EXTENSION OF THE BEST CONSTANT APPROXIMATION OPERATOR IN ORLICZ SPACES

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Abstract. In this paper we deal with the best φ-approximation operator by constants extended from an Orlicz space $L^\varphi(\Omega)$ to the space $L^{\psi^+}(\Omega)$, where ψ^+ denotes the right derivative of the function φ. We obtain pointwise convergence for a suitable class of functions. Also we consider a maximal operator which allows as to get modular convergence for a specific class of Orlicz spaces.

1. Introduction and Notations

We consider $\mathcal{S}$, the set of all non decreasing functions $\psi : [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ and $\psi(x) \to \infty$ as $x \to \infty$, $\psi(x) \to 0$ as $x \to 0$, and $\psi(x) > 0$ if $x > 0$.

A function $\psi \in \mathcal{S}$ satisfies the Δ_2 condition if there exists a constant $k > 0$ such that

$$\psi(2x) \leqslant k\psi(x),$$

for all $x \geqslant 0$. We write $\psi \in \Delta_2$ in this case.

Also we say that a function $\psi \in \mathcal{S}$ satisfies the ∇_2 condition, and denote $\psi \in \nabla_2$, if there exists a constant $\alpha > 1$ such that

$$\psi(x) \leqslant \frac{1}{2\alpha}\psi(\alpha x), \tag{1.1}$$

for all $x \geqslant 0$.

Let Ω be a bounded and Lebesgue measurable set in $\mathbb{R}^n$ and, for a function $\psi \in \mathcal{S} \cap \Delta_2$, we define $L^\psi(\Omega)$ as the class of all Lebesgue measurable functions f, defined on Ω, such that $\int_\Omega \psi(|f(x)|)dx < \infty$, where we write dx for the Lebesgue measure in $\mathbb{R}^n$.

We also consider Φ for the class of $N-$functions φ, i.e. $\varphi(x) = \int_0^x \psi(t)dt$ for some $\psi \in \mathcal{S}$. Note that if $\varphi \in \Phi$, then $\frac{\varphi(x)}{x} \to \infty$, as $x \to \infty$, and the space $L^\varphi(\Omega)$ defined above, is the classical Orlicz space. We denote by ψ^- and ψ^+ for the left and right derivatives of φ respectively, with $\psi^-(0) = 0$.

For $\varphi \in \Phi$, we have

$$\frac{x}{2}\psi^+\left(\frac{x}{2}\right) \leqslant \varphi(x) \leqslant x\psi^+(x) \leqslant \varphi(2x), \tag{1.2}$$

for all $x \geqslant 0$, and then it follows a similar inequality considering ψ^- instead of ψ^+,

$$\frac{x}{2}\psi^-\left(\frac{x}{2}\right) \leqslant \varphi(x) \leqslant x\psi^-(x) \leqslant \varphi(2x), \tag{1.3}$$

for all $x \geqslant 0$.

We point out that the function $\varphi \in \Phi$ satisfies the Δ_2 condition if and only if ψ^+ or ψ^- satisfies the Δ_2 condition.

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Also given a function $\varphi \in \Phi$ that satisfies the Δ_2 condition and since $\varphi(x) = \int_0^x \psi^-(t)\,dt = \int_0^x \psi^+(t)\,dt$, it is easy to see that

$$\psi^-(x) \leqslant \psi^+(x) \leqslant k\psi^-(x), \tag{1.4}$$

for all $x \geq 0$ and since $(x+y)\psi^+(x+y) \leq \varphi(2(x+y))$, it holds

$$\frac{1}{2}(\psi^+(x) + \psi^+(y)) \leq \psi^+(x+y) \leqslant k(\psi^+(x) + \psi^+(y)), \tag{1.5}$$

for all $x, y \geqslant 0$ and where the constant k refers to the one appearing in the Δ_2 condition on φ. We also point out that inequality (1.5) holds for ψ^- instead of ψ^+.

Given $\varphi \in \Phi \cap \Delta_2$ we use (1.2) to see that $L^\varphi(\Omega) \subseteq L^{\psi^+}(\Omega)$, for any Lebesgue measurable and bounded set Ω. Further, from (1.4) we have $L^{\psi^+}(\Omega) = L^{\psi^-}(\Omega)$.

We say that a real number c is a best φ-approximation of $f \in L^\varphi(\Omega)$ if and only if it is satisfied

$$\int_\Omega \varphi(|f(x) - c|)dx = \inf_{r \in \mathbb{R}} \int_\Omega \varphi(|f(x) - r|)dx. \tag{1.6}$$

For $f \in L^\varphi(\Omega)$, we define $\mu_\varphi(f)(\Omega) = \mu_\varphi(f)$ as the set of all real constants c that satisfy (1.6). The mapping $\mu_\varphi : L^\varphi \to 2^{\mathbb{R}}$ is called the best approximation operator.

In [1], Landers and Rogge have made a significant development of the best approximation theory in Orlicz spaces. In that paper the authors considered a $\sigma-$lattice of functions as the approximation class, which includes the constant functions considered in this manuscript. The same authors deal, in [2] , with an extension for the best approximation operator originally defined in $L^p(\Omega)$. Lately the extension of this operator for constant functions, as the approximation class, was treated in [3] where the pointwise convergence, as ε goes to 0, of the best approximation $f_\varepsilon(x)$, was obtained when Ω is the ball centered in x and radius ε, $B_\varepsilon(x)$. In this paper it was also considered a maximal function, related with the best approximation operator, but different to the one considered in [4], which allows us to get norm convergence for these extended best approximation, when ϵ goes to 0. In [5] and [6] it was considered the extension of the best approximation where the approximation class is the algebraic polynomials and for some suitable class of smooth functions. For Orlicz spaces this extension of the best approximation operator and its relation with other classical operators in harmonic analysis are considered in [1], [3], [7] and [8]. We point out also that the conditional expectation is a known example of a best approximation operator originally defined in $L^2(\Omega)$, extended to $L^1(\Omega)$.

Until now, in all cases, the extension of the best approximation operator in an Orlicz space $L^\varphi(\Omega)$, was treated with the hypothesis of differentiability of φ, which plays an important role to extend the best approximation operator. In our case we deal with a non necessarily smooth convex function φ and then the right and left derivatives ψ^- and ψ^+ are involved.

The paper is organized in the following way. In Section 2, we characterize the best φ-approximations and extend the best approximation operator by constants from the Orlicz space L^φ to the wider space L^{ψ^+}. We also obtain some estimates on the extended operator which allow us to get pointwise convergence and strong inequalities. These results generalize those, obtained in [3], for the extended operator defined in $L^{\varphi'}$, where φ' denoted the derivative function of φ.

The purpose of Section 3 is to prove a modular convergence result

$$\int_{\mathbb{R}^n} \theta(|f(y) - f_\epsilon(y)|)dy \to 0, \tag{1.7}$$

as $\epsilon \to 0$, for a suitable class of function θ function. For this aim we introduce the maximal operator $\mathcal{M}_\varphi f$ related with the best φ-approximation by constants for $f \in L^{\psi^+}_{loc}(\mathbb{R}^n)$. We also get a sufficient condition which assures strong inequalities of the type

$$\int_{\mathbb{R}^n} \theta(\mathcal{M}_\varphi(|f|)(x))dx \leqslant \widetilde{C} \int_{\mathbb{R}^n} \theta(\widetilde{C}|f(x)|)dx, \tag{1.8}$$

where $f \in L^{\psi^+}_{loc}(\mathbb{R}^n)$.

We point out that the authors in [7] obtain necessary and sufficient conditions that ensure inequalities like (1.8).

Finally, we compare the maximal operator $\mathcal{M}_\varphi|f|$ with the operator of Hardy-Littlewood $M_H f$.

2. Characterization of the best approximation operator

We start with a characterization for the best approximation constant in $\mu_\varphi(f)(\Omega)$. First, observe that given $\varphi \in \Phi$, the left and right derivatives ψ^- and ψ^+ are measurable functions.

Lemma 2.1. *Let $\varphi \in \Phi \cap \Delta_2$ and let $f \in L^\varphi(\Omega)$. Then, $c \in \mu_\varphi(f)(\Omega)$ if and only if*

$$\int_{\Omega\cap\{f>c\}} \psi^-(|f-c|)dx \leq \int_{\Omega\cap\{f\leqslant c\}} \psi^+(|f-c|)dx, \tag{2.1}$$

and

$$\int_{\Omega\cap\{f<c\}} \psi^-(|f-c|)dx \leq \int_{\Omega\cap\{f\geqslant c\}} \psi^+(|f-c|)dx. \tag{2.2}$$

Proof. Set

$$F(t) = \int_\Omega \varphi(|f-t|)dx.$$

The function F is convex, then it has a minimum at $c \in \mathbb{R}$ if and only if $0 \leqslant F^+(c)$ and $F^-(c) \leqslant 0$, where we set F^- and F^+ for the right and left derivatives functions of F respectively.

Since

$$0 \leqslant F^+(c) = \int_{\Omega\cap\{f\leqslant c\}} \psi^+(|f-c|)dx - \int_{\Omega\cap\{f>c\}} \psi^-(|f-c|)dx,$$

and we get inequality (2.1).

It remains to prove inequality (2.2).

As

$$0 \geqslant F^-(c) = \int_{\Omega\cap\{f<c\}} \psi^-(|f-c|)dx - \int_{\Omega\cap\{f\geqslant c\}} \psi^+(|f-c|)dx,$$

we get,

$$\int_{\Omega\cap\{f<c\}} \psi^-(|f-c|)dx \leq \int_{\Omega\cap\{f\geqslant c\}} \psi^+(|f-c|)dx.$$

Therefore, the proof is completed. □

Theorem 2.2. *Let $\varphi \in \Phi \cap \Delta_2$ and let $f \in L^\varphi(\Omega)$. Then statement (1) or (2) is equivalent to $c \in \mu_\varphi(f)(\Omega)$.*

(1) *(a)*

$$\int_\Omega \psi^-(|f-c|)dx \le \int_{\Omega\cap\{f\le c\}} (\psi^-(|f-c|)+\psi^+(|f-c|))dx.$$

(b)

$$\int_\Omega \psi^-(|f-c|)dx \le \int_{\Omega\cap\{f\geqslant c\}} (\psi^-(|f-c|)+\psi^+(|f-c|))dx.$$

(2) *(a) For any* $\alpha > c$*, we have*

$$\int_\Omega \psi^-(|f-c|)dx \le \int_{\Omega\cap\{f<\alpha\}} (\psi^-(|f-c|)+\psi^+(|f-c|))dx.$$

(b) For any $\alpha < c$*, we have*

$$\int_\Omega \psi^-(|f-c|)dx \le \int_{\Omega\cap\{f>\alpha\}} (\psi^-(|f-c|)+\psi^+(|f-c|))dx.$$

Proof. Let's first prove that inequalities (2.1) and (2.2) imply (1)(a) and (1)(b). Since

$$\int_\Omega \psi^-(|f-c|)dx = \int_{\Omega\cap\{f>c\}} \psi^-(|f-c|)dx + \int_{\Omega\cap\{f\leqslant c\}} \psi^-(|f-c|)dx,$$

we consider (2.1) and we get

$$\int_\Omega \psi^-(|f-c|)dx \leqslant \int_{\Omega\cap\{f\leqslant c\}} \psi^+(|f-c|)dx + \int_{\Omega\cap\{f\leqslant c\}} \psi^-(|f-c|)dx,$$

and (1)(a) follows. Now we also have

$$\int_\Omega \psi^-(|f-c|)dx = \int_{\Omega\cap\{f<c\}} \psi^-(|f-c|)dx + \int_{\Omega\cap\{f\geqslant c\}} \psi^-(|f-c|)dx,$$

and using (2.2) we get

$$\int_\Omega \psi^-(|f-c|)dx \leqslant \int_{\Omega\cap\{f\geqslant c\}} \psi^+(|f-c|)dx + \int_{\Omega\cap\{f\geqslant c\}} \psi^-(|f-c|)dx.$$

Thus we have proved (1)(b).

Now we prove that inequalities (1)(a) and (1)(b) imply (2.1) and (2.2). Since

$$\int_{\Omega\cap\{f>c\}} \psi^-(|f-c|)dx = \int_\Omega \psi^-(|f-c|)dx - \int_{\Omega\cap\{f\leqslant c\}} \psi^-(|f-c|)dx,$$

inequality (1)(a), implies

$$\int_{\Omega\cap\{f>c\}} \psi^-(|f-c|)dx \leqslant \int_{\Omega\cap\{f\leqslant c\}} \psi^+(|f-c|)dx,$$

and we obtain (2.1).

Now since

$$\int_{\Omega\cap\{f<c\}} \psi^-(|f-c|)dx = \int_\Omega \psi^-(|f-c|)dx - \int_{\Omega\cap\{f\geqslant c\}} \psi^-(|f-c|)dx,$$

and by (1)(b), we get

$$\int_{\Omega\cap\{f<c\}} \psi^-(|f-c|)dx \leqslant \int_{\Omega\cap\{f\geqslant c\}} \psi^+(|f-c|)dx,$$

which is (2.2).

Now we prove that inequalities (1) (a) and (1) (b) imply (2) (a) and (2) (b). From (1)(a) and $\alpha > c$ we have

$$\int_\Omega \psi^-(|f-c|)dx \leqslant \int_{\Omega\cap\{f\leqslant c\}} \psi^+(|f-c|)dx + \int_{\Omega\cap\{f\leqslant c\}} \psi^-(|f-c|)dx,$$

and since $\{f \leqslant c\} \subset \{f < \alpha\}$, we get (2)$(a)$

$$\int_\Omega \psi^-(|f-c|)dx \leqslant \int_{\Omega\cap\{f<\alpha\}} \psi^+(|f-c|)dx + \int_{\Omega\cap\{f<\alpha\}} \psi^-(|f-c|)dx.$$

To prove (2)(b), let $\alpha < c$. From (1)(b) we have

$$\int_\Omega \psi^-(|f-c|)dx \leqslant \int_{\Omega\cap\{f\geqslant c\}} \psi^+(|f-c|)dx + \int_{\Omega\cap\{f\geqslant c\}} \psi^-(|f-c|)dx,$$

and since $\{f \geqslant c\} \subset \{f > \alpha\}$, we get

$$\int_\Omega \psi^-(|f-c|)dx \leqslant \int_{\Omega\cap\{f>\alpha\}} \psi^+(|f-c|)dx + \int_{\Omega\cap\{f>\alpha\}} \psi^-(|f-c|)dx.$$

Next we prove the statements (2)(a) and (2)(b) imply (2.1) and (2.2).

From (2)(a) with $\alpha = c + \frac{1}{n}$ we get

$$\int_\Omega \psi^-(|f-c|)dx \leqslant \int_{\Omega\cap\{f<c+\frac{1}{n}\}} \psi^+(|f-c|)dx + \int_{\Omega\cap\{f<c+\frac{1}{n}\}} \psi^-(|f-c|)dx.$$

and the right hand of the inequality above can be written as

$$\int_{\Omega\cap\{f\leqslant c\}} \psi^+(|f-c|)dx + \int_{\Omega\cap\{c<f<c+\frac{1}{n}\}} \psi^+(|f-c|)dx$$
$$+ \int_{\Omega\cap\{f\leqslant c\}} \psi^-(|f-c|)dx + \int_{\Omega\cap\{c<f<c+\frac{1}{n}\}} \psi^-(|f-c|)dx.$$

Note that $A_n = \{c < f < c + \frac{1}{n}\}$ is a decreasing sequence of sets such that $\mu(A_n) := \int_{A_n} \psi^+(|f-c|) < \infty$. Thus $\mu(\cap_1^\infty A_n) = \lim_{n\to\infty} \mu(A_n) = 0$.

And then

$$\int_\Omega \psi^-(|f-c|)dx \leqslant \int_{\Omega\cap\{f\leqslant c\}} \psi^+(|f-c|)dx + \int_{\Omega\cap\{f\leqslant c\}} \psi^-(|f-c|)dx,$$

which is equivalent to (2.1).

Now, for $\alpha = c - \frac{1}{n}$ in (2)(b) we have

$$\int_\Omega \psi^-(|f-c|)dx \leqslant \int_{\Omega\cap\{f>c-\frac{1}{n}\}} \psi^+(|f-c|)dx + \int_{\Omega\cap\{f>c-\frac{1}{n}\}} \psi^-(|f-c|)dx,$$

which can be written as

$$\int_{\Omega\cap\{f\geqslant c\}} \psi^+(|f-c|)dx + \int_{\Omega\cap\{c-\frac{1}{n}<f\leqslant c\}} \psi^+(|f-c|)dx$$
$$+ \int_{\Omega\cap\{f\geqslant c\}} \psi^-(|f-c|)dx + \int_{\Omega\cap\{c-\frac{1}{n}<f\leqslant c\}} \psi^-(|f-c|)dx.$$

Now consider $B_n = \{c - \frac{1}{n} < f < c\}$. This is a decreasing sequence of sets such that $\mu(B_n) := \int_{B_n} \psi^+(|f-c|) < \infty$. Then $\mu(\cap_1^\infty B_n) = \lim_{n\to\infty} \mu(B_n) = 0$.

Then we obtain

$$\int_\Omega \psi^-(|f-c|)dx \leqslant \int_{\Omega\cap\{f\geqslant c\}} \psi^+(|f-c|)dx + \int_{\Omega\cap\{f\geqslant c\}} \psi^-(|f-c|)dx,$$

which implies (2.2).

Then the proof is completed.

□

We point out here that the same proof of Theorem 2.2 holds even for $f \in L^{\psi^+}(\Omega)$.

Definition 2.3. *Let $\varphi \in \Phi \cap \Delta_2$. We say that a constant c is an extended best approximation of f, where $f \in L^{\psi^+}(\Omega)$, if it is a solution of the following two inequalities*

$$\int_{\Omega\cap\{f>c\}} \psi^-(|f-c|)dx \leq \int_{\Omega\cap\{f\leq c\}} \psi^+(|f-c|)dx. \tag{2.3}$$

$$\int_{\Omega\cap\{f<c\}} \psi^-(|f-c|)dx \leq \int_{\Omega\cap\{f\geqslant c\}} \psi^+(|f-c|)dx. \tag{2.4}$$

For $f \in L^{\psi^+}(\Omega)$, we set $\mu_{\psi^+}(f)(\Omega) = \mu_{\psi^+}(f)$ as the set of all real constants c that satisfy (2.3) and (2.4). The mapping $\mu_{\psi^+} : L^{\psi^+} \to 2^{\mathbb{R}}$ is called the extended best approximation operator.

Remark 2.4. The inequalities (2.3) and (2.4) in Definition 2.3 characterizes the elements in $\mu_\varphi(f)(\Omega)$ if the function f belongs to the space $L^\varphi(\Omega)$, then $\mu_\varphi(f)(\Omega) = \mu_{\psi^+}(f)(\Omega)$, for $f \in L^\varphi(\Omega)$.

In the next lemma, we extend the best approximation operator from space $L^\varphi(\Omega)$ to the space $L^{\psi^+}(\Omega)$.

Lemma 2.5. *Let $\varphi \in \Phi \cap \Delta_2$ and let $f \in L^{\psi^+}(\Omega)$. Then, there exists a constant c which is an extended best approximation of the function f.*

Proof. To prove the existence of the best extended approximation of f, we will demonstrate that the inequalities of the Definition 2.3 are verified.

Let $f \in L^{\psi^+}(\Omega)$, and set $f_n = \min(\max(f, -n), n)$, $n \in \mathbb{N}$. Since $-n \leqslant f_n \leqslant n$, we have $f_n \in L^\infty(\Omega)$. Then, for each $n \in \mathbb{N}$, $f_n \in L^\varphi(\Omega)$ and by Theorem 2.2 (1b), there exists $c_n \in \mu_\varphi(f)(\Omega)$ such that

$$\begin{aligned}\int_\Omega \psi^-(|f_n - c_n|)dx \leq \int_{\Omega\cap\{f_n\geqslant c_n\}} &\psi^-(|f_n-c_n|)dx + \\ &\int_{\Omega\cap\{f_n\geqslant c_n\}} \psi^+(|f_n-c_n|)dx.\end{aligned} \tag{2.5}$$

We prove first that the sequence $\{c_n\}$ is bounded.

Suppose that the sequence $\{c_n\}$ is not bounded from above. Then there exists a subsequence $\{c_{n_j}\}$ such that $c_{n_j} \nearrow \infty$ as $j \to \infty$. As $\{c_{n_j}\}$ satisfies (2.5), we have

$$\begin{aligned}\int_\Omega \psi^-(|f_{n_j} - c_{n_j}|)dx \leq \int_{\Omega\cap\{f_{n_j}\geqslant c_{n_j}\}} &\psi^-(|f_{n_j}-c_{n_j}|)dx + \\ &\int_{\Omega\cap\{f_{n_j}\geqslant c_{n_j}\}} \psi^+(|f_{n_j}-c_{n_j}|)dx.\end{aligned} \tag{2.6}$$

Since ψ^- and ψ^+ are non decreasing functions from (2.6) we have

$$\begin{aligned}&\int_\Omega \psi^-(|f_{n_j} - c_{n_j}|)dx \leqslant \int_{\Omega\cap\{f_{n_j}\geqslant c_{n_j}\}} \psi^-(|f_{n_j}-\alpha|)dx + \\ &\int_{\Omega\cap\{f_{n_j}\geqslant c_{n_j}\}} \psi^+(|f_{n_j}-\alpha|)dx \leqslant \int_\Omega \psi^-(|f_{n_j}-\alpha|)dx + \int_\Omega \psi^+(|f_{n_j}-\alpha|)dx,\end{aligned} \tag{2.7}$$

for any $\alpha < c_{n_j}$.

The function ψ^- satisfies the Δ_2 condition, then there exists a constant $k > 0$ such that

$$\psi^-(|c_{n_j}|) \leqslant k\psi^-(|f_{n_j} - c_{n_j}|) + k\psi^-(|f_{n_j}|). \tag{2.8}$$

From equations (2.7) and (2.8), we have

(2.9)
$$\begin{aligned}\int_\Omega \psi^-(|c_{n_j}|)dx &\leqslant k\int_\Omega \psi^-(|f_{n_j}-c_{n_j}|)dx + k\int_\Omega \psi^-(|f_{n_j}|)dx\\ &\leqslant k\int_\Omega \psi^-(|f_{n_j}-\alpha|)dx + k\int_\Omega \psi^+(|f_{n_j}-\alpha|)dx + k\int_\Omega \psi^-(|f_{n_j}|)dx.\end{aligned}$$

Since $\lim_{n\to\infty} f_n = f$ y $|f_n| \leqslant |\max(f,-n)| \leqslant |f|$, from Equation (2.9) it follows that
$$\int_\Omega \psi^-(|c_{n_j}|)dx \leqslant 2k\int_\Omega \psi^+(|f_{n_j}-\alpha|)dx + k\int_\Omega \psi^+(|f|)dx.$$
The function ψ^+ satisfies the condition Δ_2, then there exists a constant $k>0$ such that $\psi^+(|f_{n_j}-\alpha|) \leqslant k\psi^+(|f|) + k\psi^+(|\alpha|)$.

Then, $\int_\Omega \psi^-(|c_{n_j}|)dx < C$, for some constant $C>0$, which is a contradiction.

The proof that the sequence $\{c_n\}$ is bounded from below follows similarly using $(1)(a)$ instead of $(1)(b)$ in Theorem 2.2.

Then we may assume that $\lim_{n\to\infty} c_n = c$. We will prove that the constant c satisfies the inequalities (2.3) and (2.4) of the Definition 2.3.

Since $c_n \in \mu_\varphi(f_n)$, from Lemma 2.1 we have

$$\int_{\Omega\cap\{f_n>c_n\}} \psi^-(|f_n-c_n|)dx \leqslant \int_{\Omega\cap\{f_n\leqslant c_n\}} \psi^+(|f_n-c_n|)dx. \tag{2.10}$$

Now, since
$$\psi^-(|f-c|)\,\chi_{\{f>c\}} \le \liminf_{n\to\infty} \psi^-(|f_n-c_n|)\,\chi_{\{f_n>c_n\}},$$
we use the Fatou's Lemma to get

$$\int_{\Omega\cap\{f>c\}} \psi^-(|f-c|)dx \leqslant \liminf_{n\to\infty} \int_{\Omega\cap\{f_n>c_n\}} \psi^-(|f_n-c_n|)dx. \tag{2.11}$$

By (2.10) and (2.11), we get
$$\int_{\Omega\cap\{f>c\}} \psi^-(|f-c|)dx \leqslant \liminf_{n\to\infty} \int_{\Omega\cap\{f_n\leqslant c_n\}} \psi^+(|f_n-c_n|)dx.$$
As $|c_n| \le M$, for some constant $M>0$, from (1.5) we have
$$\psi^+(|f_n-c_n|)\chi_{\{f_n\le c_n\}} \le k(\psi^+(|f|)+\psi^+(M)).$$
Moreover
$$\limsup_{n\to\infty} \psi^+(|f_n-c_n|)\,\chi_{\{f_n\le c_n\}} \le \psi^+(|f-c|)\,\chi_{\{f\le c+\frac{1}{j}\}},$$
for any integer $j>0$. So, using the Fatou-Lebesgue's Theorem we get

$$\begin{aligned}\int_{\Omega\cap\{f>c\}} \psi^-(|f-c|)dx &\leqslant \limsup_{n\to\infty} \int_{\Omega\cap\{f_n\leqslant c_n\}} \psi^+(|f_n-c_n|)dx\\ &\leqslant \int_{\Omega\cap\{f\leqslant c+\frac{1}{j}\}} \psi^+(|f-c|)dx,\end{aligned} \tag{2.12}$$

for any integer $j>0$. Now, the integral on the right side of (2.12) can be written as

$$\begin{aligned}\int_{\Omega\cap\{f\leqslant c+\frac{1}{j}\}} \psi^+(|f-c|)dx &= \int_{\Omega\cap\{f\leqslant c\}} \psi^+(|f-c|)dx\\ &\quad + \int_{\Omega\cap\{c<f\leqslant c+\frac{1}{j}\}} \psi^+(|f-c|)dx.\end{aligned} \tag{2.13}$$

Now consider $A_j = \{c < f \leqslant c + \frac{1}{j}\}$ which is a decreasing sequence of sets such that $\mu(A_j) = \int_{A_j} \psi^+(|f-c|)dx < \infty$. Thus $\mu(\cap_{j=1}^{\infty} A_j) = \lim_{j\to\infty} \mu(A_j) = 0$. Then from (2.12) and (2.13) we have

$$\int_{\Omega\cap\{f>c\}} \psi^-(|f-c|)dx \leqslant \int_{\Omega\cap\{f\leqslant c\}} \psi^+(|f-c|)dx,$$

which is inequality (2.3) of Definition 2.3.

To prove Inequality (2.4) of the Definition 2.3 we proceed as follows.

Since $c_n \in \mu_\varphi(f_n)$, then from Lemma 2.1 (2.2) we have

$$\int_{\Omega\cap\{f_n<c_n\}} \psi^-(|f_n-c_n|)dx \leqslant \int_{\Omega\cap\{f_n\geqslant c_n\}} \psi^+(|f_n-c_n|)dx. \tag{2.14}$$

Now, since

$$\psi^-(|f-c|)\,\chi_{\{f<c\}} \leq \liminf_{n\to\infty} \psi^-(|f_n-c_n|)\,\chi_{\{f_n<c_n\}}$$

and using the Fatou's Lemma we have

$$\int_{\Omega\cap\{f<c\}} \psi^-(|f-c|)dx \leqslant \liminf_{n\to\infty} \int_{\Omega\cap\{f_n<c_n\}} \psi^-(|f_n-c_n|)dx. \tag{2.15}$$

By (2.14) and (2.15), we get

$$\int_{\Omega\cap\{f<c\}} \psi^-(|f-c|)dx \leqslant \liminf_{n\to\infty} \int_{\Omega\cap\{f_n\geqslant c_n\}} \psi^+(|f_n-c_n|)dx.$$

Then

$$\int_{\Omega\cap\{f<c\}} \psi^-(|f-c|)dx \leqslant \limsup_{n\to\infty} \int_{\Omega\cap\{f_n\geqslant c_n\}} \psi^+(|f_n-c_n|)dx.$$

Now, since

$$\psi^+(|f_n-c_n|)\,\chi_{\{f_n\geq c_n\}} \leq k(\psi^+(|f|)+\psi^+(M)),$$

for some $M > 0$ and

$$\limsup_{n\to\infty} \psi^+(|f_n-c_n|)\chi_{\{f_n\geq c_n\}} \leq \psi^+(|f-c|)\chi_{\{f\geq c-\frac{1}{j}\}},$$

for any integer $j > 0$. Therefore, using Fatou-Lebesgue's Theorem, we get

$$\int_{\Omega\cap\{f<c\}} \psi^-(|f-c|)dx \leqslant \int_{\Omega\cap\{f\geqslant c-\frac{1}{j}\}} \psi^+(|f-c|)dx. \tag{2.16}$$

The integral on the right side of (2.16) can be written as

$$\begin{aligned}\int_{\Omega\cap\{f\geqslant c-\frac{1}{j}\}} \psi^+(|f-c|)dx &= \int_{\Omega\cap\{f\geqslant c\}} \psi^+(|f-c|)dx \\ &\quad + \int_{\Omega\cap\{c-\frac{1}{j}\leqslant f<c\}} \psi^+(|f-c|)dx.\end{aligned} \tag{2.17}$$

Then, for $B_j = \{c - \frac{1}{j} \leqslant f < c\}$, we have a decreasing sequence of sets such that $\tilde{\mu}(B_j) := \int_{B_j} \psi^+(|f-c|)\,dx < \infty$, then $\tilde{\mu}(\cap_{j=1}^{\infty} B_j) = \lim_{j\to\infty} \tilde{\mu}(B_j) = 0$. Then from (2.16) and (2.17) we have

$$\int_{\Omega\cap\{f<c\}} \psi^-(|f-c|)dx \leqslant \int_{\Omega\cap\{f\geqslant c\}} \psi^+(|f-c|)dx.$$

Thus we get Inequality (2.4) of the Definition 2.3.

□

The following theorem provides some properties for the set of extended best approximation $\mu_{\psi^+}(f)(\Omega)$ and its proof follows exactly the same way as Theorem 2.2.

Theorem 2.6. *Let $\varphi \in \Phi \cap \Delta_2$ and let $f \in L^{\psi^+}(\Omega)$. Then the statements (1) and (2) are equivalent to $c \in \mu_{\psi^+}(f)(\Omega)$.*

(1)

$$(a) \int_\Omega \psi^-(|f-c|)dx \leq \int_{\Omega \cap \{f \leq c\}} (\psi^-(|f-c|) + \psi^+(|f-c|))dx.$$

$$(b) \int_\Omega \psi^-(|f-c|)dx \leq \int_{\Omega \cap \{f \geqslant c\}} (\psi^-(|f-c|) + \psi^+(|f-c|))dx.$$

(2) *(a) For any $\alpha > c$, we have*

$$\int_\Omega \psi^-(|f-c|)dx \leq \int_{\Omega \cap \{f < \alpha\}} (\psi^-(|f-c|) + \psi^+(|f-c|))dx.$$

(b) For any $\alpha < c$, we have

$$\int_\Omega \psi^-(|f-c|)dx \leq \int_{\Omega \cap \{f > \alpha\}} (\psi^-(|f-c|) + \psi^+(|f-c|))dx.$$

Next we give a property for the set of extended best constant approximations.

Theorem 2.7. *Let $\varphi \in \Phi \cap \Delta_2$ and let $f \in L^{\psi^+}(\Omega)$. Then the set $\mu_{\psi^+}(f)(\Omega)$ is a closed bounded interval.*

Proof. Let f be a function in $L^{\psi^+}(\Omega)$. We will see that if a constant c_1 satisfies Inequality (2.3) of Definition 2.3, so does any constant $c \geqslant c_1$. Then

$$\begin{aligned} &\int_{\Omega \cap \{f > c\}} \psi^-(|f-c|)dx \leq \int_{\Omega \cap \{f > c\}} \psi^-(|f-c_1|)dx \\ \leq &\int_{\Omega \cap \{f > c_1\}} \psi^-(|f-c_1|)dx \leq \int_{\Omega \cap \{f \leq c_1\}} \psi^+(|f-c_1|)dx \\ \leq &\int_{\Omega \cap \{f \leq c_1\}} \psi^+(|f-c|)dx \leq \int_{\Omega \cap \{f \leq c\}} \psi^+(|f-c|)dx. \end{aligned}$$

Similarly if a constant c_2 satisfies Inequality (2.4) of Definition 2.3, so does any constant $c \leqslant c_2$. In fact

$$\begin{aligned} &\int_{\Omega \cap \{f < c\}} \psi^-(|f-c|)dx \leq \int_{\Omega \cap \{f < c\}} \psi^-(|f-c_2|)dx \\ \leq &\int_{\Omega \cap \{f < c_2\}} \psi^-(|f-c_2|)dx \leq \int_{\Omega \cap \{f \geq c_2\}} \psi^+(|f-c_2|)dx \\ \leq &\int_{\Omega \cap \{f \geqslant c_2\}} \psi^+(|f-c|)dx \leq \int_{\Omega \cap \{f \geqslant c\}} \psi^+(|f-c|)dx. \end{aligned}$$

Thus, $\mu_{\psi^+}(f)(\Omega)$ is an interval.

Now, we will see that this interval is bounded. For any $\alpha < c$, where $c \in \mu_{\psi^+}(f)(\Omega)$, we have by (1)($b$) of Theorem 2.6 and the constant k for the Δ_2 condition on φ,

$$\begin{aligned}\int_\Omega \psi^-(|c|)\,dx &\le k\Big(\int_\Omega \psi^-(|f-c|)dx + \int_\Omega \psi^-(|f|)\,dx\Big)\\ &\le k\Big(\int_{\Omega\cap\{f\ge c\}} (\psi^-(|f-c|)+\psi^+(|f-c|))dx + \int_\Omega \psi^-(|f|)\,dx\Big)\\ &\le k\Big(\int_{\Omega\cap\{f\ge c\}} (\psi^-(|f-\alpha|)+\psi^+(|f-\alpha|))dx + \int_\Omega \psi^-(|f|)\,dx\Big)\\ &\le k\Big(\int_{\Omega\cap\{f> \alpha\}} (\psi^-(|f-\alpha|)+\psi^+(|f-\alpha|))dx + \int_\Omega \psi^-(|f|)\,dx\Big).\end{aligned}$$

Then $\mu_{\psi^+}(f)(\Omega)$ has an upper bound since $\psi^-(x)\to\infty$, as $x\to\infty$.

Similarly, given $c\in\mu_{\psi^+}(f)(\Omega)$, let any $\beta > c$. Then from (1)(a) of Theorem 2.6,

$$\begin{aligned}\int_\Omega \psi^-(|f-c|)dx &\le \int_{\Omega\cap\{f\le c\}} (\psi^+(|f-c|)+\psi^-(|f-c|))dx\\ &\le \int_{\Omega\cap\{f\le c\}} (\psi^+(|f-\beta|)+\psi^-(|f-\beta|))dx\\ &\le \int_{\Omega\cap\{f\le \beta\}} (\psi^+(|f-\beta|)+\psi^-(|f-\beta|))dx.\end{aligned}$$

Thus, we deduce that the set $\mu_{\psi^+}(f)(\Omega)$ is bounded from below.

To prove that $\mu_{\psi^+}(f)(\Omega)$ is closed, let $\{c_n\}_{n\in\mathbb{N}}$be a sequence in $\mu_{\psi^+}(f)(\Omega)$ such that $c_n\to c$ as $n\to\infty$. We will prove that c satisfies inequalities (2.3) and (2.4) of the Definition 2.3. Then, since

$$\psi^-(|f-c|)\,\chi_{\{f>c\}} \le \liminf_{n\to\infty} \psi^-(|f-c_n|)\,\chi_{\{f>c_n\}},$$

we get, from Fatou's Lemma,

$$\int_{\Omega\cap\{f>c\}} \psi^-(|f-c|)dx \leqslant \liminf_{n\to\infty} \int_{\Omega\cap\{f>c_n\}} \psi^-(|f-c_n|)dx. \tag{2.18}$$

Now from (2.3) we get

$$\int_{\Omega\cap\{f>c\}} \psi^-(|f-c|)dx \leqslant \liminf_{n\to\infty} \int_{\Omega\cap\{f\leqslant c_n\}} \psi^+(|f-c_n|)dx.$$

Then, since $\psi^+(0)=0$, we have

$$\begin{aligned}\int_{\Omega\cap\{f>c\}} \psi^-(|f-c|)dx &\le \limsup_{n\to\infty} \int_{\Omega\cap\{f\leqslant c_n\}} \psi^+(|f-c_n|)dx\\ &= \limsup_{n\to\infty} \int_{\Omega\cap\{f< c_n\}} \psi^+(|f-c_n|)dx.\end{aligned}$$

As

$$\psi^+(|f_n-c_n|)\,\chi_{\{f<c\}} \le k(\psi^+(|f|)+\psi^+(M)),$$

for some $M>0$ and

$$\psi^+(|f-c|)\,\chi_{\{f\le c\}} \ge \limsup_{n\to\infty} \psi^+(|f-c_n|)\,\chi_{\{f<c_n\}},$$

we get, using Lebesgue-Fatou's Theorem,

$$\int_{\Omega\cap\{f>c\}} \psi^-(|f-c|)dx \leqslant \int_{\Omega\cap\{f\leqslant c\}} \psi^+(|f-c|)dx.$$

The proof that c satisfies (2.4) of Definition 2.3. follows in a similiar way and then the proof is completed. □

According to [1] a multivalued operator $T : L^{\psi^+}(\Omega) \to 2^{\mathbb{R}}$ is a monotone operator if given $f, g \in L^{\psi^+}(\Omega), f \leq g,\ \ g_1 \in \mu_{\psi^+}(f)$ and $g_2 \in \mu_{\psi^+}(g)$ we have $\min(g_1, g_2) \in \mu_{\psi^+}(f)$ and $\max(g_1, g_2) \in \mu_{\psi^+}(g)$. We point out that if T is an univalued operator, this definition coincides with the standard definition of monotony. Next we prove that μ_{ψ^+} is a monotone operator using this generalized notion.

Theorem 2.8. *Let $\varphi \in \Phi \cap \Delta_2$, then μ_{ψ^+} is a monotone operator on $L^{\psi^+}(\Omega)$.*

Proof. Let $f_1,\ f_2 \in L^{\psi^+}(\Omega)$, where $f_1 \leqslant f_2$ and $c_1 \in \mu_{\psi^+}(f_1)$ and $c_2 \in \mu_{\psi^+}(f_2)$. We will prove first that

$$\int_{\Omega\cap\{f_1>\min(c_1,c_2)\}} \psi^-(|f_1 - c_2|)dx \leq \int_{\Omega\cap\{f_1\leq\min(c_1,c_2)\}} \psi^+(|f_1 - c_2|)\, dx, \tag{2.19}$$

and

$$\int_{\Omega\cap\{f_1<\min(c_1,c_2)\}} \psi^-(|f_1 - c_2|)dx \leq \int_{\Omega\cap\{f_1\geq\min(c_1,c_2)\}} \psi^+(|f_1 - c_2|)\, dx, \tag{2.20}$$

but we only have to consider the case $\min(c_1, c_2) = c_2$ because the case $c_1 \leq c_2$ follows straightforward. So let $\min(c_1, c_2) = c_2$. Then

$$\begin{aligned} &\int_{\Omega\cap\{f_1>c_2\}} \psi^-(|f_1 - c_2|)dx \leq \int_{\Omega\cap\{f_2>c_2\}} \psi^-(|f_1 - c_2|)dx \\ &\leq \int_{\Omega\cap\{f_2>c_2\}} \psi^-(|f_2 - c_2|)dx \leq \int_{\Omega\cap\{f_2\leq c_2\}} \psi^+(|f_2 - c_2|)dx \\ &\leq \int_{\Omega\cap\{f_2\leq c_2\}} \psi^+(|f_1 - c_2|)dx \leq \int_{\Omega\cap\{f_1\leq c_2\}} \psi^+(|f_1 - c_2|)dx, \end{aligned}$$

and inequality (2.19) holds.

Next, we analyze inequality (2.20).

$$\begin{aligned} &\int_{\Omega\cap\{f_1<c_2\}} \psi^-(|f_1 - c_2|)dx \leq \int_{\Omega\cap\{f_1<c_2\}} \psi^-(|f_1 - c_1|)dx \\ &\leq \int_{\Omega\cap\{f_1<c_1\}} \psi^-(|f_1 - c_1|)dx \leq \int_{\Omega\cap\{f_1\geq c_1\}} \psi^+(|f_1 - c_1|)dx \\ &\leq \int_{\Omega\cap\{f_1\geq c_1\}} \psi^+(|f_1 - c_2|)dx \leq \int_{\Omega\cap\{f_1\geq c_2\}} \psi^+(|f_1 - c_2|)dx. \end{aligned}$$

Thus, $\min(c_1, c_2) \in \mu_{\psi^+}(f_1)$.

Now we prove $\max(c_1, c_2) \in \mu_{\psi^+}(f_2)$. As the case $c_1 \leq c_2$ is trivial we assume $\max(c_1, c_2) = c_1$. Then

$$\begin{aligned} &\int_{\Omega\cap\{f_2>c_1\}} \psi^-(|f_2 - c_1|)dx \leq \int_{\Omega\cap\{f_2>c_1\}} \psi^-(|f_2 - c_2|)dx \\ &\leq \int_{\Omega\cap\{f_2>c_2\}} \psi^-(|f_2 - c_2|)dx \leq \int_{\Omega\cap\{f_2\leq c_2\}} \psi^+(|f_2 - c_2|)dx \\ &\leq \int_{\Omega\cap\{f_2\leq c_2\}} \psi^+(|f_2 - c_1|)dx \leq \int_{\Omega\cap\{f_2\leq c_1\}} \psi^+(|f_2 - c_1|)dx. \end{aligned} \tag{2.21}$$

Now, we analyze the other inequality

$$
\begin{aligned}
&\int_{\Omega\cap\{f_2<c_1\}} \psi^-(|f_2-c_1|)dx \le \int_{\Omega\cap\{f_2<c_1\}} \psi^-(|f_1-c_1|)dx \\
&\le \int_{\Omega\cap\{f_1<c_1\}} \psi^-(|f_1-c_1|)dx \le \int_{\Omega\cap\{f_1\ge c_1\}} \psi^+(|f_1-c_1|)dx \\
&\le \int_{\Omega\cap\{f_1\ge c_1\}} \psi^+(|f_2-c_1|)dx \le \int_{\Omega\cap\{f_2\ge c_1\}} \psi^+(|f_2-c_1|)dx.
\end{aligned} \tag{2.22}
$$

Then, from (2.21) and (2.22), we obtain $\max(c_1, c_2) \in \mu_{\psi^+}(f_2)$. □

From now on we consider $\Omega = B_\epsilon(x)$, where $B_\epsilon(x)$ is a ball centered at $x \in \mathbb{R}^n$ with radius ϵ. We denote $\mu^\epsilon_{\psi^+}(f)(x)$ for the set of extended best constant approximations of f and we denote $f_\epsilon(x)$ for any constant $c \in \mu^\epsilon_{\psi^+}(f)(x)$.

Also, for a function f locally integrable in $L^{\psi^+}(\mathbb{R}^n)$, we write $f \in L^{\psi^+}_{loc}(\mathbb{R}^n)$.

Next we get some inequalities involving the extended best approximations that generalize Theorem 3 in [4] and Theorem 2.1 in [5]. This result allows us, on one hand, to compare the maximal operator of Hardy-Littlewood with an operator associated with the family $\{f_\epsilon(x)\}$ and on the other hand, it allows us to prove the a.e. convergence of $f_\epsilon(x) \to f(x)$ as $\epsilon \to 0$.

Theorem 2.9. *Let $\varphi \in \Phi \cap \Delta_2$ and $f \in L^{\psi^+}_{loc}(\mathbb{R}^n)$. If $f_\epsilon(x) \in \mu^\epsilon_{\psi^+}(f)(x)$, where $x \in \mathbb{R}^n$ and $\epsilon > 0$, then it holds*

$$
\frac{1}{C}\psi^+(|f_\epsilon(x)|) \le \frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)} \psi^-(|f(y)|)dy \leqslant C_1\psi^+(|f|_\epsilon(x)). \tag{2.23}
$$

$$
\frac{1}{C}\psi^+(|f_\epsilon(x)-f(x)|) \le \frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)} \psi^-(|f(y)-f(x)|)dy, \tag{2.24}
$$

where $C = 3k^3$, $C_1 = 3k^2$ and the constant k is the one appearing in the Δ_2 condition on φ.

Proof. First, we show the right side of (2.23). Without lost of generality we may assume $f \geqslant 0$ and then $f_\epsilon(x) \geqslant 0$ by Theorem 2.8.
If $f(y) = [f(y) - f_\epsilon(x)] + f_\epsilon(x)$, then $\psi^-(f(y)) = \psi^-([f(y) - f_\epsilon(x)] + f_\epsilon(x))$. We have

$$
\begin{aligned}
\frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)} \psi^-(f)dy &= \frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)} \psi^-([f-f_\epsilon(x)]+f_\epsilon(x))dy \\
&= \frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)\cap\{f>f_\epsilon\}} \psi^-([f-f_\epsilon(x)]+f_\epsilon(x))dy \\
&+ \frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)\cap\{f\leqslant f_\epsilon\}} \psi^-(f)dy.
\end{aligned}
$$

From (1.5), we have

$$
\begin{aligned}
\frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)} \psi^-(f)dy \leqslant & \frac{k}{|B_\epsilon(x)|}\int_{B_\epsilon(x)\cap\{f>f_\epsilon\}} \psi^-(f-f_\epsilon(x))dy \\
&+ \frac{k}{|B_\epsilon(x)|}\int_{B_\epsilon(x)\cap\{f>f_\epsilon\}} \psi^-(f_\epsilon(x))dy \\
&+ \frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)\cap\{f\leqslant f_\epsilon\}} \psi^-(f)dy.
\end{aligned} \tag{2.25}
$$

We apply (2.3) of Definition 2.3 and we have that

$$\begin{aligned}\frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)}\psi^-(f)dy \leqslant \frac{k}{|B_\epsilon(x)|}\int_{B_\epsilon(x)\cap\{f\leqslant f_\epsilon\}}\psi^+(f_\epsilon(x)-f)dy \\ + k\psi^-(f_\epsilon(x)) + \frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)\cap\{f\leqslant f_\epsilon\}}\psi^+(f)dy.\end{aligned} \tag{2.26}$$

Then we have

$$\begin{aligned}\frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)}\psi^-(f)dy \leqslant \frac{k}{|B_\epsilon(x)|}\int_{B_\epsilon(x)\cap\{f\leqslant f_\epsilon\}}[\psi^+(f_\epsilon(x)-f)+\psi^+(f)]dy \\ + k\psi^-(f_\epsilon(x)).\end{aligned} \tag{2.27}$$

Since in the above integral $f_\epsilon(x) - f \geqslant 0$ and $f \geqslant 0$, we apply (1.5) and we obtain

$$\psi^+(f_\epsilon(x)-f)+\psi^+(f) \leqslant 2\psi^+[(f_\epsilon(x)-f)+f] = 2\psi^+(f_\epsilon(x)).$$

Then from (2.27) we get

$$\frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)}\psi^-(f)dy \leqslant C_1\psi^+(f_\epsilon(x)),$$

with $C_1 = 3k^2$.
To prove the left hand side of (2.23) we may assume $f \geq 0$. In fact, given a constant $f_\epsilon(x)$ in the set $\mu^\epsilon_{\psi^+}(f)(x)$, there exists $c_\epsilon \in \mu^\epsilon_{\psi^+}(|f|)(x)$ such that $|f_\epsilon(x)| \leqslant c_\epsilon$. Indeed as $-|f| \leqslant f \leqslant |f|$ and the extended best approximation operator is a monotone operator, there exist constants $a_\epsilon \geqslant 0$ and $b_\epsilon \geqslant 0$ with $-a_\epsilon \in \mu^\epsilon_{\psi^+}(-|f|)(x)$ and $b_\epsilon \in \mu^\epsilon_{\psi^+}(|f|)(x)$ such that $-a_\epsilon \leqslant f_\epsilon \leqslant b_\epsilon$. As $-a_\epsilon \in \mu^\epsilon_{\psi^+}(-|f|)(x)$, then $a_\epsilon \in \mu^\epsilon_{\psi^+}(|f|)(x)$. Set $c_\epsilon = \max\{a_\epsilon, b_\epsilon\}$. Since the set $\mu^\epsilon_{\psi^+}(|f|)(x)$ is a closed interval, the maximun belongs to the set of extended best constant approximations $\mu^\epsilon_{\psi^+}(|f|)(x)$. Thus, $|f_\epsilon(x)| \leqslant c_\epsilon$.

Now $\psi^+(f_\epsilon(x))$ can be written as

$$\begin{aligned}\frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)}\psi^+(f_\epsilon(x))dy = \frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)\cap\{f<f_\epsilon\}}\psi^+([f_\epsilon(x)-f]+f)dy \\ + \frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)\cap\{f\geqslant f_\epsilon\}}\psi^+(f_\epsilon(x))dy.\end{aligned} \tag{2.28}$$

We have $f_\epsilon(x) - f > 0$ on $B_\epsilon(x) \cap \{f_\epsilon > f\}$ and since $f \geq 0$ we use (1.5) to get

$$\begin{aligned}\frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)}\psi^+(f_\epsilon(x))dy \leqslant \frac{k}{|B_\epsilon(x)|}\int_{B_\epsilon(x)\cap\{f<f_\epsilon\}}[\psi^+(f_\epsilon(x)-f)+\psi^+(f)]dy \\ + \frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)\cap\{f\geqslant f_\epsilon\}}\psi^+(f_\epsilon(x))dy.\end{aligned}$$

Also by (1.4) there exists a positive constant k such that the above expression can be estimated by

$$\frac{k^2}{|B_\epsilon(x)|}\int_{B_\epsilon(x)\cap\{f<f_\epsilon\}}[\psi^-(f_\epsilon(x)-f)+\psi^-(f)]dy + \frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)\cap\{f\geqslant f_\epsilon\}}\psi^+(f_\epsilon(x))dy.$$

Now, by (2.4) of Definition 2.3, we have

$$\begin{aligned}\frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)}\psi^+(f_\epsilon(x))dy \leqslant & \frac{k^2}{|B_\epsilon(x)|}\int_{B_\epsilon(x)\cap\{f\geqslant f_\epsilon\}}\psi^+(-f_\epsilon(x)+f)dy \\ &+\frac{k^2}{|B_\epsilon(x)|}\int_{B_\epsilon(x)\cap\{f<f_\epsilon\}}\psi^-(f)dy \\ &+\frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)\cap\{f\geqslant f_\epsilon\}}\psi^+(f_\epsilon(x))dy.\end{aligned}$$

Then assuming $k\geqslant 1$, we have

$$\begin{aligned}&\frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)}\psi^+(f_\epsilon(x))dy \\ &\leqslant \frac{k^2}{|B_\epsilon(x)|}\int_{B_\epsilon(x)\cap\{f\geqslant f_\epsilon\}}[\psi^+(-f_\epsilon(x)+f)+\psi^+(f_\epsilon(x))]dy \\ &+\frac{k^2}{|B_\epsilon(x)|}\int_{B_\epsilon(x)\cap\{f<f_\epsilon\}}\psi^-(f)dy,\end{aligned} \tag{2.29}$$

which can be bounded by

$$\frac{2k^3}{|B_\epsilon(x)|}\int_{B_\epsilon(x)\cap\{f\geqslant f_\epsilon\}}\psi^-(f)dy+\frac{k^3}{|B_\epsilon(x)|}\int_{B_\epsilon(x)\cap\{f<f_\epsilon\}}\psi^-(f)dy,$$

then we have proved

$$\frac{1}{C}\psi^+(f_\epsilon(x))\leqslant\frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)}\psi^-(f)dy,$$

with $C=3k^3$.

Now to obtain (2.24) let $f_\epsilon(x)\in\mu^\epsilon_{\psi^+}(f)(x)$, then $f_\epsilon(x)-f(x)\in\mu^\epsilon_{\psi^+}(f-f(x))$. We apply (2.23) to the function $f-f(x)$ and we have

$$\frac{1}{C}\psi^+(|f_\epsilon(x)-f(x)|)\leqslant\frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)}\psi^-(|f(y)-f(x)|)dy.$$

Thus the proof is completed.

□

The next corollary follows strightforward from Theorem 2.9.

Corollary 2.10. *Let $\varphi\in\Phi\cap\Delta_2$ and $f\in L^{\psi^+}_{loc}(\mathbb{R}^n)$. If $f_\epsilon(x)\in\mu^\epsilon_{\psi^+}(f)(x)$, where $x\in\mathbb{R}^n$ and $\epsilon>0$, then we have the following estimates*

$$\frac{1}{C}\psi^-(|f_\epsilon(x)|)\leq\frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)}\psi^-(|f(y)|)dy\leqslant C\psi^-(|f|_\epsilon(x)). \tag{2.30}$$

$$\frac{1}{C}\psi^-(|f_\epsilon(x)-f(x)|)\leq\frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)}\psi^-(|f(y)-f(x)|)dy, \tag{2.31}$$

where $C=3k^3$ and the constant k is the one appearing in the Δ_2 condition on φ.

Now, we have the next theorem which is a direct consequence of Theorem 2.9.

Theorem 2.11. *Let $\varphi\in\Phi\cap\Delta_2$ with $\psi^+(x)>0$, when $x>0$. Then for each function $f\in L^{\psi^+}_{loc}(\mathbb{R}^n)$ and for almost every $x\in\mathbb{R}^n$ we have*

$$\lim_{\epsilon\to 0}(\sup\{|f_\epsilon(x)-f(x)|:f_\epsilon(x)\in\mu^\epsilon_{\psi^+}(f)\})=0.$$

Note that last theorem can be seen as a generalization of the Lebesgue Differentiation Theorem. This is the case if $\varphi(x)=x^2$.

3. Maximal Inequalities

In this section we will study the modular convergence (1.7). Note that if $\varphi(x) = x^2$, the best approximation constant is $\frac{1}{B_\varepsilon}\int_{B_\varepsilon} f(y)\,dy$, and the modular convergence (1.7) follows from the properties of the Hardy Littlewood maximal function. In our case we consider any convex function φ and then we have to consider a different maximal function.

Definition 3.1. *Let $\mathcal{M}_\varphi$ be the maximal operator defined as*

$$\mathcal{M}_\varphi f(x) = \sup_{\epsilon>0}\{|f_\epsilon(x)| : f_\epsilon(x) \in \mu^\epsilon_{\psi^+}(f)(x)\}. \tag{3.1}$$

We will set strong inequalities for $\mathcal{M}_\varphi$ which guarantee modular convergence of $f_\varepsilon(x)$, as ε goes to 0. To do this we need to consider the Hardy Littlewood Maximal Operator.

Definition 3.2. *Let M_H be the maximal operator of Hardy-Littlewood defined as*

$$M_H(f)(x) = \sup_{\epsilon>0} \frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)} |f(y)|dy, \tag{3.2}$$

where $f \in L^1_{loc}(\mathbb{R}^n)$.

We denote $\bar\varphi$ for the generalized inverse function of ψ^- which is given by

$$\bar\varphi(x) = \sup_{\psi^-(s)\leqslant x} s, \tag{3.3}$$

and

denote by $\tilde\varphi$ for the generalized inverse function of ψ^+ which is given by

$$\tilde\varphi(x) = \sup_{\psi^+(s)\leqslant x} s. \tag{3.4}$$

From (1.4) we get

$$\bar\varphi(\frac{x}{\tilde C}) \leqslant \tilde\varphi(x) \leqslant \bar\varphi(x), \tag{3.5}$$

for some constant $\tilde C > 0$ and for every $0 < x < \infty$.

Theorem 3.3. *Let $\varphi \in \Phi \cap \delta_2$ and $A\psi^+(t) \leqslant \psi^+(Kt)$, $t \geqslant 0$, for some constants K, $A > 1$. Then there exist positive constants C_1, C and $\hat C$ such that*

$$\frac{1}{K}\tilde\varphi(\frac{1}{C_1\hat C}M_H(\psi^+(|f|))(x)) \leqslant \mathcal{M}_\varphi(|f|)(x) \leqslant \tilde\varphi(CM_H(\psi^+)(|f|)(x)). \tag{3.6}$$

Proof. First we prove the right hand side of (3.6).

By Theorem 2.9 there exists a constant $C > 0$ such that

$$\psi^+(|f|_\epsilon(x)) \leqslant \frac{C}{|B_\epsilon(x)|}\int_{B_\epsilon(x)} \psi^-(|f(y)|)dy.$$

As $|f|_\epsilon(x) \leqslant \tilde\varphi(\psi^+(|f|_\epsilon(x)))$, we have

$$|f|_\epsilon(x) \leqslant \tilde\varphi\left(\frac{C}{|B_\epsilon(x)|}\int_{B_\epsilon(x)} \psi^-(|f(y)|)dy\right)$$

$$\leqslant \tilde\varphi\left(\sup_{\epsilon>0}\frac{C}{|B_\epsilon(x)|}\int_{B_\epsilon(x)} \psi^-(|f(y)|)dy\right) \leqslant \tilde\varphi\left(\sup_{\epsilon>0}\frac{C}{|B_\epsilon(x)|}\int_{B_\epsilon(x)} \psi^+(|f(y)|)dy\right).$$

Then $|f|_\epsilon(x) \leqslant \tilde{\varphi}(CM_H(\psi^+(|f|))$ for all $\epsilon > 0$, thus

$$\mathcal{M}_\varphi(|f|)(x) = \sup_{\epsilon>0} |f|_\epsilon(x) \leqslant \tilde{\varphi}(CM_H(\psi^+(|f|)))(x). \tag{3.7}$$

To prove the left hand side of (3.6) we have, from Theorem 2.9, that there exists a constant $C_1 > 0$ such that

$$\frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)} \psi^-(|f(y)|)dy \leqslant C_1\psi^+(|f|_\epsilon(x)). \tag{3.8}$$

Since $|f|_\epsilon(x) \leqslant \mathcal{M}_\varphi(|f|)(x)$ we can estimate (3.8) by

$$\frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)} \psi^-(|f(y)|)dy \leqslant C_1\psi^+(\mathcal{M}_\varphi(|f|)(x)),$$

for all $\epsilon > 0$. As $\psi^+(x) \leqslant \hat{C}\psi^-(x)$ for some constant $\hat{C} > 0$, we have

$$\frac{1}{\hat{C}}\frac{1}{|B_\epsilon(x)|}\int_{B_\epsilon(x)} \psi^+(|f(y)|)dy \leqslant C_1\psi^+(\mathcal{M}_\varphi(|f|)(x)),$$

for all $\epsilon > 0$, thus

$$\frac{1}{\hat{C}}M_H(\psi^+(|f|))(x) \leqslant C_1\psi^+(\mathcal{M}_\varphi(|f|)(x)). \tag{3.9}$$

Using the hypothesis on ψ^+ we have $A\psi^+(t) \leqslant \psi^+(Kt)$, $t \geqslant 0$, then $0 \leqslant \psi^+(t) < A\psi^+(t) \leqslant \psi^+(Kt)$, $t > 0$ and thus $0 < \psi^+(Kt) - \psi^+(t)$ for all $t > 0$.

Now, for all $0 < \epsilon$ we have $\tilde{\varphi}(\psi^+(t) - \epsilon) \leqslant t$ for all $t \geqslant 0$. If we consider $0 < \epsilon = \psi^+(Kt) - \psi^+(t)$ for all $t > 0$, we obtain

$$\tilde{\varphi}(\psi^+(t)) = \tilde{\varphi}(\psi^+(Kt) - \epsilon) \leqslant Kt, \tag{3.10}$$

for all $t > 0$.

Using (3.9) and (3.10) we have

$$\tilde{\varphi}(\frac{1}{C_1\hat{C}}M_H(\psi^+(|f|))(x)) \leqslant \tilde{\varphi}(\psi^+(\mathcal{M}_\varphi(|f|)(x))) \leqslant K(\mathcal{M}_\varphi|f|)(x). \tag{3.11}$$

Therefore the proof is completed. □

Next we point out the following corollary.

Corollary 3.4. *Let $\varphi \in \Phi \cap \Delta_2$ and $A\psi^+(t) \leqslant \psi^+(Kt)$, $t \geqslant 0$, for some constants K, $A > 1$. Then there exists positive constants C_1, C and $\hat{C}$ such that*

$$\frac{1}{K}\bar{\varphi}(\frac{1}{C_1\hat{C}^2}M_H(\psi^+(|f|))(x)) \leqslant \mathcal{M}_\varphi(|f|)(x) \leqslant \bar{\varphi}(CM_H(\psi^+)(|f|)(x)). \tag{3.12}$$

Proof. The proof follows straightforward from Theorem 3.3 and (3.5). □

We also obtain the following inequality from Corollary 2.10.

Corollary 3.5. *Let $\varphi \in \Phi \cap \Delta_2$ and $A\psi^+(t) \leqslant \psi^+(Kt)$, $t \geqslant 0$, for some constants K, $A > 1$. Then there exists positive constants C_1 and C such that*

$$\frac{1}{K}\tilde{\varphi}(\frac{1}{C_1}M_H(\psi^-(|f|))(x)) \leqslant \mathcal{M}_\varphi(|f|)(x) \leqslant \tilde{\varphi}(CM_H(\psi^-)(|f|)(x)). \tag{3.13}$$

The following result, established in [9], will be used in the sequel and characterizes the functions $\psi \in \mathcal{S}$ which allows a strong inequality for the Hardy Littlewood maximal function in the Orlicz space L^ψ.

Theorem 3.6. *Let $\psi : [0,\infty) \to [0,\infty)$ be a non decreasing function such that $\psi(0) = 0$ and $\psi(x) \to \infty$, as $x \to \infty$. Then ψ satisfies the ∇_2 condition if and only if*

$$\int_{\mathbb{R}^n} \psi(M_H(f)(x))dx \leqslant C \int_{\mathbb{R}^n} \psi(C|f(x)|)dx,$$

where $f \in L^1_{loc}(\mathbb{R}^n)$ and the constant C is independent of f.

Theorem 3.7. *Let $\varphi \in \mathcal{S} \cap \Delta_2$ and let ψ^+ be such that $A\psi^+(t) \leqslant \psi^+(Kt)$, where $t \geqslant 0$, for some constants K,$A>1$. If $\theta \in \mathcal{S} \cap \Delta_2$ and the function $\theta \circ \tilde{\varphi} \in \nabla_2$, then there exists a constant $\widetilde{C}$ independent of f such that*

$$\int_{\mathbb{R}^n} \theta(\mathcal{M}_\varphi(|f|)(x))dx \leqslant \widetilde{C} \int_{\mathbb{R}^n} \theta(\widetilde{C}|f(x)|)dx, \tag{3.14}$$

where $f \in L^{\psi^+}_{loc}(\mathbb{R}^n)$.

Proof. Set $\psi = \theta \circ \tilde{\varphi}$. As $\psi \in \nabla_2$ we have, by Theorem 3.6, that there exists a constant $K_1 > 0$ such that

$$\int_{\mathbb{R}^n} \psi(M_H(g)(x))dx \leqslant K_1 \int_{\mathbb{R}^n} \psi(K_1 g(x))dx, \tag{3.15}$$

for all non negative function $g \in L^1_{loc}(\mathbb{R}^n)$.

By Theorem 3.3 we have

$$\mathcal{M}_\varphi(|f|)(x) \leqslant \tilde{\varphi}(CM_H(\psi^+(|f|))(x)). \tag{3.16}$$

Note that the inequalities (3.15) and (3.16) hold for the constant $K_2 = \max\{C, K_1\}$.

Now by the homogeneity of M_H, the monotony of θ and inequalities (3.15) and (3.16) we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \theta(\mathcal{M}_\varphi(|f|(x)))dx \leqslant \int_{\mathbb{R}^n} \theta(\tilde{\varphi}(K_2 M_H(\psi^+(|f|))(x)))dx \\
&= \int_{\mathbb{R}^n} \psi(K_2 M_H(\psi^+(|f|))(x))dx = \int_{\mathbb{R}^n} \psi(M_H(K_2\psi^+(|f|))(x))dx \\
&\leqslant K_2 \int_{\mathbb{R}^n} \psi(K_2K_2\psi^+(|f(x)|))dx \leqslant K_3 \int_{\mathbb{R}^n} \psi(K_3\psi^+(|f(x)|))dx,
\end{aligned} \tag{3.17}$$

where $K_3 = \max\{K_2, K_2^2\}$. Now since $A\psi^+(t) \leqslant \psi^+(Kt)$ for all $t \geqslant 0$ for some constants $K, A > 1$ and for $l \in \mathbb{N}$ such that $K_3 \leqslant A^l$ we get

$$K_3\psi^+(x) \leqslant A^l\psi^+(x) \leqslant \psi^+(K^l x) = \psi^+(K_4 x).$$

Therefore

$$\begin{aligned}
K_3 \int_{\mathbb{R}^n} \psi(K_3\psi^+(|f(x)|))dx &\leqslant K_3 \int_{\mathbb{R}^n} \psi(\psi^+(K_4|f(x)|))dx \\
&\leqslant K_5 \int_{\mathbb{R}^n} \psi(\psi^+(K_5|f(x)|))dx,
\end{aligned}$$

where $K_5 = \max\{K_3, K_4\}$. Now we use (3.10) to get

$$\tilde{\varphi}(\psi^+(t)) \leqslant Kt.$$

Then since $\psi = \theta \circ \tilde{\varphi}$, we have $(\psi \circ \psi^+)(t) = ((\theta \circ \tilde{\varphi}) \circ \psi^+)(t) \leqslant \theta(Kt)$. Since ψ^+ is a non decreasing function, we obtain

$$\psi(\psi^+(K_5|f(x)|)) \leqslant \theta(KK_5|f(x)|),$$

and then

$$K_5 \int_{\mathbb{R}^n} \psi(\psi^+(K_5|f(x)|))dx \leqslant K_5 \int_{\mathbb{R}^n} \theta(KK_5|f(x)|)dx \leqslant \widetilde{C} \int_{\mathbb{R}^n} \theta(\widetilde{C}|f(x)|)dx,$$

where $\widetilde{C} = \max\{K_5, KK_5\}$.

Thus the proof is completed. □

Corollary 3.8. *Let $\varphi \in \mathcal{S} \cap \Delta_2$ and let f be a function in $L^\theta(\mathbb{R}^n) \cap L^{\psi^+}(\mathbb{R}^n)$. Let ψ^+ be such that $A\psi^+(t) \leqslant \psi^+(Kt)$, where $t \geqslant 0$ and for some constants $K, A > 1$. If $\theta \in \mathcal{S} \cap \Delta_2$ and $\theta \circ \tilde{\varphi} \in \nabla_2$, then for $f_\varepsilon(x) \in \mu_{\psi^+}(f)$ we get*

$$\int_{\mathbb{R}^n} \theta(|f(y) - f_\epsilon(y)|)dy \to 0,$$

as $\epsilon \to 0$.

Proof. Since θ is a nondecreasing function and satisfies the Δ_2 condition, there exists a constant $K > 0$ such that

$$\theta(|f_\epsilon - f|) \leqslant \theta(|f_\epsilon| + |f|) \leqslant \theta(\mathcal{M}_\varphi|f| + |f|) \leqslant K\theta(\mathcal{M}_\varphi|f|) + K\theta(|f|),$$

for all $\epsilon > 0$.

By Theorem 3.7, there exists a constant $\widetilde{C}$ independent of f such that

$$\int_{\mathbb{R}^n} \theta(\mathcal{M}_\varphi|f|(x))dx \leqslant \widetilde{C} \int_{\mathbb{R}^n} \theta(\widetilde{C}|f(x)|)dx. \tag{3.18}$$

As $f \in L^\theta(\mathbb{R}^n) \cap L^{\psi^+}(\mathbb{R}^n)$, from (3.18) we have

$$K \int_{\mathbb{R}^n} \theta(\mathcal{M}_\varphi|f|(x))dx + K \int_{\mathbb{R}^n} \theta(|f(x)|)dx < \infty.$$

Now by Theorem 2.11 we have

$$\sup\{|f_\epsilon(x) - f(x)| : f_\epsilon(x) \in \mu^\epsilon_{\psi^+}(f)(x)\} \to 0,$$

as $\epsilon \to 0$,

and then, $\theta(|f_\epsilon(x) - f(x)|) \to 0$ as $\epsilon \to 0$. Thus, by Dominated Convergence Theorem we get

$$\int_{\mathbb{R}^n} \theta(|f(y) - f_\epsilon(y)|)dy \to 0,$$

as $\epsilon \to 0$.

□

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