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A thermodynamical gradient theory for deformation and strain localization of porous media

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ABSTRACT

In this work, a thermodynamically consistent gradient formulation for partially saturated cohesive-frictional porous media is proposed. The constitutive model includes a classical or *local* hardening law and a softening formulation with state parameters of non-local character based on gradient theory. Internal characteristic length in softening regime accounts for the strong shear band width sensitivity of partially saturated porous media regarding both governing stress state and hydraulic conditions. In this way the variation of the transition point (TP) of brittle-ductile failure mode can be realistically described depending on current confinement condition and saturation level. After describing the thermodynamically consistent gradient theory the paper focuses on its extension to the case of partially saturated porous media and, moreover, on the formulation of the gradient-based characteristic length in terms of stress and hydraulic conditions. Then the localization indicator for discontinuous bifurcation is formulated for both drained and undrained conditions.

1. Introduction

The mechanic of porous media constitutes a discipline of great relevance in several knowledge areas like Geophysics, Biomechanics and Materials Science. Its main aim is the description of the kinematic and pore pressure of porous continua when subjected to arbitrary mechanical and/or physical actions. The definitive advantages of porous mechanics to macroscopically describe or predict complex response behavior of cohesive-frictional materials based on fundamental aspects of their microstructure while accounting for the hydraulic properties and their influence in the resulting failure mechanism were recognized by several authors in the scientific community (Bary et al., 2000; Borja, 2004; Ito, 2008). Consequently, a tendency to replace the theoretical framework of classical continuum mechanics with that of non-linear porous mechanics was observed. Firstly this process took place in case of soil mechanics, (see a.o. Ehlers et al., 2004; Coussy and Monteiro, 2007), but subsequently in the field of concrete, (see a.o. Ulm et al., 2004; Pesavento et al., 2008) and, furthermore, of biomaterials, (see a.o. Naili et al., 1989; Pierre et al., 2008).

A relevant aspect of failure processes in cohesive-frictional materials is the transition from brittle to ductile response. In the realm of classical *non-porous* smeared-crack-based continua (NPSC), the concept of discontinuous bifurcation by means of the so-called localization indication (see a.o. Etse, 1994a; Jirásek and Rolshoven, 2009) gave the mathematically foundation to distinguish between diffuse and localized or brittle failure mode. Many proposals of constitutive models based on NPSC used the discontinuous bifurcation approach to accurately evaluate failure modes under different stress conditions,

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(Perić, 1990). A critical situation in this analysis is the eventual case when discontinuous bifurcation occurs in pre-peak regime before peak stress. This situation may arise in cohesive-frictional materials when subjected to monotonic compressive loading in the low confinement regime due to excessive dilatation that leads to sudden brittle failure (Etse and Willam, 1994b).

The extension of smeared-crack concept to porous media allows accounting for the influence of the saturation level in the location of the TP in the stress space. However, the extension of discontinuous bifurcation theory to porous media is not straightforward due to the difficulties related to the additional fields, basically gas and liquid, and their eventual jumps. We refer here to the works by Ehlers and Blome (2003), Schrefler and Pesavento (2004), Borja (2004), Ehlers et al. (2004), Voyiadjis et al. (2005), Schiava and Etse (2006), Hashiguchi and Tsutsumi (2007), Di Rado et al. (2009), Nicot et al. (2009). In some of these contributions (see Voyiadjis et al., 2005; Schiava and Etse, 2006) the discontinuous bifurcation theory was used to differentiate brittle from ductile failure modes of porous materials, following previous and similar works related to NPSC based material formulations.

Further development in classical continuum mechanics was the extension to non-local concepts. The main aim was the regularization of post peak response behavior regarding mesh size and element orientation in case of finite-element analyses, based on fundamental aspects of the material microstructure, (see a.o. Svedberg and Runesson, 1997; Fleck and Hutchinson, 2001; Etse and Vrech, 2006; Abu Al-Rub and Voyiadjis, 2006; Vrech and Etse, 2009).

In recent years significant progresses and relevant contributions were made in non-local gradient formulations for non-porous materials. Thermodynamic frameworks were considered in the proposals of Abu Al-Rub and Voyiadjis (2006), Abu Al-Rub et al. (2007), Polizzotto (2008, 2009a), Voyiadjis and Deliktas (2009), Gurtin and Anand (2009), Hirschberger and Steinmann (2009), Vrech and Etse (2009). Phenomenological aspects of the considered materials at the microscopic level of observation were taken into account in non-local gradient formulations by Polizzotto (2009b), Bardella (2010), Kuroda and Tvergaard (2010). Objective descriptions of the gradient internal length based on crystal-plasticity concepts were due to Bardella (2007), Kuroda and Tvergaard (2008a,b), Ertürk et al. (2009), while based on actual confining pressure in case of quasi-brittle materials like concrete as proposed by Vrech and Etse (2009). Considerations of material anisotropy in the formulation of internal variables evolution laws in case of gradient plasticity are due to Aifantis (2009), Voyiadjis et al. (2010). Geometrical analysis of bifurcation condition in case of non-local gradient formulations as proposed by Vrech and Etse (2005). Formulation of gradient enhanced coupled damage-plasticity material models and related finite-element implementations, see Svedberg and Runesson (1997), Liebe et al. (2001), Dorgan and Voyiadjis (2006).

Recently, non-local concepts were extended for the formulation of porous material models, see a.o. La Ragione et al. (2008), Kamrin (2010), Muraleetharan et al. (2009). Likewise, the consideration of microscopic aspects in the formulation of non-local constitutive theories for porous materials are due to Zhu et al. (2010), Nicot and Darve (2007), Yin et al. (2009).

In spite of the strong development of constitutive modelling for porous media there is still a need of thermodynamically consistent theoretical frameworks. This is particularly the case of non-local models for porous materials. Thermodynamic concepts should lead to dissipative stress formulations in hardening and softening regimes that allow non-constant descriptions of the internal variables of non-local character to accurately predict the sensitivity of porous material failure behavior to both confinement and saturation levels.

In this work the thermodynamically consistent formulation for gradient-based elastoplasticity by Vrech and Etse (2009) that follows general thermodynamic approach by Svedberg and Runesson (1997) for non-local damage formulation is extended for porous media. Main feature of present proposal is the definition of a gradient-based characteristic length in terms of both the governing stress and hydraulic conditions to capture the variation of the transition from brittle to ductile failure mode of cohesive-frictional porous materials with the level of confinement pressure and saturation. The paper includes the particularization of the proposed thermodynamically consistent gradient poroplastic theory to partially saturated soils, as well as the formulation and evaluation of the discontinuous bifurcation condition for different hydraulic conditions.

2. Macroscopic description of porous media

Porous media are multiphase systems with interstitial voids in the grain matrix filled with water (liquid phase), water vapor and dry air (gas phase) at microscopic level (see Fig. 1(a)).

Key argument to reconcile continuum mechanics with the intrinsic microscopic discontinuities of porous like materials composed by several interacting phases, is to consider them as thermodynamically open continuum systems (see Fig. 1(b)). Thus, their kinematics and deformations are referred to those of the skeleton. Contrarily to mixture theories based upon an averaging process (Lewis and Schrefler, 1998; Coussy et al., 1998; Huang and Zhang, 2003; Mroginski et al., 2010), the representation of porous media is made by a superposition, in time and space, of two or more continuum phases. In case of non-saturated porous continua we recognize three phases, the skeleton, the liquid and the gaseous phases.

2.1. Stress tensors

The mechanical behavior of partially saturated porous media is usually described by the effective stress tensor σ'_{ij}, as follows

$$\sigma'_{ij} = \sigma_{ij} - \delta_{ij} p^w = \sigma^n_{ij} + s_{ij} \tag{1}$$

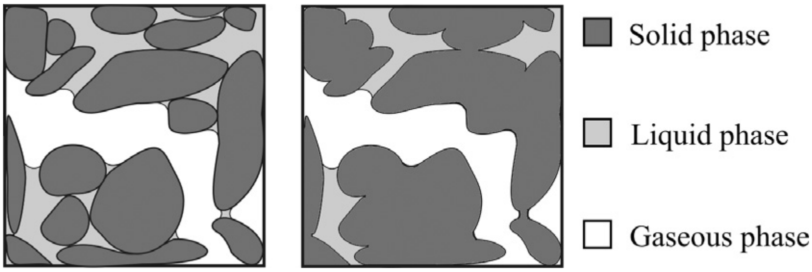

Fig. 1. Porous media description. (a) Microscopic level; (b) Macroscopic level.

with

$$s_{ij} = \delta_{ij}(p^a - p^w) \tag{2}$$
$$\sigma^n_{ij} = \sigma_{ij} - \delta_{ij}p^a \tag{3}$$

where $\sigma_{ij}, \sigma^n_{ij}$ and s_{ij} are the total, net, and suction stress tensors, respectively, while δ_{ij} is the Kronecker delta. Moreover, p^a and p^w are the gas and water pore pressures, respectively. In several geotechnical problems the gas pore pressure can be considered as a constant term that equals the atmospheric pressure. In these cases the suction tensor is counterpart to the water pore pressure, p.

2.2. Flow theory of poroplasticity

Plasticity is a property exhibited by several materials to undergo permanent strains after a complete process of loading and unloading. Hence, poroplasticity is that property of porous media which defines their ability to undergo not only permanent skeleton strains, but also permanent variations in fluid mass content due to related porosity variations. To fully characterize current stages of poroelastoplastic media and to describe their irreversible evolutions, internal variables such as the plastic porosity ϕ^p or the plastic fluid mass content m^p must be considered in addition to the plastic strain ε^p_{ij}, and the irreversible entropy density s^p.

Initial, non-deformed stage with $\varepsilon_{ij} = 0$ and $m = 0$, corresponds to initial values of the stress tensor σ^0_{ij}, the pore pressure p^0 and absolute temperature θ^0.

Small strain flow rule of poroplastic materials is based on additive decompositions of internal variables into elastic and plastic components

$$\begin{aligned} \dot{\varepsilon}_{ij} &= \dot{\varepsilon}^e_{ij} + \dot{\varepsilon}^p_{ij} \\ \dot{m} &= \dot{m}^e + \dot{m}^p \\ \dot{s} &= \dot{s}^e + \dot{s}^p \end{aligned} \tag{4}$$

For finite deformation problems the proposed theory presents some modifications, for details see Appendix A.

Both, the rate of skeleton plastic strains $\dot{\varepsilon}^p_{ij}$ and the rate of plastic fluid mass content $\dot{m}^p$ are related to the irreversible evolution of the skeleton. Thereby, the rate of plastic porosity $\dot{\phi}^p$ can be obtained as

$$\dot{\phi}^p = \frac{\dot{m}^p}{\rho^{fl}_0} \tag{5}$$

with ρ^{fl}_0 the initial fluid mass density.

3. Thermodynamic of local dissipative porous media

In this section some basic thermodynamic relationships for classical or local dissipative porous media are presented.

3.1. First law of the thermodynamic

Considering a body occupying the volume Ω, with boundary $\partial\Omega$, the first law of the Thermodynamic can be expressed as

$$\dot{E} + \dot{K} = P + Q \tag{6}$$

with

$$E = \int_\Omega (\rho e + pm)\,\mathrm{d}\Omega \tag{7}$$

$$K = \frac{1}{2}\int_{\Omega} \rho|\dot{u}_i\dot{u}_i|\mathrm{d}\Omega \tag{8}$$

$$P = \int_{\Omega} \rho b_i \dot{u}_i \mathrm{d}\Omega + \int_{\partial\Omega} \sigma_{ij} n_i \dot{u}_j \mathrm{d}\partial\Omega \tag{9}$$

$$Q = \int_{\Omega} \rho r \mathrm{d}\Omega - \int_{\partial\Omega} h_i n_i \mathrm{d}\partial\Omega \tag{10}$$

Here, E and K are the internal and the kinetic energy of the body, respectively, whereas P and Q represent the mechanical and thermal supply. Moreover, e is the internal energy density (per unit mass), b_i is the body force, σ_{ij} is the stress, r is a heat source density and h_i is the heat flux. The displacement u_i, the unit normal vector on $\partial\Omega$, n_i, and the mass density ρ, were also included.

Considering the equilibrium equation, the explicit form of the internal energy density for local dissipative porous material follows from Eq. (6) as

$$\rho\dot{e} = \sigma_{ij}\dot{\varepsilon}_{ij} + \rho r + p\dot{m} - h_{i,i} \tag{11}$$

3.2. Second law of the thermodynamic

According to the second law of thermodynamic (entropy inequality) the entropy S of a thermodynamic system can not decrease. This can be expressed as

$$\dot{S} + Q_\theta \geqslant 0 \tag{12}$$

with

$$\dot{S} = \int_{\Omega} \rho s \mathrm{d}\Omega; \quad Q_\theta = \int_{\Omega} \frac{\rho r}{\theta} \mathrm{d}\Omega - \int_{\partial\Omega} \frac{n_i h_i}{\theta} d\partial\Omega \tag{13}$$

being Q_θ the entropy flux, and s the entropy density (per unit mass). Invoking Eq. (12) in Eq. (13) and eliminating r by combining with Eq. (11), the global form of the Clausius–Duhem inequality (CDI) can be obtained as

$$\int_{\Omega} \frac{1}{\theta}\left(\sigma_{ij}\dot{\varepsilon}_{ij} + \rho\theta\dot{s} + p\dot{m} - \rho\dot{e} - \frac{h_i\theta_{,i}}{\theta}\right)\mathrm{d}\Omega \geqslant 0 \tag{14}$$

Introducing the Helmholtz's free energy $\Psi = e - \theta s^e$, the following expression is attained

$$\int_{\Omega} \frac{1}{\theta}\left[\sigma_{ij}\dot{\varepsilon}_{ij} + \rho\theta\dot{s} + p\dot{m} - \rho\left(\dot{\Psi} + \theta\dot{s}^e + \dot{\theta}s^e\right) - \frac{h_i\theta_{,i}}{\theta}\right]\mathrm{d}\Omega \geqslant 0 \tag{15}$$

4. Thermodynamically consistent gradient poroplasticity with non-local effects limited to the state variables

The thermodynamic framework of classical or local plasticity is extended to non-local gradient-based elastoplastic porous material.

Following Simo and Miehe (1992) we assume that arbitrary thermodynamic states of the dissipative material during isothermal processes are completely determined by the elastic strain $\varepsilon^e_{ij} = \varepsilon_{ij} - \varepsilon^p_{ij}$, the elastic entropy $s^e = s - s^p$ and the internal variables q_α with $\alpha = s, p$ for solid or porous phase, which are considered here as scalar variables. Isothermal condition means that θ and s become irrelevant quantities, i.e. the elastic entropy s^e does not need to be included as an additional argument of the thermodynamic state. When considering poroplastic materials the elastic mass content $m^e = m - m^p$ needs also to be included as a thermodynamic argument, see Coussy (1995). Based on Svedberg and Runesson (1997) and Vrech and Etse (2009) we further assume that the internal variables q_α, (q_s and q_p), are the only ones of non-local character. The extension to more than two scalar internal variables is straightforward. Hence, both q_α and $q_{\alpha,i}$ will appear as arguments in e or, moreover, in Ψ, such that

$$\Psi = \Psi\left(\varepsilon^e_{ij}, m^e, q_\alpha, q_{\alpha,i}\right) \tag{16}$$

Note that by restricting non-local effects to the state variables the energy balance in Eq. (11) remains unchanged. This is because the strain rate tensor remains local and the energy density is not expressed in terms of its arguments that involve non-local effects through internal variables, like in Eq. (16).

Upon differentiation of Eq. (16) and combining with Eq. (15), under consideration of Eq. (4),

$$\int_{\Omega}\left[\left(\sigma_{ij} - \rho\partial_{\varepsilon^e_{ij}}\Psi\right)\dot{\varepsilon}_{ij} + (p - \rho\partial_{m^e}\Psi)\dot{m} + \rho\partial_{\varepsilon^e_{ij}}\Psi\dot{\varepsilon}^p_{ij} + \rho\partial_{m^e}\Psi\dot{m}^p - \sum_\alpha \rho\partial_{q_\alpha}\Psi\dot{q}_\alpha - \sum_\alpha \rho\partial_{q_{\alpha,i}}\Psi\dot{q}_{\alpha,i}\right]\mathrm{d}\Omega \geqslant 0 \tag{17}$$

and integrating the gradient term by parts, it follows

$$\int_{\Omega}\left[\left(\sigma_{ij}-\rho\partial_{\varepsilon^e_{ij}}\Psi\right)\dot{\varepsilon}_{ij}+(p-\rho\partial_{m^e}\Psi)\dot{m}+\rho\partial_{\varepsilon^e_{ij}}\Psi\dot{\varepsilon}^p_{ij}+\rho\partial_{m^e}\Psi\dot{m}^p-\sum_{\alpha}\rho\partial_{q_\alpha}\Psi\dot{q}_\alpha-\sum_{\alpha}\left(\rho\partial_{q_{\alpha,i}}\Psi\right)_{,i}\dot{q}_\alpha\right]\mathrm{d}\Omega$$
$$-\int_{\partial\Omega}\rho\sum_{\alpha}n_i\partial_{q_{\alpha,i}}\Psi\dot{q}_\alpha\mathrm{d}\partial\Omega\geqslant 0 \tag{18}$$

where the following compact notation for partial derivative was adopted, $\partial_x F=\frac{\partial F}{\partial x}$. On the above equation, the Divergence Theorem was applied being n_i the (outward) unit normal to $\partial\Omega$.

Then, the dissipative stress in the domain Ω and on the boundary $\partial\Omega$ are defined as Q_α and $Q^{(b)}_\alpha$, respectively

$$Q_\alpha=-\rho\partial_{q_\alpha}\Psi+(\rho\partial_{q_{\alpha,i}}\Psi)_{,i}\quad\text{in }\Omega \tag{19}$$
$$Q^{(b)}_\alpha=-\rho\partial_{q_{\alpha,i}}\Psi n_i\quad\text{on }\partial\Omega \tag{20}$$

and the Eq. (18) can be expressed as

$$\int_{\Omega}\left[\left(\sigma_{ij}-\rho\partial_{\varepsilon^e_{ij}}\Psi\right)\dot{\varepsilon}_{ij}+(p-\rho\partial_{m^e}\Psi)\dot{m}+\rho\partial_{\varepsilon^e_{ij}}\Psi\dot{\varepsilon}^p_{ij}+\rho\partial_{m^e}\Psi\dot{m}^p+\sum_{\alpha}Q_\alpha\dot{q}_\alpha\right]\mathrm{d}\Omega+\int_{\partial\Omega}\sum_{\alpha}Q^{(b)}_\alpha\dot{q}_\alpha\mathrm{d}\partial\Omega\geqslant 0 \tag{21}$$

In standard form (as for local theory), it is postulated that the last inequality must hold for any choice of domain Ω and for any independent thermodynamic process. As a result, Coleman's equation are formally obtained like for the local continuum theory

$$\sigma_{ij}=\rho\partial_{\varepsilon^e_{ij}}\Psi \tag{22}$$

$$p=\rho\partial_{m^e}\Psi \tag{23}$$

$$\mathfrak{D}=\sigma_{ij}\dot{\varepsilon}^p_{ij}+p\dot{m}^p+\sum_{\alpha}Q_\alpha\dot{q}_\alpha\geqslant 0\quad\text{in }\Omega \tag{24}$$

$$\mathfrak{D}^{(b)}=\sum_{\alpha}Q^{(b)}_\alpha\dot{q}_\alpha\geqslant 0\quad\text{on }\partial\Omega \tag{25}$$

In the particular case of non-porous material ($p = 0$) above equations takes similar form to those obtained by Svedberg and Runesson (1997) and Vrech and Etse (2009) for isothermal situations.

From the above equations, (24) and (25) it can be concluded that the difference between this non-local theory and the local one is the additional gradient term in the expression of the dissipative stresses Q_α, and the boundary dissipation term $Q^{(b)}_\alpha$. Consequently, the dissipative stress Q_α can be decomposed into the local and non-local components

$$Q_\alpha=Q^{loc}_\alpha+Q^{nloc}_\alpha \tag{26}$$

with

$$Q^{loc}_\alpha=-\rho\partial_{q_\alpha}\Psi \tag{27}$$
$$Q^{nloc}_\alpha=\left(\rho\partial_{q_{\alpha,i}}\Psi\right)_{,i} \tag{28}$$

Remark 1. While the global inequality in Eq. (21) is necessary in order to satisfy the CDI, the inequalities Eqs. (24) and (25) are only sufficient conditions.

It is interesting to compare the rate of dissipation expression in Eq. (18), when non-porous media are considered, with that corresponding to the unified treatment of thermodynamically consistent gradient plasticity by Gudmundson (2004). When applying integration by parts, followed by the Divergence Theorem, to the gradient terms of the rate of dissipation, see Eq. (7) of Gudmundson (2004), this formulation leads to dissipative non-local stresses on the boundary, similar to the present proposal when particularized to non-porous media.

However, a relevant difference between Gudmundson formulation and the present one is that the free energy density in the first one is expressed as function of the elastic strain, plastic strain, and plastic strain gradient tensors. Consequently, the dissipation includes the differences between the rate of change of the free energy with respect to the plastic strains and plastic strain gradients, on the one hand, and the internal stresses conjugated to both kinematic fields, on the other hand. These internal stresses are denoted by Gudmundson as microstresses and moment stresses, respectively. In the present formulation, and based on Svedberg and Runesson (1997) and Vrech and Etse (2009), the free energy density is expressed in terms of the elastic strains, the internal variables and their gradients (being the only ones of non-local character). So, the rate of dissipation in Eq. (18) does not include the so-called microstresses and moment stresses.

The convex set β of plastically admissible states can then be defined as

$$\beta = \{(\sigma_{ij}, p, Q_\alpha) | \Phi(\sigma_{ij}, p, Q_\alpha) \leqslant 0\} \tag{29}$$

where $\Phi(\sigma_{ij}, p, Q_\alpha)$ is the convex yield function.

4.1. Thermodynamically consistent constitutive relations

Based on previous works (Svedberg and Runesson, 1997; Vrech and Etse, 2009), the following additive expression of the free energy corresponding to non-local gradient poroplastic materials is adopted

$$\Psi\left(\varepsilon_{ij}^e, m^e, q_\alpha, q_{\alpha,i}\right) = \Psi^e(\varepsilon_{ij}^e, m^e) + \Psi^{p,loc}(q_\alpha) + \Psi^{p,nloc}(q_{\alpha,i}) \tag{30}$$

whereby Ψ^e is the elastic energy of the porous media defined by Coussy (1995) as

$$\rho\Psi^e = \sigma_{ij}^0 \varepsilon_{ij}^e + p^0 \frac{m^e}{\rho_0^{fl}} + 3\alpha_m \theta M \frac{m^e}{\rho_0^{fl}} + \frac{1}{2}\varepsilon_{ij}^e C_{ijkl}^0 \varepsilon_{kl}^e + \frac{1}{2} M \left(B_{ij}\varepsilon_{ij}^e - \frac{m^e}{\rho_0^{fl}}\right)^2 \tag{31}$$

Whereas $\Psi^{p,loc}$ and $\Psi^{p,nloc}$ are the local and non-local gradient contributions due to dissipative hardening/softening behaviors, which are expressed in terms of the internal variables q_α and their gradient $q_{\alpha,i}$, respectively.

Once the Coleman's relations are deduced from Eqs. (22) and (23), neglecting initial stress and pressures, the following expressions can be obtained

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}^e - MB_{ij}\frac{m^e}{\rho_0^{fl}} \tag{32}$$

$$p = -MB_{ij}\varepsilon_{ij}^e + M\frac{m^e}{\rho_0^{fl}} \tag{33}$$

being M the Biot's module, $B_{ij} = b\delta_{ij}$ with b the Boit coefficient, and $C_{ijkl} = C_{ijkl}^0 + MB_{ij}B_{kl}$, whereby C_{ijkl}^0 is the fourth-order elastic tensor which linearly relates stress and strain.

4.2. Non-local plastic flow rule

Rate equations for the internal variables are introduced in the same way as for local theory. Hence, for general non-associative flow and hardening rule, we introduce the dissipative potential Φ^* such that

$$\dot{\varepsilon}_{ij}^p = \dot{\lambda}\partial_{\sigma_{ij}}\Phi^*; \quad \dot{m}^p = \dot{\lambda}\partial_p\Phi^*; \quad \dot{q_\alpha} = \dot{\lambda}\partial_{Q_\alpha}\Phi^* \tag{34}$$

To complete problem formulation in Ω, the Kuhn–Tucker complementary conditions are introduced as follow

$$\dot{\lambda} \geqslant 0; \quad \Phi(\sigma_{ij}, p, Q_\alpha) \leqslant 0; \quad \dot{\lambda}\Phi(\sigma_{ij}, p, Q_\alpha) = 0 \tag{35}$$

4.3. Rate form of constitutive equations

In the undrained condition and considering the additive decomposition of the free energy potential in Eq. (30) and the flow rule of Eq. (34), the following rate expressions of the stress tensor $\dot{\sigma}_{ij}$ and pore pressure $\dot{p}$ are obtained from Eqs. (32) and (33)

$$\dot{\sigma}_{ij} = C_{ijkl}\dot{\varepsilon}_{kl} - C_{ijkl}\dot{\lambda}\partial_{\sigma_{kl}}\Phi^* - MB_{ij}\frac{\dot{m}}{\rho_0^{fl}} + MB_{ij}\dot{\lambda}\partial_p\Phi^* \tag{36}$$

$$\dot{p} = -MB_{ij}\dot{\varepsilon}_{ij} + MB_{ij}\dot{\lambda}\partial_{\sigma_{ij}}\Phi^* + M\frac{\dot{m}}{\rho_0^{fl}} - M\dot{\lambda}\partial_p\Phi^* \tag{37}$$

After multiplying Eq. (37) by B_{ij} and combining with Eq. (36), a more suitable expression of the rate of the stress tensor for drained condition is achieved

$$\dot{\sigma}_{ij} = C_{ijkl}^0\dot{\varepsilon}_{kl} - B_{ij}\dot{p} - C_{ijkl}^0\partial_{\sigma_{kl}}\Phi^*\dot{\lambda} \tag{38}$$

while the evolution law of the local and non-local dissipative stress in Eq. (26) results

$$\dot{Q}_\alpha = \dot{Q}_\alpha^{loc} + \dot{Q}_\alpha^{nloc} \tag{39}$$

with

$$\dot{Q}_\alpha^{loc} = -\dot{\lambda} H_\alpha^{loc} \partial_{Q_\alpha} \Phi^* \tag{40}$$

$$\dot{Q}_\alpha^{nloc} = l_\alpha^2 \left(H_{\alpha ij}^{nloc} \dot{\lambda}_{,j} \partial_{Q_\alpha} \Phi^* + \dot{\lambda} H_{\alpha ij}^{nloc} Q_{\alpha,j} \partial^2_{Q_\alpha Q_\alpha} \Phi^* \right)_{,i} \tag{41}$$

Thereby, local hardening/softening module H_α^{loc} have been introduced as well as the new non-local hardening/softening tensor $H_{\alpha ij}^{nloc}$ as defined in Svedberg and Runesson (1997)

$$H_\alpha^{loc} = \rho \partial^2_{q_\alpha q_\alpha} \Psi^{p,loc}, \quad H_{\alpha ij}^{nloc} = \rho \frac{1}{l_\alpha^2} \partial^2_{q_{\alpha,i} q_{\alpha,j}} \Psi^{p,nloc} \tag{42}$$

$H_{\alpha ij}^{nloc}$ is a second order positive defined tensor. For the characteristic length l_α three alternative definitions can be given, see Pamin (1994), Svedberg (1999), Vrech and Etse (2005). On the one hand, it can be defined as a convenient dimensional parameter so as H_α^{loc} and $H_{\alpha ij}^{nloc}$ will get the same dimension. On the other hand, as a physical entity that characterizes the material microstructure. Alternatively, l_α can be interpreted as an artificial numerical stabilization mechanism for the non-local theory.

4.4. Differential equation for the plastic multiplier

The complementary Kuhn–Tucker condition in Eq. (35) together with the plastic consistency condition, leads to

$$\dot{\Phi} = \partial_{\sigma_{ij}} \Phi \dot{\sigma}_{ij} + \partial_p \Phi \dot{p} + \partial_{Q_\alpha} \Phi \dot{Q}_\alpha = 0 \tag{43}$$

From Eqs. (36), (37) and (39), the following differential equation for undrained condition can be obtained

$$\begin{aligned} \dot{\Phi} = \dot{\lambda} \Big[& -\partial_{\sigma_{ij}} \Phi C_{ijkl} \partial_{\sigma_{kl}} \Phi^* + M \partial_{\sigma_{ij}} \Phi B_{ij} \partial_p \Phi^* + M \partial_p \Phi B_{ij} \partial_{\sigma_{ij}} \Phi^* - M \partial_p \Phi \partial_p \Phi^* - H_\alpha^{loc} \partial_{Q_\alpha} \Phi \partial_{Q_\alpha} \Phi^* \Big] \\ & + \left(\partial_{\sigma_{ij}} \Phi C_{ijkl} - M \partial_p \Phi B_{kl} \right) \dot{\varepsilon}_{kl} + \left(M \partial_p \Phi - \partial_{\sigma_{ij}} \Phi B_{ij} \right) \dot{m} / \rho_0^{fl} + \partial_{Q_\alpha} \Phi \left[l_\alpha^2 \left(H_{\alpha ij}^{nloc} \dot{\lambda}_{,j} \partial_{Q_\alpha} \Phi^* + \dot{\lambda} H_{\alpha ij}^{nloc} Q_{\alpha,j} \partial^2_{Q_\alpha Q_\alpha} \Phi^* \right)_{,i} \right] = 0 \end{aligned} \tag{44}$$

And also, a more suitable differential equation for the drained condition can be obtained when Eq. (38) instead of Eq. (36) is combined with Eqs. (37) and (39).

$$\begin{aligned} \dot{\Phi} = \dot{\lambda} \Big[& -\partial_{\sigma_{ij}} \Phi C^0_{ijkl} \partial_{\sigma_{kl}} \Phi^* - H_\alpha^{loc} \partial_{Q_\alpha} \Phi \partial_{Q_\alpha} \Phi^* \Big] + \partial_{\sigma_{ij}} \Phi C^0_{ijkl} \dot{\varepsilon}_{kl} + \left(\partial_p \Phi - \partial_{\sigma_{ij}} \Phi B_{ij} \right) \dot{p} \\ & + \partial_{Q_\alpha} \Phi \left[l_\alpha^2 \left(H_{\alpha ij}^{nloc} \dot{\lambda}_{,j} \partial_{Q_\alpha} \Phi^* + \dot{\lambda} H_{\alpha ij}^{nloc} Q_{\alpha,j} \partial^2_{Q_\alpha Q_\alpha} \Phi^* \right)_{,i} \right] = 0 \end{aligned} \tag{45}$$

For the sake of clarity last equation is rewritten in compacted form

$$-\dot{\Phi}^{nloc} + (h + h^{nloc}) \dot{\lambda} = \dot{\Phi}^e - \dot{\Phi} \tag{46}$$

where $\dot{\Phi}^e$ is the local loading function, h the generalized plastic modulus, h^{nloc} the gradient plastic modulus, and $\dot{\Phi}^{nloc}$ the gradient loading function defined as

$$\dot{\Phi}^{nloc} = l_\alpha^2 \partial_{Q_\alpha} \Phi \left\{ \partial_{Q_\alpha} \Phi^* \left[H_{\alpha ij}^{nloc} \dot{\lambda}_{,ij} + H_{\alpha ij,j}^{nloc} \dot{\lambda}_{,i} \right] + 2 \partial^2_{Q_\alpha Q_\alpha} \Phi^* Q_{\alpha,i} H_{\alpha ij}^{nloc} \dot{\lambda}_{,j} \right\} \tag{47}$$

$$h^{nloc} = -l_\alpha^2 \partial_{Q_\alpha} \Phi \left\{ \partial^2_{Q_\alpha Q_\alpha} \Phi^* \left[H_{\alpha ij}^{nloc} Q_{\alpha,ij} + H_{\alpha ij,j}^{nloc} Q_{\alpha,i} \right] + \partial^3_{Q_\alpha Q_\alpha Q_\alpha} \Phi^* Q_{\alpha,i} H_{\alpha ij}^{nloc} Q_{\alpha,j} \right\} \tag{48}$$

Both, the local yield function and the generalized plastic modulus can be decomposed into the components ($\dot{\Phi}_s^e, h_s$) and ($\dot{\Phi}_p^e$ and h_p) related to the soil skeleton and to the porous, respectively. This decomposition is valid for undrained and drained conditions.

$$\dot{\Phi}^e = \dot{\Phi}_s^e + \dot{\Phi}_p^e \tag{49}$$

$$h = h_s + h_p + \overline{H} \tag{50}$$

with

$$\overline{H} = H_\alpha^{loc} \partial_{Q_\alpha} \Phi \partial_{Q_\alpha} \Phi^* \tag{51}$$

where, for drained condition, it can be obtained

$$\dot{\Phi}_s^{e,d} = \partial_{\sigma_{ij}} \Phi C_{ijkl}^0 \dot{\varepsilon}_{kl} \tag{52}$$

$$\dot{\Phi}_p^{e,d} = \left(\partial_p \Phi - \partial_{\sigma_{ij}} \Phi B_{ij}\right) \dot{p} \tag{53}$$

$$h_s^d = \partial_{\sigma_{ij}} \Phi C_{ijkl}^0 \partial_{\sigma_{kl}} \Phi^* \tag{54}$$

$$h_p^d = 0 \tag{55}$$

while for undrained condition

$$\dot{\Phi}_s^{e,u} = \partial_{\sigma_{ij}} \Phi C_{ijkl} \dot{\varepsilon}_{kl} - M \partial_p \Phi B_{ij} \dot{\varepsilon}_{ij} \tag{56}$$

$$\dot{\Phi}_p^{e,u} = \frac{\dot{m}}{\rho_0^{fl}} \left(M \partial_p \Phi - \partial_{\sigma_{ij}} \Phi B_{ij}\right) \tag{57}$$

$$h_s^u = \partial_{\sigma_{ij}} \Phi C_{ijkl} \partial_{\sigma_{kl}} \Phi^* \tag{58}$$

$$h_p^u = -M\left(\partial_{\sigma_{ij}} \Phi B_{ij} \partial_p \Phi^* + \partial_p \Phi B_{ij} \partial_{\sigma_{ij}} \Phi^* - \partial_p \Phi \partial_p \Phi^*\right) \tag{59}$$

When all state variables are spatially homogeneous, it can be assumed that $\partial^2_{Q_\alpha Q_\alpha} \Phi^* = 0$ (Svedberg and Runesson, 1997). Thereby

$$h^{nloc} = 0 \quad \text{and } \dot{\Phi}^{nloc} = l_\alpha^2 \partial_{Q_\alpha} \Phi \partial_{Q_\alpha} \Phi^* H_{\alpha ij}^{nloc} \dot{\lambda}_{,ij} \tag{60}$$

and the pertinent differential equation to evaluate $\dot{\lambda}$ in this particular case results

$$-\dot{\Phi}^{nloc} + h\dot{\lambda} = \dot{\Phi}^e - \dot{\Phi} \tag{61}$$

4.5. Gradient form of elastoplastic constitutive equations

Under consideration of plastic loading, the plastic multiplier can be easily determined from Eq. (61). Replacing it in the constitutive equations Eqs. (36) and (38) for drained and undrained porous media, respectively, leads to

$$\dot{\sigma}_{ij} = C_{ijkl}^0 \dot{\varepsilon}_{kl} - B_{ij} \dot{p} - C_{ijkl}^0 \partial_{\sigma_{kl}} \Phi^* (\dot{\Phi}^e + \dot{\Phi}^{nloc})/h \tag{62}$$

$$\dot{\sigma}_{ij} = C_{ijkl} \dot{\varepsilon}_{kl} - M B_{ij} \dot{m}/\rho_0^{fl} + \left(M B_{ij} \partial_p \Phi^* - C_{ijkl} \partial_{\sigma_{kl}} \Phi^*\right) \left(\dot{\Phi}^e + \dot{\Phi}^{nloc}\right)/h \tag{63}$$

Taking into account the definitions of $\dot{\Phi}^e$ and $\dot{\Phi}^{nloc}$ in the above equations results

$$\dot{\sigma}_{ij} = E_{ijkl}^{ep,sd} \dot{\varepsilon}_{kl} + E_{ij}^{ep,pd} \dot{p} - E_{ij}^{g,spd} \dot{f}^g \tag{64}$$

$$\dot{\sigma}_{ij} = E_{ijkl}^{ep,su} \dot{\varepsilon}_{kl} + E_{ij}^{ep,pu} \dot{m}/\rho_0^{fl} - E_{ij}^{g,spu} \dot{f}^g \tag{65}$$

being $\boldsymbol{E}^{ep,s}$ and $\boldsymbol{E}^{ep,p}$ the elastoplastic operators of the solid skeleton and porous phase, respectively, and $\boldsymbol{E}^{g,sp}$ the continuum gradient-elastoplastic tensor of both constituents. The superscript ^{d} or ^{u} indicates the considered hydraulic condition, drained or undrained, respectively. For more details see Appendix B.

5. Instability analysis in the form of discontinuous bifurcation

In this section the discontinuous bifurcation analysis for local and non-local porous media will be treated.

5.1. Discontinuous bifurcation analysis in local porous media

It has been widely accepted that when dissipative constitutive models of quasi-brittle and ductile materials are subjected to monotonic loading in the inelastic regime, they may exhibit spatial discontinuities of the kinematic fields (Hill, 1962; Rudnicki and Rice, 1975) depending on the particular boundary condition but also on the degree of non-associative, water content, inhomogeneities, etc. The occurence of these discontinuities is related to the so-called localized failure mode.

In case of non-porous constitutive theories, different authors performed numerical and theoretical analyses to obtain model predictions of localized failure modes in the form of discontinuous bifurcation (see a.o. Etse and Willam, 1994b; Pamin, 1994; Carosio et al., 2000; Fleck and Hutchinson, 2001; Voyiadjis et al., 2005; Jirásek and Rolshoven, 2009).

In case of porous media, localization analysis should not be restricted to the consideration of discontinuities taking place only in the solid phase (see Benallal and Comi, 2002; Borja, 2004; Schiava and Etse, 2006). Contrarily, discontinuities may develop in all different phases during monotonic loading and/or changes in the humidity conditions of porous media. From the mathematic stand point this assumption means that both the field of velocity gradients and the rate of fluid mass content are discontinuous and their jumps are defined as

$$[[\dot{\varepsilon}_{ij}]] = 1/2(g_i n_j + n_i g_j) \tag{66}$$

$$[[\dot{m}]] = -[[M_{i,i}]] = -n_i g_i^M \tag{67}$$

Applying Hadamard relation (Hadamard, 1903; Coussy, 1995) to the tensors of zero and first order, p and σ_{ij}, respectively, the following balance equations are obtained

$$c[[p_{,i}]] + [[\dot{p}]]n_i = 0 \tag{68}$$

$$c[[\sigma_{ij,j}]] + [[\dot{\sigma}_{ij}]]n_j = 0 \tag{69}$$

5.1.1. Drained state

In drained state instability analysis is restricted to the solid skeleton. The fluid flow in deformable porous media is governed by the Darcy's law. Thus, neglecting inertial forces, the relative flow vector of fluid mass M_i is then expressed as

$$M_i = -\rho^{fl} k_{ij} p_{,j} \tag{70}$$

where k_{ij} is the permeability tensor of porous media. During quasi-static loading the fluid subjected to strong pressure gradients may exhibit a spontaneous diffusion process, with very fast pressure degradation. Thereby, the relative flow vector of fluid mass should remain continuous. Then, from Eq. (70) follows

$$[[M_i]] = -\rho^{fl} k_{ij}[[p_{,j}]] = 0 \tag{71}$$

The last expression leads to the conclusion that pore pressure gradient does not present discontinuities $[[p_{,i}]] = 0$. Thus, Eq. (68) can only be fulfilled if the rate of pore pressure remains continuous, i.e. $[[\dot{p}]] = 0$.

Considering the momentum balance equation for quasi-static problems, applying the jump operator to the incremental constitutive equation, Eq. (64), and substituting the resulting expression into Eq. (69), we obtain

$$[[\dot{\sigma}_{ij}]]n_j = E_{ijkl}^{ep,sd}[[\dot{\varepsilon}_{kl}]]n_j = 0 \tag{72}$$

being $E_{ijkl}^{ep,sd}$ the solid skeleton elastoplastic tensor, as described in Section 4.5. Introducing Eq. (66) in Eq. (72) results

$$[[\dot{\sigma}_{ij}]]n_j = A_{ij}^{d,loc} g_j = 0 \tag{73}$$

where the elastoplastic acoustic tensor for local plasticity in porous media under drained condition is decomposed in its elastic, $A_{ij}^{e,s}$, and elastoplastic parts, $A_{ij}^{ep,s}$, as

$$A_{ij}^{d,loc} = E_{ijkl}^{ep,sd} n_l n_k = A_{ij}^{e,s} - A_{ij}^{ep,s} \tag{74}$$

being

$$\begin{aligned} A_{ij}^{e,s} &= C_{ijkl}^{0} n_l n_k \\ A_{ij}^{ep,s} &= \frac{C_{ijmn}^{0}\partial_{\sigma_{mn}}\Phi^{*}\partial_{\sigma_{pq}}\Phi C_{pqkl}^{0}}{h} n_l n_k \end{aligned} \tag{75}$$

Since trivial solutions are out of interest, the spectral properties of the local acoustic tensor $A_{ij}^{d,loc}$, must be analyzed. Then, the localization condition of drained porous media is achieved as

$$\det(A_{ij}^{d,loc}) = 0 \tag{76}$$

As porous effects has been neglected the discontinuity related to the above bifurcation condition affects only the strain velocity field. Consequently, the localization tensor in fully drained condition takes the same form as in classical elastoplastic continua.

It can be then concluded that the localization condition in Eq. (76) involves only the drained poroelastic properties. The fluid pressure is only concerned in the localization phenomenon through its potential influence on the current values of both the loading function Φ and the generalized plastic modulus h.

5.1.2. Undrained state

In undrained state the variation of fluid mass content in the solid skeleton vanishes, $\dot{m} = 0$. The pore pressure can be obtained from the solid phase kinematics, $g_i \equiv g_i^{\mathrm{M}}$.

Applying the jump operator to Eqs. (65) and (66),

$$[[\dot{\sigma}_{ij}]]n_j = A_{ij}^{u,loc} g_j = 0 \tag{77}$$

where, as well as the section before, the elastoplastic acoustic tensor for local plasticity under undrained condition is decomposed in its elastic part referred to the solid phase, $A_{ij}^{e,s}$, the elastic part referred to the porous phase, $A_{ij}^{e,p}$, and elastoplastic parts referred to both phase, $A_{ij}^{ep,sp}$, as

$$A_{ij}^{u,loc} = E_{ijkl}^{ep,su} n_l n_k = A_{ij}^{e,s} + A_{ij}^{e,p} - A_{ij}^{ep,sp} \tag{78}$$

being

$$A_{ij}^{e,p} = M B_{ij} B_{kl} n_l n_k$$

$$A_{ij}^{ep,sp} = \left[\frac{C_{ijmn} \partial_{\sigma_{mn}} \Phi^* \partial_{\sigma_{pq}} \Phi C_{pqkl}}{h} + \frac{M}{h} \left(M \partial_p \Phi^* B_{ij} B_{kl} \partial_p \Phi - C_{ijmn} \partial_{\sigma_{mn}} \Phi^* B_{kl} \partial_p \Phi - \partial_p \Phi^* B_{ij} C_{klmn} \partial_{\sigma_{mn}} \Phi \right) \right] n_l n_k \tag{79}$$

The localization condition follows from the spectral properties analysis of the acoustic tensor

$$\det(A_{ij}^{u,loc}) = 0 \tag{80}$$

From the comparison between Eqs. (76) and (80) it can be concluded that the hydraulic border conditions affect the solid–fluid coupled matrix.

5.2. Bifurcation analysis in non-local gradient-based porous media

In the previous section the discontinuous bifurcation problem of local porous medium has been studied. The aforementioned analysis is hold in case of brittle failure modes. This is the case of some cementitious sandy soils cemented with iron oxide (so-called Sandstone) as well as concrete in tensile or uniaxial compression states. In these situations strain localization is generated in a region of null thickness $l_\alpha = 0$.

In case of quasi-brittle and, moreover, ductile failure modes, shear bands or microcracking zones of non-zero thicknesses develop during failure processes. This is typically the case of cementitious and granular materials under triaxial compression with medium or high confinements, and of metals. The size of finite localization zones that develop during failure processes of quasi-brittle and ductile materials are defined by the so-called characteristic length $l_\alpha \neq 0$ (Pamin, 1994; Voyiadjis et al., 2005; Vrech and Etse, 2005).

In the following the conditions for the occurrence of localized failure modes in the form of discontinuous bifurcation in non-local gradient elastoplastic porous media are analyzed. It is assumed homogeneous fields of stress and strain rates just before the onset of localization. Contrarily to the case of local poroplastic media discussed in the previous section, the plastic consistency, see Eq. (61), is now a function of both the plastic multiplier $\dot{\lambda}$, and its second gradient $\dot{\lambda}_{,ij}$.

The jump operator of the current stress on the discontinuity surface should satisfy the equilibrium equation

$$\dot{\sigma}_{ij,j} = 0 \tag{81}$$

where the incremental stress tensor is defined by Eq. (36) or Eq. (38), depending on the assumed hydraulic border conditions.

To investigate the stability of an equilibrium state the loss of ellipticity is commonly investigated by a wave propagation analysis (Abellan and de Borst, 2006; Tsagrakis et al., 2003; Benallal and Comi, 2002; Liebe et al., 2001; Svedberg and Runesson, 1997). Thus, considering a homogeneous state before the onset of localization the following harmonic perturbation with respect to the incremental field variables, i.e. displacements, mass content and plastic multiplier, for an infinite porous medium is assumed, which corresponds to the assumption of stationary planar waves

$$\begin{bmatrix} \dot{u}(\boldsymbol{x},t) \\ \dot{\gamma}(\boldsymbol{x},t) \\ \dot{\lambda}(\boldsymbol{x},t) \end{bmatrix} = \begin{bmatrix} \dot{\mathcal{U}}(t) \\ \dot{\mathcal{M}}(t) \\ \dot{\mathcal{L}}(t) \end{bmatrix} \exp\left(\frac{i2\pi}{\delta} \boldsymbol{n} \cdot \boldsymbol{x} \right) \tag{82}$$

being $\dot{\gamma}$ the mass content, $\boldsymbol{x}$ the position vector (in Cartesian coordinates), $\boldsymbol{n}$ the wave normal direction and δ the wave length. Moreover $\dot{\mathcal{U}}, \dot{\mathcal{M}}$ and $\dot{\mathcal{L}}$ are spatially homogeneous amplitude of the wave solutions.

Replacing Eq. (82) in Eqs. (61), (81) and, (36) or Eq. (38) (depending on the assumed hydraulic conditions), that represent the differential expression of plastic consistency, the equilibrium condition, and the incremental constitutive relations, respectively, follows that the equilibrium condition on the discontinuity surface is fulfill if

$$\left(\frac{2\pi}{\delta} \right)^2 \left\{ C_{ijkl}^0 - \frac{C_{ijmn}^0 \partial_{\sigma_{mn}} \Phi^* \partial_{\sigma_{pq}} \Phi C_{pqkl}^0}{h + \bar{h}^{nloc}} \right\} n_l n_k \dot{\mathcal{U}} = 0 \tag{83}$$

in case of drained conditions, and

$$\left(\frac{2\pi}{\delta} \right)^2 \left\{ C_{ijkl} - \frac{C_{ijmn} \partial_{\sigma_{mn}} \Phi^* \partial_{\sigma_{pq}} \Phi C_{pqkl}}{h + \bar{h}^{nloc}} - M^2 \frac{\partial_p \Phi^* B_{ij} B_{kl} \partial_p \Phi}{h + \bar{h}^{nloc}} + M \left(\frac{C_{ijmn} \partial_{\sigma_{mn}} \Phi^* B_{kl} \partial_p \Phi}{h + \bar{h}^{nloc}} + \frac{\partial_p \Phi^* B_{ij} C_{mnkl} \partial_{\sigma_{mn}} \Phi}{h + \bar{h}^{nloc}} \right) \right\} n_l n_k \dot{\mathcal{U}} = 0 \tag{84}$$

in the undrained case, being $\bar{h}^{nloc}$ the generalized gradient modulus as

$$\bar{h}^{nloc} = l_\alpha^2 \left(\partial_{Q_\alpha} \Phi \partial_{Q_\alpha} \Phi^* H_{\alpha\ ij}^{nloc} \right) n_j n_i \left(\frac{2\pi}{\delta} \right)^2 \tag{85}$$

The expressions between brackets on Eqs. (83) and (84) correspond to the localization acoustic tensor for porous media under drained and undrained conditions $A_{ij}^{d,nloc}$ y $A_{ij}^{u,nloc}$, respectively. From the comparison between the bifurcation analysis carried out for local, and non-local porous media, Eqs. (76), (80) and Eqs. (83), (84), respectively, follows that the difference between both lies only in the generalized (non-local) gradient modulus $\bar{h}^{nloc}$. Precisely the effect of $\bar{h}^{nloc}$ at the finite-element level is the regularization of the post peak regime.

6. Numerical analysis

In this section some numerical results are presented to illustrate the regularization capabilities of the non-local gradient poroplastic formulation in this paper.

Plane strain condition is assumed while the modified Cam Clay criterion for saturated porous media (Borja, 2004) based on non-associated flow rule is considered in the analysis. The resulting yield condition is defined as

$$\Phi(\sigma, \tau, p, Q_\alpha) = \left(\sigma + \beta p + \frac{1}{2} Q_\alpha \right)^2 + \left(\frac{\tau}{M} \right)^2 - \frac{1}{4} Q_\alpha^2 \tag{86}$$

where $\sigma = I_1/3$ is the Cauchy hidrostatic stress, $\tau = \sqrt{3J_2}$ the shear stress, M the slope of the Critical State Line and Q_α the dissipative stress defined in Eq. (26).

The thermodynamic consistency enforces the definition of the local and non-local plastic portions of the free energy Eq. (30),

$$\Psi^p(\kappa, \kappa_{,i}) = \Psi^{p,loc}(\kappa) + \Psi^{p,nloc}(\kappa_{,i}) = -\frac{1}{\chi} p_{co}^0 \exp(\chi\kappa) - \left(\frac{1}{2} l_\alpha^2 H_\alpha^{nloc} \kappa_{,i} \right)_{,i} \tag{87}$$

whereby the corresponding internal variable κ of the Cam Clay model is the total plastic volumetric deformation ε^p, which is a function of the plastic porosity ϕ^p and the plastic volumetric deformation of the skeleton, ε_s^p (Coussy, 1995)

$$\varepsilon^p = \phi^p + (1 - \phi_0)\varepsilon_s^p \tag{88}$$

Then, the dissipative stress can be obtained by Eq. (26)

$$Q^{loc}(\kappa) = (2 - \phi_0) p_{co}^0 \exp(\chi(\phi^p + (1 - \phi_0)\varepsilon_s^p)) \tag{89}$$

$$Q^{nloc}(\kappa_{,i}) = l_s^2 H_s^{nloc} \nabla^2 \varepsilon_s^p + l_p^2 H_p^{nloc} \nabla^2 \phi^p \tag{90}$$

where l_s and l_p are the internal characteristic lengths of the solid and porous phases, respectively.

In Fig. 2 the localization condition corresponding to both classical and gradient poroplastic formulations at maximum strength capacity of the modified Cam Clay poroplastic material under different confinement levels is depicted for all possible loading directions. The results of the localization analysis presented in Fig. 2 indicate that the gradient dependent Cam Clay elastoplastic formulation that results from the proposed theory in this paper is able to suppress the potential discontinuous bifurcations of the local poroplastic material and, consequently, to regularize the post peak behavior.

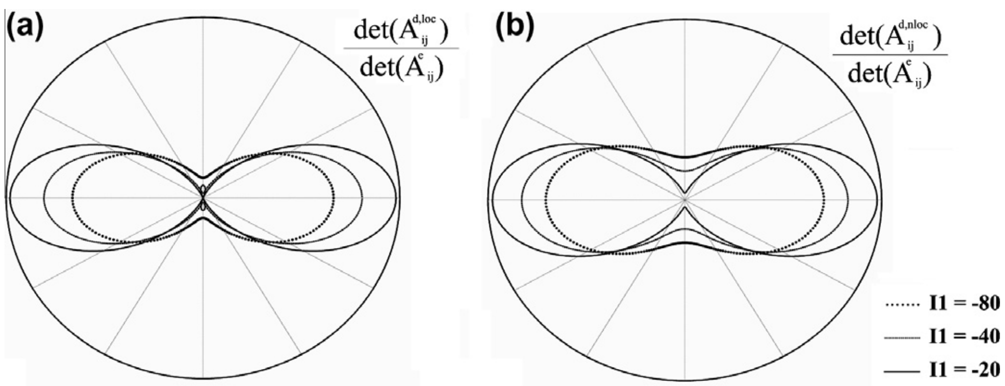

Fig. 2. Localization condition plots in polar coordinated system. (a) Classical-plasticity model; (b) Gradient-plasticity model.

7. Conclusions

In this work a general thermodynamically consistent gradient constitutive formulation to describe non-local behavior of porous media is proposed. The proposal is an extension of the gradient-based thermodynamically consistent theories by Svedberg and Runesson (1997) and Vrech and Etse (2009) for non-porous continua. Porous materials in this work are modelled from the macroscopic level of observation. They are considered to defined open thermodynamic systems characterized by the presence of occluded sub regions.

Discontinuous bifurcation theory to predict localized failure modes is consistently extended to porous media. As a result, the analytical expression of the localization tensor for gradient regularized plasticity in porous media is obtained. This failure indicator is particularized for both drained and undrained hydraulic conditions.

The thermodynamically constitutive theory and related localization indicator in this work can be applied to the analysis of failure behavior of different types of porous materials like soils, bones and concrete.

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Appendix A. Finite deformation

The present theory was developed under small deformations hypothesis nevertheless it can be easily adapted to finite deformations problems using the concept of corrotational magnitudes (Belytschko et al., 2000; Polizzotto, 2009b). The gradient-plasticity treatment in terms of corrotational magnitudes does not present much complexity since all state variables are objectively described in the corrotational configuration. In contrast, this kind of corrotational-based constitutive models leads to a somewhat awkward in numerical implementation due to the appearance of non-symmetric matrices and deformation-dependent constitutive tensors. However the non-symmetrical part of those constitutive models mainly depends on tangential stress and could be disregarded, in some cases (Di Rado et al., 2008).

As we said before, finite deformation problems require an objectivity description of the state variables. Namely, the strain tensor employed should be indifferent under rigid body motions in order to avoid the appearance of unreal stress. The infinitesimal strain tensor $\boldsymbol{\varepsilon}$ does not fulfill this postulate and is it mandatory to be replaced by another strain tensor. An appropriated strain measure is the spatial rate of deformation tensor $\mathbf{D}$ which is the symmetric part of the additively decomposition of the spatial velocity gradient $\mathbf{L}$, being its $\mathbf{W}$ the antisymmetric part,

$$\mathbf{L} = \mathbf{D} + \mathbf{W} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} = \mathbf{L}^e + \mathbf{F}^e \cdot \mathbf{L}^p \cdot \mathbf{F}^{e^{-1}} \tag{A.1}$$

with $\mathbf{L}^e = \dot{\mathbf{F}}^e \cdot \mathbf{F}^{e^{-1}}$ and $\mathbf{L}^p = \dot{\mathbf{F}}^p \cdot \mathbf{F}^{p^{-1}}$

The basic hypothesis in physical and geometrical non-lineal problem analysis is the multiplicative decomposition of the deformation gradient tensor in its elastic and plastic parts, $\mathbf{F}^e$, $\mathbf{F}^p$, respectively.

$$\mathbf{F} = \mathbf{F}^e \cdot \mathbf{F}^p \tag{A.2}$$

On the other hand, the spatial rate of deformation tensor can be expressed in a convenient form considering the relation between the deformation gradient tensor and the rate of the Green–Lagrange strain tensor, $\dot{\mathbf{E}}$, in the following way

$$\mathbf{D} = \mathbf{F}^{-T} \cdot \dot{\mathbf{E}} \cdot \mathbf{F}^{-1} \tag{A.3}$$

with

$$\dot{\mathbf{E}} = \mathbf{F}^{p^T} \left[\overset{\circ}{\mathbf{E}}{}^e + \frac{1}{2} \left(\mathbf{L}^{pr^T} \cdot \mathbf{C}^e + \mathbf{C}^e \cdot \mathbf{L}^{pr} \right) \right] \mathbf{F}^p \tag{A.4}$$

being $\overset{\circ}{\mathbf{E}}{}^e = \dot{\mathbf{E}}^e - \omega \mathbf{E}^e + \mathbf{E}^e \omega,\ \mathbf{L}^{pr} = \mathbf{L}^p - \omega,\ \mathbf{C}^e = \mathbf{F}^{e^T} \cdot \mathbf{F}^e = \mathbf{I} + 2\mathbf{E}^e$ and $\omega = \dot{\mathbf{R}}\mathbf{R}^T$

Likewise, considering the relationship between the second Piola–Kirchhoff stress tensor $\mathbf{S}$, and the Cauchy stress tensor $\boldsymbol{\sigma}$,

$$\mathbf{S} = \frac{\bar{\rho}}{\rho} \mathbf{F}^{e^{-1}} \cdot \boldsymbol{\sigma} \cdot \mathbf{F}^{e^{-T}} \tag{A.5}$$

it is possible rewrite the second principle of the thermodynamic Eq. (15) for isothermal condition

$$\int_\Omega \left[\frac{\rho}{\bar{\rho}} \mathbf{F}^e \cdot \mathbf{S} \cdot \mathbf{F}^{e^T} : \mathbf{F}^{-T} \cdot \dot{\mathbf{E}} \cdot \mathbf{F}^{-1} + p\dot{m} - \rho \dot{\Psi} \right] \mathrm{d}\Omega \geqslant 0 \tag{A.6}$$

In this gradient plasticity framework, considering corrotational magnitudes, the free energy of Helmholtz can be decomposed in the following way

$$\Psi = \Psi^e(\widehat{\mathbf{E}}^e, m^e) + \Psi^{p,loc}(q_\alpha) + \Psi^{p,nloc}(\hat{\nabla} q_\alpha) \tag{A.7}$$

where the operator $\hat{\bullet} = \mathbf{R}^T \cdot \bullet \cdot \mathbf{R}$ implies a configuration change for the magnitude $\bullet$passing from the reference, $\mathcal{C}$, to the corrotated configuration,$\widehat{\mathcal{C}}$. Note that m^e and q_α are considered here as scalar quantities, thus they are indifferent under rigid body motions.

Replacing the rate of the free energy in Eq. (A.6) and considering Eq. (A.4) it is possible to proceed in a similar way to Eq. (17), then

$$\int_\Omega \left[\left(\frac{\rho}{\bar{\rho}} \mathbf{S} - \rho \mathbf{R}^T \partial_{\widehat{\mathbf{E}^e}} \Psi \mathbf{R} \right) \overset{\circ}{\mathbf{E}}{}^e + (p - \rho \partial_{m^e} \Psi) \dot{m} + \frac{\rho}{2\bar{\rho}} \mathbf{S} \cdot \left(\mathbf{L}^{pr^T} \cdot \mathbf{C}^e + \mathbf{C}^e \cdot \mathbf{L}^{pr} \right) + \rho \partial_{m^e} \Psi \dot{m}^p + \sum_\alpha \mathbf{Q}_\alpha \dot{q}_\alpha \right] \mathrm{d}\Omega + \int_{\partial\Omega} \sum_\alpha \mathbf{Q}_\alpha^{(b)} \dot{q}_\alpha \mathrm{d}\partial\Omega \geqslant 0 \tag{A.8}$$

with

$$\mathbf{Q}_\alpha = -\rho \partial_{q_\alpha} \Psi - \nabla \cdot \left(\rho \mathbf{R}^T \partial_{\hat{\nabla} q_\alpha} \Psi \right) \quad \text{in } \Omega \tag{A.9}$$

$$\mathbf{Q}_\alpha^{(b)} = -\rho \mathbf{n} \partial_{\hat{\nabla} q_\alpha} \Psi \quad \text{in } \partial\Omega \tag{A.10}$$

Since the inequality (A.8) holds for arbitrary elastic–plastic deformation mechanisms, even when these are purely elastic, all plastic strain variable disappears (Polizzotto, 2009b), and inequality(A.8) implies

$$\mathbf{S} = \bar{\rho} \mathbf{R}^T \partial_{\widehat{\mathbf{E}^e}} \Psi \mathbf{R} = \bar{\rho} \partial_{\mathbf{E}^e} \Psi \quad p = \rho \partial_{m^e} \Psi \tag{A.11}$$

which are the pertinent elasticity laws and the dissipation expression in the domain Ω and on the boundary $\partial\Omega$ are formally obtained as

$$\mathfrak{D} = \frac{1}{2\bar{\rho}} \mathbf{S} \cdot \left(\mathbf{L}^{pr^T} \cdot \mathbf{C}^e + \mathbf{C}^e \cdot \mathbf{L}^{pr} \right) + p\dot{m}^p + \sum_\alpha \mathbf{Q}_\alpha \dot{q}_\alpha \geqslant 0 \quad \text{in } \Omega \tag{A.12}$$

$$\mathfrak{D}^{(b)} = \sum_\alpha \mathbf{Q}_\alpha^{(b)} \dot{q}_\alpha \geqslant 0 \quad \text{on } \partial\Omega \tag{A.13}$$

Appendix B. Matrix expressions of Eq. (64)

The matrix expressions of the gradient-plasticity constitutive relationship of Eq. (64) with drained conditions are

$$E_{ijkl}^{ep,sd} = C_{ijkl}^0 - \frac{C_{ijmn}^0 \partial_{\sigma_{mn}} \Phi^* \partial_{\sigma_{pq}} \Phi C_{pqkl}^0}{h} \tag{B.1}$$

$$E_{ij}^{ep,pd} = -B_{ij} - \frac{C_{ijkl}^0 \partial_{\sigma_{kl}} \Phi^* \left(\partial_p \Phi - \partial_{\sigma_{mn}} \Phi B_{mn} \right)}{h} \dot{p} \tag{B.2}$$

$$E_{ij}^{g,spd} = \frac{C_{ijkl}^0 \partial_{\sigma_{kl}} \Phi^*}{h} \tag{B.3}$$

$$\dot{f}^g = l_\alpha^2 \partial_{Q_\alpha} \Phi \partial_{Q_\alpha} \Phi^* H_{\alpha ij}^{nloc} \dot{\lambda}_{,ij} \tag{B.4}$$

In the same way, the matrix expressions of the gradient-plasticity constitutive relationship of Eq. (65) with undrained conditions are presented here as

$$E_{ijkl}^{ep,su} = C_{ijkl} - \frac{C_{ijmn} \partial_{\sigma_{mn}} \Phi^* \partial_{\sigma_{pq}} \Phi C_{pqkl}}{h} - M^2 \frac{\partial_p \Phi^* B_{ij} B_{kl} \partial_p \Phi}{h} + M \left(\frac{C_{ijmn} \partial_{\sigma_{mn}} \Phi^* B_{kl} \partial_p \Phi}{h} + \frac{\partial_p \Phi^* B_{ij} C_{mnkl} \partial_{\sigma_{mn}} \Phi}{h} \right) \tag{B.5}$$

$$E_{ij}^{ep,pu} = -M \left(B_{ij} - \frac{\partial_p \Phi^* B_{ij} \left(M \partial_p \Phi - \partial_{\sigma_{mn}} \Phi B_{mn} \right)}{h} \right) - \frac{C_{ijkl} \partial_{\sigma_{kl}} \Phi^* \left(M \partial_p \Phi - \partial_{\sigma_{mn}} \Phi B_{mn} \right)}{h} \tag{B.6}$$

$$E_{ij}^{g,spu} = \frac{C_{ijkl} \partial_{\sigma_{kl}} \Phi^* - M B_{ij} \partial_p \Phi^*}{h} \tag{B.7}$$

$$\dot{f}^g = l_\alpha^2 \partial_{Q_\alpha} \Phi \partial_{Q_\alpha} \Phi^* H_{\alpha ij}^{nloc} \dot{\lambda}_{,ij} \tag{B.8}$$

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