Nonlinear Analysis: Hybrid Systems 7 (2013) 28–38

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis: Hybrid Systems

journal homepage: www.elsevier.com/locate/nahs

On controller-driven varying-sampling-rate stabilization via Lie-algebraic solvability[☆]

Hernan Haimovich[*], Esteban N. Osella

CONICET and Departamento de Control, Escuela de Ing. Electrónica, Universidad Nacional de Rosario, Riobamba 250 bis, Rosario (2000), Argentina

ARTICLE INFO

Keywords:
Varying sampling
Stability
Switched systems
Solvable Lie algebras
Simultaneous triangularization
Networked control systems

ABSTRACT

Control systems involving shared communication networks are becoming ubiquitous. The inclusion of a communication network within a feedback loop imposes new control challenges. We consider a setting where a centralized controller/scheduler is in charge of the control of several processes and also of administering access to the shared communication network. In this setting, the controller may perform on-line variations of the sampling rate of all processes in order to accommodate for new processes requiring access to the network and to maximize performance when processes finish operation. We refer to this setting as controller-driven varying-sampling-rate (VSR). We regard a continuous-time system sampled at varying rates as a discrete-time switched system (DTSS), and aim at devising sampling-rate dependent feedback to ensure stability irrespective of the way in which the sampling rate is varied. Our feedback design strategy is based on Lie-algebraic solvability. The current paper presents two main contributions: (a) it demonstrates that control design based on Lie-algebraic solvability is much less restrictive when applied to the controller-driven VSR setting than when applied to DTSSs of arbitrary form, and (b) we give sufficient conditions for the stabilizability of the VSR-DTSS by means of the Lie-algebraic-solvability condition. As opposed to previous results, these sufficient conditions do not impose a restriction on the number of subsystems of the DTSS.

1. Introduction

Networked Control Systems (NCSs) are control systems that include some form of communication network within the feedback loop. NCSs offer many advantages, including cost efficiency, low maintenance cost, and high flexibility, but also introduce new design challenges due to the problems associated with the inclusion of a communication network. These problems may be related with information loss, time-varying delays and communication constraints. See the special issues [1,2]. A standard feature of NCSs is that component elements (sensors, actuators, controllers) are spatially distributed and may operate in an asynchronous manner.

In this paper, we consider a setting where the feedback loops of interest involve some form of shared communication network for increased flexibility and reduced cost. However, as opposed to the general NCS setting, we address the case where a centralized controller/scheduler exists that not only controls all processes sharing the same communication network but also is in charge of administering access and communication over the network. The centralized controller/scheduler and all processes operate synchronously, but the sampling rate of each control process may be varied on-line by the centralized controller. These sampling-rate variations may be needed in order to maximize performance

☆ Work partially supported by ANPCyT grant PICT 2010-0783, Argentina.

* Corresponding author.
E-mail addresses: haimo@fceia.unr.edu.ar (H. Haimovich), osella.esteban@gmail.com (E.N. Osella).

1751-570X/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.nahs.2012.04.001

and prevent information loss as new control tasks may require access and control over the network (and hence sampling rates may need to be reduced to accommodate the new tasks), or running control tasks may finish operation (and hence the sampling rates of the continuing tasks can be increased). We refer to this setting as *controller-driven varying-sampling rate*. The setting considered is akin to that in [3], where a centralized scheduler utilizes feedback from the execution-time measurements and feedforward from the workload changes to adjust the sampling periods of all control tasks.

The main problem addressed in the current paper is to devise a sampling-rate-dependent feedback strategy so that the stability of every control process is ensured irrespective of the way in which the centralized controller varies the sampling rate. The framework employed to deal with this problem will be to regard the discrete-time system resulting from sampling a continuous-time process at varying rates as a switched system. A switched system comprises several subsystems and a switching rule that orchestrates switching between subsystems. Several recent results on the stability and stabilizability of NCSs model the NCS as a switched system [4–7]. In turn, several survey papers [8–11], and books [12,13] summarize many important advances in the area of switched systems. In the context of switched systems, the control problem considered in the current paper will be equivalent to devising a feedback control law for each subsystem that ensures the stability of the closed-loop switched system under *arbitrary switching*.

The stability of switched systems has been widely studied in recent years [10,11]. Most existing stability results deal with *autonomous* switched systems, i.e. switched systems with no control inputs. A well-known result on stability under arbitrary switching states that the global exponential stability, uniform with respect to every possible switching rule, is equivalent to the existence of a Lyapunov function common to every subsystem. Another well-known result states that the existence of a quadratic Lyapunov function common to every subsystem is only a sufficient condition for stability under arbitrary switching, even when all subsystems are linear.

In contrast to the case of autonomous switched systems, results on feedback design for stabilization of switched systems (with control inputs) under arbitrary switching are relatively scarce. Usual approaches are related with quadratic Lyapunov functions and employ linear matrix inequalities (LMIs) [14–18]. LMIs can be employed for both stability analysis and feedback design, and constitute a numerically efficient method. A possible disadvantage of control design via blind use of LMIs is that no structural information is provided for the closed-loop system. The lack of transparency and interpretability of LMI-based control design has been observed by the community, for example in [19].

A structure-based condition for the stability of autonomous switched systems with linear subsystems has been presented in [20,21] for continuous-time switched systems and in [22,23] for discrete-time switched systems. This condition states that if the subsystem matrices are individually stable and generate a solvable Lie algebra, then a common quadratic Lyapunov function exists and hence the switched system is stable under arbitrary switching. This Lie-algebraic-solvability condition is equivalent to the existence of a similarity transformation that simultaneously renders each subsystem matrix upper triangular [24].

Stability analysis of autonomous switched systems by means of this Lie-algebraic-solvability condition has two main disadvantages: (a) solvability is a very restrictive condition and hence only a very limited class of switched systems can satisfy this condition, and (b) solvability is not robust with respect to parameter perturbations and hence even if a switched system satisfies the Lie-algebraic solvability condition, arbitrarily small perturbations of the system parameters may destroy Lie-algebraic solvability. As regards disadvantage (a), the situation can be radically different for switched systems with control inputs. Indeed, the existence of feedback matrices for every subsystem so that the closed-loop switched system satisfies the Lie-algebraic solvability condition can even become a generic property, i.e. valid for almost every set of system parameters, as was shown in [25,26]. However, such genericity holds when the switched system has a substantial number of control inputs, as compared with the number of states, and less subsystems than the number of states. As regards disadvantage (b), although small parameter perturbations may destroy solvability, the stability of the switched system is ensured by the existence of a quadratic Lyapunov function, and hence stability is indeed robust. This observation is the rationale behind control design based on the Lie-algebraic solvability property, which was pursued in previous work [27–29].

In the current paper, we analyze the application of control design based on the aforementioned Lie-algebraic solvability property to the controller-driven varying-sampling-rate setting. Previous work along this line appears in [30–32]. In [30], control design for this type of system is pursued and achieved for some restricted cases: (i) pairwise commuting closed-loop system matrices, (ii) simultaneous triangularization, where the triangularizing transformation can be directly obtained from the open-loop continuous-time system matrix, and (iii) second-order systems. In [31], the results of [30] were enhanced with a robustness and performance analysis using the fact that a diagonal Lyapunov function always exists for stable subsystems in triangular form. In [32], control design through iterative common eigenvector assignment with stability is only suggested as a possible control and observer design methodology. The analysis and implementation of this suggestion appears in [28].

In this context, the main contribution of the current paper is twofold: (a) we show that control design based on the Lie-algebraic-solvability property in the controller-driven varying-sampling-rate setting can be much less restrictive than when applied to switched systems of arbitrary form, and (b) we give sufficient conditions for the existence of feedback matrices so that the closed-loop switched system satisfies the Lie-algebraic-solvability condition. These sufficient conditions are not limited by the number of subsystems of the switched system. The current paper contains substantial extensions of the preliminary results presented in [33,34].

The remainder of the paper is organized as follows. In Section 2, we specify the problem to be addressed and introduce preliminary concepts. The main results of the paper are contained in Section 3. Section 4 illustrates the main results with a numerical example and conclusions are drawn in Section 5.

Notation: p' denotes the transpose of a matrix or vector p. The spectral radius of a square matrix M is denoted $\boldsymbol{\rho}(M)$. img B denotes the vector space spanned by the columns of the matrix B.

2. Problem statement

Consider a continuous-time linear time-invariant (LTI) system of the form

$$\dot{x}(t) = \tilde{A}^c x(t) + \tilde{B}^c u(t), \tag{1}$$

where $x(t) \in \mathbb{R}^{n_c}$, $\tilde{A}^c \in \mathbb{R}^{n_c \times n_c}$, $\tilde{B}^c \in \mathbb{R}^{n_c \times m_c}$, $u(t) \in \mathbb{R}^{m_c}$, and $(\tilde{A}^c, \tilde{B}^c)$ is stabilizable. As previously explained, in the setting considered the controller may vary the sampling rate. Consequently, at the sampling instant t_k, the controller can not only apply feedback but also select the next sampling instant, t_{k+1}. We impose the design condition that the sampling period $h_{i(k)} \doteq t_{k+1} - t_k$ has to be selected from a finite set of possible sampling periods:

$$h_{i(k)} \in \mathcal{H} \doteq \{h_1, \dots, h_N\}. \tag{2}$$

Let x_k represent the state at instant t_k, i.e. $x_k \doteq x(t_k)$. The state evolution at the sampling instants is given by

$$x_{k+1} = \tilde{A}^d_{i(k)} x_k + \tilde{B}^d_{i(k)} u_k \quad \text{where} \tag{3}$$

$$i(k) \in \underline{N} \doteq \{1, \dots, N\}, \quad \forall k, \tag{4}$$

$$\tilde{A}^d_i = e^{\tilde{A}^c h_i}, \qquad \tilde{B}^d_i = \int_0^{h_i} e^{\tilde{A}^c t} dt \tilde{B}^c, \quad \forall i \in \underline{N} \tag{5}$$

The discrete-time system (3)–(5) can be interpreted as a discrete-time switching system (DTSS), where $\tilde{A}^d_i$ and $\tilde{B}^d_i$ identify its i-th subsystem, for $i \in \underline{N}$, and $i(k)$ is its switching signal. The discrete-time system (3) switches whenever the controller selects a new sampling period from the set $\mathcal{H}$. For example, if at time t_k the sampling period $t_{k+1} - t_k$ is selected as h_1, then $x(t_{k+1}) = x_{k+1}$ will satisfy, according to (3)–(5), $x_{k+1} = \tilde{A}^d_1 x_k + \tilde{B}^d_1 u_k$. Since the resulting DTSS arises from sampling a single continuous-time system at different rates, the possible matrices $\tilde{A}^d_i$ and $\tilde{B}^d_i$ are not arbitrary: they must satisfy (5). We thus will employ the following definition.

Definition 1 (*VSR-DTSS*)**.** A DTSS (3) whose subsystem matrices $\tilde{A}^d_i$ and $\tilde{B}^d_i$ satisfy (5) for some matrices $\tilde{A}^c$ and $\tilde{B}^c$ and some positive numbers $h_1, \dots, h_N$, will be called a Varying-Sampling-Rate DTSS (VSR-DTSS). The matrices $\tilde{A}^c$ and $\tilde{B}^c$ will be called the continuous-time (CT) matrices of the VSR-DTSS.

We are interested in devising a state-feedback strategy so that stability of the DTSS (3) is ensured for every possible choice of sampling instants, provided that $t_{k+1} - t_k \in \mathcal{H}$ for all k. At the sampling instant t_k, the discrete-time control input will satisfy

$$u_k = \tilde{K}^d_{i(k)} x_k, \tag{6}$$

where $\tilde{K}^d_i \in \mathbb{R}^{m_c \times n_c}$ for every $i \in \underline{N}$. Employing zero-order hold for the continuous-time system (1), the continuous-time control input thus satisfies

$$u(t) = \tilde{K}^d_{i(k)} x(t_k), \quad \text{for } t \in [t_k, t_{k+1}). \tag{7}$$

The discrete-time closed-loop system is then

$$x_{k+1} = \tilde{A}^{d,\text{CL}}_{i(k)} x_k \tag{8}$$

$$\tilde{A}^{d,\text{CL}}_i = \tilde{A}^d_i + \tilde{B}^d_i \tilde{K}^d_i \quad \text{for all } i \in \underline{N}. \tag{9}$$

In order for the state feedback (6) to be applied, the controller must know the switching state of the system, $i(k)$. This implies that, at the sampling instant t_k, the controller must already know the next sampling instant, t_{k+1}, and hence know $h_{i(k)} = t_{k+1} - t_k$, which identifies the matrices $\tilde{A}^d_{i(k)}$ and $\tilde{B}^d_{i(k)}$ that are active at time t_k. This prior knowledge from the controller is only possible because, in the setting considered, it is the controller itself that decides what the next sampling instant will be.

As mentioned in the Introduction, our control design strategy is based on a stability result for switching systems that requires the simultaneous triangularizability (solvable Lie algebra condition) of the closed-loop subsystem matrices $\tilde{A}^{d,\text{CL}}_i$ in (8) and (9). We thus will require the following definition.

Definition 2 (*SLASF*)**.** A DTSS of the form (3) and (4) for which there exist feedback matrices $\tilde{K}^d_i$ that make $\tilde{A}^{d,\text{CL}}_i$ stable and an invertible $T \in \mathbb{C}^{n_c \times n_c}$ so that $T^{-1}\tilde{A}^{d,\text{CL}}_i T$ is upper triangular for all $i \in \underline{N}$ will henceforth be called Solvable-Lie-Algebra Stabilizable by Feedback (SLASF).

It is well-known that the condition that the matrices $\tilde{A}^{d,\text{CL}}_i$ for all $i \in \underline{N}$ be simultaneously triangularizable (i.e. that these matrices generate a solvable Lie algebra) is highly restrictive. In other words, the possible entries for the matrices

$\tilde{A}_i^{d,\text{CL}}$ that will render the latter simultaneously triangularizable form a set of measure zero within the set of all possible entries. However, when control inputs are present, the existence of feedback matrices $\tilde{K}_i^d$ so that $\tilde{A}_i^{d,\text{CL}}$ become stable and simultaneously triangularizable (i.e. so that the DTSS is SLASF) need not be so restrictive. Indeed, for the case of arbitrary DTSSs, not necessarily VSR-DTSSs, Haimovich and Braslavsky [25,26] have shown that when the number of inputs m_c is sufficiently large (but may still be less than the number of states) and the number of subsystems N is less than the number of states n_c, the SLASF property may even become generic, i.e. valid for almost every possible choice of entries of the DTSS matrices $\tilde{A}_i^d$ and $\tilde{B}_i^d$.

In this context, the main contributions of the current paper are (a) to show that VSR-DTSSs may be SLASF even when the number of inputs is not very large as compared with the number of states, and (b) to provide sufficient conditions that ensure a SLASF VSR-DTSS, both without limitations on the number of subsystems (number of possible sampling periods).

3. Stabilization via simultaneous triangularization

In this section, we provide the contributions of the paper. A first contribution is given in Section 3.1 and our main contribution is given in Section 3.2.

3.1. Dimension reduction

Let n_s denote the number of stable eigenvalues of $\tilde{A}^c$ and n the number of unstable eigenvalues, such that $n_c = n_s + n$. We know that there exists an invertible matrix $\tilde{T} \in \mathbb{R}^{n\times n}$ such that

$$A^c \doteq \tilde{T}^{-1}\tilde{A}^c\tilde{T} = \begin{bmatrix} A_s & 0 \\ 0 & A \end{bmatrix} \quad \text{and} \quad B^c \doteq \tilde{T}^{-1}\tilde{B}^c = \begin{bmatrix} B_s \\ B \end{bmatrix} \tag{10}$$

with $A_s \in \mathbb{R}^{n_s\times n_s}$, $A \in \mathbb{R}^{n\times n}$, $B_s \in \mathbb{R}^{n_s\times m_c}$, $B \in \mathbb{R}^{n\times m_c}$, and such that A_s is Hurwitz. Operating on (5) and (10) yields

$$\tilde{A}_i^d = e^{\tilde{A}^c h_i} = \tilde{T} e^{A^c h_i}\tilde{T}^{-1}, \qquad \tilde{B}_i^d = \tilde{T}\int_0^{h_i} e^{A^c t}dtB^c, \tag{11}$$

and defining

$$A_i^d \doteq e^{A^c h_i}, \qquad B_i^d \doteq \int_0^{h_i} e^{A^c t}dtB^c, \tag{12}$$

it follows that $\tilde{A}_i^d = \tilde{T}A_i^d\tilde{T}^{-1}$, $\tilde{B}_i^d = \tilde{T}B_i^d$, with

$$A_i^d = \begin{bmatrix} A_{s,i} & 0 \\ 0 & A_i \end{bmatrix}, \qquad A_{s,i} = e^{A_s h_i}, \qquad A_i = e^{Ah_i}, \tag{13}$$

$$B_i^d = \begin{bmatrix} B_{s,i} \\ B_i \end{bmatrix}, \qquad B_{s,i} = \int_0^{h_i} e^{A_s t}B_s dt, \qquad B_i = \int_0^{h_i} e^{At}Bdt. \tag{14}$$

Since A_s is Hurwitz, then $A_{s,i} = e^{A_s h_i}$ satisfies $\boldsymbol{\rho}(A_{s,i}) < 1$. Note that the above derivation shows that if a DTSS identified by the matrices $(\tilde{A}_i^d, \tilde{B}_i^d)$ for $i \in \underline{N}$ is a VSR-DTSS, then the derived reduced-dimension DTSSs identified respectively by $(A_{s,i}, B_{s,i})$ and (A_i, B_i) are both VSR-DTSSs.

The following result establishes that if the reduced-dimension DTSS identified by (A_i, B_i) is SLASF, then the full-dimension VSR-DTSS identified by $(\tilde{A}_i^d, \tilde{B}_i^d)$ also is SLASF.

Theorem 1. *Suppose that there exist $K_i \in \mathbb{R}^{m_c\times n}$ and an invertible $T \in \mathbb{C}^{n\times n}$ such that, for $i \in \underline{N}$, $\boldsymbol{\rho}(A_i + B_iK_i) < 1$ and $T^{-1}(A_i + B_iK_i)T$ are upper triangular. Let $\tilde{K}_i^d = [0\ K_i]\tilde{T}^{-1}$, where $\tilde{T}$ is the matrix employed to separate the system into its stable and unstable parts [recall (10)], and consider $\tilde{A}_i^{d,\text{CL}}$ as in (9). Then, $\boldsymbol{\rho}(\tilde{A}_i^{d,\text{CL}}) < 1$ and there exists $\bar{T} \in \mathbb{C}^{n_c\times n_c}$ such that $\bar{T}^{-1}\tilde{A}_i^{d,\text{CL}}\bar{T}$ are upper triangular.*

Proof. Recalling (11)–(14), it follows that

$$\tilde{T}^{-1}\tilde{A}_i^{d,\text{CL}}\tilde{T} = A_i^d + B_i^d[0\ K_i] = \begin{bmatrix} A_{s,i} & B_{s,i}K_i \\ 0 & A_i + B_iK_i \end{bmatrix}. \tag{15}$$

Let $T_s \in \mathbb{C}^{n_s\times n_s}$ be such that $T_s^{-1}A_sT_s$ is upper triangular. By (13), then $T_s^{-1}A_{s,i}T_s$ also are upper triangular for all $i \in \underline{N}$. Define

$$A_i^{\text{CL}} \doteq A_i + B_iK_i, \qquad \bar{T} \doteq \tilde{T}\begin{bmatrix} T_s & 0 \\ 0 & T \end{bmatrix}. \tag{16}$$

Combining (15) and (16), it follows that

$$\bar{T}^{-1}\tilde{A}_i^{d,\text{CL}}\bar{T} = \begin{bmatrix} T_s^{-1}A_{s,i}T_s & T_s^{-1}B_{s,i}K_iT \\ 0 & T^{-1}A_i^{\text{CL}}T \end{bmatrix} \tag{17}$$

Note that $\boldsymbol{\rho}(T_s^{-1}A_{s,i}T_s) = \boldsymbol{\rho}(A_{s,i})$, $\boldsymbol{\rho}(T^{-1}A_i^{\text{CL}}T) = \boldsymbol{\rho}(A_i^{\text{CL}})$ and $\boldsymbol{\rho}(\bar{T}^{-1}\tilde{A}_i^{d,\text{CL}}\bar{T}) = \boldsymbol{\rho}(\tilde{A}_i^{d,\text{CL}})$. From (17) and the latter considerations, we have

$$\boldsymbol{\rho}(\tilde{A}_i^{d,\text{CL}}) = \max\{\boldsymbol{\rho}(A_{s,i}), \boldsymbol{\rho}(A_i^{\text{CL}})\} < 1.$$

Since both $T_s^{-1}A_{s,i}T_s$ and $T^{-1}A_i^{\text{CL}}T$ are upper triangular, then $\bar{T}^{-1}\tilde{A}_i^{d,\text{CL}}\bar{T}$ is upper triangular. □

Theorem 1 states that the stabilization problem based on Lie-algebraic solvability for a VSR-DTSS (3)–(5) can be reduced to the problem corresponding to the derived reduced-dimension VSR-DTSS which corresponds to the unstable part of the underlying continuous-time system. We can immediately state the following consequences.

Corollary 1. *If the rank of the input matrix corresponding to the unstable part of the system equals the number of unstable eigenvalues of $\tilde{A}^c$, i.e. if $\text{rank}B = n$, and no pathological sampling [35] occurs for the sampling periods in $\mathcal{H}$, then the VSR-DTSS (3)–(5) is SLASF.*

Proof. Since no pathological sampling occurs, then $\text{rank}B_i = \text{rank}B = n$, and hence $B_i \in \mathbb{R}^{n\times m_c}$ has a right inverse, for every $i \in \underline{N}$. Therefore, there exists K_i so that $A_i + B_iK_i$ is upper triangular and satisfies $\boldsymbol{\rho}(A_i + B_iK_i) < 1$. Applying Theorem 1 with $T = \text{I}$, the result follows. □

Corollary 2. *If the continuous-time system (1) is stabilizable, no pathological sampling occurs for the sampling periods in $\mathcal{H}$, and $\tilde{A}^c$ has only one unstable eigenvalue, then the corresponding VSR-DTSS (3)–(5) is SLASF.*

Proof. Applying the transformation (10) to system (1) produces $A \in \mathbb{R}^{1\times 1}$ and $B \in \mathbb{R}^{1\times m_c}$. Since $(\tilde{A}^c, \tilde{B}^c)$ is stabilizable, then (A, B) is controllable. Since no pathological sampling occurs, the pairs (A_i, B_i) are controllable, and hence $B_i \neq 0$ for every $i \in \underline{N}$. Since the A_i are scalars and the B_i are nonzero, we can find $K_i \in \mathbb{R}^{m_c\times 1}$ such that $\boldsymbol{\rho}(A_i + B_iK_i) = |A_i + B_iK_i| < 1$. Being scalars, $A_i + B_iK_i$ are trivially upper triangular. By Theorem 1, the VSR-DTSS (3)–(5) is SLASF. □

Theorem 1 and its corollaries show that stabilization by state feedback based on the Lie-algebraic solvability condition has a much wider application than for arbitrary DTSSs. This is so because in the considered VSR-DTSS case, control needs only be applied to the unstable part of the system and, in addition, the simultaneous triangularization of the unstable part of the system carries over to the whole system. Moreover, Theorem 1 also shows how stabilizing feedback that renders the closed-loop matrices for the whole system simultaneously upper triangular can be computed from the feedback applied to the unstable part of the system.

In the next subsection, we identify other properties of VSR-DTSSs and exploit these properties in order to derive further sufficient conditions for stability based on Lie-algebraic solvability.

3.2. Sufficient conditions for SLASF VSR-DTSSs

Recall that the unstable part of the VSR-DTSS considered is identified by the matrices A_i and B_i [see (10)–(14)], and the corresponding CT matrices are A and B. Define

$$m \doteq \text{rank}\, B. \tag{18}$$

To simplify notation and without loss of generality, we will henceforth assume that the matrix B has full rank. Otherwise, B refers to a full rank matrix so that $\text{img}\, B$ coincides with the image of the input matrix corresponding to the unstable part of the system.

The following lemmas state properties of VSR-DTSSs that will be exploited in the derivation of our main results in Theorems 2 and 3.

Lemma 1. *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ be the CT matrices of a given VSR-DTSS with matrix pairs (A_i, B_i). Then, every vector $v \in \mathbb{R}^n$ that satisfies*

$$(A + BK)v = 0 \tag{19}$$

for some $K \in \mathbb{R}^{m\times n}$ also satisfies

$$[I - (A_i + B_iK)]\, v = 0, \quad \textit{for all } i \in \underline{N}. \tag{20}$$

Proof. By (19), then $Av = -BKv$. Premultiplying the latter equation by $\int_0^{h_i} e^{At}dt$ yields

$$\int_0^{h_i} e^{At}dtAv = -\int_0^{h_i} e^{At}dtBKv, \tag{21}$$

$$(A_i - I)v = -B_iKv, \tag{22}$$

where we have used $A_i = e^{Ah_i}$, $B_i = \int_0^{h_i} e^{At}dtB$, and $\int_0^{h_i} e^{At}dtA = (A_i - I)$. Since (22) holds for each $i \in \underline{N}$, then (20) follows. □

Lemma 1 shows that every feedback-assignable eigenvector for the CT system, whose corresponding eigenvalue is zero, can be assigned by feedback as an eigenvector with eigenvalue 1 for every subsystem by means of the same feedback matrix as for the CT system.

The following result is not restricted to VSR-DTSSs and is a consequence of particularizing the iterative eigenvector assignment algorithm introduced in [27,28] to the case of only 1 iteration and real eigenvalues. We include it here for the sake of completeness and improved readability.

Lemma 2. *Consider an arbitrary DTSS with matrices $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times n-1}$, for $i \in \underline{N}$. Suppose that there exist a nonzero vector $v \in \mathbb{R}^n$, feedback matrices $F_i \in \mathbb{R}^{n-1\times n}$, and scalars $\lambda_i \in \mathbb{R}$, such that, for all $i \in \underline{N}$,*

$$(A_i + B_iF_i)v = \lambda_i v, \quad |\lambda_i| < 1, \tag{23}$$

$$v \notin \operatorname{img} B_i. \tag{24}$$

That is, suppose that a real stable feedback-assignable eigenvector common to every subsystem exists, which is not contained in any img B_i. *Then, the DTSS is SLASF.*

Proof. Let $\bar{v} = v/\|v\|$, so that $\bar{v}'\bar{v} = 1$. Select a matrix $U \in \mathbb{R}^{n\times n-1}$ such that $U'U = \mathrm{I}$ and $U'\bar{v} = 0$. The matrix $T \doteq [\bar{v}|U]$ thus satisfies $T'T = \mathrm{I}$. By (24), it follows that $\operatorname{rank}(U'B_i) = \operatorname{rank}B_i = n-1$. Then, the matrix $U'B_i$ has a right-inverse and there exists $G_i \in \mathbb{R}^{n-1\times n-1}$ such that $M_i \doteq U'(A_i + B_iF_i)U + U'B_iG_i$ is upper triangular and $\boldsymbol{\rho}(M_i) < 1$. Let $K_i = F_i + G_iU'$. We have

$$T^{-1}(A_i + B_iK_i)T = \begin{bmatrix} \bar{v}' \\ U' \end{bmatrix} (A_i + B_iF_i + B_iG_iU') \begin{bmatrix} \bar{v} & U \end{bmatrix} = \begin{bmatrix} \lambda_i & * \\ 0 & M_i \end{bmatrix}$$

Since M_i is upper triangular, $\boldsymbol{\rho}(M_i) < 1$, and $|\lambda_i| < 1$, then $T^{-1}(A_i + B_iK_i)T$ is upper triangular and satisfies $\boldsymbol{\rho}(T^{-1}(A_i + B_iK_i)T) < 1$, for all $i \in \underline{N}$. Therefore, the DTSS is SLASF. □

Lemma 2 gives a sufficient condition for an arbitrary DTSS (not necessarily a VSR-DTSS) to be SLASF. This lemma can only be applied when the number of effective inputs of the DTSS (i.e. $\operatorname{rank}B_i$) is one less than the number of its states. We next derive sufficient conditions for a VSR-DTSS to be SLASF, which are based on the results of Lemmas 1 and 2. These conditions are applicable only when the number of effective inputs to the unstable part of the system, namely $\operatorname{rank}B$, is one less than the number of unstable eigenvalues of the system, n.

In the sequel, we assume that the matrix $\tilde{A}^c$ in (1) is nonsingular. This assumption implies that the unstable part of $\tilde{A}^c$, namely A, also is nonsingular. Under this assumption, we will focus on conditions for a stable feedback-assignable eigenvector to exist, according to the hypotheses of Lemma 2. We thus need conditions to ensure the existence of a vector $v^\star$ satisfying

$$(A_i + B_iF_i)v^\star = \lambda_i v^\star \tag{25}$$

for some F_i and λ_i with $|\lambda_i| < 1$. Define $u_i \doteq F_iv^\star$. Operating on (25) yields

$$B_iu_i = (\lambda_i\mathrm{I} - A_i)v^\star. \tag{26}$$

Since, A is nonsingular, then all eigenvalues of A_i have magnitude greater than 1 and hence, $(\lambda_i\mathrm{I} - A_i)$ is invertible whenever $|\lambda_i| \le 1$. In this case, it follows from (26) that

$$v^\star = (\lambda_i\mathrm{I} - A_i)^{-1}B_iu_i. \tag{27}$$

Define the following matrices:

$$A(h) \doteq e^{Ah} \qquad B(h) \doteq \int_0^h e^{At}dtB, \tag{28}$$

so that

$$A_i = A(h_i) \qquad B_i = B(h_i), \tag{29}$$

and the following vector-valued functions:

$$v(\lambda, h, u) \doteq (\lambda \mathrm{I} - A(h))^{-1} B(h) u \tag{30}$$

$$v_i(\lambda, u) \doteq v(\lambda, h_i, u). \tag{31}$$

Note that $v(\lambda, h, u)$ is well-defined at every (λ, h, u) satisfying $|\lambda| \le 1$. By Lemma 1 and since A is nonsingular, it follows that $(\mathrm{I} - A(h))^{-1}B(h) = -A^{-1}B$ for all h non-pathological. Therefore,

$$\bar{v}(u) \doteq v(1, h, u) = -A^{-1}Bu, \tag{32}$$

whenever h is non-pathological. For future reference, define

$$\bar{\mathcal{V}} \doteq \left\{ \bar{v}(u) : u \in \mathbb{R}^{n-1} \right\} = \operatorname{img}(A^{-1}B) \tag{33}$$

and note that $\bar{\mathcal{V}}$ is a vector subspace of dimension $n-1$. Let $\mathcal{V}(h_i)$ denote the set of stable feedback-assignable eigenvectors for subsystem i:

$$\mathcal{V}(h_i) \doteq \{ v(\lambda, h_i, u) : |\lambda| < 1, u \neq 0 \}. \tag{34}$$

Thus, the set of stable feedback-assignable eigenvectors common to all subsystems can be expressed as

$$\mathcal{V}_0 \doteq \bigcap_{i=1}^{N} \mathcal{V}(h_i). \tag{35}$$

From Lemma 2, it follows that the considered VSR-DTSS is SLASF if there exists $v^\star \in \mathcal{V}_0$ such that $v^\star \notin \operatorname{img} B_i$ for all $i \in \underline{N}$. Note that if we slightly expand the sets $\mathcal{V}(h_i)$ to the sets $\bar{\mathcal{V}}(h_i)$ defined as follows

$$\bar{\mathcal{V}}(h_i) \doteq \{ v(\lambda, h_i, u) : |\lambda| \le 1, u \neq 0 \}, \tag{36}$$

$$\bar{\mathcal{V}}_0 \doteq \bigcap_{i=1}^{N} \bar{\mathcal{V}}(h_i), \tag{37}$$

according to (32) then $\bar{v}(u) \in \bar{\mathcal{V}}_0$ and hence $\bar{\mathcal{V}}_0$ is nonempty.

We are now ready to present our main results.

Theorem 2. *Consider the VSR-DTSS (3)–(5) with CT matrices $\tilde{A}^c$ and $\tilde{B}^c$, with $(\tilde{A}^c, \tilde{B}^c)$ stabilizable. Suppose that $\tilde{A}^c$ is nonsingular. Recall the separation of the system into its stable and unstable parts in (10)–(14). Suppose that $m = n - 1$, i.e. the rank of the input matrix corresponding to the unstable part of the system equals the number of unstable eigenvalues of the system minus one. Consider the set $\bar{\mathcal{V}}$ defined in (33) and let $p \in \mathbb{R}^n$ be nonzero and satisfy $p'v = 0$ for all $v \in \bar{\mathcal{V}}$. Suppose that there exists $\bar{u} \in \mathbb{R}^{n-1}$ so that the real numbers*

$$\psi_i \doteq p' \frac{\partial v_i}{\partial \lambda}(1, \bar{u}) \tag{38}$$

satisfy $\psi_i \psi_j > 0$ for all $i, j \in \underline{N}$, with v_i as defined in (30) and (31). Then, the VSR-DTSS (3)–(5) is SLASF.

Proof. Since $\tilde{A}^c$ is nonsingular, then the unstable part of $\tilde{A}^c$, namely A, also is nonsingular. As a consequence, the corresponding discrete-time matrix $A(h)$ [see (28)] has no unit eigenvalues whenever h is non-pathological. By (30) and (31), then $v_i(\lambda, u)$ is differentiable at $(1, \bar{u})$, and hence ψ_i is well-defined for every $i \in \underline{N}$.

Claim 1. *The Jacobian of $v_i(\lambda, u)$ at the point $(1, \bar{u})$ is nonsingular.*

Proof (Claim 1)**.** Since h_i is non-pathological by assumption, we have

$$\frac{\partial v_i}{\partial u}(1, \bar{u}) = (\mathrm{I} - A(h_i))^{-1} B(h_i) = -A^{-1}B \tag{39}$$

and since B has full rank, then the columns of the matrix (39) are linearly independent. By (30)–(33) and the definition of p, we must have

$$p'(\mathrm{I} - A(h_i))^{-1} B(h_i) u = -p' A^{-1} B u = 0 \quad \text{for all } u \in \mathbb{R}^{n-1} \tag{40}$$

which implies that

$$p'(\mathrm{I} - A(h_i))^{-1} B(h_i) = -p' A^{-1} B = p' \frac{\partial v_i}{\partial u}(1, \bar{u}) = 0. \tag{41}$$

In addition, by assumption we have

$$\psi_i = p' \frac{\partial v_i}{\partial \lambda}(1, \bar{u}) \neq 0. \tag{42}$$

It follows that the vector $\frac{\partial v_i}{\partial \lambda}(1, \bar{u})$ and the $n - 1$ columns of (39) are linearly independent. Therefore, the claim is established. □

For every subsystem $i \in \underline{N}$, the Inverse Function Theorem then establishes that $v_i(\lambda, u)$ is invertible in some neighborhood of $(1, \bar{u})$. Recall that $v_i(1, \bar{u}) = \bar{v}(\bar{u})$ for every $i \in \underline{N}$ and let v_i^{-1} denote the inverse of v_i for every $i \in \underline{N}$. Consider the subsystem $i = 1$ and the stable feedback-assignable eigenvector $v^\star = v_1(1 - \epsilon_1, \bar{u})$ for some $\epsilon_1 > 0$ small enough so that $v^\star$ lies within the neighborhood where every v_i^{-1} is defined. For every $i \in \underline{N}$, compute $(\lambda_i^\star, u_i^\star)$ so that $v_i(\lambda_i^\star, u_i^\star) = v^\star$. Note that $\lambda_1^\star = 1 - \epsilon_1$ and $u_1^\star = \bar{u}$.

Claim 2. *For $\epsilon_1 > 0$ sufficiently small, the scalars $\lambda_i^\star$ satisfy $|\lambda_i^\star| < 1$.*

Proof (Claim 2)**.** Write $\lambda_i^\star = 1 - \epsilon_i$. Performing a Taylor series expansion of the real-valued function $p' v_i(\lambda, u)$ about $(1, \bar{u})$, we have

$$p' v_i(\lambda_i^\star, u_i^\star) = -\epsilon_i \underbrace{p' \frac{\partial v_i}{\partial \lambda}(1, \bar{u})}_{\psi_i} + o(\epsilon_i^2) + o(\|u_i^\star - \bar{u}\|^2) = p' v^\star \neq 0 \tag{43}$$

where we have used (41) and (42). Note that we can make ϵ_i as close to 0 as desired and $u_i^\star$ as close to $\bar{u}$ as desired by selecting $\epsilon_1 > 0$ sufficiently small. Then, for $\epsilon_1 > 0$ sufficiently small, the sign of $p' v^\star$ will equal the sign of $-\epsilon_i \psi_i$ for every $i \in \underline{N}$, and hence $\epsilon_i \psi_i \epsilon_j \psi_j > 0$ for all $i, j \in \underline{N}$. Since $\psi_i \psi_j > 0$ by assumption, then $\epsilon_i \epsilon_j > 0$, and since $\epsilon_1 > 0$, then $\epsilon_i > 0$. The claim is established by recalling that $\lambda_i^\star = 1 - \epsilon_i$ and that ϵ_i is small enough. □

Applying Claim 2 for every subsystem, it follows that $v^\star$ is a stable feedback-assignable eigenvector common to every subsystem. For a contradiction, suppose that $\bar{v}(\bar{u}) \in \operatorname{img} B(h_i)$ for some $i \in \underline{N}$. Then $\bar{v}(\bar{u}) = B(h_i) w$ for some w, and $\frac{\partial v_i}{\partial \lambda}(1, \bar{u}) = -(I - A(h_i))^{-1} B(h_i) w = v_i(1, w)$. Consequently, $p' \frac{\partial v_i}{\partial \lambda}(1, \bar{u}) = 0$, contradicting the fact that $\psi_i \neq 0$. Since we have established that $\bar{v}(\bar{u}) \notin \operatorname{img} B(h_i)$, and since we can select $v^\star$ as close to $\bar{v}(\bar{u})$ as desired, it follows that $v^\star$ can be chosen so that $v^\star \notin \operatorname{img} B(h_i)$. Applying Lemma 2, it follows that the VSR-DTSS with matrices A_i and B_i is SLASF. The result follows from application of Theorem 1. □

The following corollary shows that the verification of the hypotheses of Theorem 2 is greatly simplified when the system has only two unstable eigenvalues. Specifically, this corollary shows that in such a case, no effort is needed in order to find $\bar{u}$.

Corollary 3. *Consider a VSR-DTSS (3)–(5) with $n = 2$ unstable eigenvalues and such that the hypotheses of Theorem 2 are satisfied for some nonzero $\bar{u} \in \mathbb{R}$. Then, the hypotheses of Theorem 2 are also satisfied for every nonzero $\bar{u} \in \mathbb{R}$.*

Proof. For the given $\bar{u}$, Eq. (38) yields

$$\psi_i = p' \frac{\partial v_i}{\partial \lambda}(1, \bar{u}) = -p'(I - A_i)^{-2} B_i \bar{u}$$

and, by assumption, $\psi_i \psi_j > 0$. Let $\bar{u}_2 = \alpha \bar{u}$ for some real number $\alpha \neq 0$. We have

$$\psi_{i,2} \doteq p' \frac{\partial v_i}{\partial \lambda}(1, \bar{u}_2) = \alpha p' \frac{\partial v_i}{\partial \lambda}(1, \bar{u}) = \alpha \psi_i.$$

Therefore, $\psi_i \psi_j > 0$ if and only if $\psi_{i,2} \psi_{j,2} = \alpha^2 \psi_i \psi_j > 0$. □

We next provide an additional result that, though more restrictive than Theorem 2, may be verified in a simpler way

Theorem 3. *Consider the VSR-DTSS (3)–(5) with CT matrices $\tilde{A}^c$ and $\tilde{B}^c$, with $(\tilde{A}^c, \tilde{B}^c)$ stabilizable. Suppose that $\tilde{A}^c$ is nonsingular. Recall the separation of the system into its stable and unstable parts in (10)–(14). Suppose that $m = n - 1$, i.e. the rank of the input matrix corresponding to the unstable part of the system, equals the number of unstable eigenvalues of the system minus one. Let $h_l = \min(\mathcal{H})$ and $h_u = \max(\mathcal{H})$, where $\mathcal{H}$ was defined in (2). If every $h \in [h_l, h_u]$ is non-pathological, then the VSR-DTSS (3)–(5) is SLASF.*

Proof. Consider the function $\psi(h) : [h_l, h_u] \to \mathbb{R}$ defined as

$$\psi(h, u) := p' \frac{\partial v}{\partial \lambda}(1, h, u) \tag{44}$$

with p as in Theorem 2 and $v(\lambda, h, u)$ defined in (30). If for some $\bar{u}$ and all $h \in [h_l, h_u]$, $\psi(h, \bar{u})$ is either positive or negative, then the conditions required by Theorem 2 are satisfied because, according to (31) and (38), we have $\psi_i = \psi(h_i, \bar{u})$, and $h_i \in [h_l, h_u]$. A sufficient condition for the sign of $\psi(h, \bar{u})$ to be the same for every $h \in [h_l, h_u]$ is that $\psi(h, \bar{u})$ be continuous and nonzero in $[h_l, h_u]$. Since $\tilde{A}^c$ is nonsingular, then A is nonsingular. Since there is no pathological sampling in $[h_l, h_u]$, recalling (44) and (30), there are no discontinuities in $\psi(h, u)$ on the defined interval. We next show that there exists $\bar{u}$ such that $\psi(h, \bar{u}) \neq 0$ for all $h \in [h_l, h_u]$. For a contradiction, suppose that

$$\psi(\bar{h}, u) = 0, \quad \text{for some } \bar{h} \in [h_l, h_u] \text{ and all } u \neq 0. \tag{45}$$

From (30), (32), and by definition of p, it follows that

$$p' v(1, h, u) = p' \bar{v}(u) = 0, \quad \forall h \in [h_l, h_u],\ \forall u \neq 0. \tag{46}$$

Claim 3. *There exists $\bar{u} \neq 0$ so that $\frac{\partial v}{\partial \lambda}(1, \bar{h}, \bar{u}) = 0$.*

Proof (Claim 3)**.** If this were not the case, then $\frac{\partial v}{\partial \lambda}(1, \bar{h}, u) \neq 0$ for all $u \neq 0$. From (30), we have

$$\frac{\partial v}{\partial \lambda}(1, \bar{h}, u) = -(\mathrm{I} - A(\bar{h}))^{-2} B(\bar{h}) u \tag{47}$$

and hence the $n - 1$ columns of the matrix $(\mathrm{I} - A(\bar{h}))^{-2} B(\bar{h})$ must be linearly independent. Combining this fact with (45), then the aforementioned columns are a basis for the one-dimensional subspace $\mathrm{Span}\{p\}^{\perp}$. By definition of p, we also have that the columns of the matrix $(\mathrm{I} - A(\bar{h}))^{-1} B(\bar{h})$ are a basis for $\mathrm{Span}\{p\}^{\perp}$. Therefore,

$$\mathrm{img}\,[(\mathrm{I} - A(\bar{h}))^{-2} B(\bar{h})] = \mathrm{img}\,[(\mathrm{I} - A(\bar{h}))^{-1} B(\bar{h})] \tag{48}$$

which happens if and only if

$$\mathrm{img}\, B(\bar{h}) = \mathrm{img}\,[(\mathrm{I} - A(\bar{h})) B(\bar{h})]. \tag{49}$$

Again, this happens only if $\mathrm{img}\,[A(\bar{h}) B(\bar{h})] \subset \mathrm{img}\, B(\bar{h})$ and hence the pair $(A(\bar{h}), B(\bar{h}))$ is not controllable, contradicting the fact that $\bar{h}$ is non-pathological. Therefore, the claim is established. □

Let $\bar{u}$ be as per Claim 3. By (47), then the columns of the matrix $(\mathrm{I} - A(\bar{h}))^{-2} B(\bar{h})$ must be linearly dependent, which implies that the columns of $B(\bar{h})$ must be linearly dependent. This contradicts the fact that $\bar{h}$ is non-pathological. We have thus established that $\psi(h, \bar{u}) \neq 0$ for all $h \in [h_l, h_u]$ and some $\bar{u} \neq 0$. Theorem 2 thus establishes that the considered VSR-DTSS is SLASF. □

The following fact is a direct consequence of Theorem 3.

Corollary 4. *Consider the VSR-DTSS (3)–(5) with CT matrices $\tilde{A}^c$ and $\tilde{B}^c$, with $(\tilde{A}^c, \tilde{B}^c)$ stabilizable. Recall the separation of the system into its stable and unstable parts in (10)–(14). Suppose that $m = n - 1$, i.e. the rank of the input matrix corresponding to the unstable part of the system, equals the number of unstable eigenvalues of the system minus one. If all the unstable eigenvalues of the CT matrix $\tilde{A}^c$ are real and positive, then the VSR-DTSS (3)–(5) is SLASF for every possible (finite) set of sampling periods $\mathcal{H}$.*

Proof. Since $\tilde{A}^c$ has no zero eigenvalues by assumption, it is nonsingular. Since no complex eigenvalues exist, no pathological sampling occurs. Thus, Theorem 3 establishes that the considered VSR-DTSS is SLASF. □

Theorems 2 and 3, and Corollaries 3 and 4 give sufficient conditions for a VSR-DTSS of the form (3)–(5) to be SLASF. These sufficient conditions are based on finding a stable feedback-assignable eigenvector as required by Lemma 2. When these sufficient conditions are satisfied and such a feedback-assignable eigenvector is found, Lemma 2 and Theorem 1 show how the feedback matrices required can be computed. Once the given sufficient conditions hold, the search for the required feedback-assignable eigenvector can be performed by means of the algorithm given in [27,28]. In the next section, we illustrate the results by means of a numerical example.

4. An example

Consider a system of the form (1), with unstable part [recall (10)] given by the matrices

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.5 \\ 2 \end{bmatrix}. \tag{50}$$

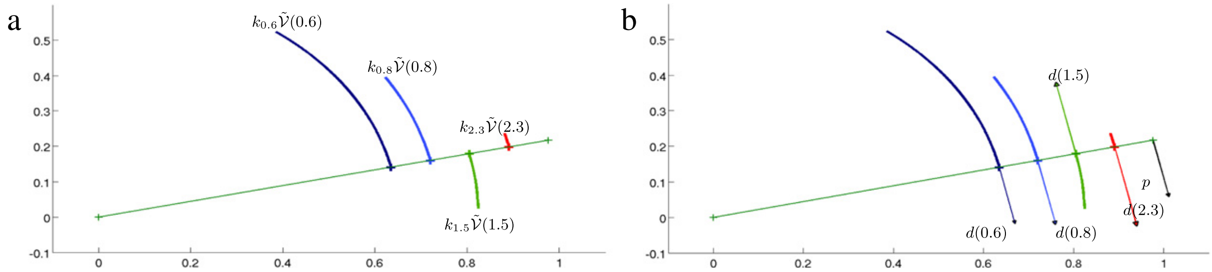

Fig. 1. (a) Left: scaled stable feedback-assignable unit eigenvector sets $\tilde{\mathcal{V}}(a)$ for $a \in \mathcal{A}$. (b) Right: the vector p and the auxiliary vectors $d(a)$.

The eigenvalues of A are $\lambda = 1 \pm 2j$. Hence, pathological sampling occurs at integer multiples of $h_p \doteq \pi/2$. Consider a set of candidate sampling periods $\mathcal{H}_c$ as

$$\mathcal{H}_c \doteq \{ah_p : a \in \mathcal{A}\} \quad \text{with} \tag{51}$$

$$\mathcal{A} \doteq \{2.3,\ 1.5,\ 0.8,\ 0.6\}. \tag{52}$$

Sampling period sets $\mathcal{H} \subset \mathcal{H}_c$ that produce SLASF VSR-DTSSs will be determined using Theorems 2 and 3. Since the system has only 2 unstable eigenvalues, Corollary 3 states that every choice of nonzero $\bar{u}$ is equivalent in order to apply Theorem 2. Define

$$w(a) \doteq \frac{\partial v}{\partial \lambda}(1, ah_p, 1), \quad \text{for every } a \in \mathcal{A}, \tag{53}$$

with $v(\lambda, h, u)$ as in (30). Evaluating $w(a)$ for every $a \in \mathcal{A}$ yields

$$w(2.3) = \begin{bmatrix} 0.0049 \\ -0.0220 \end{bmatrix}, \qquad w(1.5) = \begin{bmatrix} -0.0204 \\ 0.0917 \end{bmatrix}, \tag{54}$$

$$w(0.8) = \begin{bmatrix} 0.0236 \\ -0.1060 \end{bmatrix}, \qquad w(0.6) = \begin{bmatrix} 0.0578 \\ -0.2598 \end{bmatrix}. \tag{55}$$

From (32), we compute $\bar{v}(1) = -A^{-1}B = [0.9762 \quad 0.2169]'\|A^{-1}B\|$. Note that $\bar{v}(\bar{u})$ is a scalar multiple of $\bar{v}(1)$ for every $\bar{u} \in \mathbb{R}$. Application of Theorem 2 hence requires a nonzero vector p orthogonal to $\bar{v}(1)$. We select $p = [0.2169 \quad -0.9762]'$. Note that if a sampling period ah_p is named h_i, then the number ψ_i required by Theorem 2 equals $p'w(a)$. We thus compute $p'w(a)$, for every $a \in \mathcal{A}$, which yields

$$p'w(2.3) = 0.0225, \qquad p'w(1.5) = -0.0939,$$
$$p'w(0.8) = 0.1085, \qquad p'w(0.6) = 0.2662.$$

As can be observed, the values of $p'w(a)$ associated with the sampling periods

$$\mathcal{H}_1 = \{ah_p, a \in \mathcal{A}_1\} \qquad \mathcal{A}_1 = \{2.3\ 0.8\ 0.6\} \tag{56}$$

have the same sign and, by Theorem 2, the corresponding VSR-DTSS (3)–(5), with sampling period set $\mathcal{H} = \mathcal{H}_1$, is SLASF.

Additionally, since no pathological sampling period is contained in the interval $[0.6h_p, 0.8h_p]$, Theorem 3 ensures that the VSR-DTSS (3)–(5), with sampling period set $\mathcal{H} = \mathcal{H}_2 \doteq \{0.8h_p, \quad 0.6h_p\}$ is SLASF. In this example, it is clear that the application of Theorem 3 requires simpler calculations than those required by Theorem 2, although the result of Theorem 3 is less general.

We can give further insight into the results of Theorems 2 and 3 by computing and plotting the unit norm feedback-assignable eigenvectors $\tilde{v}(\lambda, ah_p) \doteq v(\lambda, ah_p, 1)/\|v(\lambda, ah_p, 1)\|$ for values of $\lambda \in (-1, 1)$ for all $a \in \mathcal{A}$. For every $a \in \mathcal{A}$, define the set

$$\tilde{\mathcal{V}}(a) = \{\tilde{v}(\lambda, ah_p) : \lambda \in (-1, 1)\}. \tag{57}$$

Note that $\tilde{\mathcal{V}}(a) \subset \mathcal{V}(ah_p)$, with $\mathcal{V}(ah_p)$ as in (34), for every $a \in \mathcal{A}$. Since each vector in $\tilde{\mathcal{V}}(a)$ is of unit norm, the sets $\tilde{\mathcal{V}}(a)$ are arcs on the unit circle. Note also that, regardless their associated sampling period, the arcs corresponding to different $a \in \mathcal{A}$ share one extreme point (not contained in the sets), namely $\bar{v}(1)/\|\bar{v}(1)\|$ [recall (32) and Lemma 1]. Fig. 1(a) shows the sets $\tilde{\mathcal{V}}(a)$ for every $a \in \mathcal{A}$. For ease of visualization, each set is scaled with a different scale factor, identified by the letter k with a subscript indicating the corresponding sampling period. Note that $\bigcap_{a\in\mathcal{A}} \tilde{\mathcal{V}}(a) = \emptyset$ and $\bigcap_{a\in\mathcal{A}_1} \tilde{\mathcal{V}}(a) \neq \emptyset$. In Fig. 1(b), the vector p is displayed, jointly with auxiliary vectors $d(a)$, which point in the same direction as p when $p'w(a)$ is positive and in the opposite direction when $p'w(a)$ is negative. Observe the relationship between the direction of these auxiliary vectors and the direction of the arcs $\tilde{\mathcal{V}}(a)$ away from the point $\bar{v}(1)/\|\bar{v}(1)\|$.

5. Conclusions

We have addressed controller-driven varying-sampling-rate stabilization of linear systems based on a discrete-time switched system approach. The control aim is to devise a sampling-rate-dependent state-feedback strategy that ensures that the discrete-time switched system obtained by evaluating the system state at the sampling instants is stable under arbitrary switching. The stability result employed for achieving this goal is based on the stability of the individual subsystems jointly with the solvability of the Lie algebra generated by the subsystems' closed-loop matrices. We have shown that stabilization via this technique is not as restrictive as for discrete-time switched systems of arbitrary form, and we have given sufficient conditions for the strategy to be successful irrespective of the number of different sampling periods employed by the controller. Future work will be aimed at generalizing the given conditions and at comparing the given feedback strategy with LMI-based results.

References

[1] P. Antsaklis, J. Baillieul, Guest Eds. Special issue on networked control systems, IEEE Transactions on Automatic Control 49 (9) (2004).
[2] P. Antsaklis, J. Baillieul, Guest Eds. Special issue on technology of networked control systems, Proceedings of the IEEE 95 (1) (2007).
[3] A. Cervin, J. Eker, B. Bernhardsson, K.-E. Arzén, Feedback-feedforward scheduling of control tasks, Real-Time Systems 23 (2002) 25–53.
[4] J. Yu, L. Wang, M. Yu, Switched system approach to stabilization of networked control systems, International Journal of Robust and Nonlinear Control 21 (17) (2011) 1925–1946.
[5] W.-A. Zhang, L. Yu, S. Yin, A switched system approach to H_∞ control of networked control systems with time-varying delays, Journal of the Franklin Institute 348 (2) (2011) 165–178.
[6] D. Xie, Y. Wu, Stabilisation of time-delay switched systems with constrained switching signals and its applications in networked control systems, IET Control Theory and Applications 4 (10) (2010) 2120–2128.
[7] S.-L. Dai, H. Lin, S. Ge, A switched system approach to scheduling of networked control systems with communication constraints, in: 48th IEEE Conf. on Dec. and Control, Shanghai, China, 2009, pp. 4991–4996.
[8] D. Liberzon, S. Morse, Basic problems in stability and design of switched systems, Control Systems Magazine (1999) 59–70.
[9] R. Decarlo, M. Branicky, S. Pettersson, B. Lennartson, Perspectives and results on the stability and stabilizability of hybrid systems, Proceedings of the IEEE 88 (7) (2000) 1069–1082.
[10] R. Shorten, F. Wirth, O. Mason, K. Wulff, C. King, Stability criteria for switched and hybrid systems, SIAM Review 49 (4) (2007) 545–592.
[11] H. Lin, P.J. Antsaklis, Stability and stabilisability of switched linear systems: a survey of recent results, IEEE Transactions on Automatic Control 54 (2) (2009) 308–322.
[12] A.V. Savkin, R.J. Evans, Hybrid Dynamical Systems. Controller and Sensor Switching Problems, Birkhäuser, 2002.
[13] D. Liberzon, Switching in Systems and Control, Birkhäuser, 2003.
[14] J. Daafouz, P. Riedinger, C. Iung, Stability analysis and control synthesis for switched systems: a switched Lyapunov function approach, IEEE Transactions on Automatic Control 47 (11) (2002) 1883–1887.
[15] Z. Ji, L. Wang, G. Xie, F. Hao, Linear matrix inequality approach to quadratic stabilisation of switched systems, IEE Proceedings: Control Theory and Applications 151 (3) (2004) 289–294.
[16] A. Sala, Computer control under time-varying sampling period: an LMI gridding approach, Automatica 41 (12) (2005) 2077–2082.
[17] Y. Suh, Stability and stabilization of nonuniform sampling systems, Automatica 44 (12) (2008) 3222–3226.
[18] M. Cloosterman, L. Hetel, N. van de Wouw, W. Heemels, J. Daafouz, H. Nijmeijer, Controller synthesis for networked control systems, Automatica 46 (10) (2010) 1584–1594. http://dx.doi.org/10.1016/j.automatica.2010.06.017.
[19] K. Wulff, F. Wirth, R. Shorten, A control design method for a class of switched linear systems, Automatica 45 (2009) 2592–2596.
[20] D. Liberzon, J. Hespanha, A. Morse, Stability of switched systems: a Lie-algebraic condition, Systems and Control Letters 37 (1999) 117–122.
[21] A. Agrachev, D. Liberzon, Lie-algebraic stability criteria for switched systems, SIAM Journal on Control and Optimization 40 (1) (2001) 253–269.
[22] L. Gurvits, Stability of discrete linear inclusion, Linear Algebra and its Applications 231 (1995) 47–85.
[23] J. Theys, Joint spectral radius: theory and approximations, Ph.D. Thesis, Center for Systems Engineering and Applied Mechanics, Université catholique de Louvain, 2005.
[24] K. Erdmann, M.J. Wildon, Introduction to Lie Algebras, Springer, 2006.
[25] H. Haimovich, J. Braslavsky, Conditions for the genericity of feedback stabilisability of discrete-time switching systems based on Lie-algebraic solvability, in: Australian Control Conf., Melbourne, Australia, 2011.
[26] H. Haimovich, J. Braslavsky, Sufficient conditions for the genericity of feedback stabilisability of switching systems via Lie-algebraic solvability, in: XIV Reunión de Trabajo en Procesamiento de la Información y Control, RPIC, Oro Verde, Entre Ríos, Argentina, 2011, pp. 346–351. Extended version available as arxiv.org:1109.3070v1 [cs.SY].
[27] H. Haimovich, J. Braslavsky, F. Felicioni, On feedback stabilisation of switched discrete-time systems via Lie-algebraic techniques, in: 48th IEEE Conf. on Dec. and Control, Shanghai, China, 2009, pp. 1118–1123.
[28] H. Haimovich, J. Braslavsky, F. Felicioni, Feedback stabilisation of switching discrete-time systems via Lie-algebraic techniques, IEEE Transactions on Automatic Control 56 (5) (2011) 1129–1135.
[29] H. Haimovich, J. Braslavsky, Feedback stabilization of switched systems via iterative approximate eigenvector assignment, in: 49th IEEE Conf. on Dec. and Control, Atlanta, GA, USA, 2010, pp. 1269–1274. Available at arXiv:1009.2032.
[30] F. Felicioni, S. Junco, A Lie algebraic approach to design of stable feedback control systems with varying sampling rate, in: IFAC World Congress, Seoul, South Korea, 2008.
[31] F. Felicioni, S. Junco, L'approche de l'algèbre de Lie pour l'adaptation des paramètres de la commande au changement de la période: stabilité, performance, et robustesse, in: Conf. Internationale Francophone d'Automatique, CIFA, Bucarest, 2008 (in French).
[32] F. Felicioni, S. Junco, H. Haimovich, Control de sistemas discretos LTI conmutados en base a métodos Lie-algebraicos, con aplicación al control sobre redes de comunicación, in: Congr. Argentino de Control Automático, AADECA, Buenos Aires, Argentina, 2008.
[33] H. Haimovich, E.N. Osella, J.H. Braslavsky, On Lie-algebraic-solvability-based feedback stabilization of systems with controller-driven sampling, in: IFAC World Congress, Milano, Italy, 2011, pp. 8731–8736.
[34] E. Osella, H. Haimovich, Sufficient Lie-algebraic-solvability-based conditions for stabilization of VSR-DTSSs with two unstable eigenvalues, in: XIV Reunión de Trabajo en Procesamiento de la Información y Control - Oro Verde, Argentina, 2011, pp. 863–868.
[35] T. Chen, B. Francis, Optimal Sampled-data Control Systems, Springer-Verlag, 1995.