On the Existence of Critical Clique-Helly Graphs

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Abstract

A graph is *clique-Helly* if any family of mutually intersecting cliques has non-empty intersection. Dourado, Protti and Szwarcfiter conjectured that every clique-Helly graph contains a vertex whose removal maintains it as a clique-Helly graph. We will present a counterexample to this conjecture.

Keywords: Helly property, Clique-Helly graphs, clique graphs.

1 Introduction

A set family $\mathcal{F}$ satisfies the **Helly property** if the intersection of all the members of any pairwise intersecting subfamily of $\mathcal{F}$ is non-empty. This property, originated in the famous work of Eduard Helly on convex sets in the Euclidean space, has been widely study in diverse areas of theoretical and applied mathematics such as extremal hypergraph theory, logic, optimization, theoretical computer science, computational biology, data bases, image processing and,

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clearly, graphs theory. A few surveys have been written on the Helly property, see for instance [1,2,4,5].

From the computational and algorithm point of view, the relevance of the Helly property has been highlighted in the survey [3]. In the section *Proposed Problems* of that work, the authors posed the following open question:

Conjecture 1.1 (Dourado, Protti and Szwarcfiter) *Every clique-Helly graph (the family of maximal cliques of the graph satisfies the Helly property) contains a vertex whose removal maintains it as a clique-Helly graph.*

In this work, we prove the conjecture is false: in Section 3 we will exhibit a clique-Helly graph G such that $G-v$ (the graph obtained from G by removing vertex v) is not clique-Helly for every vertex v of G.

2 Definitions and preliminary results

Given a finite and simple graph G, we let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively.

The **open** and the **closed neighborhood** of a vertex $v \in V(G)$ are denoted by $N_G(v)$ and $N_G[v]$ respectively. The **degree** of v is the cardinality of $N_G(v)$.

If $S \subseteq V(G)$ then the subgraphs of G induced by S and by $V(G) \setminus S$ are denoted by $G[S]$ and $G - S$, respectively. When S contains a unique vertex v, we write $G - v$ for $G - \{v\}$.

The **complete graph** on n vertices is denoted by K_n. A **complete set** of G is a subset of $V(G)$ inducing a complete subgraph. A **clique** is a maximal (with respect to the inclusion relation) complete set. We let $\mathcal{C}(G)$ be the family of cliques of G. When $\mathcal{C}(G)$ satisfies the Helly property, we say that G is a **clique-Helly graph**. The **clique graph** $K(G)$ of G is the intersection graph of $\mathcal{C}(G)$: the vertices of $K(G)$ are the cliques of G and two different cliques of G are adjacent in $K(G)$ if and only if they have non-empty intersection.

A **chordless cycle** in G is a sequence of at least three distinct vertices $v_1, v_2, \ldots, v_k$ of G such that two of them are adjacent in G if and only if they are consecutive in the sequence or they are v_1 and v_k. The positive integer k is the **length** of the cycle. The chordless cycle of length k is denoted by C_k. The **girth** $g(G)$ of G is the length of a shortest chordless cycle in G (if G has no cycles, then $g(G) = \infty$). The **local girth** of G **at a vertex** $v \in V(G)$ is the girth of the subgraph induced by the open neighborhood of v in G, i.e $lg_v(G) = g(G[N(v)])$. The minimum of the local girths at the different vertices

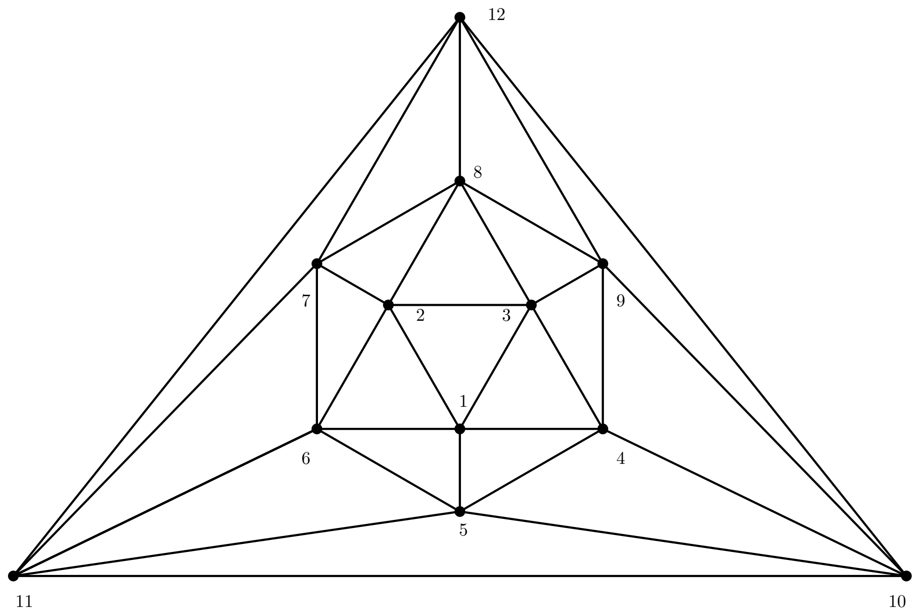

Fig. 1. The icosahedron.

of G is denoted by $lg(G)$ and named the **local girth** of G, i.e.

$$lg(G) = \min\{lg_v(G) : v \in V(G)\}.$$

Theorem 2.1 *[6] If the local girth of the graph G is greater than 6 (i.e. $lg(G) \geq 7$) then $K(G)$ is clique-Helly.*

Definition 2.2 A graph G is **critical clique-Helly** if G is clique-Helly and $G - v$ is not clique-Helly for every $v \in V(G)$.

Notice that in terms of the previous definition the conjecture of Dourado, Protti and Szwarcfiter postulates that there are no critical clique-Helly graphs. In what follows, a counterexample to that conjecture will be obtained as the clique graph of the tensor product of the icosahedron and the complete graph with three vertices K_3 (which is also called a **triangle**).

The **icosahedron** I is the graph with vertex set $\{1, 2, \ldots, 12\}$ depicted in Figure 1. The following properties of I can be easily checked.

Proposition 2.3 (i) *Every vertex of I has degree 5.*

(ii) *The open neighborhood of each vertex of I induces a C_5.*

(iii) *The cliques of I are precisely its faces which are all triangles.*

(iv) *Every vertex of I is in exactly 5 cliques.*

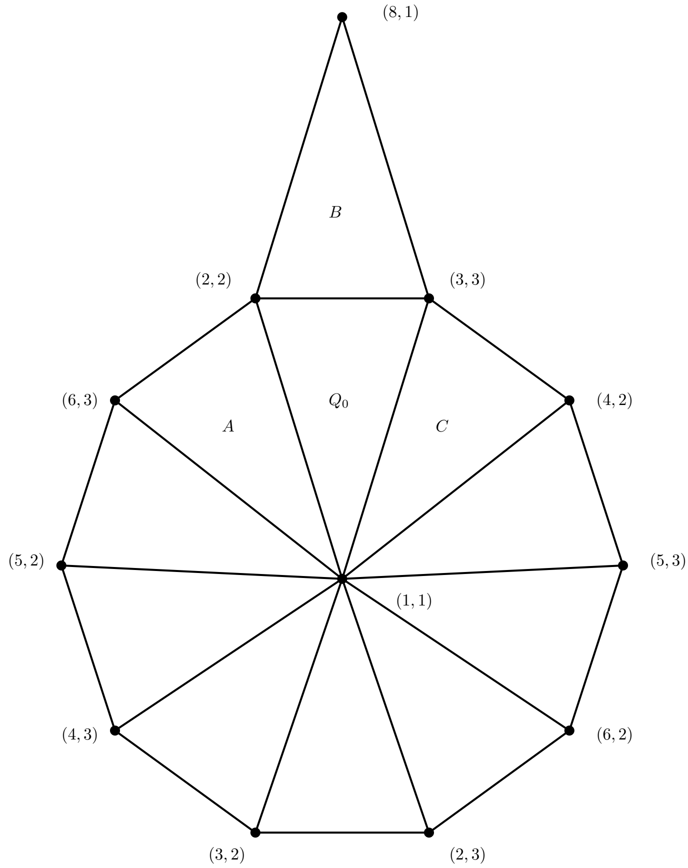

Fig. 2. A partial drawing of $I \times K_3$.

The **tensor product** $I \times K_3$ is the graph with $V(I \times K_3) = V(I) \times \{1, 2, 3\}$ and $E(I \times K_3)$ defined as follows: two vertices (i, j) and (i', j') are adjacent in $I \times K_3$ if and only if i is adjacent to i' in I and $j \neq j'$. Clearly, $I \times K_3$ is a graph on 36 vertices. Figure 2 shows an induced subgraphs of $I \times K_3$ including the neighborhood of the vertex $(1, 1)$.

Lemma 2.4 (i) *Every vertex of $I \times K_3$ has degree 10.*

(ii) *The open neighborhood of each vertex of $I \times K_3$ induces a C_{10}.*

(iii) *The cliques of $I \times K_3$ are triangles $\{(i, 1), (j, 2), (k, 3)\}$ for $\{i, j, k\}$ any triangle of I.*

(iv) *Every vertex of $I \times K_3$ is in exactly ten cliques; and any other clique of $I \times K_3$ (i.e. any clique which does not contain the given vertex) intersects at most three of those ten cliques.*

Proof. 1. Consider the vertex $(1,1)$ of $I \times K_3$. Since $N_I(1) = \{2,3,4,5,6\}$ (see Figure 1), we have that $N_{I\times K_3}((1,1)) = \{(i,j) : i \in \{2,3,4,5,6\}$ and $j \in \{2,3\}\}$. The regularity of I extends the proof to any other vertex of $I \times K_3$.

2. Again consider the vertex $(1,1)$ of $I \times K_3$ and its ten neighbors. It is easy to check that the adjacencies between them are exactly the ones depicted in Figure 2; thus $N_{I\times K_3}((1,1))$ induces a C_{10} in $I \times K_3$. The regularity and symmetry of I extends the proof to any other vertex of $I \times K_3$.

3. It is a clear consequence of the previous two items.

4. One more time, without loss of generality, consider the vertex $(1,1)$ of $I \times K_3$. That $(1,1)$ is in exactly ten cliques follows from items 1 and 2, see Figure 2. On the other hand, if Q is a clique which does not contain the vertex $(1,1)$ then Q contains at most two consecutive vertices of the cycle induced by the neighbors of $(1,1)$ which implies that Q intersect at most three of the ten cliques containing $(1,1)$. □

3 The main theorem

Theorem 3.1 *The graph $K(I \times K_3)$ is critical clique-Helly.*

Proof. By item 2 of Lemma 2.4, the local girth of $I \times K_3$ equals 10, therefore, by Theorem 2.1, $K(I \times K_3)$ is clique-Helly.

Let Q_0 be any vertex of $K(I \times K_3)$, i.e. Q_0 is a clique of $I \times K_3$. Without loss of generality assume $Q_0 = \{(1,1),(2,2),(3,3)\}$ (see Figure 2). We will prove that $K(I \times K_3) - Q_0$ is not clique-Helly.

For $i \in \{1,2,3\}$, let D_i be the set of vertices of $K(I \times K_3) - Q_0$ corresponding to the cliques of $I \times K_3$ containing the vertex (i,i), that is

$$D_i = \{Q \in \mathcal{C}(I \times K_3) : (i,i) \in Q\} \setminus \{Q_0\}.$$

By item 4 of Lemma 2.4, D_i is a clique of $K(I \times K_3) - Q_0$ for $i \in \{1,2,3\}$. We claim these three cliques are pairwise intersecting but the intersection of all three of them is empty: Indeed, the vertices of $K(I \times K_3) - Q_0$ corresponding to the cliques $\{(1,1),(2,2),(6,3)\}$, $\{(2,2),(3,3),(8,1)\}$ and $\{(1,1),(3,3),(4,2)\}$ of $I \times K_3$ (named A, B and C respectively in Figure 2) belong to $D_1 \cap D_2$, $D_2 \cap D_3$ and $D_1 \cap D_3$, respectively. And, finally, assume in order to obtain a contradiction that a vertex Q of $K(I \times K_3) - Q_0$ belongs to $D_1 \cap D_2 \cap D_3$, then, by definition of theses sets, Q is a clique of $I \times K_3$ such that $(i,i) \in Q$ for $i \in \{1,2,3\}$. Thus, by item 3 of Lemma 2.4, $Q = \{(1,1),(2,2),(3,3)\} = Q_0$ which contradicts the fact that Q is a vertex of $K(I \times K_3) - Q_0$. □

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