On an explicit representation of the Łukasiewicz sum as a quantum operation

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Abstract

The aim of this work is to introduce a quantum operation able to implement, in an approximate way, the Łukasiewicz sum in the framework of quantum computation. Different techniques for improving this approximation are studied, in particular the use of quantum cloning machine is considered.

Introduction

Since its inception, in the middle of sixties when L. A. Zadeh published his seminal work "*Fuzzy Sets*" [30], the notions of fuzzy set and fuzzy logic (as technical tools to represent and manipulate concepts with a certain degree of ambiguity) have inspired great interest because of their potential applications. On this basis, fuzzy logic has found many fruitful applications in soft computing [31], neural networks [16, 32] and in the general context of classical information theory. These fields have achieved important advances in the last decades becoming active research areas and supporting many engineering applications [25].

On the other hand, from the developing of the quantum computation [23], during the last decades other relevant research fields - such as quantum information theory and quantum neural networks [26] - naturally emerged. These studies are suggesting promising developments that, if successfully implemented, could be the origin of a prominent technological evolution. This leads to investigate the possibility to extend the use of fuzzy logic to the context of quantum information processing. This work provides a little advanced in this direction, by introducing an explicit representation of a fuzzy connective - the Łukasiewicz sum - in the framework of quantum computation.

The *Łukasiewicz sum* is defined as the binary operation on the real interval $[0, 1]$ given by

$$x \oplus y = \min\{1, x + y\} \tag{1}$$

and it is the natural interpretation for the disjunction in the infinite-valued Łukasiewicz calculus [5]. This calculus belongs to an important family of logical systems named fuzzy logic of continuous t-norm [17].

Let us remark that the Łukasiewicz logic plays an important role in different mathematical contexts. As an example, it is well known the relation between the infinite-valued Łukasiewicz logic and the game theory applied to the theory of classical communication with feedback; more specifically, the Łukasiewicz calculus is related to the Ulam's game with lies [22]. This argument suggests possible applications of the Łukasiewcz logic in the context of quantum error correction [20]. In theoretical physics, the infinite-valued Łukasiewicz calculus is also studied in virtue of its relation with the AF C^*-algebras [21] and the effects theory that plays an important role in the study of the unsharp quantum measurement [11, 12]. We also remark that some attempts to represent many valued connectives in quantum computation with potential applications have been already introduced in [27].

The aim of this paper is to implement the Łukasiewicz sum as a quantum gate, mathematically represented by a quantum operation. In order to reach this goal we refer to the probabilistic character of the quantum computational process. Indeed, quantum computational processing is probabilistic in its very essence and this is the fact that suggests a natural relation with many valued logical systems. On this basis, we provide a probabilistic representation of the Łukasiewicz sum in the framework of quantum computation with mixed states.

The paper is organized as follows. In Section 1 we briefly summarize some basic notion of quantum computation. In Section 2 we outline all the mathematical ingredients we need to represent the Łukasiewicz sum as a quantum operation. Next, in Sections 3 and 4, by appealing to numerical methods, we exhibit an explicit form of the quantum operation that probabilistically approximates the Łukasiewicz sum. In Section 5 we finally show different techniques to improve this approximation by appealing to the theory of quantum cloning.

1 Basic notions

In classical computation information is encoded by a sequence of bits. A bit is viewed as a kind of physical object which can assume one of two distinct classical states, represented by the binary numbers 0 or 1. Bits are manipulated via an ensemble of logical gates like *NOT, OR, AND*, etc, which are arranged in circuits in order to obtain the desired results of a computation.

Standard quantum computing is based on quantum systems described by finite dimensional Hilbert spaces, starting from the two-dimensional space $\mathbb{C}^2$. A *quantum bit* or *qubit* is a pure state in the Hilbert space $\mathbb{C}^2$. From a logical point of view we refer to an the orthonormal basis $\{|0\rangle, |1\rangle\}$ of $\mathbb{C}^2$, where $|0\rangle = (1, 0)^\dagger$ and $|1\rangle = (0, 1)^\dagger$, which is generally called *logical basis*. This name refers to the

fact that the logical truth is related to $|1\rangle$ and the falsity to $|0\rangle$. Thus, pure states $|\psi\rangle$ in $\mathbb{C}^2$ are linear combinations of the basis vectors $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ where c_0 and c_1 are complex numbers such that $|c_0|^2 + |c_1|^2 = 1$. Recalling the Born rule, any qubit $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ may be regarded as a piece of information, where the number $|c_0|^2$ corresponds to the probability-value of the information described by the basic state $|0\rangle$; while $|c_1|^2$ corresponds to the probability-value of the information described by the basic state $|1\rangle$. The two basis-elements $|0\rangle$ and $|1\rangle$ are usually taken as the encoding of the classical bit-values 0 and 1, respectively. In this way, the qubit probability value we are interested on, is $p(|\psi\rangle) = |c_1|^2$ that is related to the basis vector associated with truth.

In the following, we will use the tensor product space $\otimes^n \mathbb{C}^2 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$ (n times), which is 2^n-dimensional. A special basis, called the 2^n-*logical basis*, is chosen for $\otimes^n \mathbb{C}^2$. More precisely, it consists on the 2^n orthogonal states $|\iota\rangle$, $0 \leq \iota \leq 2^n$, where ι is in binary representation and $|\iota\rangle$ is a tensor product of states $|\iota\rangle = |\iota_1\rangle \otimes |\iota_2\rangle \otimes \ldots \otimes |\iota_n\rangle$, whit $\iota_j \in \{0,1\}$. A pure state $|\psi\rangle \in \otimes^n \mathbb{C}^2$ can be written as a superposition of the basis vectors as follows $|\psi\rangle = \sum_{\iota=1}^{2^n} c_\iota |\iota\rangle$, with $\sum_{\iota=1}^{2^n} |c_\iota|^2 = 1$. Let $\mathcal{L}(H)$ be the space of linear operators in the complex Hilbert space H. The evolution of a pure state in $\otimes^n \mathbb{C}^2$ is dictated by quantum gates, that are mathematically represented by unitary operators acting on $\otimes^n \mathbb{C}^2$.

But in general, a quantum system is not in a pure state. This may be caused, for example, by the noncomplete efficiency in the preparation procedure or by the fact that systems cannot be completely isolated from the environment, undergoing decoherence of their states. In view of these facts, several authors [1, 4, 9, 15, 27] have paid attention to a more general model of quantum computational processes, where pure states are replaced by mixed states. In what follows we give a short description of this mathematical model.

A *mixed state* is an Hermitian operator $\rho \in \mathcal{L}(H)$ that is positive semidefinite ($\rho \geq 0$) and has unit trace ($tr(\rho) = 1$).

In this powerful model, the notion of quantum gate is mathematically generalized by the notion of quantum operation.

A *quantum operation* is a linear map $\mathcal{E} : \mathcal{L}(H_1) \to \mathcal{L}(H_2)$ that is trace-preserving and *completely positive*. Roughly speaking, complete positivity means that if we embed H into a larger system, the standard lifting of $\mathcal{E}$ to the larger system preserves positive definiteness, and thus states get mapped into states. Formally, this means that for any Hilbert space K, the linear map $\mathcal{E} \otimes \mathcal{I}_K : \mathcal{L}(H_1 \otimes K) \to \mathcal{L}(H_2 \otimes K)$ where $\mathcal{I}_K$ is the identity in $\mathcal{L}(H)$, satisfies that for any $\rho \in \mathcal{L}(H_1 \otimes K)$, if $\rho > 0$ then $(\mathcal{E} \otimes \mathcal{I}_K)(\rho) > 0$. Each quantum operation $\mathcal{E}$ can be expressed as

$$\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger \tag{2}$$

where A_i are linear operators satisfying $\sum_i A_i^\dagger A_i = I$. This is known as the *Kraus representation* [19] of a quantum operation. Let us notice that every unitary operator U on a Hilbert space $\otimes^m \mathbb{C}^2$ gives rise to a quantum operation $\mathcal{O}_\mathcal{U}$ such that $\mathcal{O}_U(\rho) = U\rho U^\dagger$ for each $\rho \in \mathcal{L}(H)$.

We may associate to each vector of the logical basis of $\mathbb{C}^2$ two projection operators $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ that represent, in this more general framework, the falsity-property and the truth-property, respectively. Let us consider the operator $P_1^{(n)} = \otimes^{n-1}I \otimes P_1$ on $\otimes^n\mathbb{C}^2$. By applying the Born rule, we obtain the (logical) probability that a density operator $\rho = [\rho_{i,j}]$ on $\otimes^n\mathbb{C}^2$ is true, as follows:

$$p(\rho) = Tr(P_1^{(n)}\rho) = \sum_{i=1}^{2^{n-1}} \rho_{2i,2i}. \tag{3}$$

Notice that the probability value $p(\rho)$ depends only on the odd diagonal elements of ρ. In the particular case in which $\rho = |\psi\rangle\langle\psi|$, where $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$, we obtain that $p(\rho) = |c_1|^2$. Thus, this probability value associated to ρ is the generalization of the probability value considered for qubits.

2 Łukasiewicz sum and quantum circuits

Łukasiewicz sum exhibits intersting relations with the quantum structures [11, 12, ?, 22]. In particular, several logical and algebraic structures based on the relationship between the Łukasiewicwicz logic and quantum computational logic, were studied in the literature [2, 9, 6, 13, 15]. One may be tempted to define a probabilistic representation of the Łukasiewicwicz sum "$\oplus$" in the framework of quantum computational logic as

$$p(\mathcal{O}(\rho \otimes \sigma)) = p(\rho) \oplus p(\sigma) \tag{4}$$

where $\mathcal{O}$ should be a quantum operation and ρ and σ are arbitrary density operators on $\mathbb{C}^2$. But such a quantum operation $\mathcal{O}$ does not exist due to the following no-go theorem (see also [13]):

Theorem 2.1 *There is no quantum operation $\mathcal{O}$ such that $p(\mathcal{O}(\rho \otimes \sigma)) = p(\rho) \oplus p(\sigma)$ holds for all ρ and σ.*

Proof: Suppose that there exists a quantum operation $\mathcal{O}$ satisfying Eqn. 4. Then, for arbitrary ρ and σ, by using the Kraus representation we have that

$$\mathcal{O}(\rho \otimes \sigma) = \sum_i A_i(\rho \otimes \sigma)A_i^\dagger.$$

Writing $\rho = [x_{j,k}]$, $\sigma = [y_{l,m}]$ and $A_i = [a_{r_i,s_i}]$ (with coordinates expressed in the logical basis), by matrix product and Eq. (3), $p(\mathcal{O}(\rho \otimes \sigma))$ can be seen as a polynomial in the variables $x_{2,2}$, $y_{2,2}$ assuming the form:

$$p(\mathcal{O}(\rho \otimes \sigma)) = \sum_{\alpha,\beta} f_{\alpha,\beta}(a_{r_i,s_i}, x_{j,k}, y_{l,m})x_{2,2}^\alpha y_{2,2}^\beta \tag{5}$$

where $\alpha, \beta \in \{0, 1\}$. But, since the Łukasiewicz sum is not a polynomial, it is not representable as a quantum operation in the sense of Eq.(4).

Figure 1: Schematic representation of the action of the quantum operation $\mathcal{E}$.

$\square$

However, the expression given in Eq.(5) suggests us to look for a quantum operation $\mathcal{E}$ such that the polynomial $p(\mathcal{E}(\rho \otimes \sigma))$ approximates the value $p(\rho) \oplus p(\sigma)$. The action of $\mathcal{E}$ as a quantum operation is depicted in Fig.1.

In order to find the approximation, first notice that the probability value $p(\mathcal{E}(\rho \otimes \sigma))$ does not only depend on $p(\rho)$ and $p(\sigma)$ but it also depends on the anti-diagonal elements of ρ and σ among the coefficients $f_{\alpha,\beta}(a_{r_i,s_i}, \rho_{j,k}, \sigma_{l,m})$. To solve this problem we need to introduce a quantum operation that deletes the anti-diagonal elements of a density operator on $\mathbb{C}^2$, preserving at the same time its probability values. To do this, we first define the following Kraus operators

$$D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad D_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{6}$$

and the quantum operation

$$\mathcal{A}nt(\rho) = D_1 \rho D_1^\dagger + D_2 \rho D_2^\dagger. \tag{7}$$

In this way, the action of $\mathcal{A}nt$ is given by

$$\rho = \begin{bmatrix} 1-x & r \\ r^\dagger & x \end{bmatrix} \mapsto \mathcal{A}nt(\rho) = \begin{bmatrix} 1-x & 0 \\ 0 & x \end{bmatrix} \tag{8}$$

and solves the problem posed above.

It is important to remark that the polynomial degree of $p(\mathcal{E}(\rho \otimes \sigma))$ is equal to 2 (in the sense that it has crossed terms of the form xy). One could attain a better approximation of the right hand side of Eq.(4) by appealing to a higher order polynomial. The general properties of convergence (for the limiting case) of higher order approximations are discussed in [13]. Given that we are looking for a simple example of an explicit representation of the quantum operation $\mathcal{E}$, we stick here to low degree polynomials.

In general, in order to obtain a polynomial $p(\mathcal{E}(\rho, \sigma))$ of arbitrary degree, the power of the variables $x_{2,2}$ and $y_{2,2}$ has to be increased. This task can be

achieved by involving tensor powers of both, ρ and σ. Indeed, for an arbitrary density operator $\rho = \begin{bmatrix} 1-x & r \\ r^\dagger & x \end{bmatrix}$ acting on $\mathbb{C}^2$, by induction on k, we can prove that $Diag(\otimes^n \rho) = \{(1-x)^\alpha x^\beta : \alpha + \beta = n\}$.

Taking into account the above discussion, and in order to obtain a good approximation for the Łukasiewicz sum, our stategy will be to consider a new quantum operation of the form $\mathcal{E}^{(n,m)}((\otimes^n \rho) \otimes (\otimes^m \sigma))$ (instead of $\mathcal{E}(\rho \otimes \sigma)$; notice that $\mathcal{E} = \mathcal{E}^{(1,1)}$). By using Eqn. 6, we can assume — without losing generality — that ρ and σ are diagonal. In this way, we look for an approximation of the form $p(\mathcal{E}^{(n,m)}((\otimes^n \rho) \otimes (\otimes^m \sigma)) \approx p(\rho) \oplus p(\sigma)$. In the following, we show how to obtain $\mathcal{E}^{(n,m)}$ for the particular case $n = m = 2$.

3 Polynomial approximation for the Łukasiewicz sum

In this Section we investigate a family of approximant polynomials for the Łukasiewicz sum obtained by using binomial series. The key idea is to reduce the problem to a one-variable approximation function. Let us consider the function $h : [0,2] \to [0,\frac{1}{2}]$ defined as

$$h(z) = \begin{cases} \frac{z}{2}, & \text{if } z \in [0,1], \\ 1 - \frac{z}{2}, & \text{if } z \in (1,2]. \end{cases} \tag{9}$$

Next, define

$$g(z) = \frac{z}{2} + h(z). \tag{10}$$

Notice that if $z = x + y$ (with $x, y \in [0,1]$), then $g(x+y) = x \oplus y$. We will approximate $h(z)$ by using the symmetric functions $z^i(2-z)^i$ (notice that $h(z)$ is also a symmetric function with respect to the point $z = 1$). We look for coefficients c_i such that:

$$\frac{z}{2} = \sum_{i=1}^{\infty} c_i z^i (2-z)^i, \qquad z \in [0,1]. \tag{11}$$

Now, we are going to consider the partial sums

$$h_n(z) = \sum_{i=1}^{n} c_i z^i (2-z)^i, \qquad z \in [0,2] \tag{12}$$

using the same coefficients as in Eq.(11). Notice that the function $h(z)$ and the polynomials $z^i(2-z)^i$ are symmetric in the interval $[0,2]$ with respect to the point $z = 1$. If the coefficients c_i in Eq. (11) are positive, then the series $h_n(z)$ is monotone and increasing for each $z \in [0,2]$. Thus, by Dini's Theorem, it follows that $h_n(z)$ uniformly converges to $h(z)$. Indeed, the coefficients c_i can

be obtained by using the binomial series as follows. By appealing to the change of variables $w = z(2 - z)$ in $[0, 1]$, we obtain that Eq.(11) can be written as $\frac{1-\sqrt{1-w}}{2} = \sum_i c_i w^i$, where $w \in [0, 1]$. Let us notice that the binomial series $\sqrt{1 - w} = \sum_{i=0}^{\infty} (-1)^i \binom{1/2}{i} w^i$ - with $\binom{1/2}{i} = \frac{(-1)^{i+1}}{2^{2i}(2i-1)} \binom{2i}{i}$ - is uniformly convergent for $|w| \leq 1$. Thus $c_0 = 0$ and $c_i = \frac{(-1)^{i+1}}{2} \binom{1/2}{i} > 0$ for $i > 0$. Hence, the approximants $P_n(x, y) = g_n(x + y)$ for the Łukasiewicz $x \oplus y$ assume the form:

$$P_n(x, y) = \frac{x + y}{2} + \sum_{i=1}^{n} \frac{(-1)^{i+1}}{2} \binom{1/2}{i} (x + y)^i \big((1 - x) + (1 - y)\big)^i. \quad (13)$$

Taking into account the above family of polynomials, the rest of the paper is devoted to provide a quantum operation able to represent the Łkasiewicz sum by only involving two qubits input state.

4   Łukasiewicz sum as 16-dimensional quantum operation

In this section we provide an explicit form of the quantum operation that represents an approximation of the Łukasiewicz sum when the input is given by a two qubit state. As it will be shown in the second part of the section, when the degree of the approximant is n, then the input of the corresponding quantum operation needs to assume the form $(\otimes^n \rho) \otimes (\otimes^n \sigma)$ (with ρ and σ qubits). As it is well known, larger is n and more serious are the physical problems in order to produce the states $\otimes^n \rho$ and $\otimes^n \sigma$. We attack this problem in the next section but now we confine in the simplest case where $n = 2$.

By Eq.(13) the approximant polynomial assumes the following form

$$P_2(x, y) = \frac{5}{12}(x + y)(1 - x) + \frac{5}{12}(x + y)(1 - y) + \frac{1}{2}(x + y).$$

We want to estimate the error for this approximation. It is given by the following two-variables constrained maximization problem:

$$E(P_2) = \max_{(x,y) \in [0,1]^2} |x \oplus y - P_2(x, y)|.$$

Taking into account the geometry of the function $x \oplus y$, if we divide the square $[0, 1]^2$ in two regions

$$\mathbf{I} = \begin{cases} 0 \leq x, y < 1, \\ x + y \leq 1. \end{cases} \qquad \mathbf{II} = \begin{cases} 0 \leq x, y \leq 1, \\ x + y \geq 1 \end{cases}$$

then the function $x \oplus y$ is differentiable in the interior of the regions $\mathbf{I}$ and $\mathbf{II}$. This allows us to apply the standard maximization method to obtain $E(P_2)$ separately. By doing this we obtain $E(P_2) = \frac{1}{12}$ in correspondence to the point $(x, y) = (\frac{1}{2}, \frac{1}{2})$.

Remarkable enough, let us notice that it is also possible to obtain a further improvement of the above approximation. Using the same maximization argument, we can see that $\max_{(x,y)\in[0,1]^2}(x \oplus y - P_2(x,y)) = \frac{1}{12}$ for $(x,y) = (\frac{1}{2}, \frac{1}{2})$ and $\max_{(x,y)\in[0,1]^2}(P_2(x,y) - x \oplus y) = \frac{1}{15}$ for $(x,y) = (\frac{3}{16}, \frac{17}{80})$. Thus, if we add the number $\frac{1}{2}(\frac{1}{12} - \frac{1}{15}) = \frac{1}{120}$ to P_2 i.e.

$$
\begin{aligned}
\tilde{P}_2(x,y) &= P_2 + \frac{1}{120} \hspace{5cm} (14)\\
&= \frac{5}{12}(x+y)(1-x) + \frac{5}{12}(x+y)(1-y) + \frac{1}{2}(x+y) + \frac{1}{120}
\end{aligned}
$$

then, we can see that $E(\tilde{P}_2) = \frac{3}{40} < \frac{1}{12}$ in the points $(\frac{3}{16}, \frac{17}{80})$ and $(\frac{1}{2}, \frac{1}{2})$, hence providing a further reduction of the error in the approximation.

On this basis, we now build a quantum operation L_2 in $\otimes^4 \mathbb{C}^2$ such that

$$p(L_2(\otimes^2\rho, \otimes^2\sigma)) \approx p(\rho) \oplus p(\sigma).$$

Let ρ and σ are density operators in $\mathbb{C}^2$ with vanishing off-diagonal entries. This quantum operation will be based on the approximant $\tilde{P}_2$ defined in Eq.(14). First, by appealing to a standard two-variable maximization argument, we can see that

$$
\max_{(x,y)\in[0,1]^2} \tilde{P}_2(x,y) = \frac{43}{40} > 1
$$

and this maximum value is attained in $(x,y) = (\frac{63}{80}, \frac{13}{16})$. The above inequality implies that there is not a quantum operation $\mathcal{E}$ such that $p(\mathcal{E}(\rho \otimes \sigma)) = \tilde{P}_2(p(\rho), p(\sigma))$. Indeed, as an example, if $p(\rho) = \frac{63}{80}$ and $p(\sigma) = \frac{13}{16}$, then $p(\mathcal{E}(\rho \otimes \sigma)) = \frac{40}{43} > 1$. But this is not possible, because $p(-)$ represents a probability value.

Thus, in order to avoid this problem, we instead consider the normalized polynomial $\frac{40}{43}\tilde{P}_2$. In this way, the approximation of the Łukasiewicz sum reads:

$$
P_{L_2}(x,y) = \frac{40}{43}\left(\frac{5}{12}(x+y)(1-x) + \frac{5}{12}(x+y)(1-y) + \frac{1}{2}(x+y) + \frac{1}{120}\right) \hspace{0.5cm} (15)
$$

satisfying the normalization condition, i.e. $0 \leq P_{L_2}(x,y) \leq 1$ in $[0,1]^2$. Once avoided the above problem, from now on we will focus our attention on the polynomial given by Eq.(15) in order to build the quantum operation that approximately represents the Łukasiewicz sum.

Let $\rho = \begin{bmatrix} 1-x & 0 \\ 0 & x \end{bmatrix}$ and $\sigma = \begin{bmatrix} 1-y & 0 \\ 0 & y \end{bmatrix}$ be two diagonal density operators acting on $\mathbb{C}^2$ (hence, $p(\rho) = x$ and $p(\sigma) = y$). Let us notice that the matrix

$$(z_{i,j})_{1\leq i,j\leq 16} = (\rho \otimes \rho) \otimes (\sigma \otimes \sigma)$$

is the diagonal matrix whose diagonal coefficients — for the sake of the simplicity denoted by z_i — are given by:

8

$$z_1 = (1-x)^2(1-y)^2, \qquad z_9 = (1-x)x(1-y)^2,$$

$$z_2 = (1-x)^2(1-y)y, \qquad z_{10} = (1-x)x(1-y)y,$$

$$z_3 = (1-x)^2(1-y)y, \qquad z_{11} = (1-x)x(1-y)y,$$

$$z_4 = (1-x)^2 y^2, \qquad z_{12} = (1-x)xy^2,$$

$$z_5 = (1-x)x(1-y)^2, \qquad z_{13} = x^2(1-y)^2,$$

$$z_6 = (1-x)x(1-y)y, \qquad z_{14} = x^2(1-y)y,$$

$$z_7 = (1-x)x(1-y)y, \qquad z_{15} = x^2(1-y)y,$$

$$z_8 = (1-x)xy^2, \qquad z_{16} = x^2 y^2.$$

Therefore, a quantum operation $\mathsf{L}_2(-) = \sum_k A_k(-)A_k^\dagger$ acting on $\otimes^4 \mathbb{C}^2$ such that $p(\mathsf{L}_2(\rho,\sigma)) = P_{\mathsf{L}_2}(p(\rho),p(\sigma)) = P_{\mathsf{L}_2}(x,y)$ needs to satisfy

$$P_{\mathsf{L}_2}(x,y) = \sum_{i=1}^{16} a_i(A_1,\ldots,A_k)z_i \qquad (16)$$

where $a_i(A_1,\ldots,A_k)$ are real numbers depending on the elements of the matrices A_k for $1 \le k \le 16$. Thus, by Eq.(16) the family of matrices $(A_k)_k$ of $\mathsf{L}_2(-)$ will be defined.

For this reason we first need to rewrite $P_{\mathsf{L}_2}(x,y)$ in the base $(z_i)_{1\le i\le 16}$, as shown in the following proposition.

Proposition 4.1 *The polynomial $P_{L_2}(x,y)$ introduced in Eq.(15) can be rewritten as follows*

$$\begin{aligned}
P_{L_2}(x,y) &= \frac{40}{43}\Big[\frac{1}{120}z_1 + \frac{27}{20}z_2 + \frac{111}{120}z_4 + \frac{27}{120}z_5 + \frac{51}{20}z_6 + \frac{141}{60}z_8 \\
&+ \frac{111}{120}z_{13} + \frac{141}{60}z_{14} + \frac{121}{120}z_{16}\Big].
\end{aligned}$$

Proof: The key of idea is to write each term of the polynomial of the Eq.(15) in terms of $(z_i)_{1\le i\le 16}$. By a straightforward calculation we have that

$$\begin{aligned}
\frac{5}{12}(x+y)(1-x) &= \frac{5}{12}x(1-x) + \frac{5}{12}y(1-x) \\
&= \frac{5}{12}(z_5 + 2z_6 + z_8) + \frac{5}{12}(z_4 + z_2 + z_8 + z_6).
\end{aligned}$$

$$\begin{aligned}
\frac{5}{12}(x+y)(1-y) &= \frac{5}{12}x(1-y) + \frac{5}{12}y(1-y) \\
&= \frac{5}{12}(z_{13} + z_5 + z_{14} + z_6) + \frac{5}{12}(z_2 + 2z_6 + z_{14}).
\end{aligned}$$

$$\frac{1}{2}(x+y) \quad = \quad \frac{1}{2}x + \frac{1}{2}y$$
$$= \quad \frac{1}{2}(z_{13} + z_5 + 2z_{14} + z_{16} + z_5 + 2z_6 + z_8) +$$
$$+ \quad \frac{1}{2}(z_4 + 2z_8 + z_{16} + z_3 + 2z_6 + z_{14}).$$

$$\frac{1}{120} \quad = \quad \frac{1}{120}\sum_{i=1}^{16} z_i.$$

Then,

$$P_{L_2}(x,y) \quad = \quad \frac{40}{43}[\frac{1}{120}z_1 + \frac{27}{20}z_2 + \frac{111}{120}z_4 + \frac{27}{120}z_5 + \frac{51}{20}z_6 + \frac{141}{60}z_8$$
$$+ \quad \frac{111}{120}z_{13} + \frac{141}{60}z_{14} + \frac{121}{120}z_{16}].$$

$\square$

Theorem 4.2 *Let us consider the following family of* 16×16 *matrices for* $1 \leq k \leq 8$:

$[\mathbf{1}]_{(2k,j)}$ *having* 1 *in the* $(2k,j)$-*entry and* 0 *in any other entry,*

$[\mathbf{1}]_{(2k-1,j)}$ *having* 1 *in the* $(2k-1,j)$-*entry and* 0 *in any other entry.*

Let us define the following family of matrices, for $1 \leq k \leq 8$

$$L_{(2k,1)} = \sqrt{\frac{1}{129}}[\mathbf{1}]_{(2k,1)} \qquad L_{(2k-1,1)} = \sqrt{\frac{1}{16} - \frac{1}{129}}[\mathbf{1}]_{(2k-1,1)}$$

$$L_{(2k,2)} = \sqrt{\frac{54}{43}}[\mathbf{1}]_{(2k,2)} \qquad L_{(2k-1,2)} = \sqrt{\frac{1}{16} - \frac{54}{43}}[\mathbf{1}]_{(2k-1,1)}$$

$$L_{(2k,4)} = \sqrt{\frac{37}{43}}[\mathbf{1}]_{(2k,4)} \qquad L_{(2k-1,4)} = \sqrt{\frac{1}{16} - \frac{37}{43}}[\mathbf{1}]_{(2k-1,1)}$$

$$L_{(2k,5)} = \frac{3}{\sqrt{43}}[\mathbf{1}]_{(2k,5)} \qquad L_{(2k-1,5)} = \sqrt{\frac{1}{16} - \frac{9}{43}}[\mathbf{1}]_{(2k-1,1)}$$

$$L_{(2k,6)} = \sqrt{\frac{102}{43}}[\mathbf{1}]_{(2k,6)} \qquad L_{(2k-1,6)} = \sqrt{\frac{1}{16} - \frac{102}{43}}[\mathbf{1}]_{(2k-1,1)}$$

$$L_{(2k,8)} = \sqrt{\frac{94}{43}}[\mathbf{1}]_{(2k,8)} \qquad L_{(2k-1,8)} = \sqrt{\frac{1}{16} - \frac{94}{43}}[\mathbf{1}]_{(2k-1,1)}$$

$$L_{(2k,13)} = \sqrt{\frac{37}{43}}[\mathbf{1}]_{(2k,13)} \qquad L_{(2k-1,13)} = \sqrt{\frac{1}{16} - \frac{37}{43}}[\mathbf{1}]_{(2k-1,1)}$$

$$L_{(2k,14)} = \sqrt{\frac{94}{43}}[\mathbf{1}]_{(2k,14)} \qquad L_{(2k-1,14)} = \sqrt{\frac{1}{16} - \frac{94}{43}}[\mathbf{1}]_{(2k-1,1)}$$

$$L_{(2k,16)} = \frac{11}{\sqrt{129}}[\mathbf{1}]_{(2k,16)} \qquad L_{(2k-1,16)} = \sqrt{\frac{1}{16} - \frac{121}{129}}[\mathbf{1}]_{(2k-1,1)}$$

Then, L_2 is defined as the operator acting on $\otimes^4 \mathbb{C}^2$ given by

$$
\begin{aligned}
L_2(\varphi) \;=\; & \sum_{k=1}^{8} [L_{(2k,1)}\varphi L^{\dagger}_{(2k,1)} + L_{(2k,2)}\varphi L^{\dagger}_{(2k,2)} + L_{(2k,4)}\varphi L^{\dagger}_{(2k,4)} + \\
& + \; L_{(2k,5)}\varphi L^{\dagger}_{(2k,5)} + L_{(2k,6)}\varphi L^{\dagger}_{(2k,6)} + L_{(2k,8)}\varphi L^{\dagger}_{(2k,8)} + \\
& + \; L_{(2k,13)}\varphi L^{\dagger}_{(2k,13)} + L_{(2k,14)}\varphi L^{\dagger}_{(2k,14)} + L_{(2k,16)}\varphi L^{\dagger}_{(2k,16)} + \\
& + \; L_{(2k-1,1)}\varphi L^{\dagger}_{(2k-1,1)} + L_{(2k-1,2)}\varphi L^{\dagger}_{(2k-1,2)} + L_{(2k-1,4)}\varphi L^{\dagger}_{(2k-1,4)} + \\
& + \; L_{(2k-1,5)}\varphi L^{\dagger}_{(2k-1,5)} + L_{(2k-1,6)}\varphi L^{\dagger}_{(2k-1,6)} + L_{(2k-1,8)}\varphi L^{\dagger}_{(2k-1,8)} + \\
& + \; L_{(2k-1,13)}\varphi L^{\dagger}_{(2k-1,13)} + L_{(2k-1,14)}\varphi L^{\dagger}_{(2k-1,14)} + L_{(2k-1,16)}\varphi L^{\dagger}_{(2k-1,16)}]
\end{aligned}
$$

is a quantum operation such that, for any diagonal density operators ρ and σ acting on $\mathbb{C}^2$ is

$$
p(L_2((\otimes^2 \rho) \otimes (\otimes^2 \sigma))) = P_{L_2}(p(\rho), p(\sigma)).
$$

Proof: Let ρ and σ be two arbitrary diagonal density operators acting on $\mathbb{C}^2$ and let $(z_i)_{1 \leq i \leq 16}$ be the diagonal of the matrix $\psi = (\rho \otimes \rho) \otimes (\sigma \otimes \sigma)$. Let us denote by a_k the coefficient of z_k in the polynomial $P_{L_2}(x, y)$. Notice that $k \in \{1, 2, 4, 5, 6, 8, 13, 14, 16\}$. By a straightforward calculation we can see that

$$
\sum_{s=1}^{8} L_{(2k,s)}\psi L^{\dagger}_{(2k,s)} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & a_k z_k & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & a_k z_k & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}
$$

and

$$
\sum_{s=1}^{8} L_{(2k-1,s)}\psi L^{\dagger}_{(2k-1,s)} = \begin{pmatrix} (\frac{1}{16} - a_k)z_k & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & (\frac{1}{16} - a_k)z_k & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.
$$

Thus,

$$
\begin{aligned}
L_2(\psi) \;=\; & \sum_{k}\sum_{1=s}^{16} L_{(2k,s)}\psi L^{\dagger}_{(2k,s)} + \sum_{k}\sum_{1=s}^{16} L_{(2k-1,s)}\psi L^{\dagger}_{(2k-1,s)} \\
\;=\; & (\otimes^3 I) \otimes \begin{pmatrix} 1 - \sum_k a_k z_k & 0 \\ 0 & \sum_k a_k z_k \end{pmatrix}
\end{aligned}
$$

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where $k \in \{1, 2, 4, 5, 6, 8, 13, 14, 16\}$.

Now, all we need is to verify that $\mathbb{L}_2$ is a quantum operation, by checking that it satisfies the Kraus representation introduced in Section 1.

Let us notice that the matrix $\mathbb{L}^{\dagger}_{(2k,s)}\mathbb{L}_{(2k,s)}$ has the value a_k just in the (k, k)-th entry and 0 in any other entry. Therefore, the matrix $\sum_{s=1}^{16} \mathbb{L}^{\dagger}_{(2k,s)}\mathbb{L}_{(2k,s)}$ has the value $16a_k$ in the (k, k)-th entry and all the other entries are equal to 0. Hence:

$$\sum_{k}\sum_{s=1}^{16} \mathbb{L}^{\dagger}_{(2k,s)}\mathbb{L}_{(2k,s)} = \begin{pmatrix} 16a_1 & 0 & 0 & \dots \\ 0 & 16a_2 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In a similar way, the matrix $\mathbb{L}^{\dagger}_{(2k-1,s)}\mathbb{L}_{(2k-1,s)}$ has the value $\frac{1}{16} - a_k$ just in the (k, k)-th entry and 0 in any other entry. Therefore, the matrix $\sum_{s=1}^{16} \mathbb{L}^{\dagger}_{(2k-1,s)}\mathbb{L}_{(2k-1,s)}$ has the value $1 - 16a_k$ in the (k, k)-th entry, while all the other entries are equal to 0. Hence:

$$\sum_{k}\sum_{s=1}^{16} \mathbb{L}^{\dagger}_{(2k-1,s)}\mathbb{L}_{(2k-1,s)} = \begin{pmatrix} 1 - 16a_1 & 0 & 0 & \dots \\ 0 & 1 - 16a_2 & 0 & \dots \\ 0 & 0 & 1 - 16a_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus $\sum_{k}\sum_{s=1}^{16} \mathbb{L}^{\dagger}_{(2k,s)}\mathbb{L}_{(2k,s)} + \sum_{k}\sum_{s=1}^{16} \mathbb{L}^{\dagger}_{(2k-1,s)}\mathbb{L}_{(2k-1,s)} = \otimes^4 I$. Hence, $\mathbb{L}_2$ is a quantum operation.

Taking into account the matrix representation of $\mathbb{L}_2$, we can conclude that

$$p(\mathbb{L}_2((\otimes^2\rho) \otimes (\otimes^2\sigma))) = P_{\mathbb{L}_2}(p(\rho), p(\sigma)),$$

as required.

$\square$

Finally, let us notice that, in virtue of Eq.(8), the argument above can be easily generalized to the case where the input states are not represented by diagonal matrices.

5 Łukasiewicz sum and symmetric universal quantum cloning machine for qubits

As mentioned at the beginning of the Section 4, when the degree of the approximant polynomial n is large, that leads to physical problems in order to prepare the input state $\otimes^n\rho$.

As is well known, the No Cloning Theorem [8, 29] forbids the existence of a quantum operation capable of making perfect copies of an arbitrary input

state; on this basis there are objective physical limitations in order to prepare the state $\otimes^n \rho$ from an unknown state ρ. But, given a state ρ, it is possible to obtain approximate copies whose "distance" (generally expressed in terms of the quantum fidelity) should be maximal [3, 10, 14, 28]. The devices capable of performing this approximate cloning operation are called *Quantum Cloning Machines* (see [7, 24] for empirical implementations). Quantum information theory includes different protocols to provide an approximate cloning of a state ρ.

Here we consider the protocol presented in [18] and named Symmetric UQCM (universal quantum cloning machine), that is characterized by the fact that the approximate copies of the initial state are identical.

Formally, given a qubit state $\omega = |\psi\rangle\langle\psi|$ (with $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ in $\mathbb{C}^2$) the protocol in represented by the following transformation (see Eq.(50) in [18])

$$\omega \mapsto \omega^{clon} \otimes \omega^{clon}$$

where

$$\omega^{clon} = \frac{2}{3}|\psi\rangle\langle\psi| + \frac{1}{6}I = \begin{pmatrix} 1 - \frac{1}{6} - \frac{2}{3}|c_1|^2 & \frac{1}{6} + \frac{2}{3}c_1 c_0 \\ \frac{1}{6} + \frac{2}{3}(c_1 c_0)^\dagger & \frac{1}{6} + \frac{2}{3}|c_1|^2 \end{pmatrix}. \qquad (17)$$

Let us now apply the cloning operation (17) to each of the input pure states $\rho = |\alpha\rangle\langle\alpha|$ with $|\alpha\rangle = a_0|0\rangle + a_1|1\rangle$ and $\sigma = |\beta\rangle\langle\beta|$ with $|\beta\rangle = b_0|0\rangle + b_1|1\rangle$. Let $x = |a_1|^2$ and $y = |b_1|^2$. Thus, we obtain two approximate copies of ρ and σ (say ρ_x^{clon} and σ_x^{clon}, respectively). Appling the quantum operation $\mathcal{A}nt$ (defined in Eq.(8)) to ρ_x^{clon} and σ_x^{clon} we obtain

$$\rho_x = \mathcal{A}nt(\rho_x^{clon}) = \begin{pmatrix} 1 - \frac{1}{6} - \frac{2}{3}x & 0 \\ 0 & \frac{1}{6} + \frac{2}{3}x \end{pmatrix}$$

$$\sigma_y = \mathcal{A}nt(\sigma_y^{clon}) = \begin{pmatrix} 1 - \frac{1}{6} - \frac{2}{3}y & 0 \\ 0 & \frac{1}{6} + \frac{2}{3}y \end{pmatrix}$$

then

$$
\begin{aligned}
p(\rho_x) \oplus p(\sigma_y) &= \min\left\{p(\rho_x) + p(\sigma_y), 1\right\} \\
&= \min\left\{\frac{1}{6} + \frac{2}{3}x + \frac{1}{6} + \frac{2}{3}y, 1\right\} \\
&= \min\left\{\frac{1}{3} + \frac{2}{3}(x + y), 1\right\} \\
&= \frac{1}{3}\min\left\{\frac{2}{3}(x + y), 1 - \frac{1}{3}\right\} \\
&= \frac{1}{3} + \frac{2}{3}\min\{x + y, 1\} \\
&= \frac{1}{3} + \frac{2}{3}\left(p(\rho) \oplus p(\sigma)\right).
\end{aligned}
$$

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In this way, we can show that the quantum operation described in Fig. 1 (that we denote by $\mathcal{E}$) satisfies

$$p(\mathcal{E}(\rho \otimes \sigma)) = \frac{2}{3} P_{\text{Ł}_2}(p(\rho), p(\sigma)) + \frac{1}{6} \approx \frac{2}{3}(p(\rho) \oplus p(\sigma)) + \frac{1}{6}. \qquad (18)$$

Finally, in order to obtain the best approximation of $\tilde{P}_2(x, y)$ provided in Eq.(14) (that is equal to $\frac{3}{40}$), we obtain

$$p(\rho) \oplus p(\sigma) \approx \frac{43}{40} \frac{3}{2}(p(\mathcal{E}(\rho \otimes \sigma)) - \frac{1}{6}).$$

In this way we obtain the probabilistic value of the Łukasiewicz sum between ρ and σ up to an affine transformation.
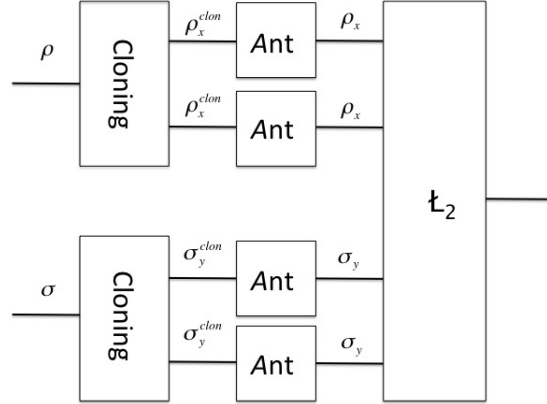

Figure 2: Quantum operation using a cloning machine.

6 Conclusions

In this paper we have studied the problem of representing the probabilistic version of the Łukasiewicz sum by using quantum operations. After showing that it is not possible to find an exact representation for this probabilistic version, we have provided an approximation of this in terms of quantum operations. We have explicitly shown this construction for the case of two qubit states (using two copies of each input state). Furthermore, in Section 5 we have reduced the number of available initial states to only two and, by appealing to quantum cloning machines, we have obtained the same degree of approximation of the Łukasiewicz sum between them (up to an affine transformation).

Our method can be extended to higher dimensional problems in a natural way. This allows to increase the accuracy of the approximation due to the use of

higher degree polynomials. It is also important to remark that the method presented in this paper could be used, in principle, for representing other functions in the quantum setting.

Our work shows the problems linked with the task of finding quantum versions of the operations associated to Fuzzy logic. As explained in the Introduction, this line of research could shed some light into the problem of how to develop soft computing techniques in the framework of quantum computation.

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