# WEIGHTED INEQUALITIES OF FEFFERMAN-STEIN TYPE FOR RIESZ-SCHRÖDINGER TRANSFORMS

B. BONGIOANNI, E. HARBOURE AND P. QUIJANO

Abstract. In this work we are concerned with Fefferman-Stein type inequalities. More precisely, given an operator $T$ and some $p$, $1 < p < \infty$, we look for operators $\mathcal{M}$ such that the inequality

$$\int |Tf|^p w \leq C \int |f|^p \mathcal{M} w$$

holds true for any weight $w$. Specifically, we are interested in the case of $T$ being any first or second order Riesz transform associated to the Schrödinger operator $L = -\Delta + V$, with $V$ a non-negative function satisfying an appropriate reverse-Hölder condition. For the Riesz-Schrödinger transforms $\nabla L^{-1/2}$ and $\nabla^2 L^{-1}$ we make use of a result due to C. Pérez where this problem is solved for classical Calderón-Zygmund operators.

## 1. Introduction

In the theory of weighted $L^p$-inequalities a relevant question is the following: given an operator $T$ and $1 < p < \infty$, to find a positive operator $\mathcal{M}$ such that inequalities of the form

$$\int |Tf|^p w \;\leq\; \int |f|^p \mathcal{M} w, \tag{1}$$

hold for some reasonable set of functions $f$ defined on $\mathbb{R}^d$, $d \geq 1$, and a general weight $w$, i.e. $w \in L^1_{\text{loc}}(\mathbb{R}^d)$, $w \geq 0$. However, the above inequality becomes more interesting when $\mathcal{M}w$ is finite a.e. and to that end it is desirable to get the operator $\mathcal{M}$ as small as possible.

The first appearance of such inequality goes back to the classical result of Fefferman-Stein ([7]) for $T = \mathcal{M} = M$, the Hardy-Littlewood maximal operator, namely

$$\int_{\mathbb{R}^d} |Mf|^p w \;\leq\; \int_{\mathbb{R}^d} |f|^p M w,$$

for $1 < p < \infty$.

When $T$ is a singular integral operator, Córdoba and Fefferman showed in [4] that inequality (1) holds taking $\mathcal{M} = M_r = (M(w^r))^{1/r}$, for any $1 < r < \infty$. However, it is known that for the Hilbert transform that inequality fails for $r = 1$.

Later, Wilson in [11] obtained inequalities for $1 < p < 2$ and $\mathcal{M} = M \circ M$ improving the result in [4] since $M \circ M(w) \leq (M(w^r))^{1/r}$, for all $r > 1$.

In 1995, C. Pérez provided a full answer to this question with different techniques including weak type inequalities for $p = 1$. He deals with maximal operators

2010 *Mathematics Subject Classification.* Primary 42B20, Secondary 35J10.
*Key words and phrases.* Schrödinger operator, singular integral, weights.

associated to averages with respect to a Young function which can be smaller than $M_r$.

Below, we state the precise statements since they are essential to our work.

By a Young function $A$ we mean $A : [0,\infty) \to [0,\infty)$ continuous, convex, increasing and such that $A(0) = 0$. To define a maximal operator associated to a Young function $A$ we introduce the $A$-average of a function $f$ over a ball $B$ as

$$\|f\|_{A,B} = \inf\left\{\lambda > 0 : \frac{1}{|B|}\int_B A\left(\frac{|f(t)|}{\lambda}\right) dt \le 1\right\}.$$

Then, the maximal operator associated to a Young function $A$ is

$$M_A f(x) = \sup_{B\ni x} \|f\|_{A,B}.$$

For $1 < p < \infty$, we define $\mathcal{D}_p$ as the class of Young functions such that

$$\int_c^\infty \left(\frac{t}{A(t)}\right)^{p'-1} \frac{dt}{t} < \infty \tag{2}$$

for some $c > 0$.

The following theorem appears as Theorem 1.5 in [9]. There it is stated for singular integral operators. But according to the comment in Section 3 there, it also holds for Calderón-Zygmund operators as it is stated next.

**Theorem 1.** *Let $1 < p < \infty$, and let $T$ be a Calderón-Zygmund operator. Suppose that $A \in \mathcal{D}_p$. Then there exists a constant $C$ such that for each weight $w$*

$$\int |Tf|^p w \le C \int |f|^p M_A w.$$

The following theorem deals with the endpoint case $p = 1$ and it is also due to C. Pérez. Here we state a version that can be found in [5] as Theorem 9.31.

**Theorem 2.** *Let $T$ be a Calderón-Zygmund operator and let $A \in \bigcup_{p>1} \mathcal{D}_p$. Then there exists a constant $C$ such that for each weight $w$ and for all $\lambda > 0$ we have*

$$w(\{y \in \mathbb{R}^d : |Tf(y)| > \lambda\}) \le \frac{C}{\lambda}\int |f(y)| M_A w(y) dy.$$

Some examples of functions on the class $\mathcal{D}_p$ are $A(t) = t\log^{p-1+\varepsilon}(1+t)$ or $A(t) = t\log^{p-1}(1+t)\log^{p-1+\varepsilon}(\log(1+t))$ for any $\varepsilon > 0$. As for the class $\bigcup_{p>1}\mathcal{D}_p$, we can take $A(t) = t\log^{\varepsilon}(1+t)$ for any $\varepsilon > 0$.

In this work we attempt to provide results of this type for the first and second order Riesz transforms associated to the Schrödinger differential operator $L = -\Delta + V$ on $\mathbb{R}^d$, $d \ge 3$ and with $V$ satisfying a reverse Hölder inequality of order $q$, $q > d/2$, that is, there exists $C$ such that

$$\left(\frac{1}{|B|}\int_B V^q\right)^{1/q} \le C\frac{1}{|B|}\int_B V, \tag{3}$$

holds for every ball $B$ in $\mathbb{R}^d$. From now on, if a function $V$ satisfy (3) above we will say that $V \in RH_q$.

The study of these operators under such assumptions on $V$, was started by Shen in [10], where he proves $L^p$ boundedness for most of the operators we will be concerned with. As he observed, when $q > d$, the first order Riesz transforms $\nabla L^{-1/2}$

are standard Calderón-Zygmund operators. Otherwise, they are not necessarily bounded on $L^p$ for all $p$, $1 < p < \infty$. The case of the second order Riesz transforms given by $\nabla^2 L^{-1}$ is even worse since one can assure boundedness only for $1 < p < q$. However, even in the case that they are Calderón-Zygmund operators, we may expect in inequality (1) a smaller operator $\mathcal{M}$ than those given by Pérez, since Schrödinger Riesz transforms have kernels with a better decay at infinity. Also, in this context, kernels may have no symmetry and hence we might obtain different results for $T$ and its adjoint.

Essentially, we will consider two types of first and second order Riesz transforms: one involving only derivatives $\nabla L^{-1/2}$ and $\nabla^2 L^{-1}$, and the others involving the potential $V$, as $V^{1/2}L^{-1/2}$, $VL^{-1}$ and $V^{1/2}\nabla L^{-1}$. In the first case we will get our results by locally comparing with the classical Riesz transforms, allowing us to apply the results of C. Pérez. Let us point out that for $\nabla L^{-1/2}$ such comparison estimate appeared already in [10] but that is not the case for $\nabla^2 L^{-1}$, so it must be provided. We do that in Lemma 7 and we believe that it might be useful for other purposes. As for those operators involving $V$ we shall require only estimates on the size of their kernels.

We would like to make a remark about the values of $p$ for which inequalities like (1) will be obtained. In all instances the operator $\mathcal{M}$ on the right hand side satisfies $\mathcal{M}(1) \leq 1$ and therefore our results would imply boundedness on $L^p$, so the range of $p$ should be limited as in the original work of Shen.

The paper is organized as follows. In the next section we state some general theorems in a somehow abstract framework but having in mind the Schrödinger Riesz transforms mentioned above, leaving all the proofs and technical lemmas to Section 3

The results include strong type $(p, p)$ inequalities like (1) as well as weak type $(1, 1)$ estimates for a suitable class of operators and their adjoints. Let us remark that inequalities for the adjoint operators are not obtained by duality. In fact, if we proceed in that way, we would not arrive to an inequality with an arbitrary weight on the left hand side as we wanted.

Section 4 is devoted to apply the general theorems of Section 2 to specific operators associated to Schrödinger semigroup: $\nabla L^{-1/2}$, $\nabla^2 L^{-1}$, $V^\alpha L^{-\alpha}$, $V^{\alpha-1/2}\nabla L^{-\alpha}$, with $\alpha$ in a range depending on the operator. In order to check that their kernels satisfy the required assumptions, sometimes we make use of known estimates but in other occasions we must prove them. In particular we prove a local comparison between the kernels of $\nabla^2(-\Delta)^{-1}$ and $\nabla^2 L^{-1}$ stated in Lemma 7.

Finally in the last section we use the above results to get sufficient conditions on a function $f$ to ensure local $L^p$ integrability of $Tf$, where $T$ is any of the operators of Section 4. Consequently we obtain a large class of functions $f$ such that $Tf$ is finite a.e.. In fact, $f$ is allowed to increase polynomially. When these results are applied to the Riesz-Schrödinger transforms they provide qualitative information about some solutions of differential equations involving $L$.

## 2. General Results

In this section we will consider the space $\mathbb{R}^d$ equipped with a *critical radius function* $\rho : \mathbb{R}^d \to (0, \infty)$, that is, a function whose variation is controlled by the existence of $C_0$ and $N_0 \geq 1$ such that

$$C_0^{-1}\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N_0} \leq \rho(y) \leq C_0\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{N_0}{N_0+1}}. \tag{4}$$

It is worth noting that if $\rho$ is a critical radius function , then for any $\gamma > 0$ the function $\gamma\rho$ is also a critical radius function. Moreover, if $0 < \gamma \leq 1$ then $\gamma\rho$ satisfies (4) with the same constants as $\rho$.

Let us remark that in [10] a function $\rho$ satisfying (4) was introduced related with the potential $V$. But, once such a function $\rho$ is defined, there is not need of any further reference to $V$. So we choose to work in this frame in order to emphasize that fact.

Very often we will refer to critical balls, meaning balls of the type $B(x_0, \rho(x_0))$, and we shall call subcritical balls to those $B(x_0, r)$ with $r \leq \rho(x_0)$. Observe that from (4), $\rho(y) \simeq \rho(x_0)$ whenever $y \in B(x_0, \rho(x_0))$.

The next lemma is a useful consequence of (4).

**Lemma 1** (see [3], Corollary 1 )**.** *Let $y \in B(x_0, R)$. Then, there exists a constant $C > 0$ such that*

$$1+\frac{R}{\rho(y)} \leq C\left(1+\frac{R}{\rho(x_0)}\right)^{N_0+1}.$$

*Proof.* From (4) and the fact that $y \in B(x_0, R)$ we have

$$\frac{1}{\rho(y)} \leq \frac{C_0}{\rho(x_0)}\left(1+\frac{R}{\rho(x_0)}\right)^{N_0}.$$

Multiplying by $R$ and adding 1, we get

$$1+\frac{R}{\rho(y)} \leq C_0\left(\frac{R}{\rho(x_0)}\left(1+\frac{R}{\rho(x_0)}\right)^{N_0}+1\right) \leq C\left(1+\frac{R}{\rho(x_0)}\right)^{N_0+1},$$

where we used that $C_0 \geq 1$.

□

Associated to a critical radius function $\rho$ we can define the following maximal operators. First, let us denote $\mathcal{F}_\rho$ the set of all balls $B(x,r)$ such that $r \leq \rho(x)$. Then, for $f$ a locally integrable function, and $A$ a Young function, we set

$$M_A^{\text{loc}} f(x) = \sup_{\substack{B \ni x \\ B \in \mathcal{F}_\rho}} \|f\|_{A,B},$$

and for $\theta \geq 0$,

$$M_A^{\theta} f(x) = \sup_{B(x_0,r_0) \ni x} \left(1+\frac{r_0}{\rho(x_0)}\right)^{-\theta} \|f\|_{A,B}.$$

As usual, when $A(t) = t^r$ we use the notation $M_r^{\text{loc}}$ and $M_r^{\theta}$ respectively.

Now, we are in position to state our main theorems.

**Theorem 3.** *Let $T$ be a linear operator with associated kernel $K$. Suppose that for some $s > 1$, $K$ satisfies the following estimates*

$(a_s)$ *For each $N > 0$ there exists $C_N$ such that*

$$\left(\int_{R<|x_0-x|<2R} |K(x,y)|^s dx\right)^{1/s} \leq C_N R^{-d/s'}\left(1+\frac{R}{\rho(x_0)}\right)^{-N},$$

*whenever* $|y - x_0| < R/2$.

($b_s$) *There exists a Calderón-Zygmund operator* $T_0$ *with kernel* $K_0$ *such that, for some* $C$ *and* $\delta > 0$,

$$\left(\int_{R<|x_0-x|<2R} |K(x,y) - K_0(x,y)|^s dx\right)^{1/s} \leq CR^{-d/s'} \left(\frac{R}{\rho(x_0)}\right)^{\delta},$$

*whenever* $|y - x_0| < R/2$ *with* $R \leq \rho(x_0)$.

*Then, for each* $\theta \geq 0$, *the operator* $T$ *and its adjoint* $T^\star$ *satisfy the following inequalities for any weight* $w$,

$$\int |Tf|^p w \leq C_\theta \int |f|^p M_r^\theta w, \tag{5}$$

*for* $1 < p < s$ *and* $r = (s/p)'$,

$$\int |T^\star f|^p w \leq C_\theta \int |f|^p (M_A^{\text{loc}} + M^\theta) w, \tag{6}$$

*for* $s' < p < \infty$ *and any Young function* $A \in \mathcal{D}_p$.

*Remark* 1. Assumption ($a_s$) can be seen as a size condition with a kind of "decay at infinity", while condition ($b_s$) tells us that $K$ has the same singularity as a Calderón-Zygmund kernel. Nevertheless, both conditions on $K$ are not symmetric since integration is always made in the first variable. Consequently we do not get the same kind of estimates for $T$ and $T^\star$.

If the kernel $K$ satisfies point-wise estimates we obtain a sharper result for $T$, as a corollary of the previous theorem.

**Corollary 1.** *Let* $T$ *be a linear operator with associated kernel* $K$ *and* $T_0$ *be a Calderón-Zygmund operator with kernel* $K_0$. *Suppose that* $K$ *satisfy the following estimates.*

($a_\infty$) *For each* $N > 0$ *there exists* $C_N$ *such that*

$$|K(x,y)| \leq \frac{C_N}{|x-y|^d} \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N}.$$

($b_\infty$) *There exist* $C$ *and* $\delta > 0$ *such that*

$$|K(x,y) - K_0(x,y)| \leq \frac{C}{|x-y|^d} \left(\frac{|x-y|}{\rho(y)}\right)^{\delta}.$$

*Then,* $T$ *and its adjoint* $T^\star$ *satisfy* (6) *for* $1 < p < \infty$ *and any Young function* $A \in \mathcal{D}_p$.

Corollary 1 follows immediately from Theorem 3 since conditions ($a_\infty$) and ($b_\infty$) imply conditions ($a_s$) and ($b_s$) for all $1 < s < \infty$, and they can be made symmetric in $x$ and $y$ due to Lemma 1. For the limiting case $p = 1$ we obtain the following weak-type inequalities.

**Theorem 4.** *Let* $T$ *be a linear operator with associated kernel* $K$ *and let* $T_0$ *be a Calderón Zygmund operator with kernel* $K_0$. *Suppose that for some* $s > 1$, $K$ *satisfies conditions* ($a_s$) *and* ($b_s$) *Then, for* $\theta \geq 0$ *and* $w \in L^1_{loc}$, $w \geq 0$, $T$ *satisfies*

$$w(\{|Tf| > \lambda\}) \leq \frac{C_\theta}{\lambda} \int |f| M_{s'}^\theta(w), \text{ for } \lambda > 0. \tag{7}$$

*Further, if $T$ satisfies $(a_\infty)$ and $(b_\infty)$, then, for any Young function $A \in \bigcup_{p>1} \mathcal{D}_p$,*

$$w(\{|Tf| > \lambda\}) \leq \frac{C_\theta}{\lambda} \int |f| \left(M_A^{\text{loc}} + M^\theta\right) w, \ \text{for } \lambda > 0. \tag{8}$$

*Moreover, inequality (8) also holds for $T^\star$.*

The associated kernels of some operators related to $L$ satisfy condition $(b_s)$ without subtracting $K_0$ and hence condition $(a_s)$ and $(b_s)$ can be unified. For this type of operators we can get sharper inequalities stated in the following theorem.

**Theorem 5.** *Let $T$ be a linear operator with associated kernel $K$. Suppose that for some $s > 1$ and $\delta > 0$, $K$ satisfies the following condition:*

*$(c_s)$ For each $N > 0$, there exists $C_N$ such that for any $x_0 \in \mathbb{R}^d$ and $R > 0$,*

$$\left(\int_{R<|x_0-x|<2R} |K(x,y)|^s dx\right)^{1/s} \leq C_N R^{-d/s'} \left(1 + \frac{\rho(x_0)}{R}\right)^{-\delta} \left(1 + \frac{R}{\rho(x_0)}\right)^{-N},$$

*whenever $|y - x_0| < R/2$.*

*Then, for any $\theta \geq 0$ and any weight $w$, there exists $C_\theta$ such that $T$ satisfies (5) for $1 \leq p < s$ and*

$$\int |T^\star f|^p w \leq C_\theta \int |f|^p M^\theta w, \tag{9}$$

*for $s' < p < \infty$.*

## 3. Proofs

Before giving the proofs of the theorems above we need to state some technical lemmas that will be useful in the sequel. The first one gives a nice covering of $\mathbb{R}^d$ with critical balls. It is a consequence of inequality (4) and can be found in [6].

In some proofs we will use the notation $\lesssim$ instead of $\leq$ to denote that the right hand side of the inequality is greater up to multiplicative constants that may depend on some parameters specified when necessary.

**Proposition 1.** *There exists a sequence of points $\{x_j\}_{j\in\mathbb{N}}$ such that the family of critical balls $Q_j = B(x_j, \rho(x_j))$ satisfies*

*i)* $\bigcup_{j\in\mathbb{N}} Q_j = \mathbb{R}^d$

*ii) There exist constants $C$ and $N_1$ such that for any $\sigma \geq 1$, $\sum_{j\in\mathbb{N}} \chi_{\sigma Q_j} \leq C\sigma^{N_1}$.*

In general, maximal operators can not be controlled point-wisely by localized ones. Nevertheless, this is possible if we are considering functions supported on sub-critical balls and for points close enough to the support. In the next lemma we determine how much a critical ball must be contracted in order to have that kind of control. Such contraction of critical balls is needed to arrive to inequality (6) of Theorem 3.

**Lemma 2.** *Let $A$ be a Young function and $B_0$ any critical ball. There exists $\gamma_0 > 0$ such that if $0 < \gamma \leq \gamma_0$ then for any measurable function $f$,*

$$M_A(f\chi_{\gamma B_0})(x) \leq C M_A^{\text{loc}}(f)(x),$$

*for all* $x \in 2\gamma B_0$. *Here, the constant* $C$ *only depends on the dimension* $d$ *and the Young function* $A$.

*Proof.* Assume $x \in 2\gamma B_0$ with $\gamma$ to be determined later. It is enough to consider balls centered at $x$; in fact, it is not difficult to see that if $M_A^c$ is the centered maximal function, then $M_A(f)(x) \leq CM_A^c(f)(x)$ for any function $f$ with $C$ that only depends on $d$ and $A$. Let $x_0$ be the center of $B_0$ and suppose first that $r > 3\gamma\rho(x_0)$. Therefore $B(x,r) \supset B(x, 3\gamma\rho(x_0)) \supset \gamma B_0$ and thus, for any non-negative function $g$,

$$\frac{1}{|B(x,r)|}\int_{B(x,r)\cap\gamma B_0} g \ \leq \ \frac{1}{|B(x,3\gamma\rho(x_0))|}\int_{\gamma B_0} g \leq \frac{1}{|B(x,3\gamma\rho(x_0))|}\int_{B(x,3\gamma\rho(x_0))} g.$$

Now, if $\lambda > 0$, applying the above inequality to $g = A(|f|/\lambda)$ we have, for $r \geq 3\gamma\rho(x_0)$,

$$\|f\chi_{\gamma B_0}\|_{A,B(x,r)} \leq \|f\|_{A,B(x,3\gamma B_0)}.$$

Therefore, if $x \in 2\gamma B_0$,

$$M_A^c(f\chi_{\gamma B_0})(x) \ \leq \sup_{r\leq 3\gamma\rho(x_0)} \|f\|_{A,B(x,r)}.$$

To complete the proof, it is enough to take $\gamma$ such that $3\gamma\rho(x_0) \leq \rho(x)$ for all $x \in 2\gamma B_0$.

From inequality (4), we have $\rho(x_0) \leq \rho(x)C_0(1+2\gamma)^{N_0}$ and thus $\gamma$ should be taken such that

$$3\gamma C_0(1+2\gamma)^{N_0} \leq 1. \tag{10}$$

Since the left hand side goes to 0 when $\gamma$ goes to 0, there exists $\gamma_0$ such that for $0 < \gamma \leq \gamma_0$ the above inequality holds.

□

Conditions $(a_s)$ and $(b_s)$ are written in a suitable way to prove inequalities concerning $T^\star$. To prove the inequalities for $T$ it will be easier to use the following equivalent conditions.

**Lemma 3.** *For any* $s > 1$, *conditions* $(a_s)$ *and* $(b_s)$ *are equivalent, respectively, to the following conditions.*

$(a_s')$ *For each* $N > 0$ *there exists* $C_N$ *such that*

$$\left(\int_{B(x_0,R/2)} |K(x,y)|^s dx\right)^{1/s} \leq C_N R^{-d/s'}\left(1+\frac{R}{\rho(x_0)}\right)^{-N},$$

*whenever* $R < |y-x_0| < 2R$.

$(b_s')$ *There exist* $C$ *and* $\delta > 0$ *such that*

$$\left(\int_{B(x_0,R/2)} |K(x,y)-K_0(x,y)|^s dx\right)^{1/s} \leq CR^{-d/s'}\left(\frac{R}{\rho(x_0)}\right)^{\delta}$$

*whenever* $R < |y-x_0| < 2R$ *and* $R \leq \rho(x_0)$.

*Remark* 2. Observe that $(a_s)$ holds true replacing the ring, $R < |x-x_0| < 2R$ with $R < |x-x_0| < c_0R$ for any constant $c_0 > 1$, with the constant $C_N$ depending on $c_0$. Similarly in $(a_s')$ the ring $R < |y-x_0| < 2R$ may be replaced by $R < |y-x_0| < c_0R$. In fact, it is only a matter of applying $(a_s)$ or $(a_s')$ a finite number of times depending on $c_0$.

The same comment applies to $(b_s)$ and $(b'_s)$.

*Proof of Lemma 3.* We will show first that $(a_s)$ implies $(a'_s)$. Let $K$ be a kernel satisfying $(a_s)$ for some $s > 1$, and let $x_0 \in \mathbb{R}^d$, $R > 0$ and $y$ such that $R < |x_0 - y| < 2R$. It is easy to check that $B(x_0, R/2) \subset \{x : R/2 < |x - y| < 4R\}$. So, applying condition $(a_s)$ and the previous remark we get that

$$\begin{aligned}\left(\int_{B(x_0,R/2)} |K(x,y)|^s dx\right)^{1/s} &\leq \left(\int_{R/2<|y-x|<4R} |K(x,y)|^s dx\right)^{1/s} \\ &\leq C_N R^{-d/s'}\left(1+\frac{R}{\rho(y)}\right)^{-N} \\ &\leq C_N R^{-d/s'}\left(1+\frac{R}{\rho(x_0)}\right)^{-\tilde{N}},\end{aligned}$$

where in the last inequality we used Lemma 1.

To see that $(a'_s)$ implies $(a_s)$ let $x_0 \in \mathbb{R}^d$, $R > 0$ and $y \in B(x_0, R/2)$. The ring $\{x : R < |x - x_0| < 2R\}$ can be covered by $M$ balls (depending on $d$), of radius $R/4$ and centres $x_i$, with $R < |x_i - x_0| < 2R$, for $i = 1, \ldots, M$. For each of these balls we can check that $R/2 < |x_i - y| < 5R/2$. Applying condition $(a'_s)$ and Remark 2 on each ball,

$$\begin{aligned}\left(\int_{R<|y-x|<2R} |K(x,y)|^s dx\right)^{1/s} &\leq \sum_{i=1}^{M}\left(\int_{B(x_i,R/4)} |K(x,y)|^s dx\right)^{1/s} \\ &\leq \sum_{i=1}^{M} C_N R^{-d/s'}\left(1+\frac{R}{\rho(x_i)}\right)^{-N} \\ &\leq C_N R^{-d/s'}\left(1+\frac{R}{\rho(x_0)}\right)^{-\tilde{N}},\end{aligned}$$

where we used again Lemma 1 in the last inequality.

We can omit the proof of the equivalence of $(b_s)$ and $(b'_s)$ since it follows the same lines as above.

□

*Proof of Theorem 3.* Let $T$ be a linear operator with kernel $K$ satisfying $(a_s)$ and $(b_s)$, for some $s > 1$ and some Calderón-Zygmund operator $T_0$ with kernel $K_0$. Let $w \geq 0$, $w \in L^1_{loc}$, $\theta \geq 0$, $1 < p < s$ and let A be a Young function satisfying (2).

We will prove first inequality (5). Let $\gamma_0$ be as in Lemma 2. For some $\gamma \leq \gamma_0$, to be chosen later, let $\{Q_n\}$ be the decomposition of the space given in Proposition 1 for the critical radius function $\gamma\rho$. Then we write

(11)
$$\begin{aligned}\int |Tf|^p w &\le \sum_{n\in\mathbb{N}} \int_{Q_n} |Tf|^p w\\ &= \sum_{n\in\mathbb{N}} \int_{Q_n} |T(f\chi_{2Q_n}) + T(f\chi_{(2Q_n)^c}) \pm T_0(f\chi_{2Q_n})|^p w\\ &\lesssim \sum_{n\in\mathbb{N}} \int_{Q_n} |T(f\chi_{2Q_n}) - T_0(f\chi_{2Q_n})|^p w \ + \sum_{n\in\mathbb{N}} \int_{Q_n} |T(f\chi_{(2Q_n)^c})|^p w\\ &\qquad + \sum_{n\in\mathbb{N}} \int_{Q_n} |T_0(f\chi_{2Q_n})|^p w = I + II + III.\end{aligned}$$

For $III$, since $T_0$ is a Calderón-Zygmund operator, we apply Theorem 1 and Lemma 2 to get

$$\begin{aligned}III &= \sum_{n\in\mathbb{N}} \int |T_0(f\chi_{2Q_n})|^p w\chi_{Q_n}\\ &\lesssim \sum_{n\in\mathbb{N}} \int |f\chi_{2Q_n}| M_A(w\chi_{Q_n})\\ &\lesssim \sum_{n\in\mathbb{N}} \int_{2Q_n} |f|^p M_A^{\mathrm{loc}} w\\ &\lesssim \int |f|^p M_A^{\mathrm{loc}} w,\end{aligned}$$

for any Young function $A \in \mathcal{D}_p$.

For $k \in \mathbb{Z}$ we denote $Q_n^k = 2^k Q_n$. To estimate $II$ we use Minkowski's and Hölder's inequalities to obtain

$$\begin{aligned}II &= \sum_{n\in\mathbb{N}} \int_{Q_n} |T(f\chi_{(2Q_n)^c})|^p w\\ &= \sum_{n\in\mathbb{N}} \int_{Q_n} \left[\int_{(2Q_n)^c} |K(x,y)||f(y)|dy\right]^p w(x)dx\\ &\le \sum_{n\in\mathbb{N}} \left[\int_{(2Q_n)^c} |f(y)| \left(\int_{Q_n} |K(x,y)|^p w(x)dx\right)^{1/p} dy\right]^p\\ &\le \sum_{n\in\mathbb{N}} \left[\sum_{k\in\mathbb{N}} \int_{Q_n^{k+1}\setminus Q_n^k} |f(y)| \left(\int_{Q_n} |K(x,y)|^s dx\right)^{1/s} \left(\int_{Q_n} w^r(x)dx\right)^{1/rp} dy\right]^p,\end{aligned}$$

where $r = (s/p)'$.

Next we apply condition $(a_s')$ for $K$ since by Lemma 3 is equivalent to $(a_s)$. Then for each $N$ we have

$$\begin{aligned} II &\lesssim \sum_{n\in\mathbb{N}} \left[ \sum_{k\in\mathbb{N}} |Q_n^k|^{-1/s'} 2^{-kN} \int_{Q_n^k} |f(y)| \left( \int_{Q_n} w^r \right)^{1/rp} dy \right]^p \\ &\lesssim \sum_{n\in\mathbb{N}} \left[ \sum_{k\in\mathbb{N}} |Q_n^k|^{-1/s'+1/p'} 2^{-kN} \left( \int_{Q_n^k} |f(y)|^p \left( \int_{Q_n^k} w^r \right)^{1/r} dy \right)^{1/p} \right]^p \\ &\lesssim \sum_{n\in\mathbb{N}} \left[ \sum_{k\in\mathbb{N}} 2^{-k(N-\theta/p)} \left( \int_{Q_n^k} |f(y)|^p \, 2^{-k\theta} \left( \frac{1}{|Q_n^k|} \int_{Q_n^k} w^r \right)^{1/r} dy \right)^{1/p} \right]^p \\ &\lesssim \sum_{n\in\mathbb{N}} \left[ \sum_{k\in\mathbb{N}} 2^{-k(N-\theta/p)} \left( \int_{Q_n^k} |f(y)|^p \, M_r^\theta w(y) dy \right)^{1/p} \right]^p, \end{aligned}$$

with constants that may depend on $N$.

Finally, using Hölder's inequality in the sum over $k$ and choosing $N = N_1 + \theta/p + 1$, where $N_1$ is the constant appearing in Proposition 1, we arrive to

$$\begin{aligned} II &\lesssim \sum_{n\in\mathbb{N}} \left[ \sum_{k\in\mathbb{N}} 2^{-k(N_1+1)} \int_{Q_n^k} |f|^p \, M_r^\theta w \right] \left[ \sum_{k\in\mathbb{N}} 2^{-k(N_1+1)} \right]^{p/p'} \\ &\lesssim \sum_{k\in\mathbb{N}} 2^{-k(N_1+1)} \int_{\mathbb{R}^d} \left( \sum_{n\in\mathbb{N}} \chi_{Q_n^k} \right) |f|^p \, M_r^\theta w \\ &\lesssim \int_{\mathbb{R}^d} |f|^p \, M_r^\theta w, \end{aligned}$$

with constants depending on $N_1$ and $\theta$ and $p$.

It only remains to estimate $I$. Let $D(x, y) = K(x, y) - K_0(x, y)$. For $x \in Q_n$, we have $2Q_n \subset B(x, \rho(x))$ due to our choice of $\gamma$ (see inequality (10)), therefore we may write

$$\begin{aligned} I &= \sum_{n\in\mathbb{N}} \int_{Q_n} |T(f\chi_{2Q_n}) - T_0(f\chi_{2Q_n})|^p w \\ &\leq \sum_{n\in\mathbb{N}} \int_{Q_n} \left[ \int_{2Q_n} |D(x, y)||f(y)| dy \right]^p w(x) dx \\ &\leq \sum_{n\in\mathbb{N}} \int_{Q_n} \left[ \int_{B(x,\rho(x))} |D(x, y)||f(y)| dy \right]^p w(x) dx \\ &\leq \int_{\mathbb{R}^d} |h(x)|^p w(x) dx = \|h\|_{L^p(\mathbb{R}^d, w)}^p, \end{aligned} \tag{12}$$

where

$$h(x) = \int_{B(x,\rho(x))} |D(x, y)||f(y)| dy.$$

For a fixed $k$ and for any $n$ we can take $2^{dk}$ disjoint balls of the form $B^l_{n,k} = B(x^l_{n,k}, 2^{-k}\gamma\rho(x_n))$ such that for $\sigma > \sqrt{d}$,

$$Q_n \subset \bigcup_{l=1}^{2^{dk}} \sigma B^l_{n,k} \subset 2\sigma Q_n.$$

Moreover, there exists a constant depending only on $\sigma$ and $d$ such that,

$$\sum_{l=1}^{2^{dk}} \chi_{\sigma B^l_{n,k}} \leq C_{d,\sigma}\chi_{2\sigma Q_n}.$$

Therefore, from Proposition 1, the family of balls $\{\sigma B^l_{n,k}\}_{l,n}$ covers $\mathbb{R}^d$ and

$$\sum_{l,n} \chi_{\sigma B^l_{n,k}} \leq C_{d,\sigma,\rho}.$$

Let us fix $\sigma = 2\sqrt{d}$. It is possible to choose $\gamma$ small enough such that if $x \in \sigma B^l_{n,k}$ and $2^{-k-1}\rho(x) \leq |y - x| \leq 2^{-k}\rho(x)$ then

$$y \in E^l_{n,k} = \{y : 4\sqrt{d}\gamma 2^{-k}\rho(x_n) \leq |y - x^l_{n,k}| \leq \beta\gamma 2^{-k}\rho(x_n)\}.$$

for some constant $\beta > 4\sqrt{d}$ depending only on $\rho$ and $d$ [1].

Now, we write $h$ in the following way

$$h(x) = \sum_{k=0}^{\infty} h_k(x) = \sum_{k=0}^{\infty} \int_{B(x,2^{-k}\rho(x))\setminus B(x,2^{-k-1}\rho(x))} |D(x,y)||f(y)|dy.$$

So, for the covering of the space described above, we may write

$$\begin{aligned}
\|h_k\|^p_{L^p(w)} &\leq \sum_{n,l} \int_{2\sqrt{d}B^l_{n,k}} \left[\int_{B(x,2^{-k}\rho(x))\setminus B(x,2^{-k-1}\rho(x))} |D(x,y)||f(y)|dy\right]^p w(x)dx \\
&\leq \sum_{n,l} \int_{2\sqrt{d}B^l_{n,k}} \left[\int_{E^l_{n,k}} |D(x,y)||f(y)|dy\right]^p w(x)dx \\
&\leq \sum_{n,l} \left[\int_{E^l_{n,k}} |f(y)| \left(\int_{2\sqrt{d}B^l_{n,k}} |D(x,y)|^p w(x)dx\right)^{1/p} dy\right]^p \\
&\leq \sum_{n,l} \left[\int_{E^l_{n,k}} |f(y)| \left(\int_{2\sqrt{d}B^l_{n,k}} |D(x,y)|^s dx\right)^{1/s} \left(\int_{2\sqrt{d}B^l_{n,k}} w^r(x)dx\right)^{1/(rp)} dy\right]^p,
\end{aligned}$$

[1]For example, it works taking $\gamma = \frac{1}{2C_0(5\sqrt{d})^{N_0+1}}$ and $\beta = 2C_0^2(5\sqrt{d})^{N_0+2}$

where we have used Minkowski's and Hölder's inequalities in the last two steps. Now, using condition $(b_s')$ for $D(x,y)$ (See Remark 2), we arrive to

$$\begin{aligned}
\|h_k\|^p_{L^p(w)} &\lesssim \sum_{n,l}(2^{-k}\rho(x_n))^{-dp/s'}2^{-k\delta p}\left[\int_{\beta B^l_{n,k}}|f(y)|\left(\int_{\beta B^l_{n,k}}w^r(x)dx\right)^{1/(rp)}dy\right]^p\\
&\lesssim 2^{-k\delta p}\sum_{n,l}\int_{\beta B^l_{n,k}}|f(y)|^p\left(\frac{1}{|\beta B^l_{n,k}|}\int_{\beta B^l_{n,k}}w^r(x)dx\right)^{1/r}dy\\
&\lesssim 2^{-k\delta p}\sum_{n,l}\int_{\beta B^l_{n,k}}|f(y)|^p M^\theta_r w(y)dydy\\
&\lesssim 2^{-k\delta p}\|f\|^p_{L^p(M^\theta_r w)}.
\end{aligned}$$

Finally,

$$\|h\|_{L^p(w)} \le \sum_{k\ge 0}\|h_k\|_{L^p(w)} \lesssim \sum_{k\ge 0}2^{-k\delta}\|f\|_{L^p(M^\theta_r w)} \lesssim \|f\|_{L^p(M^\theta_r w)}. \tag{13}$$

Using the estimates obtained for $I$, $II$ and $III$ we arrive to inequality (5).

Now, let us prove inequality (6). Proceeding as in (11) we get `preguntar por el` $T_0^\star$

$$\begin{aligned}
\int|T^\star f|^p w \lesssim \sum_{n\in\mathbb{N}}\int_{Q_n}|T^\star(f\chi_{2Q_n})-T_0^\star(f\chi_{2Q_n})|^p w \;+\; \sum_{n\in\mathbb{N}}\int_{Q_n}|T^\star(f\chi_{(2Q_n)^c})|^p w\\
+\;\sum_{n\in\mathbb{N}}\int_{Q_n}|T_0^\star(f\chi_{2Q_n})|^p w = I^\star + II^\star + III^\star,
\end{aligned}$$

and we can estimate $III^\star$ in the same way as $III$, since $T^\star$ `acá va` $T_0^\star$`?` is also a Calderón-Zygmund operator.

For $II^\star$, we write

$$\begin{aligned}
II^\star &= \sum_{n\in\mathbb{N}}\int_{Q_n}|T^\star(f\chi_{(2Q_n)^c})|^p w\\
&= \sum_{n\in\mathbb{N}}\int_{Q_n}\left(\int_{(2Q_n)^c}|K(x,y)||f(x)|dx\right)^p w(y)dy.
\end{aligned}$$

If $y \in Q_n$ we may use Hölder inequality and condition $(a_s)$ to obtain

$$\begin{aligned}
&\int_{(2Q_n)^c} |K(x,y)||f(x)|dx \\
&\quad \leq \sum_{k\geq 1} \left( \int_{Q_n^{k+1}\setminus Q_n^k} |K(x,y)|^{p'} dx \right)^{1/p'} \left( \int_{Q_n^{k+1}} |f|^p \right)^{1/p} \\
&\quad \lesssim \sum_{k\geq 1} \left( \int_{Q_n^{k+1}\setminus Q_n^k} |K(x,y|^{s} dx \right)^{1/s} \left( \int_{Q_n^{k+1}} |f|^p \right)^{1/p} |Q_n^k|^{1/s'-1/p} \\
&\quad \lesssim \sum_{k\geq 1} 2^{-kN} \left( \frac{1}{|Q_n^{k+1}|} \int_{Q_n^{k+1}} |f|^p \right)^{1/p} \\
&\quad \lesssim \left[ \sum_{k\geq 1} \frac{2^{-kN}}{|Q_n^{k+1}|} \int_{Q_n^{k+1}} |f|^p \right]^{1/p} \left[ \sum_{k\geq 1} 2^{-kN} \right]^{1/p'} \\
&\quad \lesssim \left[ \sum_{k\geq 1} \frac{2^{-kN}}{|Q_n^{k+1}|} \int_{Q_n^{k+1}} |f|^p \right]^{1/p},
\end{aligned}$$

with constants independent of $n$.

Therefore,

$$\begin{aligned}
II^\star &\lesssim \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} 2^{-kN} \frac{1}{|Q_n^{k+1}|} \int_{Q_n^{k+1}} |f|^p \int_{Q_n^{k+1}} w(y)dy \\
&\lesssim \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} 2^{-k(N-\theta)} \int_{Q_n^{k+1}} |f|^p M^\theta w \\
&\lesssim \sum_{k\in\mathbb{N}} 2^{-k(N-\theta)} \int_{\mathbb{R}^d} \left( \sum_{n\in\mathbb{N}} \chi_{Q_n^{k+1}} \right) |f|^p M^\theta w \\
&\lesssim \int_{\mathbb{R}^d} |f|^p M^\theta w,
\end{aligned}$$

choosing $N = N_1 + \theta + 1$, where $N_1$ is the exponent given in Proposition 1.

It only remains to estimate $I^\star$. Proceeding as in (12), we have

$$I^\star = \sum_{n\in\mathbb{N}} \int_{Q_n} |T^\star(f\chi_{2Q_n}) - T_0^\star(f\chi_{2Q_n})|^p w \leq \|h^\star\|_{L^p(w)}^p,$$

where

$$h^\star(y) = \int_{B(y,\rho(y))} |D(x,y)||f(x)|dx,$$

and write

$$h^\star(y) = \sum_{k=0}^{\infty} h_k^\star(y) = \sum_{k=0}^{\infty} \int_{B(y,2^{-k}\rho(y))\setminus B(y,2^{-k-1}\rho(y))} |D(x,y)||f(x)|dx.$$

Now, for a fixed $k$, using Hölder's inequality and denoting $B(y, 2^{-k}\rho(y)) = B_y^k$, we have

$$h_k^\star(y) \le C \left( \int_{B_y^k \setminus B_y^{k-1}} |D(x,y)|^s dx \right)^{1/s} \left( \int_{B_y^k} |f|^p \right)^{1/p} (2^{-k}\rho(y))^{d/((s/p')'p')}.$$

Again for a fixed $k$, we consider the covering $\{B(x_{n,k}^l, 2\sqrt{d}\gamma 2^{-k}\rho(x_n))\}_{n,l}$. Using condition $(b_s)$, we obtain

$$\begin{aligned}
\|h_k^\star\|_{L^p(w)}^p &\le \sum_{n,l} \int_{B_{n,k}^l} |h_k^\star(y)|^p w(y) dy \\
&\lesssim \sum_{n,l} \left( \int_{\beta B_{n,k}^l} |f|^p \right) (2^{-k}\rho(y_j))^{dp(1/s'-1/p)} \int_{B_{n,k}^l} \left( \int_{E_{n,k}^l} |Q(x,y)|^s dx \right)^{p/s} w(y) dy \\
&\lesssim \sum_{n,l} \left( \int_{\beta B_{n,k}^l} |f|^p \right) \frac{2^{-kp\delta}}{(2^{-k}\rho(y_n))^d} \int_{\beta B_{n,k}^l} w \\
&\lesssim \sum_{n,l} 2^{-kp\delta} \int_{\beta B_{n,k}^l} |f(x)|^p \left( \frac{1}{|\beta B_{n,k}^l|} \int_{\beta B_{n,k}^l} w \right) dx \\
&\lesssim 2^{-kp\delta} \sum_{n,l} \int_{\beta B_{n,k}^l} |f|^p M^\theta w \\
&\lesssim 2^{-kp\delta} \|f\|_{L^p(M^\theta w)}^p.
\end{aligned}$$

So, as it was done in (13),

$$\|h^\star\|_{L^p(w)} \le C_\theta \sum_k \|h_k^\star\|_{L^p(w)} \lesssim \|f\|_{L^p(M^\theta w)}.$$

Using the estimates obtained for $I^\star$, $II^\star$ and $III^\star$ we arrive to inequality (6). □

*Remark* 3. It is worth noting that the estimates obtained for $I$ and $II$ also hold for the case $p = 1$. Following the same ideas as above we arrive to

$$\sum_{n\in\mathbb{N}} \int_{Q_n} |T(f\chi_{(2Q_n)^c})| w \le C_\theta \int_{\mathbb{R}^d} |f| M_{s'}^\theta(w),$$

$$\sum_{n\in\mathbb{N}} \int_{Q_n} |T(f\chi_{2Q_n}) - T_0(f\chi_{2Q_n})| w \le C_\theta \int_{\mathbb{R}^d} |f| M_{s'}^\theta(w).$$

Now we prove the weak-type inequalities stated in Theorem 4.

*Proof of Theorem 4.* Let $T$ be a linear operator with kernel $K$ and $w \in L^1_{loc}$, $w \ge 0$. Suppose first that $K$ satisfy conditions $(a_s)$ and $(b_s)$ for some $1 < s < \infty$ and $\delta > 0$. Consider again $\{Q_n\}_{n\in\mathbb{N}}$, the partition of the space associated to $\gamma\rho$, with $\gamma$ chosen as in the proof of Theorem 3. For $\lambda > 0$, we may write

$$
\begin{aligned}
w(\{|Tf| > \lambda\}) &\leq \sum_{n\in\mathbb{N}} w(\{x \in Q_n : |Tf(x)| > \lambda\}) \\
&\leq \sum_{n\in\mathbb{N}} w(\{x \in Q_n : |T(f\chi_{2Q_n})(x) - T_0(f\chi_{2Q_n})(x)| > \lambda/3\}) \\
&\qquad + \sum_{n\in\mathbb{N}} w(\{x \in Q_n : |T(f\chi_{(2Q_n)^c})(x)| > \lambda/3\}) \\
&\qquad\quad + \sum_{n\in\mathbb{N}} w(\{x \in Q_n : |T_0(f\chi_{2Q_n})(x)| > \lambda/3\}) \\
&= I + II + III.
\end{aligned} \tag{14}
$$

To estimate $III$ we can use this time Theorem 2 together with Lemma 2 to get

$$
\begin{aligned}
III &= \sum_{n\in\mathbb{N}} w(\{x \in Q_n : |T_0(f\chi_{2Q_n})(x)| > \lambda/3\}) \\
&\leq \sum_{n\in\mathbb{N}} w\chi_{Q_n}(\{x : |T_0(f\chi_{2Q_n})(x)| > \lambda/3\}) \\
&\lesssim \frac{1}{\lambda} \sum_{n\in\mathbb{N}} \int_{2Q_n} |f| M_A(w\chi_{Q_n}) \\
&\lesssim \frac{1}{\lambda} \sum_{n\in\mathbb{N}} \int_{2Q_n} |f| M_A^{loc}(w) \\
&\lesssim \frac{1}{\lambda} \int_{\mathbb{R}^d} |f| M_A^{loc}(w),
\end{aligned}
$$

for any Young function $A \in \bigcup_{p>1} \mathcal{D}_p$. In particular we can take $A(t) = t^{s'}$ since we will not get anything better for the other terms.

As for $I$ and $II$ we use the strong type inequalities for $p = 1$ stated on Remark 3. In this way we obtain (7).

Now, suppose that the kernel $K$ satisfy conditions $(a_\infty)$ and $(b_\infty)$. Let $\lambda > 0$, we use the same decomposition as in (14) to get

$$
w(\{|Tf > \lambda\}) \leq I + II + III.
$$

We deal with $III$ in the same way, obtaining

$$
III \lesssim \frac{1}{\lambda} \int_{\mathbb{R}^d} |f| M_A^{loc}(w),
$$

for any $A \in \bigcup_{p>1} \mathcal{D}_p$.

For $k \in \mathbb{Z}$ we set $Q_n^k = B(x_j, \gamma 2^k \rho(x_j))$. To estimate the term $II$ by the Tchebysheff's inequality we may write

$$
\begin{aligned}
II &= \sum_{n\in\mathbb{N}} w(\{x \in Q_n : |T(f\chi_{(2Q_n)^c})(x)| > \lambda/3\}) \\
&\leq \sum_{n\in\mathbb{N}} \frac{3}{\lambda} \int_{Q_n} |T(f\chi_{(2Q_n)^c})|(x) w(x) dx \\
&\leq \sum_{n\in\mathbb{N}} \frac{3}{\lambda} \int_{Q_n} \left( \sum_{k\in\mathbb{N}} \int_{Q_n^{k+1}\setminus Q_n^k} |K(x,y)||f(y)| dy \right) w(x) dx.
\end{aligned}
$$

Now, using condition $(a_\infty)$,

$$\begin{aligned} II &\lesssim \frac{1}{\lambda}\sum_{n\in\mathbb{N}}\int_{Q_n}\sum_{k\in\mathbb{N}}\frac{2^{-kN}}{(2^k\rho(x_n))^d}\left(\int_{Q_n^{k+1}}|f(y)|dy\right)w(x)dx \\ &\lesssim \frac{1}{\lambda}\sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}}2^{-kN}\int_{Q_n^{k+1}}|f(y)|\left(\frac{1}{|Q_n^{k+1}|}\int_{Q_n^{k+1}}w(x)dx\right)dy \\ &\lesssim \frac{1}{\lambda}\sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}}2^{-k(N-\theta)}\int_{Q_n^{k+1}}|f(y)|M^\theta(w)dy \\ &\lesssim \sum_{k\in\mathbb{N}}2^{-k(N-\theta)}\int_{\mathbb{R}^d}\left(\sum_{n\in\mathbb{N}}\chi_{Q_n^{k+1}}\right)|f(y)|M^\theta(w)dy \\ &\lesssim \int_{\mathbb{R}^d}|f(y)|M^\theta(w)dy, \end{aligned}$$

choosing $N = N_1 + \theta + 1$.

Next, to estimate $I$ we use the Tchebysheff's inequality and condition $(b_\infty)$.

$$\begin{aligned} I &= \sum_{j\in\mathbb{N}} w(\{x\in Q_n : |(T-T_0)(f\chi_{2Q_n})(x)| > \lambda/3\}) \\ &\le \sum_{n\in\mathbb{N}}\frac{3}{\lambda}\int_{Q_n}\left(\int_{2Q_n}|K(x,y)-K_0(x,y)||f(y)|dy\right)w(x)dx \\ &\lesssim \frac{1}{\lambda}\sum_{j\in\mathbb{N}}\int_{Q_n}\left(\int_{2Q_n}\frac{|f(y)|}{|x-y|^d}\left(\frac{|x-y|}{\rho(x)}\right)^\delta dy\right)w(x)dx \\ &\lesssim \frac{1}{\lambda}\sum_{n\in\mathbb{N}}\rho(x_n)^{-\delta}\int_{2Q_n}|f(y)|\int_{Q_n}|x-y|^{\delta-d}w(x)dx\,dy. \end{aligned}$$

Now, if $y\in 2Q_n$, and calling $B_n^y = B(y, 3\gamma\rho(x_n))$, then $Q_n\subset B_n^y$ and hence

$$\begin{aligned} &\int_{Q_n}|x-y|^{\delta-d}w(x)dx \\ &\qquad\le \sum_{k\in\mathbb{N}}\int_{2^{-k+1}B_n^y\setminus 2^{-k}B_n^y}|x-y|^{\delta-d}w(x)dx \\ &\qquad\le \rho(x_n)^\delta\sum_{k\in\mathbb{N}}\frac{2^{-k\delta}}{(2^{-k}\rho(x_n))^d}\int_{2^{-k+1}B_n^y}w \\ &\qquad\le \rho(x_n)^\delta M^{\mathrm{loc}}w(y), \end{aligned}$$

since $\rho(y)\simeq\rho(x_n)$. Therefore, we obtain

$$I \lesssim \frac{1}{\lambda}\sum_n\int_{2Q_n}|f|M^{\mathrm{loc}}w \lesssim \frac{1}{\lambda}\int_{\mathbb{R}^d}|f(y)|M^{\mathrm{loc}}w.$$

Altogether we obtain inequality (8). The same estimate is obtained for $T^\star$ since conditions $(a_\infty)$ and $(b_\infty)$ are symmetric on $x$ and $y$.

□

Finally, we end this section with the proof of Theorem 5.

*Proof of Theorem 5.* Let $T$ be a linear operator with associated kernel $K$ satisfying $(c_s)$. First, observe that condition $(c_s)$ implies both conditions $(a_s)$ and $(b_s)$ with $K_0 = 0$. Then, proceeding as in equation (11) we can write

$$\begin{aligned}\int |Tf|^p w &= \sum_{n\in\mathbb{N}} \int_{Q_n} |T(f\chi_{2Q_n}) + T(f\chi_{2Q_n^c})|^p w \\ &\lesssim \sum_{n\in\mathbb{N}} \int_{Q_n} |T(f\chi_{2Q_n})|^p w + \sum_{n\in\mathbb{N}} \int_{Q_n} |T(f\chi_{2Q_n^c})|^p w \\ &= I + II.\end{aligned}$$

Then, inequality (5) holds for $1 \le p < s$ following the same lines as in the proof of Theorem 3 and taking into account Remark 3 for $p = 1$.

To obtain estimate (9) we proceed as above to get

$$\int |T^\star f|^p w \le \sum_{n\in\mathbb{N}} \int_{Q_n} |T^\star(f\chi_{2Q_n})|^p w + \sum_{n\in\mathbb{N}} \int_{Q_n} |T^\star(f\chi_{2Q_n^c})|^p w = I^\star + II^\star,$$

and we deal with $I^\star$ and $II^\star$ as in the proof of Theorem 3.

□

## 4. Application to Schrödinger operators

In this section we apply our general results to operators associated to the semigroup generated by the Schrödinger differential operator $L = -\Delta + V$ on $\mathbb{R}^d$ with $d \ge 3$. We will always suppose that the potential $V$ is a non-negative function, non-identically zero, satisfying a reverse Hölder condition of order $q > d/2$. Under these assumptions the function $\rho$ defined by

$$\rho(x) = \sup\left\{ r > 0 : \frac{1}{r^{d-2}} \int_{B(x,r)} V \le 1 \right\},\ x \in \mathbb{R}^d,$$

is a critical radius function, that is, property (4) is satisfied for some constants $C_0$ and $N_0$.

It is known that $V \in RH_q$, $q > 1$ implies that $V$ is a doubling measure, i.e. there exists $C_1$ such that

$$\int_{B(x,2r)} V \le C_1 \int_{B(x,r)} V.$$

In fact, if $V \in RH_q$, $q > 1$, then $V$ belongs to the $A_\infty$ class of Muckenhoupt.

The following is an useful inequality for $V \in RH_q$ with $q > d/2$ that follows easily from Lemma 1.2 and Lemma 1.8 in [10].

**Lemma 4.** *Let $V \in RH_q$ for some $q > d/2$. Let $N_2 = \log_2 C_1 + 2 - d$, where $C_1$ is the doubling constant of $V$. Then, for any $x_0 \in \mathbb{R}^d$, $R > 0$,*

$$\frac{1}{R^{d-2}} \int_{B(x_0,R)} V(y)dy \le C \left(1 + \frac{R}{\rho(x_0)}\right)^{N_2} \left(1 + \frac{\rho(x_0)}{R}\right)^{d/q-2}.$$

*Remark* 4. Observe that when $R < \rho(x_0)$ the leading term is the second factor while the latter is bounded by a constant when $R \ge \rho(x_0)$.

For the fundamental solution of $L$, the following estimate was shown in [10].

**Lemma 5.** *Let $V \in RH_q$, with $q > d/2$ and $\Gamma$ the fundamental solution of the operator $L$ in $\mathbb{R}^d$. Then for each $N > 0$, there exists a constant $C_N$ such that*

$$|\Gamma(x,y)| \leq C_N \frac{1}{|x-y|^{d-2}} \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N}.$$

4.1. **Riesz-Schrödinger transforms.** We consider the operators $\mathcal{R}_1 = \nabla L^{-1/2}$ and $\mathcal{R}_2 = \nabla^2 L^{-1}$, the Riesz-Schrödinger transforms of order 1 and 2 respectively. Let $K_1$ and $K_2$ be their associated kernels.

The size condition $(a_s)$ can be found in [1], for both $K_1$ and $K_2$. In fact, $(a_s)$ holds for $K_1$ with $s = p_0$, where $\frac{1}{p_0} = \frac{1}{q} - \frac{1}{d}$, in the case $d/2 < q < d$ (see page 28 in [1]), and $s = \infty$ for $q \geq d$ (see inequality (6.5) in [10]). Regarding $K_2$, it satisfies $(a_s)$ for $s = q$ (see Proposition 8 in [1]).

To prove that these kernels satisfy also condition $(b_s)$ we will compare them with the classical Riesz transforms $R_1 = \nabla(-\Delta)^{-1/2}$ and $R_2 = \nabla^2(-\Delta)^{-1}$ and their associated kernels $K_{0,1}$ and $K_{0,2}$.

Before proving condition $(b_s)$ we present the following lemmas that provide us point-wise estimates for the difference between the kernels associated to the Riesz-Schrödinger transforms and the classical ones. For the Riesz-Schrödinger transform of order 1 such result was already obtained by Shen. On the other hand, the estimate corresponding to the second order operator is new and we believe is interesting in its own right.

**Lemma 6.** *[See [10], inequality (5.9)] Let $V \in RH_q$ for $d/2 < q < d$. There exists $C$ such that*

$$|K_1(x,y) - K_{0,1}(x,y)| \leq \frac{C}{|x-y|^{d-1}} \left( G(x,y) + \frac{1}{|x-y|} \left(\frac{|x-y|}{\rho(x)}\right)^{2-d/q} \right),$$

*where*

$$G(x,y) = \int_{B(x,|x-y|/4)} \frac{V(u)}{|u-x|^{d-1}} du. \tag{15}$$

**Lemma 7.** *Let $x$, $y_0 \in \mathbb{R}^d$ and $R > 0$ such that $R \leq |y - x_0| \leq \rho(x_0)$. Let $x \in B(x_0, R/8)$. Then there exists a constant $C$ such that*

$$|K_2(x,y) - K_{0,2}(x,y)| \leq C|R_2(V\Gamma(y,\cdot)\chi_{B(x_0,R/4)})(x)| + \frac{C}{R^d}\left(\frac{R}{\rho(x_0)}\right)^{\delta},$$

*with $\delta = \min\{1, 2 - d/q\}$.*

*Proof.* Let $\Gamma$ and $\Gamma_0$ be the fundamental solution of $L$ and $-\Delta$ respectively. As it was shown in [10], page 540,

$$\Gamma(x,y) - \Gamma_0(x,y) = -\int_{\mathbb{R}^d} \Gamma_0(x,\xi)V(\xi)\Gamma(y,\xi)d\xi.$$

From this we get the following expression for the difference of the kernels.

$$K_2(x,y) - K_{0,2}(x,y) = \nabla_1^2\Gamma(x,y) - \nabla_1^2\Gamma_0(x,y) = -\nabla_1^2 \int_{\mathbb{R}^d} \Gamma_0(x,\xi)V(\xi)\Gamma(y,\xi)d\xi.$$

Next, we define the domains $J_1 = B(x_0, R/4)$, $J_2 = B(y, R/4)$ and $J_3 = (J_1 \cup J_2)^c$. The term corresponding to the integral over $J_1$ is, upon a sign, the classical second order Riesz transform applied to function in $L^q$ with compact support, that is

$$|\nabla_1^2 \int_{J_1} \Gamma_0(x,\xi)V(\xi)\Gamma(y,\xi)d\xi| = |R_2(V\Gamma(y,\cdot)\chi_{B(x_0,R/4)})(x)|.$$

On $J_2$, since we are away from the singularity of $\Gamma_0$, we can use the size estimate for $\Gamma$ given in Lemma 5, together with Hölder's inequality to obtain

$$\begin{aligned}\left|\int_{J_2} \nabla_1^2 \Gamma_0(x,\xi)V(\xi)\Gamma(y,\xi)d\xi\right| &\\ &\leq \frac{C}{R^d}\int_{B(y,R/4)} \frac{V(\xi)}{|y-\xi|^{d-2}}d\xi \\ &\leq \frac{C}{R^d}\left(\int_{B(y,R/4)} V^q(\xi)d\xi\right)^{\frac{1}{q}}\left(\int_{B(y,R/4)} \frac{d\xi}{|y-\xi|^{(d-2)q'}}\right)^{\frac{1}{q'}}.\end{aligned}$$

For the first integral we can use the reverse Hölder condition for $V$ together with Lemma 4, while on the second integral $q > d/2$ implies that $(d-2)q' < d$. Then

$$\left|\int_{J_2} \nabla_1^2\Gamma_0(x,\xi)V(\xi)\Gamma(y,\xi)d\xi\right| \lesssim \frac{1}{R^d}\left(\frac{R}{\rho(x_0)}\right)^{2-d/q},$$

since $y \in B(x_0, \rho(x_0))$.

To estimate the integral on $J_3$ we divide $J_3 = J_{31} \cup J_{32}$, where $J_{31} = \{\xi \in \mathbb{R}^d : R/4 \leq |y-\xi| < 2R \wedge |x_0 - \xi| \geq R/4\}$ and $J_{32} = \{\xi \in \mathbb{R}^d : |y-\xi| \geq 2R\}$. On $J_{31}$ we are away from the singularities of both $\Gamma$ y $\Gamma_0$, then

$$\begin{aligned}\left|\int_{J_{31}} \nabla_1^2\Gamma_0(x,\xi)V(\xi)\Gamma(y,\xi)d\xi\right| &\lesssim \int_{J_{31}} \frac{V(\xi)}{|x-\xi|^d|y-\xi|^{d-2}}d\xi \\ &\lesssim \frac{1}{R^{2d-2}}\int_{B(y,2R)} V(\xi)d\xi \\ &\lesssim \frac{1}{R^d}\left(\frac{R}{\rho(x_0)}\right)^{2-d/q},\end{aligned}$$

where in the last inequality we have used again Lemma 4.

Regarding $J_{32}$ it is easy to check that $|x-\xi| \geq 3|y-\xi|/8$. Then, using Lemma 5 again,

(16)
$$\begin{aligned}\left|\int_{J_{32}} \nabla_1^2\Gamma_0(x,\xi)V(\xi)\Gamma(y,\xi)d\xi\right| &\leq C_N\int_{J_{32}} \frac{V(\xi)}{|x-\xi|^d|y-\xi|^{d-2}}\left(1+\frac{|y-\xi|}{\rho(y)}\right)^{-N}d\xi \\ &\leq C_N\int_{J_{32}} \frac{V(\xi)}{|y-\xi|^{2d-2}}\left(1+\frac{|y-\xi|}{\rho(y)}\right)^{-N}d\xi.\end{aligned}$$

Assume firts that $2R < \rho(y)$. We split the integral in $J_{321} = \{\xi \in \mathbb{R}^d : 2R \leq |y-\xi| < \rho(y)\}$ and $J_{322} = \{\xi \in \mathbb{R}^d : |y-\xi| \geq \rho(y)\}$. For the integral on $J_{321}$, let $k_0 \in \mathbb{N}$ such that $2^{k_0-1}R \leq \rho(y) \leq 2^{k_0}R$. Then using again Lemma 4 and that

$d > 2 - d/q$,

$$\begin{aligned}\int_{2R\le|y-\xi|<\rho(y)} \frac{V(\xi)}{|y-\xi|^{2d-2}}d\xi &\le \sum_{k=2}^{k_0}\int_{2^{k-1}R\le|y-\xi|<2^kR}\frac{V(\xi)}{|y-\xi|^{2d-2}}\\ &\lesssim \sum_{k=1}^{k_0}\frac{1}{(2^kR)^d}\frac{1}{(2^kR)^{d-2}}\int_{B(y,2^kR)}V(\xi)d\xi\\ &\lesssim \frac{1}{R^d}\sum_{k=1}^{k_0}2^{-kd}\left(\frac{2^kR}{\rho(y)}\right)^{2-d/q}\\ &\lesssim \frac{1}{R^d}\left(\frac{R}{\rho(x_0)}\right)^{2-d/q},\end{aligned}$$

since $y \in B(x_0, \rho(x_0))$, and hence $\rho(y) \simeq \rho(x_0)$.

On $J_{322}$, let $\mu = \log_2 C_1$, where $C_1$ is the doubling constant of the potential $V$. Then we bound the right hand side of (16) by a constant (that may depend on $N$) times

$$\begin{aligned}&\int_{|x-\xi|\ge\rho(y)}\frac{V(\xi)}{|y-\xi|^{2d-2}}\left(\frac{\rho(y)}{|y-\xi|}\right)^N d\xi\\ &\lesssim \sum_{k=1}^{\infty}\frac{1}{2^{kN}}\int_{2^{k-1}\rho(y)|y-\xi|<2^{k+}\rho(y)}\frac{V(\xi)}{|y-\xi|^{2d-2}}\\ &\le \sum_{k=1}^{\infty}\frac{1}{2^{k(2d-2+N)}\rho(y)^{2d-2}}\int_{B(y,2^k\rho(y))}V(\xi)d\xi\\ &\lesssim \frac{1}{\rho(y)^d}\sum_{k=1}^{\infty}\frac{1}{2^{k(2d-2+N-\mu)}\rho(y)^{d-2}}\int_{B(y,\rho(y))}V(\xi)d\xi\\ &\lesssim \frac{1}{\rho(y)^d}\le\frac{1}{R^d}\left(\frac{R}{\rho(x_0)}\right)^{2-d/q},\end{aligned}$$

choosing $N$ big enough and using that $\rho(y) \simeq \rho(x_0)$, $R < \rho(x_0)$ and $2 - d/q < d$.

□

For $\mathcal{R}_1$ different inequalities hold true depending on $q$. For $q > d$, Shen showed in [10] that $\mathcal{R}_1$ and $\mathcal{R}_1^\star$ are Calderón-Zygmund operators. Moreover, their associated kernels satisfy the stronger size condition ($(a_\infty)$) (see inequality (6.5) there). Later on, condition ($(b_\infty)$) was proved for the difference between $K_1$ and $K_{1,0}$ (see [2] Lemma 3).

Therefore, as an application of Theorem 3, Corollary 1 and Theorem 4 we obtain the following result.

**Theorem 6.** *Let $V \in RH_q$ with $q > d/2$, $\theta \ge 0$ and $p_0$ such that $1/p_0 = (1/q - 1/d)^+$. Then for any weight $w$ the following inequalities hold.*

$$\int |\mathcal{R}_1 f|^p w \le C_\theta \int |f|^p M_r^\theta w, \tag{17}$$

*when $d/2 < q < d$, for $1 < p < p_0$ and $r = (p_0/p)'$.*

$$\int |\mathcal{R}_1^\star f|^p w \le C_\theta \int |f|^p (M_A^{loc} w + M^\theta w), \tag{18}$$

*when $q > d/2$, for $p_0' < p < \infty$, and any Young function $A \in \mathcal{D}_p$, and*

$$w(\{|\mathcal{R}_1 f| > \lambda\}) \leq \frac{C_\theta}{\lambda} \int |f| M^\theta_{p_0'} w, \tag{19}$$

*for any $q > d/2$.*

*Moreover, if $q > d$, we have*

$$\int |\mathcal{R}_1 f|^p w \leq C_\theta \int |f|^p (M^{loc}_A w + M^\theta w), \tag{20}$$

*for $1 < p < \infty$ and any Young function $A \in \mathcal{D}_p$. Also*

$$w(\{|\mathcal{R}_1 f| > \lambda\}) \leq \frac{C_\theta}{\lambda} \int |f| M^\theta_A w, \tag{21}$$

*and*

$$w(\{|\mathcal{R}^\star_1 f| > \lambda\}) \leq \frac{C_\theta}{\lambda} \int |f| M^\theta_A w, \tag{22}$$

*for any Young function $A \in \bigcup_{p>1} \mathcal{D}_p$.*

*Proof.* Let $V \in RH_q$ for $q > d/2$. Suppose first that $q < d$. As we said, condition $(a_s)$ was proved in [1] for $s = p_0$ with $1/p_0 = (1/q - 1/d)$. Therefore it is enough to check $(b'_{p_0})$ which is equivalent to $(b_{p_0})$.

Let $x_0 \in \mathbb{R}^d$, $0 < R \leq \rho(x_0)$ and $R < |y - x_0| < 2R$. First, we make use of Lemma 6. Due to the boundedness of the classical fractional integral operator $I_1$ and the reverse Hölder property of $V$ we get that, for $G$ defined in (15),

(23)

$$\begin{aligned}
\left(\int_{B(x_0,R/2)} \left(\frac{G(x,y)}{|x-y|^{d-1}}\right)^{p_0} dx\right)^{1/p_0} &\leq \frac{C}{R^{d-1}} \left(\int_{B(x_0,R/2)} \left(\int_{B(x_0,R)} \frac{V(u)}{|u-x|} du\right)^{p_0} dx\right)^{1/p_0} \\
&\leq \frac{C}{R^{d-1}} \left(\int_{\mathbb{R}^d} |I_1(\chi_{B(x_0,R)} V)|^{p_0}\right)^{1/p_0} \\
&\leq \frac{C}{R^{d-1}} \left(\int_{B(x_0.R)} V^q\right)^{1/q} \\
&\leq C \frac{R^{d/q-d}}{R^{d-1}} \int_{B(x_0,R)} V \\
&\leq C R^{-d/p_0'} \left(\frac{R}{\rho(x_0)}\right)^{2-d/q},
\end{aligned}$$

where, in the last inequality, we have used Lemma 4. As for the second term appearing in Lemma 6, the same estimate holds easily. Therefore, inequalities (17), (18) and (19) follow as an application of Theorems 3 and 4.

Next, suppose that $q > d$. In this case, as we mentioned it is known that $K_1$ satisfy the point-wise estimates $(a_\infty)$ and $(b_\infty)$. For the size condition we refer to inequality (6.5) in [10]. Condition $(b_\infty)$ was stated and proved in [2], Lemma 3. Thus, applying now Corollary 1 and Theorem 4 we obtain inequalities (20), (21) and (22).

□

As an application of Lemma 7, Theorem 3 and Theorem 4 we obtain the following inequalities for $\mathcal{R}_2$.

**Theorem 7.** *Let $V \in RH_q$ for $q > d/2$, and $\theta \geq 0$. Then, for any weight $w$ the following inequalities hold.*

$$\int |\mathcal{R}_2 f|^p w \leq C_\theta \int |f|^p M_r^\theta w, \tag{24}$$

*for $1 < p < q$ and $r = (q/p)'$,*

$$\int |\mathcal{R}_2^\star f|^p w \leq C_{\theta,A} \int |f|^p (M_A^{loc} + M^\theta) w, \tag{25}$$

*for $q' < p < \infty$ and any Young function $A \in \mathcal{D}_p$,*

$$w(\{|\mathcal{R}_2 f| > \lambda\}) \leq \frac{C_\theta}{\lambda} \int |f| M_{q'}^\theta w. \tag{26}$$

*Proof.* As we said before, it only remains to check condition $(b_s')$ for the kernel $K_2$. Let $x_0$, $y \in \mathbb{R}^d$ and $R > 0$ such that $R < |y - x_0| < 2R$ and $R \leq \rho(x_0)$. We are going to check condition $(b_s')$ with $s = q$ using Lemma 7,

$$\left( \int_{B(x_0,R/2)} |K_2(x,y) - K_{2,0}(x,y)|^q dx \right)^{1/q} \leq \left( \int_{B(x_0,R/2)} \left( |R_2(V\Gamma(y,\cdot)\chi_{B(x_0,R/4)})(x)| + \frac{C}{R^d} \left( \frac{R}{\rho(x_0)} \right)^\delta \right)^q dx \right)^{1/q}.$$

Dividing the integral in two terms it is straightforward that the second one gives us the desired estimate. For the first one, recalling that $R_2$ is a bounded operator on $L^q$ for $1 < q < \infty$, and applying Lemma 4,

$$\begin{aligned} \left( \int_{B(x_0,R/2)} |R_2(V\Gamma(y,\cdot)\chi_{B(x_0,R/4)})(x)|^q dx \right)^{1/q} &\lesssim \left( \int_{B(x_0,R/4)} V^q(x) |\Gamma(y,x)|^q dx \right)^{1/q} \\ &\lesssim \frac{1}{R^{d-2}} \left( \int_{B(x_0,R/4)} V^q \right)^{1/q} \\ &\lesssim R^{-d/q'} \left( \frac{R}{\rho(x_0)} \right)^{2-d/q}. \end{aligned}$$

□

*Remark* 5. Observe that except for $\mathcal{R}_1$ in the case $q > d$, the maximal operators on the right hand side are better for the adjoints $\mathcal{R}_1^\star$, $\mathcal{R}_2^\star$, even for common values of $p$. Also, the maximal operators appearing in (17) and (24), get closer to those in (18) and (25) as $q$ goes to $d$ or infinity, respectively.

4.2. **Operators $V^\gamma L^{-\gamma}$.** We consider, for $V \in RH_q$, $q > d/2$, the family of operators of type $V^\gamma L^{-\gamma}$ for $0 < \gamma < d/2$. For each $\gamma$, we can write $K_\gamma$, the kernel of $V^\gamma L^{-\gamma}$, as

$$K_\gamma(x,y) = V^\gamma(x) J_\gamma(x,y),$$

where $J_\gamma$ is the corresponding kernel of the fractional integral operator $L^{-\gamma}$. For $J_\gamma$ we have the following estimate that can be found in [8], page 587. For each $N > 0$ there exists $C_N$ such that

$$|J_\gamma(x,y)| \leq \frac{1}{|x-y|^{d-2\gamma}} C_N \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N}. \tag{27}$$

We will show next that the size estimate for $J_\gamma$ gives us condition $(c_s)$ for $K_\gamma$ with $s = q/\gamma$. In fact, let $x_0$, $y \in \mathbb{R}^d$ and $R$ such that $|y - x_0| < R/2$. Applying Lemma 1 and Lemma 4 we get

(28)

$$\begin{aligned}\left(\int_{R<|x-x_0|<2R} |K_\gamma(x,y)|^{q/\gamma} dx\right)^{\gamma/q} \\ &\leq \frac{C_N}{R^{d-2\gamma}} \left(1 + \frac{R}{\rho(x_0)}\right)^{-N/(N_0+1)} \left(\int_{B(x_0,2R)} V^q\right)^{\gamma/q} \\ &\lesssim R^{-d/(q/\gamma)'} \left(1 + \frac{R}{\rho(x_0)}\right)^{-N/(N_0+1)+\gamma N2} \left(1 + \frac{\rho(x_0)}{R}\right)^{-\gamma(2-d/q)}.\end{aligned}$$

The above estimate together with Theorem 5 give us the following result.

**Theorem 8.** *Let $V \in RH_q$ for $q > d/2$, $0 < \gamma < d/2$ and $\theta \geq 0$. Then, for any weight $w$,*

$$\int |V^\gamma L^{-\gamma} f|^p w \leq C_\theta \int |f|^p M_r^\theta w,$$

*for $1 \leq p < q/\gamma$, $r = (q/(\gamma p))'$ and*

$$\int |L^{-\gamma} V^\gamma f|^p w \leq C_\theta \int |f|^p M^\theta w,$$

*for $(q/\gamma)' < p < \infty$.*

4.3. **Operators $V^{\gamma-1/2}\nabla L^{-\gamma}$.** We consider the family of operators $V^{\gamma-1/2}\nabla L^{-\gamma}$ for $1/2 < \gamma \leq 1$ that includes the operator $L^{-1}\nabla V^{1/2}$ which appeared first in [10]. As a consequence of the results in Section 4.2 in [1], the associated kernel $\mathcal{K}^\gamma$ can be written as the product $\mathcal{K}^\gamma(x,y) = V^{\gamma-1/2}(x) K_{2\gamma-1}(x,y)$, with $K_{2\gamma-1}$ a fractional kernel of order $2\gamma - 1$, satisfying for each $N$,

$$|K_{2\gamma-1}(x,y)| \leq \frac{C_N}{|x-y|^{d-2\gamma+1}} \left(1 + \frac{|x-y|}{\rho(y)}\right)^{-N}, \tag{29}$$

as long as $V \in RH_q$ with $q > d$, and

$$\left(\int_{R<|x-y|<2R} |K_{2\gamma-1}(x,y)|^{p_0} dx\right)^{1/p_0} \leq C R^{-d/p_0'+2\gamma-1} \left(1 + \frac{R}{\rho(y)}\right)^{-N}, \tag{30}$$

when $d/2 < q < d$, with $p_0$ such that $1/p_0 = 1/q - 1/d$. In fact, estimates of this type were shown in [1] (see estimates (66) and (67) there) for the kernel of the adjoint operator $L^{-\gamma}\nabla V^{\gamma-1/2}$.

We will show now that these inequalities imply condition $(c_s)$ for $s = p_\gamma$ such that

$$\frac{1}{p_\gamma} = \left(\frac{1}{q} - \frac{1}{d}\right)^+ + \frac{2\gamma - 1}{2q}. \tag{31}$$

Let $x_0,\ y \in \mathbb{R}^d$ and $R > 0$ such that $|y - x_0| < R/2$. If $q > d$, and hence $\frac{1}{p_\gamma} = \frac{2\gamma-1}{2q}$, we may use estimate (29), Lemma 4 and the reverse Hölder inequality to get, as in (28),

$$\begin{aligned}
&\left(\int_{R<|x-x_0|<2R} |K_\gamma(x,y)V^{\gamma-1/2}(x)|^{\frac{2q}{2\gamma-1}}\,dx\right)^{\frac{2\gamma-1}{2q}} \\
&\qquad \le \frac{C_N}{R^{d-2\gamma+1}}\left(\int_{B(x_0,2R)} V^q\right)^{\frac{2\gamma-1}{2q}} \left(1+\frac{R}{\rho(x_0)}\right)^{-N/(N_0+1)} \\
&\qquad \lesssim R^{-d/p'_\gamma}\left(1+\frac{\rho(x_0)}{R}\right)^{-(\gamma-1/2)(2-d/q)} \left(1+\frac{R}{\rho(x_0)}\right)^{-N/(N_0+1)+N_2(\gamma-1/2)}.
\end{aligned}$$

If $d/2 < q < d$, now we have $\frac{1}{p_\gamma} = \frac{1}{p_0} + \frac{2\gamma-1}{2q}$. Then, by Holder's inequality together with (30) and Lemma 4 as above we obtain

$$\begin{aligned}
&\left(\int_{R<|x-x_0|<2R} |K_\gamma(x,y)V^{\gamma-1/2}(x)|^{p_\gamma}\,dx\right)^{1/p_\gamma} \\
&\qquad \le \left(\int_{R<|x-x_0|<2R} |K_\gamma(x,y)|^{p_0}\,dx\right)^{1/p_0} \left(\int_{B(x_0,2R)} V^q\right)^{\frac{2\gamma-1}{2q}} \\
&\qquad \lesssim R^{-d/p'_\gamma}\left(1+\frac{\rho(x_0)}{R}\right)^{-(\gamma-1/2)(2-d/q)} \left(1+\frac{R}{\rho(x_0)}\right)^{-N+N_1(\gamma-1/2)}.
\end{aligned}$$

Applying the above estimates and Theorem 5 we obtain the following result.

**Theorem 9.** *Let $V \in RH_q$ for $q > d/2$, $1/2 < \gamma \le 1$, and $\theta \ge 0$. Then if $p_\gamma$ is given by* (31)*, for any weight $w$ we have*

$$\int |V^{\gamma-1/2}\nabla L^{-\gamma} f|^p w \le C_\theta \int |f|^p M_r^\theta w,$$

*for $1 \le p < p_\gamma$ with $r = (p_\gamma/p)'$, and*

$$\int |L^{-\gamma}\nabla V^{\gamma-1/2} f|^p w \le C_\theta \int |f|^p M^\theta w,$$

*for $p'_\gamma < p < \infty$.*

## 5. On local integrability of $Tf$ and $T^\star f$

In this section we are going to apply the general results of Section 2 to weights of the form $w = \chi_B$. Studying maximal operators like $M_\phi^\theta$ acting on such weights we are going to get sufficient conditions on $f$ to assume some local integrability of $Tf$. We do that in the next lemma.

**Lemma 8.** *Let $\theta \geq 0$, $\phi$ a Young function and $Q = B(x_0, \rho(x_0))$ a critical ball. Then there exist positive constants $c_1$, $c_2$, such that*

$$c_1\left(1+\frac{|x-x_0|}{\rho(x_0)}\right)^{-\theta} \leq M_\phi^\theta \chi_Q(x) \leq c_2\left(1+\frac{|x-x_0|}{\rho(x_0)}\right)^{-\theta/(N_0+1)} \tag{32}$$

*where $N_0$ is the exponent appearing in* (4).

*Proof.* Let $Q = B(x_0, \rho(x_0))$ be a critical ball, $\theta \geq 0$ and $\phi$ a Young function. We may suppose without loss of generality that $\phi(1) = 1$. Recalling that

$$M_\phi^\theta \chi_Q(x) = \sup_{B(x_B,r_B)\ni x}\left(1+\frac{r_B}{\rho(x_B)}\right)^{-\theta}\|\chi_Q\|_{\phi,B},$$

it is enough to consider $B$ such that $Q \cap B \neq \emptyset$, otherwise $\|\chi_Q\|_{\phi,B} = 0$, since

$$\begin{aligned}\|\chi_Q\|_{\phi,B} &= \inf\left\{\lambda : \frac{1}{|B|}\int_B \phi\left(\frac{\chi_Q}{\lambda}\right) \leq 1\right\}\\ &= \inf\left\{\lambda : \frac{1}{|B|}\int_{B\cap Q} \phi\left(\frac{1}{\lambda}\right) \leq 1\right\}.\end{aligned}$$

Let us consider first a ball $B = B(x_B, r_B)$ with $r_B \leq \rho(x_B)$, and take $x \in B$. Then choosing $y \in B \cap Q$,

$$|x-x_0| \leq |x-y| + |y-x_0| \leq 2r_B + \rho(x_0) \leq 2\rho(x_B) + \rho(x_0).$$

Also, since $B$ is sub-critical, $Q$ is critical and $B \cap Q \neq \emptyset$ we have that $\rho(x_B) \simeq \rho(y) \simeq \rho(x_0)$. Then,

$$|x-x_0| \leq \tilde{C}\rho(x_0),$$

for some $\tilde{C} > 0$. Then if $x \notin \tilde{Q} = B(x_0, \tilde{C}\rho(x_0))$ we have

$$M_\phi^{\mathrm{loc}}(\chi_Q)(x) = \sup_{\substack{B\ni x\\ r_B\leq\rho(x_B)}} \|\chi_Q\|_{\phi,B} = 0.$$

Now if $x \in \tilde{Q}$ and $B \cap Q \neq \emptyset$,

$$\begin{aligned}\|\chi_Q\|_{\phi,B} &= \inf\left\{\lambda : \frac{|B\cap Q|}{|B|}\phi\left(\frac{1}{\lambda}\right) \leq 1\right\}\\ &\leq \inf\left\{\lambda : \phi\left(\frac{1}{\lambda}\right) \leq 1\right\}\\ &= 1/\phi^{-1}(1) = 1.\end{aligned}$$

So, taking the supremum over all balls we have for $x \in \tilde{Q}$,

$$M_\phi^{\mathrm{loc}}(\chi_Q)(x) \leq \left(1+\frac{r_B}{\rho(x_B)}\right)^{-\sigma},$$

for any $\sigma > 0$.

Next, we consider the operator

$$M_\phi^{\theta,\mathrm{glob}}(\chi_Q)(x) = \sup_{\substack{B\ni x\\ r_B\geq\rho(x_B)}}\left(1+\frac{r_B}{\rho(x_B)}\right)^{-\theta}\|\chi_Q\|_{\phi,B}.$$

We take a ball $B$ with $r_B \geq \rho(x_B)$ and such that $Q \cap B \neq \emptyset$. For $y \in Q \cap B$, we have $\rho(y) \simeq \rho(x_0)$. Using also Lemma 1,

$$\left(1+\frac{r_B}{\rho(x_B)}\right)^{-\theta} \leq C\left(1+\frac{r_B}{\rho(y)}\right)^{-\theta/(N_0+1)} \leq C\left(1+\frac{r_B}{\rho(x_0)}\right)^{-\theta/(N_0+1)}.$$

Let $x \in B$ and suppose first that $x \notin 2Q$, then

$$|x-x_0| \leq |x-y|+|y-x_0| \leq 2r_B+\rho(x_0) \leq 2r_B+|x-x_0|/2,$$

and hence $|x-x_0| \leq 4r_B$. Therefore,

$$\left(1+\frac{r_B}{\rho(x_B)}\right)^{-\theta} \leq C\left(1+\frac{|x-x_0|}{\rho(x_0)}\right)^{-\theta/(N_0+1)}.$$

As before, we have $\|\chi_Q\|_{\phi,B} \leq 1$. Then, if $x \notin 2Q$

$$M_\phi^{\theta,\mathrm{glob}}(\chi_Q)(x) \leq C\left(1+\frac{|x-x_0|}{\rho(x_0)}\right)^{-\sigma},$$

where $\sigma = \theta/(N_0+1)$.

On the other hand, if $x \in 2Q$,

$$M_\phi^{\theta,\mathrm{glob}}(\chi_Q)(x) \leq M_\phi(\chi_Q)(x) \leq 1.$$

Then, since $|x-x_0|/\rho(x_0) \leq 2$

$$M_\phi^{\theta,\mathrm{glob}}(\chi_Q)(x) \leq C\left(1+\frac{|x-x_0|}{\rho(x_0)}\right)^{-\sigma}.$$

Using that $M_\phi^\theta \leq M_\phi^{\mathrm{loc}} + M^{\theta,\mathrm{glob}}$ and collecting the obtained estimates we arrive to the right hand side of (32). For the boundedness by below, given $x$ we consider $B_x = B(x, |x-x_0|+\rho(x_0))$. Then $x \in B_x$ and $\|\chi_Q\|_{\phi,B_x} = 1$. Therefore,

$$\begin{aligned} M_\phi^\theta(x) &\geq \left(1+\frac{|x-x_0|+\rho(x_0)}{\rho(x_0)}\right)^{-\theta} \|\chi_Q\|_{\phi,B_x} \\ &\geq 2^\theta\left(1+\frac{|x-x_0|}{\rho(x_0)}\right)^{-\theta}. \end{aligned}$$

□

*Remark* 6. We observe that in particular Lemma 8 holds for all maximal operators appearing in Theorem 3 and Theorem 4. Hence they satisfy inequality (32) for some constants $c_1$, $c_2$, $\sigma_1$ and $\sigma_2$ when applied to the function $\chi_B$, with $B$ a critical ball.

**Proposition 2.** *Let $p \geq 1$ and $\phi$ a Young function. There exists $\theta \geq 0$ such that for any ball $Q = B(x_0, \rho(x_0))$*

$$\int |f|^p M_\phi^\theta(\chi_Q) < \infty \tag{33}$$

*if and only if there exists $\sigma > 0$ such that*

$$\int \frac{|f|^p}{(1+|x|)^\sigma} < \infty. \tag{34}$$

*Proof.* Let $p \geq 1$ and $\phi$ a Young function. Let $Q = B(x_0, \rho(x_0))$ a critical ball. It is a straightforward verification that there are constants $c$ and $\tilde{c}$ depending on $x_0$ and $\rho$ such that

$$\frac{c}{1 + \frac{|x-x_0|}{\rho(x_0)}} \leq \frac{1}{1+|x|} \leq \frac{\tilde{c}}{1 + \frac{|x-x_0|}{\rho(x_0)}}. \tag{35}$$

Then, the equivalence between conditions (33) and (34) follows from equation (35) above and Lemma 8.

□

**Theorem 10.** *Let $1 \leq p < \infty$ and $T$ an operator such that for some Young function $\phi$ and for all $\theta$ there exists a constant $C$ such that*

$$\int |Tf|^p w \leq C \int |f|^p M_\phi^\theta w, \tag{36}$$

*for any weight $w$. Then, if a function $f$ satisfy* (34)*, $Tf \in L^p_{loc}$. In particular $Tf$ is finite almost everywhere.*

*Proof.* Let $1 \leq p < \infty$ and $T$ as stated. Let $f$ be a function satisfying (34) for some $\sigma > 0$. Then, applying Proposition 2, there exists some $\theta \geq 0$ such that (33) holds for any critical ball $Q$.

Let $B$ be a ball in $\mathbb{R}^d$. According to Proposition 1 we can cover $B$ by a finite number of critical balls $B_1, \dots B_N$. Using the hypothesis on the operator for such $\theta$,

$$\begin{aligned}\int_B |Tf|^p &\leq \sum_{i=1}^N \int |Tf|^p \chi_{B_i} \\ &\leq C \sum_{i=1}^N \int |f|^p M_\phi^\theta \chi_{B_i} < \infty.\end{aligned}$$

□

For operators that satisfy a weak type inequality for $p = 1$ we obtain an analogous result following the same lines as in the proof of Theorem 10.

**Theorem 11.** *Let $T$ be an operator such that for some Young function $\phi$ and for all $\theta$ there exists a constant $C$ such that*

$$w(\{|Tf| > \lambda\}) \leq C \int |f| M_\phi^\theta w, \text{ for all } \lambda > 0,$$

*for all weight $w$. Then, if a function $f$ satisfy* (34) *with $p = 1$, $Tf \in L^{1,\infty}_{loc}$. In particular $Tf$ is finite almost everywhere.*

The above results can be applied to all operators considered in Section 4 since, as it was shown there, theorems of Section 2 hold in those cases. In particular we point out that for $\mathcal{R}_1$ and $\mathcal{R}_1^\star$ we can apply Theorem 10, for $1 < p < \infty$, and Theorem 11, if $V \in RH_q$ with $q > d$. As for the case $d/2 < q < d$, the conclusion holds for $1 < p < p_0$ and $p > p_0'$ respectively. On the other hand, Theorem 10 and Theorem 11 can be applied to $\mathcal{R}_2$ for $1 < p < q$, when $q > d/2$.

Similarly $VL^{-1}$, $V^{1/2}\nabla L^{-1}$ and $V^{1/2}L^{-1/2}$ fall under the scope of Theorem 10 for $1 \leq p < q$, $1 \leq p \leq p_1$ and $1 \leq p < 2q$, respectively (see Theorem 8 and Theorem 9).

In [10], Shen obtained $L^p$-estimates for derivatives of solutions of differential equations related to Schrödinger operator as a consequence of $L^p$-continuity of Riesz-Schrödinger Transforms (see Corollary 0.9 and Corollary 0.10). Here, with our results, we can give qualitative information on their local integrability.

**Corollary 2.** *Suppose $V \in RH_q$ for some $q > d/2$. Assume that $-\Delta u + Vu = f$ in $\mathbb{R}^d$, with $f$ satisfying* (34) *for some $\sigma > 0$ and some $p \geq 1$. Then,*

*(1) if $1 < p < q$, $\nabla^2 u \in L^p_{loc}$,*
*(2) if $1 \leq p < q$, $Vu \in L^p_{loc}$ ,*
*(3) if $1 \leq p < p_1$, $V^{1/2}\nabla u \in L^p_{loc}$,*

*with $p_1$ such that $1/p_1 = (1/q - 1/d)^+ + 1/2q$.*

*Proof.* If we set $u = L^{-1}f$ we have $\nabla^2 u = \nabla^2 L^{-1} f$, $Vu = VL^{-1}f$ and $V^{1/2}\nabla u = V^{1/2}\nabla L^{-1} f$. Therefore we only have to apply Theorem 10 to the operators $\nabla^2 L^{-1}$, $VL^{-1}$ and $V^{1/2}\nabla L^{-1}$ to get the result.

□

**Corollary 3.** *Suppose $V \in RH_q$ for some $q > d/2$ and let $p_0' < p < p_0$, with $p_0$ such that $1/p_0 = (1/q - 1/d)^+$. Assume that $-\Delta u + Vu = \nabla \cdot F$ in $\mathbb{R}^d$, for a vector field $F$ with $|F|$ satisfying* (34) *for some $\sigma > 0$.*

*(1) If $p_0' < p < p_0$, then $u \in L^p_{loc}$.*
*(2) If $p_0' < p < 2q$, then $V^{1/2} u \in L^p_{loc}$.*

*Proof.* We will show only item (1). The proof of (2) is similar. Let $u = L^{-1}\nabla \dot{F}$. Then $\nabla u = \mathcal{R}_1(\mathcal{R}_1^\star \cdot F)$. Then in order to get that $\nabla u \in L^p_{\text{loc}}$, due to Theorem 10, it will be enough to check that the operators $T_j = \mathcal{R}_1 \circ (\mathcal{R}_1^\star)_j$ satisfy inequality (36). In fact, if $p_0' < p < p_0$, then

$$\begin{aligned} \int |T_j f|^p w &\lesssim \int |(\mathcal{R}_1^\star)_j f|^p M_r^\theta w \\ &\lesssim \int |f|^p M_\nu^\theta M_r^\theta w. \end{aligned}$$

for any $\nu > 1$. Choosing $\nu > r$, it follows easily $M_\nu^\theta(M_r^\theta w) \leq M_\nu^\theta w$, and then (36) holds.

□

## References


[1] B. Bongioanni, E. Harboure, and P. Quijano. Weighted inequalities for schrödinger type singular integrals. *Journal of Fourier Analysis and Applications*, 25(3):595–632, 2019.

[2] B. Bongioanni, E. Harboure, and O. Salinas. Riesz transforms related to Schrödinger operators acting on *BMO* type spaces. *J. Math. Anal. Appl.*, 357(1):115–131, 2009.

[3] B. Bongioanni, E. Harboure, and O. Salinas. Weighted inequalities for commutators of Schrödinger-Riesz transforms. *J. Math. Anal. Appl.*, 392(1):6–22, 2012.

[4] A. Cordoba and C. Fefferman. A weighted norm inequality for singular integrals. *Studia Math.*, 57(1):97–101, 1976.

[5] David V. Cruz-Uribe, José Maria Martell, and Carlos Pérez. *Weights, extrapolation and the theory of Rubio de Francia*, volume 215 of *Operator Theory: Advances and Applications*. Birkhäuser/Springer Basel AG, Basel, 2011.

[6] J. Dziubański and J. Zienkiewicz. Hardy spaces $H^1$ associated to Schrödinger operators with potential satisfying reverse Hölder inequality. *Revista Matemática Iberoamericana*, 15(2):279–296, 1999.

[7] C. Fefferman and E. M. Stein. Some maximal inequalities. *Amer. J. Math.*, 93:107–115, 1971.

[8] Tao Ma, Pablo Raúl Stinga, José L. Torrea, and Chao Zhang. Regularity estimates in Hölder spaces for Schrödinger operators via a $T1$ theorem. *Ann. Mat. Pura Appl. (4)*, 193(2):561–589, 2014.

[9] C. Pérez. Weighted norm inequalities for singular integral operators. *J. London Math. Soc. (2)*, 49(2):296–308, 1994.

[10] Z. Shen. $L^p$ estimates for Schrödinger operators with certain potentials. *Ann. Inst. Fourier (Grenoble)*, 45(2):513–546, 1995.

[11] J. Michael Wilson. Weighted norm inequalities for the continuous square function. *Trans. Amer. Math. Soc.*, 314(2):661–692, 1989.

*Email address*: bbongio@santafe-conicet.gov.ar

Instituto de Matemática Aplicada del Litoral CONICET-UNL, and Facultad de Ingeniería y Ciencias Hídricas, UNL. Colectora Ruta Nac. N 168, Paraje El Pozo.

*Email address*: harbour@santafe-conicet.gov.ar

Instituto de Matemática Aplicada del Litoral, CONICET-UNL, and Facultad de Ingeniería Química, UNL. Colectora Ruta Nac. N 168, Paraje El Pozo. 3000 Santa Fe, Argentina.

*Email address*: pquijano@santafe-conicet.gov.ar

Instituto de Matemática Aplicada del Litoral, CONICET-UNL, and Facultad de Ingeniería Química, UNL. Colectora Ruta Nac. N 168, Paraje El Pozo. 3000 Santa Fe, Argentina.