem (9), then $\|\bar{p}\| = \bar{t}$ and we have the following associated Lagrange multipliers set:*

(1) if $(\bar{p}, \bar{t}) \neq (0, 0)$,

$$\widetilde{\mathcal{M}}(\bar{x}, \bar{p}, \bar{t}) = \left\{ \left(\nu, -\frac{\partial L}{\partial x}(\bar{x}, \nu), \nu - \mu, -r\bar{t}^{s-1} \right) \right\},$$

where $0 \in \frac{\partial L}{\partial x}(\bar{x}, \nu) + \mathcal{N}_\Omega(\bar{x})$ and $\|\nu - \mu\| = r\bar{t}^{s-1}$, with

$$\nu = \Pi_Q(\mu + r\bar{t}^{s-2}g(\bar{x})).$$

(2) If $(\bar{p}, \bar{t}) = (0, 0)$,

$$\widetilde{\mathcal{M}}(\bar{x}, 0, 0) = \left\{ (\nu, -\frac{\partial L}{\partial x}(x, \mu), \nu - \mu, -\delta_{1s}r) \mid \nu \in \mathcal{M}(\bar{x}) \cap B(\mu, \delta_{1s}r) \right\},$$

where δ_{ij} is the Kronecker delta and $B(\mu, r) = \{\nu \mid \|\nu - \mu\| \leq r\}$.

Proof. By Proposition 3.1 we obtain that $\widetilde{\mathcal{M}}(\bar{x}, \bar{p}, \bar{t}) \neq \emptyset$ and for any $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \widetilde{\mathcal{M}}(\bar{x}, \bar{p}, \bar{t})$ it holds that

$$0 = f'(\bar{x}) + g'(x)^\top \lambda_1 + \lambda_2, \quad (11)$$

$$0 = \mu - \lambda_1 + \lambda_3, \quad (12)$$

$$0 = r\bar{t}^{s-1} + \lambda_4. \quad (13)$$

where for $\bar{t} = 0$ we consider $\bar{t}^0 = 1$, and

$$\lambda_1 \in \mathcal{N}_{Q^\circ}(g(\bar{x}) - \bar{p}), \quad \lambda_2 \in \mathcal{N}_\Omega(\bar{x}), \quad (\lambda_3, \lambda_4) \in \mathcal{N}_S(\bar{p}, \bar{t}). \quad (14)$$

Taking $\nu = \lambda_1$, from (11)-(13) we have that

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \left(\nu, -(f'(\bar{x}) + g'(\bar{x})^\top \nu), \nu - \mu, -r\bar{t}^{s-1} \right).$$

Now, from (14) we obtain that $-(f'(\bar{x}) + g'(\bar{x})^\top \nu) \in \mathcal{N}_\Omega(\bar{x})$. Additionally, $(\nu - \mu, -r\bar{t}^{s-1}) \in \mathcal{N}_S(\bar{p}, \bar{t})$, or equivalently $S \ni (\bar{p}, \bar{t}) \perp (\nu - \mu, -r\bar{t}^{s-1}) \in S^\circ$. Thus

$$0 = \langle \bar{p}, \nu - \mu \rangle - \bar{t}^s r, \quad \|\nu - \mu\| \leq r\bar{t}^{s-1}.$$

Since $\|\bar{p}\| \leq \bar{t}$, because $(\bar{p}, \bar{t}) \in S$, we have that

$$r\bar{t}^s = \langle \bar{p}, \nu - \mu \rangle \leq \|\bar{p}\| \|\nu - \mu\| \leq r\bar{t}^s.$$

Thus, $\|\bar{p}\| = \bar{t}$. If $\bar{t} \neq 0$ we conclude that $\nu - \mu$ and \bar{p} are linearly dependent with $\nu - \mu = r\bar{t}^{s-2}\bar{p}$. Hence $\|\nu - \mu\| = r\bar{t}^{s-1}$. Moreover, from (14) and the fact that $\bar{p} = \bar{t}^{2-s}(\nu - \mu)/r$,

$$\begin{aligned} \nu \in \mathcal{N}_{Q^\circ} \left(g(\bar{x}) - \bar{t}^{2-s} \frac{1}{r} (\nu - \mu) \right) &\Leftrightarrow g(\bar{x}) - \bar{t}^{2-s} \frac{1}{r} (\nu - \mu) \in \mathcal{N}_Q(\nu) \\ &\Leftrightarrow \mu + r\bar{t}^{s-2} g(\bar{x}) - \nu \in \mathcal{N}_Q(\nu) \\ &\Leftrightarrow \nu = \Pi_Q(\mu + r\bar{t}^{s-2} g(\bar{x})). \end{aligned}$$

This shows all relations for the singleton in item 1.

For the set in item 2, if $\bar{t} = 0$ and $s = 1$ we have that $\|\nu - \mu\| \leq r$ with $-(f'(\bar{x}) + g'(\bar{x})^\top \nu) \in \mathcal{N}_\Omega(\bar{x})$ and $\nu \in \mathcal{N}_{Q^\circ}(g(\bar{x}))$, i.e., $\nu \in \mathcal{M}(\bar{x}) \cap B(\mu, r)$.

If $\bar{t} = 0$ and $s = 2$ we have that $\|\nu - \mu\| = 0$. Then $-(f'(\bar{x}) + g'(\bar{x})^\top \mu) \in \mathcal{N}_\Omega(\bar{x})$ and $\mu \in \mathcal{N}_{Q^\circ}(g(\bar{x}))$, i.e., $\mu \in \mathcal{M}(\bar{x})$. □

In order to study the second order optimality conditions for the auxiliary problem (9), we shall compute the necessary elements. First, note that taking $z = (x, p, t)$ and

$\tilde{L}(z, \lambda) = \tilde{f}(z) + \langle \lambda, \tilde{g}(z) \rangle$, the Hessian of the Lagrangian associated to problem (9) satisfy

$$\left\langle \frac{\partial^2 \tilde{L}}{\partial z^2}(\bar{z}, \lambda)v, v \right\rangle = \left\langle \frac{\partial^2 L}{\partial x^2}(\bar{x}, \lambda_1)u, u \right\rangle + \delta_{2s} r \tau^2, \quad (15)$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $v = (u, \omega, \tau)$.

The critical cone associated to problem (9) will be denoted by $\tilde{\mathcal{C}}$ or by $\tilde{\mathcal{C}}_{(\mu, r, s)}$ to emphasize its dependence on the parameter (μ, r, s) . Hence, at a stationary point $(\bar{x}, \bar{p}, \bar{t})$ is defined by

$$\tilde{\mathcal{C}}(\bar{x}, \bar{p}, \bar{t}) = \left\{ (u, \omega, \tau) \in \mathcal{T}_\Omega(\bar{x}) \times \mathcal{T}_S(\bar{p}, \bar{t}) \mid \begin{array}{l} \langle f'(\bar{x}), u \rangle + \langle \mu, \omega \rangle + r\bar{t}^{s-1}\tau \leq 0, \\ g'(\bar{x})u - \omega \in \mathcal{T}_{Q^\circ}(g(\bar{x}) - \bar{p}). \end{array} \right\}, \quad (16)$$

where, from [14, Lemma 25], for $(p, t) \in S$

$$\mathcal{T}_S(p, t) = \begin{cases} \mathbb{R}^n \times \mathbb{R}, & \|p\| < t, \\ S, & (p, t) = (0, 0), \\ \{(\omega, \tau) \mid \langle p, \omega \rangle - t\tau \leq 0\}, & \|p\| = t \neq 0. \end{cases}$$

Since S is not polyhedral, second order optimality conditions for the auxiliary problem (9), at a stationary point $\bar{z} = (\bar{x}, \bar{p}, \bar{t})$, involve the support function of the (outer) second order tangent set

$$\mathcal{T}_K^2(\tilde{g}(\bar{z}), \tilde{g}'(\bar{z})v) = \mathcal{T}_{Q^\circ}^2(g(\bar{x}), g'(\bar{x})u - \omega) \times \mathcal{T}_\Omega^2(\bar{x}, u) \times \mathcal{T}_S^2((\bar{p}, \bar{t}), (\omega, \tau)), \quad (17)$$

where $v = (u, \omega, \tau) \in \tilde{\mathcal{C}}(\bar{z})$ and, from [14, Lemma 27], for $(p, t) \in S$ and $(\omega, \tau) \in \mathcal{T}_S(p, t)$

$$\mathcal{T}_S^2((p, t), (\omega, \tau)) = \begin{cases} \mathbb{R}^n \times \mathbb{R}, & (\omega, \tau) \in \text{int}(\mathcal{T}_S(p, t)), \\ \mathcal{T}_S(\omega, \tau), & (p, t) = (0, 0), \\ \{(\vartheta, \varsigma) \mid \langle p, \vartheta \rangle - t\varsigma \leq \tau^2 - \|\omega\|^2\}, & \text{otherwise.} \end{cases}$$

The equality in (17) follows from [11, page 168] and the fact that Q° and Ω are polyhedral. For nonpolyhedral sets (17) may not hold. This polyhedrality also guarantee the convexity of the second order tangent set to Q° and Ω . By the definition of the second order tangent set to S , we conclude that the set in (17) is convex. Another important property, shown in [11, (3.109)], is that for all $\lambda \in \tilde{\mathcal{M}}(\bar{z})$ and $v \in \tilde{\mathcal{C}}(\bar{z})$

$$\sigma(\lambda, \mathcal{T}_K^2(\tilde{g}(\bar{z}), \tilde{g}'(\bar{z})v)) := \sup_{\zeta \in \mathcal{T}_K^2(\tilde{g}(\bar{z}), \tilde{g}'(\bar{z})v)} \langle \lambda, \zeta \rangle \leq 0. \quad (18)$$

4. A unified second-order condition

Taking minimum over the unit sphere, it can be seen that SOS (4) holds at \bar{x} for $\mu \in \mathcal{M}(\bar{x})$ if and only if there exists $\beta > 0$ such that

$$\left\langle \frac{\partial^2 L}{\partial x^2}(\bar{x}, \mu)u, u \right\rangle \geq \beta \|u\|^2 \quad \forall u \in \mathcal{C}(\bar{x}).$$

For a unified analysis, we will state a similar expression for the weak second order optimality condition.

Proposition 4.1. *The WSOSC (5) holds at \bar{x} if and only if for a given $\mu \in \mathbb{R}^m$ there exist $\hat{r} > 0$ and $\beta > 0$ such that*

$$\max_{\nu \in \mathcal{M}(\bar{x}) \cap B(\mu, \hat{r})} \left\langle \frac{\partial^2 L}{\partial x^2}(\bar{x}, \nu)u, u \right\rangle \geq \beta \|u\|^2 \quad \forall u \in \mathcal{C}(\bar{x}). \quad (19)$$

Proof. Clearly, if (19) holds then WSOSC (5) holds. So, we only must prove the other implication.

By contradiction, suppose that exist $r_k \rightarrow +\infty$ and $u^k \in \mathcal{C}(\bar{x})$ with $\|u^k\| = 1$ such that

$$\max_{\nu \in \mathcal{M}(\bar{x}) \cap B(\mu, r_k)} \left\langle \frac{\partial^2 L}{\partial x^2}(\bar{x}, \nu)u^k, u^k \right\rangle < \frac{1}{k}.$$

Since $\{u^k\}$ is bounded, taking subsequence if necessary, assume that $u^k \rightarrow \bar{u}$. Then $\bar{u} \in \mathcal{C}(\bar{x})$ with $\|\bar{u}\| = 1$. By WSOSC, there exists $\bar{\mu} \in \mathcal{M}(\bar{x})$ such that

$$\left\langle \frac{\partial^2 L}{\partial x^2}(\bar{x}, \bar{\mu})\bar{u}, \bar{u} \right\rangle > 0.$$

For k large enough we have that $\bar{\mu} \in \mathcal{M}(\bar{x}) \cap B(\mu, r_k)$. Then

$$\left\langle \frac{\partial^2 L}{\partial x^2}(\bar{x}, \bar{\mu})u^k, u^k \right\rangle < \frac{1}{k}.$$

Taking limits for $k \rightarrow \infty$ we obtain a contradiction. □

Hence, let us consider the following second order condition: there exist $\hat{r} > 0$ and $\beta > 0$ such that

$$\max_{\nu \in \mathcal{M}(\bar{x}) \cap B(\mu, \delta_{1s} \hat{r})} \left\langle \frac{\partial^2 L}{\partial x^2}(\bar{x}, \nu)u, u \right\rangle \geq \beta \|u\|^2 \quad \forall u \in \mathcal{C}(\bar{x}). \quad (20)$$

Clearly, for $s = 1$ we have WSOSC and for $s = 2$ we have SOSC at μ . Also note that this inequality only has meaning if $\mathcal{M}(\bar{x}) \neq \emptyset$.

Now, let us show the link between second order conditions for problem (1) and the auxiliary problem (9).

Lemma 4.2. *If the second order condition (20) holds, then there exist $r_1 > 0$ and $\beta_1 > 0$ such that for any $r \geq r_1$*

$$\max_{\nu \in \mathcal{M}(\bar{x}) \cap B(\mu, \delta_{1s} r)} \left\langle \frac{\partial^2 L}{\partial x^2}(\bar{x}, \nu)u, u \right\rangle + \delta_{2s} r \tau^2 \geq \beta_1 (\|u\|^2 + \|\omega\|^2 + \tau^2), \quad (21)$$

for all $(u, \omega, \tau) \in \tilde{\mathcal{C}}(\bar{x}, 0, 0)$.

Proof. Suppose that there exist $r_k \rightarrow +\infty$ and $(u^k, \omega^k, \tau_k) \in \tilde{\mathcal{C}}_{(\mu, r_k, s)}(\bar{x}, 0, 0)$ such that $\psi_{\hat{r}}(u^k) + \delta_{2s} r_k \tau_k^2 < \frac{1}{k} \eta_k^2$, where $\eta_k^2 = \|u^k\|^2 + \|r_k^{1/s} \omega^k\|^2 + (r_k^{1/s} \tau_k)^2$,

$$\psi_{\hat{r}}(u) = \max_{\nu \in \mathcal{M}(\bar{x}) \cap B(\mu, \delta_{1s} \hat{r})} \left\langle \frac{\partial^2 L}{\partial x^2}(\bar{x}, \nu) u, u \right\rangle,$$

and

$$\tilde{\mathcal{C}}_{(\mu, r_k, s)}(\bar{x}, 0, 0) = \left\{ (u, \omega, \tau) \in \mathcal{T}_{\Omega}(\bar{x}) \times S \mid \begin{array}{l} \langle f'(\bar{x}), u \rangle + \langle \mu, \omega \rangle + \delta_{1s} r_k \tau \leq 0, \\ g'(\bar{x})u - \omega \in \mathcal{T}_{Q^\circ}(g(\bar{x})). \end{array} \right\}.$$

Taking subsequence if necessary, assume that

$$\frac{1}{\eta_k} (u^k, r_k^{1/s} \omega^k, r_k^{1/s} \tau_k) \rightarrow (\bar{u}, \bar{\omega}, \bar{\tau}), \quad \text{with} \quad \|\bar{u}\|^2 + \|\bar{\omega}\|^2 + \bar{\tau}^2 = 1.$$

Dividing by η_k in the definition of the critical cone, we have

$$\begin{aligned} \frac{1}{\eta_k} u^k &\in \mathcal{T}_{\Omega}(\bar{x}), & \frac{r_k^{1/s}}{\eta_k} (\omega^k, \tau_k) &\in S, \\ \left\langle f'(\bar{x}), \frac{1}{\eta_k} u^k \right\rangle + \frac{1}{r_k^{1/s}} \left\langle \mu, \frac{1}{\eta_k} r_k^{1/s} \omega^k \right\rangle + \delta_{1s} \frac{1}{\eta_k} r_k \tau_k &\leq 0, \\ g'(\bar{x}) \frac{1}{\eta_k} u^k - \frac{1}{r_k^{1/s}} \frac{1}{\eta_k} r_k^{1/s} \omega^k &\in \mathcal{T}_{Q^\circ}(g(\bar{x})). \end{aligned}$$

Thus, taking limits, we obtain

$$(\bar{\omega}, \bar{\tau}) \in S, \tag{22}$$

and

$$\bar{u} \in \mathcal{T}_{\Omega}(\bar{x}), \tag{23}$$

$$\langle f'(\bar{x}), \bar{u} \rangle + \delta_{1s} \bar{\tau} \leq 0, \tag{24}$$

$$g'(\bar{x}) \bar{u} \in \mathcal{T}_{Q^\circ}(g(\bar{x})). \tag{25}$$

Since $\bar{\tau} \geq 0$, from (23)-(25) and (3), we conclude that $\bar{u} \in \mathcal{C}(\bar{x})$. Using the fact that $\psi_{\hat{r}}(tu) = t^2 \psi_{\hat{r}}(u)$, that the function $\psi_{\hat{r}}$ is lower semicontinuous (see [3, Proposition 1.26(a)]) and (20), we obtain

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} \frac{1}{k} \\ &\geq \liminf_{k \rightarrow \infty} \psi_{\hat{r}}\left(\frac{1}{\eta_k} u^k\right) + \delta_{2s} \frac{1}{\eta_k^2} (r_k^{1/2} \tau_k)^2 \\ &\geq \psi_{\hat{r}}(\bar{u}) + \delta_{2s} \bar{\tau}^2 \\ &\geq \beta \|\bar{u}\|^2 + \delta_{2s} \bar{\tau}^2. \end{aligned}$$

Hence, $\bar{u} = 0$. Thus, since $s \in \{1, 2\}$, from the previous inequality and (24) we have that either $\bar{\tau}^2 \leq 0$ or $\bar{\tau} \leq 0$. Hence $\bar{\tau} = 0$. Now, by (22), we obtain $\bar{\omega} = 0$. This contradicts the fact that $(\bar{u}, \bar{\omega}, \bar{\tau})$ is nonzero. Then, there exist positive constants \tilde{r}

and $\tilde{\gamma}$ such that

$$\psi_{\tilde{r}}(u) + \delta_{2s}\tilde{r}\tau^2 \geq \tilde{\gamma}(\|u\|^2 + \tilde{r}^{2/s}\|\omega\|^2 + \tilde{r}^{2/s}\tau^2),$$

for all $(u, \omega, \tau) \in \tilde{\mathcal{C}}_{(\mu, \tilde{r}, s)}(\bar{x}, 0, 0)$. Take $\beta_1 = \min\{1, \tilde{r}^{2/s}\}\tilde{\gamma}$ and $r_1 = \max\{\tilde{r}, \tilde{r}\}$. Thus, for any $r \geq r_1$ we have $\tilde{\mathcal{C}}(\bar{x}, 0, 0) = \tilde{\mathcal{C}}_{(\mu, r, s)}(\bar{x}, 0, 0) \subset \tilde{\mathcal{C}}_{(\mu, \tilde{r}, s)}(\bar{x}, 0, 0)$ and $B(\mu, \delta_{1s}\hat{r}) \subset B(\mu, \delta_{1s}r)$. Hence, $\psi_r(u) \geq \psi_{\tilde{r}}(u)$ and for all $(u, \omega, \tau) \in \tilde{\mathcal{C}}(\bar{x}, 0, 0)$,

$$\psi_r(u) + \delta_{2s}r\tau^2 \geq \psi_{\tilde{r}}(u) + \delta_{2s}\tilde{r}\tau^2 \geq \beta_1(\|u\|^2 + \|\omega\|^2 + \tau^2).$$

Concluding that (21) holds. \square

5. Necessary and sufficient conditions

Now, jointing subsidiary results, we obtain that second order optimality conditions guarantee quadratic growth condition of augmented Lagrangians.

Theorem 5.1. *Consider the problem (1).*

- (1) *If SOSC (4) holds at \bar{x} for $\mu \in \mathcal{M}(\bar{x})$, then there exist $r_2 > 0$ and $\beta_2 > 0$ such that for any $r \geq r_2$*

$$\bar{L}_2(x; \mu, r) \geq f(\bar{x}) + \beta_2\|x - \bar{x}\|^2,$$

for all $x \in \Omega$ in a neighborhood of \bar{x} .

- (2) *If WSOSC (5) holds at \bar{x} , then for a given $\mu \in \mathbb{R}^m$ there exist $r_2 > 0$ and $\beta_2 > 0$ such that for any $r \geq r_2$*

$$\bar{L}_1(x; \mu, r) \geq f(\bar{x}) + \beta_2\|x - \bar{x}\|^2,$$

for all $x \in \Omega$ in a neighborhood of \bar{x} .

Proof. If SOSC or WSOSC hold at \bar{x} we have that (20) holds for $s = 2$ or $s = 1$, respectively. From Lemma 4.2 there exist $r_1 > 0$ and $\beta_1 > 0$ such that (21) holds. By Proposition 3.2 item 2 and (15), taking $r \geq r_1$ and $\bar{z} = (\bar{x}, 0, 0)$ we obtain

$$\max_{\lambda \in \tilde{\mathcal{M}}(\bar{z})} \left\langle \frac{\partial^2 \tilde{L}}{\partial z^2}(\bar{z}, \lambda)v, v \right\rangle \geq \beta_1\|v\|^2,$$

for all $v \in \tilde{\mathcal{C}}(\bar{z})$. Thus, from [11, Theorem 3.63], we obtain the following quadratic growth condition:

$$\tilde{f}(x, p, t) \geq \tilde{f}(\bar{x}, 0, 0) + \beta_2(\|x - \bar{x}\|^2 + \|p\|^2 + t^2),$$

for all (x, p, t) feasible for the auxiliary problem (9) such that $\|x - \bar{x}\| + \|p\| + t \leq \varepsilon$ with $\varepsilon > 0$ and $\beta_2 > 0$. Thus, for $\varepsilon_2 = \varepsilon/3$ and $r \geq r_1$, we have

$$f(x) + \langle \mu, p \rangle + \frac{1}{s}r\|p\|^s = \tilde{f}(x, p, \|p\|) \geq f(\bar{x}) + \beta_2\|x - \bar{x}\|^2, \quad (26)$$

for all $x \in \Omega \cap B(\bar{x}, \varepsilon_2)$ and $p \in B(0, \varepsilon_2)$ such that $g(x) - p \in Q^\circ$.

Let

$$\alpha = \min_{x \in \Omega \cap B(\bar{x}, \varepsilon_2)} \bar{L}_s(x; \mu, r_1) - \beta_2 \|x - \bar{x}\|^2.$$

If $\alpha \geq f(\bar{x})$, then $\bar{L}_s(x; \mu, r_1) \geq f(\bar{x}) + \beta_2 \|x - \bar{x}\|^2$ for all $x \in \Omega \cap B(\bar{x}, \varepsilon_2)$. If $\alpha < f(\bar{x})$, take

$$r_2 = r_1 + \frac{s}{\varepsilon_2^s} (f(\bar{x}) - \alpha).$$

Fix $x \in \Omega \cap B(\bar{x}, \varepsilon_2)$ and consider p such that $g(x) - p \in Q^\circ$. If $\|p\| > \varepsilon_2$ we have

$$\begin{aligned} f(x) + \langle \mu, p \rangle + \frac{1}{s} r_2 \|p\|^s &\geq \bar{L}_s(x; \mu, r_1) + \frac{r_2 - r_1}{s} \|p\|^s \\ &> \alpha + \beta_2 \|x - \bar{x}\|^2 + \frac{r_2 - r_1}{s} \varepsilon_2^s \\ &= f(\bar{x}) + \beta_2 \|x - \bar{x}\|^2. \end{aligned}$$

Using (26) for $\|p\| \leq \varepsilon_2$, we conclude that

$$f(x) + \langle \mu, p \rangle + \frac{1}{s} r_2 \|p\|^s \geq f(\bar{x}) + \beta_2 \|x - \bar{x}\|^2,$$

for all p such that $g(x) - p \in Q^\circ$. Taking infimum over p we have

$$\bar{L}_s(x; \mu, r_2) \geq f(\bar{x}) + \beta_2 \|x - \bar{x}\|^2,$$

for all $x \in \Omega \cap B(\bar{x}, \varepsilon_2)$. The result follows from the fact that the augmented Lagrangian is nondecreasing in r . \square

In the previous proof the extension for inequality (26) for a feasible p in a neighborhood to any feasible p , follows the lines in the proof of [3, Theorem 11.61].

The result in the first item is well-known from the literature, see for instance [7, Theorem 7.4(b)].

Let us show that quadratic growth condition of augmented Lagrangians are sufficient for the second order optimality conditions.

Theorem 5.2. *Let $\mu \in \mathbb{R}^m$ and $r > 0$.*

(1) *If*

$$\bar{L}_2(x; \mu, r) \geq f(\bar{x}) + \gamma \|x - \bar{x}\|^2,$$

for some $\gamma > 0$ and all $x \in \Omega$ in a neighborhood of \bar{x} , then $\mu \in \mathcal{M}(\bar{x})$ and SOSC

(4) holds at \bar{x} for μ .

(2) *If*

$$\bar{L}_1(x; \mu, r) \geq f(\bar{x}) + \gamma \|x - \bar{x}\|^2,$$

for some $\gamma > 0$ and all $x \in \Omega$ in a neighborhood of \bar{x} , then $\mathcal{M}(\bar{x}) \cap B(\mu, r) \neq \emptyset$ and WSOSC (5) holds at \bar{x} .

Proof. For any (x, p, t) feasible for problem (9) and close enough to $(\bar{x}, 0, 0)$ we have

$$\tilde{f}(x, p, t) \geq \bar{L}_s(x; \mu, r) \geq \tilde{f}(\bar{x}, 0, 0) + \gamma \|x - \bar{x}\|^2.$$

Hence, from Proposition 3.2 we have that $\mathcal{M}(\bar{x}) \cap B(\mu, \delta_{1s}r) \neq \emptyset$.

Now, taking $\bar{z} = (\bar{x}, 0, 0)$, we have that \bar{z} is a local minimizer for problem (9) with objective function $(x, p, t) \mapsto \tilde{f}(x, p, t) - \gamma \|x - \bar{x}\|^2$. From [11, Theorem 3.45] we have that for every $v = (u, \omega, \tau) \in \tilde{\mathcal{C}}(\bar{z})$

$$\max_{\lambda \in \tilde{\mathcal{M}}(\bar{z})} \left\{ \left\langle \frac{\partial^2 \tilde{L}}{\partial z^2}(\bar{z}, \lambda)v, v \right\rangle - 2\gamma \|u\|^2 - \sigma(\lambda, \mathcal{T}_K^2(\tilde{g}(\bar{z}), \tilde{g}'(\bar{z})v)) \right\} \geq 0.$$

Since Q° and Ω are polyhedral and $0 \in \mathcal{T}_S^2((0, 0), (\omega, \tau))$, from (17) we have that $0 \in \mathcal{T}_K^2(\tilde{g}(\bar{z}), \tilde{g}'(\bar{z})v)$. Thus, from (18) we obtain that

$$\sigma(\lambda, \mathcal{T}_K^2(\tilde{g}(\bar{z}), \tilde{g}'(\bar{z})v)) = 0.$$

Hence, from Proposition 3.2 item 2 and (15) we conclude that

$$\max_{\nu \in \mathcal{M}(\bar{x}) \cap B(\mu, \delta_{1s}r)} \left\langle \frac{\partial^2 L}{\partial x^2}(\bar{x}, \nu)u, u \right\rangle + \delta_{2s}r\tau^2 \geq 2\gamma \|u\|^2,$$

for all $(u, \omega, \tau) \in \tilde{\mathcal{C}}(\bar{x}, 0, 0)$. From (3) and (16), we have that if $u \in \mathcal{C}(\bar{x})$ then $(u, 0, 0) \in \tilde{\mathcal{C}}(\bar{x}, 0, 0)$. Thus, from the previous inequality, we conclude that

$$\max_{\nu \in \mathcal{M}(\bar{x}) \cap B(\mu, \delta_{1s}r)} \left\langle \frac{\partial^2 L}{\partial x^2}(\bar{x}, \nu)u, u \right\rangle \geq 2\gamma \|u\|^2,$$

for all $u \in \mathcal{C}(\bar{x})$. That is, SOSC holds for $s = 2$ and WSOSC holds for $s = 1$. \square

6. Other consequences

We mention some improvements that can be obtained from the previous results.

6.1. Equivalent expression for the sharp Lagrangian

From [3, Example 11.57] we know that the classical augmented Lagrangian can be written as

$$\bar{L}_2(x; \mu, r) = f(x) + \frac{1}{2r} \left(\|\Pi_Q(\mu + rg(x))\|^2 - \|\mu\|^2 \right).$$

This is a continuously differentiable function in the primal variable, which is an important property for the implementation of the augmented Lagrangian method. In this method the minimization of $x \mapsto \bar{L}_2(x; \mu, r)$ over Ω is performed at each iteration, and continuous differentiability is the lowest degree of smoothness required in most computational methods to solve this problem. It can be seen that the expressions obtained in Proposition 3.2 for the Lagrange multipliers of the auxiliary problem are the key to obtain the previous expression and the following result.

Proposition 6.1. For any $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$ and $r > 0$, it holds that

$$\bar{L}_1(x; \mu, r) = \inf_{t>0} \left\{ \bar{L}_2(x; \mu, \frac{r}{t}) + \frac{r}{2}t \right\} \quad (27)$$

Moreover, $\bar{L}_1(x; \mu, r) = \inf_{t>0} \{f(x) + \phi(t)\}$ for a function ϕ continuously differentiable on $(0, +\infty)$ with derivative

$$\phi'(t) = \frac{r}{2} - \frac{1}{2r} \left\| \Pi_Q \left(\mu + \frac{r}{t}g(x) \right) - \mu \right\|^2.$$

Proof. For $t > 0$, define

$$\phi(t) = \frac{t}{2r} (\|\nu\|^2 - \|\mu\|^2) + \frac{r}{2}t,$$

where

$$\nu = \Pi_Q \left(\mu + \frac{r}{t}g(x) \right).$$

Note that $\bar{L}_2(x; \mu, \frac{r}{t}) + \frac{r}{2}t = f(x) + \phi(t)$.

Let p satisfies $g(x) - p \in Q^\circ$. Take

$$\nu_p = \mu + \frac{r}{t}p,$$

for $t > 0$. Since $\mu + rt^{-1}g(x) - \nu_p = rt^{-1}(g(x) - p) \in Q^\circ$, we have

$$\|\nu\| = \|\nu - \Pi_Q(\mu + rt^{-1}g(x) - \nu_p)\| \leq \|\nu_p\|.$$

Then, if $p \neq 0$ taking $t = \|p\|$ we obtain

$$\begin{aligned} \langle \mu, p \rangle + r\|p\| &= \frac{t}{r} \langle \mu, \nu_p - \mu \rangle + rt \\ &= \frac{t}{2r} (\|\nu_p\|^2 - \|\mu\|^2 - \|\nu_p - \mu\|^2) + rt \\ &= \frac{t}{2r} (\|\nu_p\|^2 - \|\mu\|^2 - r^2) + rt \\ &= \frac{t}{2r} (\|\nu_p\|^2 - \|\mu\|^2) + \frac{r}{2}t \\ &\geq \frac{t}{2r} (\|\nu\|^2 - \|\mu\|^2) + \frac{r}{2}t \\ &\geq \inf_{\tau>0} \phi(\tau). \end{aligned}$$

If $p = 0$, for any $t > 0$ we have $\|\nu\| \leq \|\nu_p\| = \|\mu\|$. Then, $\phi(t) \leq \frac{r}{2}t$ and

$$\langle \mu, p \rangle + r\|p\| = 0 = \inf_{t>0} \frac{r}{2}t \geq \inf_{t>0} \phi(t).$$

Thus, adding $f(x)$ and taking infimum over p we have

$$\bar{L}_1(x; \mu, r) \geq \inf_{t>0} f(x) + \phi(t).$$

On the other hand, from the definition of ν we have $\mu + \frac{r}{t}g(x) - \nu \in Q^\circ$. Then

$g(x) - \frac{t}{r}(\nu - \mu) \in Q^\circ$. Taking $p = \frac{t}{r}(\nu - \mu)$ we have

$$\begin{aligned} \langle \mu, p \rangle + r\|p\| &= \frac{t}{r} \langle \mu, \nu - \mu \rangle + t\|\nu - \mu\| \\ &= \frac{t}{2r} (2\langle \mu, \nu - \mu \rangle + \|\nu - \mu\|^2) - \frac{t}{2r} \|\nu - \mu\|^2 + t\|\nu - \mu\| \\ &= \frac{t}{2r} (\|\nu\|^2 - \|\mu\|^2) + \frac{r}{2}t - \frac{t}{2r} (\|\nu - \mu\| - r)^2 \\ &\leq \phi(t). \end{aligned}$$

Thus $\bar{L}_1(x; \mu, r) \leq f(x) + \phi(t)$ for any $t > 0$.

Since Q is a cone, the function $y \rightarrow \frac{1}{2}\|\Pi_Q(y)\|^2$ is continuously differentiable with derivative $y \rightarrow \Pi_Q(y)$. Hence, ϕ is continuously differentiable on $(0, +\infty)$ with derivative

$$\begin{aligned} \phi'(t) &= \frac{1}{2r} (\|\nu\|^2 - \|\mu\|^2) - \frac{1}{r} \left\langle \nu, \frac{r}{t}g(x) \right\rangle + \frac{r}{2} \\ &= \frac{1}{2r} (\|\nu\|^2 - \|\mu\|^2) - \frac{1}{r} \|\nu\|^2 + \frac{1}{r} \langle \nu, \mu \rangle + \frac{r}{2} \\ &= -\frac{1}{2r} (\|\nu\|^2 - 2\langle \nu, \mu \rangle + \|\mu\|^2) + \frac{r}{2} \\ &= -\frac{1}{2r} \|\nu - \mu\|^2 + \frac{r}{2}, \end{aligned}$$

where we use that, since Q is a cone, $\|\nu\|^2 = \langle \nu, \mu + \frac{r}{t}g(x) \rangle$. \square

Note that from the equivalent expression, the sharp augmented Lagrangian can be thought as the classical augmented Lagrangian with a suitable scaled penalty parameter r/t . We should stress that, in contrast to the standard definition of the sharp augmented Lagrangian (8), the formulation in (27) is the minimization of the function ϕ which is continuously differentiable. This new reformulation may be useful for practical implementations of the method.

6.2. Second order for nondifferentiable merit function

In the study of merit functions for problem (1), the function

$$\theta_r(x) = f(x) + r\|\Pi_Q(g(x))\|_P,$$

is the classical example of an exact penalty function. That is, \bar{x} is a local minimizer of problem (1) if \bar{x} is a (feasible) local minimizer of θ_r . Popular choices for $\|\cdot\|_P$ are the ℓ_1 and ℓ_∞ norms. Thus, involving second order information, necessary and sufficient conditions for the exactness are the following.

Proposition 6.2. *Consider problem (1) and let β_l and β_u be positive constants such that $\beta_l\|\nu\| \leq \|\nu\|_P \leq \beta_u\|\nu\|$ for all $\nu \in \mathbb{R}^m$. Then,*

- (1) *If \bar{x} is a local minimizer of θ_r over Ω , with \bar{x} feasible for problem (1), then given $u \in \mathcal{C}(\bar{x})$ there exists $\hat{\nu} \in \mathcal{M}(\bar{x})$ with $\|\hat{\nu}\| \leq \beta_u r$ such that*

$$\left\langle \frac{\partial^2 L}{\partial x^2}(\bar{x}, \hat{\nu})u, u \right\rangle \geq 0.$$

(2) If \bar{x} is a local minimizer of problem (1) and WSOSC (5) holds at \bar{x} , then there exist $\bar{r} > 0$ and $\gamma > 0$ such that

$$\theta_r(x) \geq \theta_r(\bar{x}) + \gamma\|x - \bar{x}\|^2,$$

for any $r \geq \bar{r}$ and all $x \in \Omega$ in a neighborhood of \bar{x} .

Proof. The proof of item 1 follows from the proof of Theorem 5.2 with $\gamma = 0$ and the fact that for x in a neighborhood of \bar{x} ,

$$\bar{L}_1(x; 0, \beta_u r) \geq \theta_r(x) \geq \theta_r(\bar{x}) = f(\bar{x}).$$

Item 2 is a direct consequence of Theorem 5.1 taking $\bar{r} = r_2/\beta_l$, $\gamma = \beta_2$ and the fact that for $r \geq \bar{r}$ and $x \in \Omega$ in a neighborhood of \bar{x} we have

$$\theta_r(x) \geq \theta_{\bar{r}}(x) \geq \bar{L}_1(x; 0, \beta_l \bar{r}) \geq f(\bar{x}) + \gamma\|x - \bar{x}\|^2 = \theta_r(\bar{x}) + \gamma\|x - \bar{x}\|^2.$$

□

We stress that although the function θ_r is nondifferentiable we obtain a second order necessary optimality condition. Additionally, sufficient conditions in the literature using WSOSC, for example [11, Theorem 3.113], guarantee the quadratic growth of θ_r if $r > \|\nu\|$ for all ν in a compact subset of $\mathcal{M}(\bar{x})$. We obtain that the quadratic growth of θ_r holds for any r such that (19) holds taking maximum over the set $\mathcal{M}(\bar{x}) \cap B(0, r)$.

6.3. Quadratic growth of the objective function

As mentioned before, from [11, Theorem 3.86] we have that under the Robinson constraint qualification at \bar{x} , the WSOSC (5) is equivalent to the quadratic growth of the objective function, i.e., there exists $\beta > 0$ such that

$$f(x) \geq f(\bar{x}) + \beta\|x - \bar{x}\|^2, \tag{28}$$

for all $x \in D$ in a neighborhood of \bar{x} , where $D = \{x \in \Omega \mid g(x) \in Q^\circ\}$. Moreover, the equivalence is valid under any constraint qualification satisfying the following error bound:

$$\text{dist}(x, D) \leq \kappa \text{dist}(g(x), Q^\circ), \tag{29}$$

for $\kappa > 0$ and all $x \in \Omega$ in a neighborhood of \bar{x} . An example of such constraint qualification, different from Robinson, is the constant rank of the subspace component (CRSC) that can be found in [15].

We can see that our result subsumes this equivalence. First, note that for $x \in D$,

$$\bar{L}_1(x; \mu, r) \leq f(x).$$

Then, by Theorem 5.1, WSOSC at \bar{x} implies (28). For the converse, we need the following auxiliary result [11, Proposition 3.111] that we rewrite for completeness.

Proposition 6.3. *If conditions (28) and (29) hold, then there exists $r > 0$ such that*

$$\bar{L}_1(x; 0, r) \geq f(\bar{x}) + \frac{\beta}{2}\|x - \bar{x}\|^2,$$

for all $x \in \Omega$ in a neighborhood of \bar{x} .

Proof. Let $L > 0$ be the Lipschitz constant for f in $B(\bar{x}, 2\varepsilon)$ with $\varepsilon > 0$. Take $x \in \Omega \cap B(\bar{x}, \varepsilon)$ and let $\hat{x} \in D$ satisfies $\|x - \hat{x}\| = \text{dist}(x, D)$. Thus, $\hat{x} \in B(x, \varepsilon) \subset B(\bar{x}, 2\varepsilon)$ and

$$\begin{aligned} \frac{\beta}{2}\|x - \bar{x}\|^2 &\leq \beta\|x - \hat{x}\|^2 + \beta\|\hat{x} - \bar{x}\|^2 \\ &\leq \beta\|x - \hat{x}\|^2 + f(\hat{x}) - f(\bar{x}) \\ &\leq \beta\|x - \hat{x}\|^2 + f(x) - f(\bar{x}) + L\|\hat{x} - x\| \\ &\leq f(x) - f(\bar{x}) + (\varepsilon\beta + L)\|x - \hat{x}\| \\ &\leq f(x) - f(\bar{x}) + (\varepsilon\beta + L)\kappa \text{dist}(g(x), Q^\circ) \\ &= f(x) - f(\bar{x}) + (\varepsilon\beta + L)\kappa \|\Pi_Q(g(x))\| \\ &= \bar{L}_1(x; 0, r) - f(\bar{x}), \end{aligned}$$

where $r = (\varepsilon\beta + L)\kappa$. □

Then, by Theorem 5.2 and the previous Proposition, (29) and (28) imply that WSOSC holds at \bar{x} .

7. Concluding Remarks

We characterize the weak second order sufficient optimality condition through the primal quadratic growth condition of the sharp augmented Lagrangian function. Also, we characterize the second order sufficient optimality condition through the primal quadratic growth condition of the classical augmented Lagrangian function. As a consequence of this analysis, we provide a new expression for the sharp augmented Lagrangian function as an infimum of a continuously differentiable function. Besides that, we improve some results about local behaviour of the nondifferentiable penalty function.

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