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AFFILIATIONS

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ABSTRACT

In this comment, we show that the eigenvalues of a quartic anharmonic oscillator obtained recently by means of the asymptotic iteration method may not be as accurate as the authors claim them to be.

In a recent paper, Ismail and Saad[1] revisited the asymptotic iteration method (AIM) with the purpose of deriving conditions for its validity. They first discussed some textbook examples of no interest whatsoever and later applied the approach to the quartic anharmonic oscillator,

$$\psi''(x) + \left(x^2 + Ax^4\right)\psi(x) = E\psi(x). \tag{1}$$

They claimed to have obtained several eigenvalues "accurate to fifty decimals" for $A = 0.1$.

Our earlier experience with the AIM suggested that this approach is less reliable and less accurate than other alternative approaches,[2,3] even with the improvement of an adjustable parameter.[2] For this reason, we were extremely surprised by the accuracy attained by Ismail and

TABLE I. Convergence of the lower bounds ($d = 0$) and upper bounds ($d = 1$) for the ground state of the quartic anharmonic oscillator with $A = 0.1$.

D	$d = 0$	$d = 1$
2	1.065 165 589 106 464 508 643 143 086 785 809 633 638	1.065 291 556 141 124 441 135 238 488 833 718 516 162
3	1.065 285 181 369 961 298 428 253 752 818 854 854 099	1.065 285 528 386 575 099 263 974 255 031 454 998 383
4	1.065 285 508 412 319 469 577 830 463 652 960 342 502	1.065 285 509 614 182 897 898 433 926 644 472 417 431
5	1.065 285 509 539 192 592 585 488 356 076 943 661 193	1.065 285 509 544 015 954 376 247 941 103 453 239 435
6	1.065 285 509 543 697 581 134 961 945 367 065 150 205	1.065 285 509 543 719 071 409 991 810 845 081 902 423
7	1.065 285 509 543 717 592 126 285 592 017 093 474 034	1.065 285 509 543 717 695 730 809 861 353 737 125 740
8	1.065 285 509 543 717 688 361 781 653 356 859 779 602	1.065 285 509 543 717 688 893 236 061 269 397 044 334
9	1.065 285 509 543 717 688 854 423 603 909 817 897 904	1.065 285 509 543 717 688 857 290 648 469 407 292 386
10	1.065 285 509 543 717 688 857 076 639 211 237 627 062	1.065 285 509 543 717 688 857 092 767 847 161 100 393
11	1.065 285 509 543 717 688 857 091 541 516 086 073 081	1.065 285 509 543 717 688 857 091 635 527 212 915 542
12	1.065 285 509 543 717 688 857 091 628 265 134 465 940	1.065 285 509 543 717 688 857 091 628 830 109 065 701
13	1.065 285 509 543 717 688 857 091 628 785 862 167 126	1.065 285 509 543 717 688 857 091 628 789 349 091 100
14	1.065 285 509 543 717 688 857 091 628 789 072 686 577	1.065 285 509 543 717 688 857 091 628 789 094 718 040
15	1.065 285 509 543 717 688 857 091 628 789 092 952 804	1.065 285 509 543 717 688 857 091 628 789 093 094 939

TABLE II. Lower bound, AIM,[1] and upper bound for the first states of the quartic anharmonic oscillator with $A = 0.1$.

$n = 0$
1.065 285 509 543 717 688 857 091 628 789 092 952 804
1.065 285 509 543 717 688 856 877 962 022 551 287 191 163 282 841 44
1.065 285 509 543 717 688 857 091 628 789 093 094 939
$n = 1$
3.306 872 013 152 913 507 128 121 684 692 867 756 592
3.306 872 013 152 913 507 126 866 993 202 085 609 482 310 246 676 21
3.306 872 013 152 913 507 128 121 684 692 869 154 624
$n = 2$
5.747 959 268 833 563 304 733 503 118 475 917 140 926
5.747 959 268 833 563 304 734 474 846 968 694 805 582 344 997 674 23
5.747 959 268 833 563 304 733 503 118 477 229 464 674
$n = 3$
8.352 677 825 785 754 712 155 257 734 637 775 310 436
8.352 677 825 785 754 712 154 419 082 681 400 254 841 719 288 378 95
8.352 677 825 785 754 712 155 257 734 644 178 775 630

Saad.[1] Since these authors did not show a convergence test in their paper, we decided to test their results by means of the Riccati–Padé method (RPM) that provides tight upper and lower bounds in the case of the quartic anharmonic oscillator.[4]

In order to make this comment sufficiently self-contained, we outline the main ideas of the RPM. In the case of the Schrödinger equation with an even potential, we define $\Phi(x) = x^{-s}\psi(x)$, where $s = 0$ or $s = 1$ for even or odd states, respectively. Then, we expand the logarithmic derivative $f(x) = -\Phi'(x)/\Phi(x)$ in a Taylor series about $x = 0$, $f(x) = f_0 x + f_1 x^3 + \cdots$, and obtain the Hankel determinants $H_D^d(E)$ with matrix elements $f_{i+j+d-1}$, $i, j = 1, 2, \ldots, D$. It was proved that there are sequences of roots $E^{[D,d]}$, $D = 2, 3, \ldots$, of $H_D^d(E) = 0$ that converge toward the actual eigenvalues from below ($d = 0$) or above ($d = 1$).[4] In this way, one obtains increasingly accurate lower and upper bounds, respectively.

Table I shows the remarkable (in fact, it is exponential) rate of convergence of the bounds for the ground state of the quartic anharmonic oscillator with $A = 0.1$. Table II compares the present bounds with the results of Ismail and Saad.[1] Our bounds suggest that more than half of the significant figures reported by those authors may not be correct.

Curiously, the 13 significant digits reported by Ismail and Saad[1] for the case $A = 2$ are consistent with our more accurate bounds $E^{[15,0]} = 1.607\,541\,302\,468\,547\,538\,708\,171\,929\,41 < E^{[15,1]} = 1.607\,541\,302\,468\,547\,538\,708\,171\,929\,48$.

In summary, more than half of the decimal figures shown by Ismail and Saad[1] for the quartic anharmonic oscillator with $A = 0.1$ do not appear to be correct. It may be due to the lack of convergence of the AIM or to round off errors caused by insufficient digits in the calculation.

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