Existence and Multiplicity of Periodic Solutions for Dynamic Equations with Delay and singular φ-laplacian of Relativistic Type

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Abstract

We study the existence and multiplicity of periodic solutions for singular φ-laplacian equations with delay on time scales. We prove the existence of multiple solutions using topological methods based on the Leray-Schauder degree. A special case is the T-periodic problem for the forced pendulum equation with relativistic effects.

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1 Introduction

In this work, we study the existence and multiplicity of T-periodic solutions $x : \mathbb{T} \to \mathbb{R}$ to the following equation with delay on time scales

$$(\varphi(x^{\Delta}(t)))^{\Delta} + h(x(t))x^{\Delta}(t) + g(x(t-r)) = p(t), \quad t \in \mathbb{T}, \tag{1}$$

where $\mathbb{T}$ is an arbitrary T-periodic nonempty closed subset of $\mathbb{R}$ (*time scale*), $\varphi : (-a, a) \to \mathbb{R}$ is an increasing homeomorphism with $0 < a < +\infty$ such that $\varphi(0) = 0$, and $h, g : \mathbb{R} \to \mathbb{R}$ are continuous functions. Moreover, we assume that $r \geq 0$ and $T > 0$ are real numbers, and that $p(t + T) = p(t)$ is continuous in $\mathbb{T}$ with $\overline{p} := \frac{1}{T}\int_0^T p(t)\Delta t = 0$.

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The time scales theory was introduced in 1988, in the PhD thesis of Stefan Hilger [9], as an attempt to unify discrete and continuous calculus. The time scale $\mathbb{R}$ corresponds to the continuous case and, hence, yields results for ordinary differential equations. If the time scale is $\mathbb{Z}$, then the results apply to difference equations. However, the generality of the set $\mathbb{T}$ produces many different situations in which the time scales formalism is useful in several applications. For example, in the study of hybrid discrete-continuous dynamical systems, see [5].

The methods usually employed to explore the existence of periodic solutions for dynamic equations in time scales are: fixed point theory [11, 12], Mawhins continuation theorem [6, 7, 10], lower and upper solutions [17, 18], among many other works. Some of the above cited references correspond to the *semilinear* case, that is, $\varphi(x) = x$ and some others to the *p-laplacian operator*, namely $\varphi_p(x) := |x|^{p-2}x$. For instance, in [6] Cao, Hang and Sun studied, by means of Mawhins continuation theorem, the existence of periodic solutions of

$$(\varphi_p(x^\Delta(t)))^\Delta + f(x(t))x^\Delta(t) + g(x(t)) = e(t), \quad t \in \mathbb{T},$$

with $p > 2$, f, g continuous real functions and $e \in C(\mathbb{T}, \mathbb{R})$ with period $T > 0$. However, the literature concerning singular φ-laplacian operators in time scales is more scarce. A special case of (1) with $\mathbb{T} = \mathbb{R}$ is the forced pendulum equation with relativistic effects, namely,

$$\left(\frac{x'}{\sqrt{1 - \frac{x'^2}{c^2}}} \right)' + kx'(t) + b \sin x = p(t), \quad t \in \mathbb{R}, \tag{2}$$

where $c > 0$ is the speed of light in the vacuum, $k > 0$ is a possible viscous friction coefficient and p is a continuous and T-periodic forcing term with mean value zero. This equation has received much attention by several authors, see e.g. [4, 13, 19]. In particular in [19], employing the Schauder fixed point theorem, Torres proved the existence of at least one T-periodic solution, provided that $2cT \leq 1$. This result was later improved in [20] and finally in [2], where the sharper condition $cT < \sqrt{3}\pi$ was obtained.

In this work, we generalize several aspects of the results in [6] and [19]. On the one hand, our problem consist of dynamical equations on time scales; on the other hand, the functions f, g are general and the equation may also include a delay. This implies that the use of the Poincaré operator does not reduce the problem to a finite-dimensional one, and requires the use of accurate topological methods such as the Leray-Schauder degree. Moreover, our main theorem is in fact a multiplicity result, which intuitively can be motivated as follows. If we observe for example problem (2), it is clear that the periodicity implies that if x is a T-periodic solution, then $x + 2k\pi$ is also a T-periodic solution for all $k \in \mathbb{Z}$. Such solutions are usually called in the literature *geometrically equivalent*. However, if the term kx' is replaced by $h(x)x'$ for some continuous function h close to a constant, then the problem still admits infinitely many solutions, which may be geometrically distinct if h is not a 2π-periodic function.

With this idea in mind, it shall be shown that if the nonlinear term has a more general oscillatory behaviour, then multiple solutions exist.

More specifically, our main result reads as follows:

Theorem 1.1 *Assume that there exists a strictly increasing sequence* $\{\alpha_j\}_{j=0}^n$ *such that*

$$(-1)^j \int_0^T h(x(t))x^\Delta(t) + g(x(t))\Delta t < 0 \text{ if } x(0) = \alpha_j, \ \left\|x^\Delta\right\|_\infty < a.$$

for every j *and each smooth* T*-periodic function* $x(t)$*. Then, for any continuous* T*-periodic function* $p(t)$ *with mean value zero, problem (1) has at least* n *different* T*-periodic solutions.*

In particular, if g is oscillatory over $\mathbb{R}$ and h satisfies some suitable conditions then (1) has infinitely many different T-periodic solutions, provided that the oscillations are sufficiently slow. The proof of the theorem shall be based on the search for fixed points of an appropriate compact operator defined on the Banach space of all continuous T-periodic functions on $\mathbb{T}$. The singular nature of φ shall be of help in the obtention of the required a priori bounds, thus making possible a Leray-Schauder degree approach. We highlight that, in contrast with the continuous case, the treatment of Liénard-like equations on time scales is more delicate because the average of the term $h(x(t))x^\Delta(t)$ with T-periodic x is not necessarily equal to 0. This is due to the fact that the standard chain rule does not hold and, consequently, some extra conditions are required in order to avoid this difficulty.

The paper is organized as follows. In Section 2, we set the notation, terminology, and several preliminary results which will be used throughout this paper. In Section 3, we adapt Mawhin's continuation theorem to the context of times scales in order to prove the existence of at least one T-periodic solution of (1). We remark that this first result can be deduced from the multiplicity result although, for the sake of clarity, this simpler case was analyzed separately. In Section 4, we prove our main theorem with the help of the arguments introduced in the preceding section. Some examples illustrating the results are presented in Section 5.

2 Notation and preliminaries

For fixed $T > 0$, we shall assume that $\mathbb{T}$ is T-periodic, that is, $\mathbb{T} + T = \mathbb{T}$. we denote by $C_T = C_T(\mathbb{T}, \mathbb{R})$ the Banach space of all continuous T-periodic functions on $\mathbb{T}$ endowed with the uniform norm $\|x\|_\infty = \sup_{\mathbb{T}} \|x(t)\| = \sup_{[0,T]_{\mathbb{T}}} \|x(t)\|$ and the closed subspace

$$\tilde{C_T} = \left\{ x \in C_T : \int_0^T x(s)\Delta s = 0 \right\}.$$

For an element $x \in C_T$ its maximum and minimum values shall be denoted respectively by x_M and x_m.

Moreover, denote by $C_T^1 = C_T^1(\mathbb{T}, \mathbb{R})$ the Banach space of all continuous T-periodic functions on $\mathbb{T}$ that are Δ-differentiable with continuous Δ-derivatives, endowed with the usual norm

$$\|x\|_1 = \sup_{[0,T]_{\mathbb{T}}} \|x(t)\| + \sup_{[0,T]_{\mathbb{T}}} \left|x^{\Delta}(t)\right|.$$

In this paper we shall consider T-periodic time scales $\mathbb{T}$ such that $0 \in \mathbb{T}$ and $\mathbb{T} - r \subset \mathbb{T}$.

We introduce the following operators and functions:

- The *Nemytskii operator* $N_f : C_T^1 \to C_T$, given by

$$N_f(z)(t) = f(t, x(t), x^{\Delta}(t), x(t-r)),$$

  where $f : \mathbb{T} \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function;

- The *integration operator* $H : \tilde{C_T} \to C_T^1$,

$$H(z)(t) = \int_0^t z(s)\Delta s,$$

- The continuous linear projectors:

$$Q : C_T \to C_T, \quad Q(x)(t) = \tfrac{1}{T}\int_0^T x(s)\Delta s,$$

$$P : C_T \to C_T, \quad P(x)(t) = x(0)$$

  where, for convenience, we omitted the isomorphism between $\mathbb{R}$ and the subspace of constant functions of C_T.

The above equation (1) can be written as follows:

$$(\varphi(x^{\Delta}(t)))^{\Delta} = f(t, x(t), x^{\Delta}(t), x(t-r)), \quad t \in \mathbb{T}, \tag{3}$$

A function $x \in C_T^1$ is said to be a solution of (3) if $\varphi(x^{\Delta})$ is of class C^1 and verifies $(\varphi(x^{\Delta}(t)))^{\Delta} = f(t, x(t), x^{\Delta}(t), x(t-r))$, for all $t \in \mathbb{T}$.

The following lemma is an adaptation of a result of [3] to time scales.

Lemma 2.1 *For each $x \in C_T$, there exists a unique $Q_\varphi = Q_\varphi(x) \in [x_m, x_M]$ such that*

$$\int_0^T \varphi^{-1}(x(t) - Q_\varphi(x))\Delta t = 0.$$

Moreover, the function $Q_\varphi : C_T \to \mathbb{R}$ is continuous and sends bounded sets into bounded sets.

Proof: Let $x \in C_T$ and define the continuous application $G_x : [x_m, x_M] \to \mathbb{R}$ by

$$G_x(s) = \int_0^T \varphi^{-1}(x(t) - s)\Delta t.$$

We claim that the equation

$$G_x(s) = 0 \tag{4}$$

has a unique solution $Q_\varphi(x)$. Indeed, Let $r, s \in [x_m, x_M]$ be such that

$$\int_0^T \varphi^{-1}(x(t) - r)\Delta t = 0 = \int_0^T \varphi^{-1}(x(t) - s)\Delta t,$$

then using the fact that φ^{-1} is strictly increasing we deduce that $r = s$. Moreover, It is seen that

$$\int_0^T \varphi^{-1}(x(t) - x_M)\Delta t \leq 0 \leq \int_0^T \varphi^{-1}(x(t) - x_m)\Delta t,$$

whence

$$G_x(x_m)G_x(x_M) \leq 0.$$

Thus, there exists $s \in [x_m, x_M]$ such that $G_x(s) = 0$, that is, equation (4) has a unique solution. It follows that function $Q_\varphi : C_T \to \mathbb{R}$ given by $Q_\varphi(x) = s$ is well defined and, furthermore, because $s \in [x_m, x_M]$ we deduce that

$$|Q_\varphi(x)| \leq \|x\|_\infty.$$

Therefore, the function Q_φ sends bounded sets into bounded sets.

Finally, let us verify that Q_φ is continuous on C_T. Let $(x_n)_n \subset C_T$ be a sequence such that $x_n \to x$ in C_T. Since the function Q_φ sends bounded sets into bounded sets, the sequence $(Q_\varphi(x_n))_n$ is bounded in $\mathbb{R}$ and, consequently, without loss of generality we may assume that it converges to some $\tilde{a}$. Because

$$\int_0^T \varphi^{-1}(x_n(t) - Q_\varphi(x_n))\Delta t = 0$$

for all n, by the dominated convergence theorem on time scales [5], we deduce that

$$\int_0^T \varphi^{-1}(x(t) - \widetilde{a})\Delta t = 0,$$

so $Q_\varphi(h) = \widetilde{a}$. Thus, we conclude that the function Q_φ is continuous. $\square$

Now, we define the fixed point operator, which is similar to the one employed in [3] (see also [1] for an elementary introduction). In order to transform problem (3) into a fixed point problem we use the operators H, Q, N_f, P and Lemma 2.1. The proof of this result is similar to the continuous case and shall not repeated here.

Lemma 2.2 *$x \in C_T^1$ is a solution of (3) if and only if x is a fixed point of the operator M_f defined on C_T^1 by*

$$M_f(x) = P(x) + Q(N_f(x)) +$$
$$H\left(\varphi^{-1}\left[H(N_f(x) - Q(N_f(x))) - Q_\varphi(H(N_f(x) - Q(N_f(x))))\right]\right).$$

As the function f is continuous, using the Arzelà-Ascoli theorem it is not difficult to see that M_f is completely continuous.

Using Lemma 2.2, the existence of a T-periodic solution for (3) is reduced to the study of the fixed points of the operator M_f. To this end, we will use topological degree theory.

Consider the following family of problems defined for $\lambda \in [0,1]$:

$$(\varphi(x^{\Delta}(t)))^{\Delta} = \lambda N_f(x)(t) + (1-\lambda)Q(N_f(x)), \tag{5}$$

where the operator N_f is defined by

$$\begin{gathered} N_f(x)(t) = f(t, x(t), x^{\Delta}(t), x(t-r)) := \\ -h(x(t))x^{\Delta}(t) - g(x(t-r)) + p(t), \quad t \in \mathbb{T}. \end{gathered}$$

For each $\lambda \in [0,1]$, consider the nonlinear operator $M(\lambda, \cdot)$, where M is defined on $[0,1] \times C_T^1$ by

$$\begin{aligned} M(\lambda, x) = {} & P(x) + Q(N_f(x)) + \\ & H(\varphi^{-1}\left[\lambda H(N_f(x) - Q(N_f(x))) - Q_{\varphi}(\lambda H(N_f(x) - Q(N_f(x))))\right]). \end{aligned} \tag{6}$$

Observe that $M(1,x) = M_f$ and, similarly as above, it is easy to see that M is completely continuous and, for $\lambda > 0$, the existence of solution to equation (5) is equivalent to the problem

$$x = M(\lambda, x).$$

We claim that the previous assertion is true also for $\lambda = 0$. Indeed, because $Q_{\varphi}(c) = c$ for any constant c, it is clear that $M(0,x) = P(x) + Q(N_f(x))$. If $x = M(0,x)$ then x is constant and $x = P(x)$, that is, $Q(N_f(x)) = 0$ and (5) with $\lambda = 0$ is trivially satisfied. Conversely, if $(\varphi(x^{\Delta}(t)))^{\Delta} \equiv Q(N_f(x))$ then we obtain, upon integration, $\int_0^T Q(N_f(x))\Delta t = 0$ which, in turn, implies that $Q(N_f(x)) = 0$. Thus x^{Δ} is constant and, by periodicity, $x^{\Delta} \equiv 0$, that is, x is constant and, consequently, $x = P(x) = P(x) + Q(N_f(x)) = M(0,x)$.

Remark 2.3 *It is worthy to notice that, for any $\lambda \in [0,1]$, if x is a fixed point of M then $Q(N_f(x)) = 0$.*

3 Existence of periodic solutions

In this section, we establish the existence of at least one $T-$periodic solution to problem (1). Let us denote by deg_B and deg_{LS} the Brouwer and Leray-Schauder degrees respectively.

The following continuation theorem is obtained.

Theorem 3.1 *Assume that Ω is an open bounded set in C_T^1 such that the following conditions hold:*

1. *For each* $\lambda \in (0,1)$ *the problem*

$$(\varphi(x^{\Delta}(t)))^{\Delta} = \lambda N_f(x) \tag{7}$$

has no solution on $\partial\Omega$.

2. *The equation*

$$g(y) = 0,$$

has no solution on $\partial\Omega \cap \mathbb{R}$, *where we consider the natural identification of* $\mathbb{R}$ *with the subspace of constant functions of* C^1_T.

3. *The Brouwer degree of* g *satisfies:*

$$deg_B(g, \Omega \cap \mathbb{R}, 0) \neq 0.$$

Then, for any continuous T*-periodic function* $p(t)$ *with mean value zero, the problem (1) has at least one* $T-$*periodic solution.*

Proof: Let $\lambda \in (0,1]$. If x is a solution of (7), then $Q(N_f(x)) = 0$, hence x is a solution of problem (5). On the other hand, for $\lambda \in (0,1]$, let x be a solution of (5) and since

$$Q\left(\lambda N_f(x) + (1-\lambda)Q(N_f(x))\right) = Q(N_f(x)),$$

we have that $Q(N_f(x)) = 0$, then x is a solution of (7). It follows that, for $\lambda \in (0,1]$, problems (5) and (7) have the same solutions. We assume that (5) has no solutions on $\partial\Omega$ for $\lambda = 1$, since otherwise we are done with the proof. It follows that (5) has no solutions for $(\lambda, x) \in (0,1] \times \partial\Omega$. If x is a solution of (5) for $\lambda = 0$, then we conclude as before that $Q(N_f(x)) = 0$ and $x(t) \equiv b \in \mathbb{R}$. Thus, using the fact that $\int_0^T p(t)\Delta t = 0$

$$0 = \frac{1}{T}\int_0^T f(t,b,0,b)\Delta t = -g(b),$$

which, together with hypothesis 2, implies that $b \notin \partial\Omega$.

Summarizing, we proved that (5) has no solution on $\partial\Omega$ for all $\lambda \in [0,1]$. Thus, for each $\lambda \in [0,1]$, the Leray-Schauder degree $deg_{LS}(I - M(\lambda,\cdot),\Omega,0)$ is well defined and, by the homotopy invariance property,

$$deg_{LS}(I - M(0,\cdot),\Omega,0) = deg_{LS}(I - M(1,\cdot),\Omega,0).$$

On the other hand,

$$deg_{LS}(I - M(0,\cdot),\Omega,0) = deg_{LS}(I - (P + QN_f),\Omega,0).$$

But the range of the mapping

$$z \mapsto P(z) + QN_f(z)$$

is contained in the subspace of constant functions of C^1_T, identified with $\mathbb{R}$. Thus, using the reduction property of the Leray-Schauder degree [8, 14]

$$\begin{aligned} deg_{LS}(I-(P+QN_f),\Omega,0) &= deg_B\left(I-(P+QN_f)\left|_{\overline{\Omega\cap\mathbb{R}}}\right.,\Omega\cap\mathbb{R},0\right) \\ &= deg_B(g,\Omega\cap\mathbb{R},0)\neq 0. \end{aligned}$$

Then, $deg_{LS}(I-M(1,\cdot),\Omega,0)\neq 0$ and, in consequence, there exists $x\in\Omega$ such that $M_f(x)=M(1,x)=x$, which is a solution of (3) and therefore a solution of (1). $\square$

With the help of Theorem 3.1 we shall be able to prove the existence of fixed points of M_f With this aim, for $\lambda\in(0,1)$ we consider the equation

$$\left(\varphi\left(x^{\Delta}(t)\right)\right)^{\Delta}+\lambda h(x(t))x^{\Delta}(t)+\lambda g(x(t-r))=\lambda p(t),\quad t\in\mathbb{T}, \tag{8}$$

which is the explicit expression of problem (7).

For the reader's convenience, we start with the following simple existence theorem which, in fact, can be obtained as a particular case of our main result in the next section:

Theorem 3.2 *Assume there exists $d>0$ such that*

$$\int_0^T[h(x(t))x^{\Delta}(t)+g(x(t))]\Delta t>0 \quad \textit{if}\;\; x_m\geq d,\;\; \left\|x^{\Delta}\right\|_{\infty}<a,$$

$$\int_0^T[h(x(t))x^{\Delta}(t)+g(x(t))]\Delta t<0 \quad \textit{if}\;\; x_M\leq -d,\;\; \left\|x^{\Delta}\right\|_{\infty}<a.$$

Then problem (1) has at least one T-periodic solution.

Proof: Let $\lambda\in(0,1)$. If $x\in C^1_T(\mathbb{T},\mathbb{R})$ is a solution of (8) then on the one hand, it is clear that

$$|x^{\Delta}(s)|<a, \tag{9}$$

for all $s\in\mathbb{T}$.

On the other hand, integrating both sides of (8) from 0 to T and since $\int_0^T p(s)\Delta s=0$, we have that

$$\int_0^T[h(x(t))x^{\Delta}(t)+g(x(t-r))]\Delta t=0.$$

From the periodicity of x, using (8), (9) and the assumptions, we obtain:

$$x_M>-d,\quad x_m<d. \tag{10}$$

By the inequality $x_M\leq x_m+\int_0^T\left|x^{\Delta}(s)\right|\Delta s$ and (10), it follows that

$$x_M<d+\int_0^T\left|x^{\Delta}(s)\right|\Delta s.$$

Analogously, it can be shown that

$$x_m > -\left(d + \int_0^T \left|x^\Delta(s)\right| \Delta s\right).$$

Hence,

$$-\left(d + \int_0^T \left|x^\Delta(s)\right| \Delta s\right) < x_m \le x_M < \left(d + \int_0^T \left|x^\Delta(s)\right| \Delta s\right),$$

and using (9) we deduce that

$$\|x\|_\infty < d + aT. \tag{11}$$

From (9) and (11) it follows that $\|x\|_1 < d + aT + a = d + a(1+T)$.

Let $\rho := d + a(1+T)$. Then using the hypothesis and the fact that $\rho > d$, we see that $g(\rho)g(-\rho) < 0$, which implies that $deg_B(g, \Omega \cap \mathbb{R}, 0) \neq 0$, where $\Omega := B_\rho(0) \subset C_T^1$ and $\Omega \cap \mathbb{R} = (-\rho, \rho)$. Therefore, the conditions $1, 2$ and 3 of Theorem 3.1 are satisfied and the proof is complete. □

The next example shows that the $\int_0^T h(x(t))x^\Delta(t)\Delta t$ is not always equal to zero. This is due to the fact that the standard chain rule does not hold for time scales.

Example 3.3 *Let $\mathbb{T}$ be 3-periodic with $[0,3]_\mathbb{T} = [0,1] \cup \{2,3\}$, let $h(x) = x$, and let $x : \mathbb{T} \to \mathbb{R}$ be the 3-periodic function defined by*

$$x(t) = \begin{cases} t & if \quad 0 \le t \le 1 \\ \\ 2 & if \quad\quad t = 2. \end{cases}$$

It follows by direct computation that $\int_0^3 x(t)x^\Delta(t)\Delta t = -\frac{5}{2}$.

For the next result, let us observe that the set S of right scattered points on $[0,T]_\mathbb{T}$,

$$S = \{t \in [0,T]_\mathbb{T} : \sigma(t) > t\}$$

is countable. For simplicity, we shall assume that the set $L(S)$ of limit points of S is finite. The following corollary is thus obtained:

Corollary 3.4 *Suppose there exists $d > 0$ such that $g(s) > 0$ for $s \ge d$ and $g(s) < 0$ for $s \le -d$. Moreover, assume that h is nonincreasing over $(d, +\infty)$ and nondecreasing over $(-\infty, d)$. If $L(S)$ is finite, then problem (3) has at least one $T-$periodic solution.*

Proof: Let $\mathcal{H} : \mathbb{R} \to \mathbb{R}$ be the continuous function defined by

$$\mathcal{H}(x) = \int_0^x h(s)ds.$$

Assume for example that $x \in C^1_T$ is such that $x_m \geq d$ and $\|x^\Delta\|_\infty < a$, then $g(x(t-\tau)) > 0$ for all t and therefore $\int_0^T g(x(t-\tau))\Delta t > 0$. We claim that $\int_0^T h(x(t))x^\Delta(t)\Delta t \geq 0$ and, consequently, the first condition of the previous theorem is satisfied.

The proof will follow several steps.

Step 1. Let $a < b \in \mathbb{T}$ such that $[a,b] \cap S$ is finite, then

$$\int_a^b h(x(t))x^\Delta(t)\Delta t \geq \mathcal{H}(x(b)) - \mathcal{H}(x(a)).$$

Indeed, write $[a,b] \cap S = \{t_0, \ldots, t_N\}$ with $t_j < t_{j+1}$ and compute

$$\int_{t_0}^{t_N} h(x(t))x^\Delta(t)\Delta t = \sum_{j=0}^{N-1} \left(h(x(t_j))[x(\sigma(t_j)) - x(t_j)] + \int_{\sigma(t_j)}^{t_{j+1}} h(x(t))x^\Delta(t)\Delta t \right)$$

$$= \sum_{j=0}^{N-1} h(x(t_j))[x(\sigma(t_j)) - x(t_j)] + \sum_{j=0}^{N-1} [\mathcal{H}(x(t_{j+1})) - \mathcal{H}(x(\sigma(t_j)))]$$

$$= \sum_{j=0}^{N-1} \Big(h(x(t_j))[x(\sigma(t_j)) - x(t_j)] - [\mathcal{H}(x(\sigma(t_j))) - \mathcal{H}(x(t_j))] \Big) + \mathcal{H}(x(t_N)) - \mathcal{H}(x(t_0))$$

$$= \sum_{j=0}^{N-1} (h(x(t_j)) - h(\xi_j))[x(\sigma(t_j)) - x(t_j)] + \mathcal{H}(x(t_N)) - \mathcal{H}(x(t_0))$$

for some ξ_j between $x(t_j)$ and $x(\sigma(t_j))$. Because h is nonincreasing over the range of x, it follows that $\int_{t_0}^{t_N} h(x(t))x^\Delta(t)\Delta t \geq \mathcal{H}(x(t_N)) - \mathcal{H}(x(t_0))$. Moreover,

$$\int_a^{t_0} h(x(t))x^\Delta(t)\Delta t = \mathcal{H}(x(t_0)) - \mathcal{H}(x(a))$$

and the result follows if $t_N = b$. If otherwise $t_N < b$, then

$$\int_{t_N}^b h(x(t))x^\Delta(t)\Delta t = h(x(t_N))(x(\sigma(t_N)) - x(t_N)) + \mathcal{H}(x(b)) - \mathcal{H}(x(\sigma(t_N)))$$

$$= h(x(t_N))(x(\sigma(t_N))-x(t_N)) - [\mathcal{H}(x(\sigma(t_N))) - \mathcal{H}(x(t_N))] + \mathcal{H}(x(b)) - \mathcal{H}(x(t_N))$$
$$\geq \mathcal{H}(x(b)) - \mathcal{H}(x(t_N))$$

and hence $\int_a^b h(x(t))x^\Delta(t)\Delta t \geq \mathcal{H}(x(b)) - \mathcal{H}(x(a))$.

Step 2. Let $a < b$ be two consecutive limit points of S, then

$$\int_a^b h(x(t))x^\Delta(t)\Delta t \geq \mathcal{H}(x(b)) - \mathcal{H}(x(a)).$$

Indeed, if $[a,b] \cap S$ is finite, then the proof follows from the previous step. Otherwise, there are three possible situations:

1. $(a,b)\cap S=\{t_j\}_{j\in\mathbb{N}_0}$ with $t_j \searrow a$. Then

$$\int_a^{t_0} h(x(t))x^\Delta(t)\Delta t=\sum_{j=1}^{\infty}\int_{t_j}^{t_{j-1}} h(x(t))x^\Delta(t)\Delta t$$

   and, from Step 1,

$$\int_a^{t_0} h(x(t))x^\Delta(t)\Delta t\geq\sum_{j=1}^{\infty}[\mathcal{H}(x(t_{j-1}))-\mathcal{H}(x(t_j))]=\mathcal{H}(x(t_0))-\mathcal{H}(x(a)).$$

   It follows that

$$\begin{aligned}\int_a^{b} h(x(t))x^\Delta(t)\Delta t&\geq\mathcal{H}(x(t_0))-\mathcal{H}(x(a))+\int_{t_0}^{b} h(x(t))x^\Delta(t)\Delta t\\&=\mathcal{H}(x(b))-\mathcal{H}(x(a)).\end{aligned}$$

2. $(a,b)\cap S=\{t_j\}_{j\in\mathbb{N}_0}$ with $t_j \nearrow b$. As before, it is seen that

$$\begin{aligned}\int_{t_0}^{b} h(x(t))x^\Delta(t)\Delta t&=\sum_{j=0}^{\infty}\int_{t_j}^{t_{j+1}} h(x(t))x^\Delta(t)\Delta t\\&\geq\mathcal{H}(x(b))-\mathcal{H}(x(t_0))\end{aligned}$$

   and the proof follows because $\int_a^{t_0} h(x(t))x^\Delta(t)\Delta t=\mathcal{H}(x(t_0))-\mathcal{H}(x(a))$.

3. $(a,b)\cap S=\{t_j\}_{j\in\mathbb{N}_0}\cup\{s_j\}_{j\in\mathbb{N}_0}$ with $t_j\searrow a$, $t_j\nearrow b$ and $t_0=s_0$. As before, we deduce that

$$\int_a^{t_0} h(x(t))x^\Delta(t)\Delta t\geq\mathcal{H}(x(t_0))-\mathcal{H}(x(a)),$$

$$\int_{t_0}^{b} h(x(t))x^\Delta(t)\Delta t\geq\mathcal{H}(x(b))-\mathcal{H}(x(s_0))$$

   and the proof follows.

Step 3.
If $L(S)$ is empty, then S is finite, then by Step 1

$$\int_0^{T} h(x(t))x^\Delta(t)\Delta t\geq\mathcal{H}(x(T))-\mathcal{H}(x(0))=0.$$

Next, suppose $L(S)=\{L_j\}_{j=0,\dots,N}$ with $L_j<L_{j+1}$ for all j. By periodicity we may assume, without loss of generality, that $L_0=0$ and $L_N=T$. It follows from the previous steps that

$$\int_0^{T} h(x(t))x^\Delta(t)\Delta t\geq\sum_{j=0}^{N-1}[\mathcal{H}(x(L_{j+1}))-\mathcal{H}(x(L_j))]=\mathcal{H}(x(T))-\mathcal{H}(x(0))=0$$

and the claim is proved.

In the same way, it is seen that if $x \in C_T$ satisfies $x_M \leq -d$ and $\|x^\Delta\|_\infty < a$, then $\int_0^T g(x(t-\tau))\Delta t + \int_0^T h(x(t))x^\Delta(t)\Delta t < 0$ and so completes the proof. $\square$

Remark 3.5 *If $S = \emptyset$ then the conditions on the function h in Corollary 3.4 are not needed. Obviously, this is also true if h is constant.*

Corollary 3.6 *Assume that*

$$\limsup_{x \to -\infty} g(x) < 0 < \liminf_{x \to +\infty} g(x).$$

Assume, furthermore, that $L(S)$ is finite and there exists $R > 0$ such that

$$h(y) \leq h(x) \qquad \text{for } y \geq x \geq R \text{ or } y \leq x \leq -R.$$

Then problem (3) has at least one T-periodic solution.

Proof: The proof is immediate from the previous corollary. $\square$

Corollary 3.7 *Assume there exists $R > 0$ and constants $\gamma^{\pm} \in \mathbb{R}$ such that*

$$g(x) > a|h(x) + \gamma^+| \quad \text{for } x \geq R, \qquad g(x) < -a|h(x) + \gamma^-| \quad \text{for } x \leq -R$$

Then problem (3) has at least one T-periodic solution.

Proof: Suppose for example that $x_m \geq R$ and $\|x^\Delta\|_\infty < a$, then

$$(h(x(t)) + \gamma^+)x^\Delta(t) + g(x(t)) \geq g(x(t)) - a|h(x(t)) + \gamma^+| > 0.$$

Since $\int_0^T \gamma^+ x^\Delta(t)\Delta t = 0$, we deduce that

$$\int_0^T [h(x(t))x^\Delta(t) + g(x(t))]\Delta t > 0.$$

Similarly, it is verified that if $x_M \leq -R$ then

$$\int_0^T [h(x(t))x^\Delta(t) + g(x(t))]\Delta t < 0$$

and the proof follows from Theorem 3.2. $\square$

Remark 3.8 *In particular, the assumptions in the previous corollary are satisfied when h is constant and $g(x) > 0 > g(-x)$ for $x \geq R$.*

4 Multiplicity of periodic solutions

In this section we establish the existence of at least n different solutions of problem (1). We remark that in the case $n = 1$, the assumptions are more general that those in Theorem 3.2.

Theorem 4.1 *Assume that there exists a strictly increasing sequence $\{\alpha_j\}_{j=0}^n$ such that for all j and $x \in C_T^1$,*

$$(-1)^j \int_0^T [h(x(t))x^\Delta(t) + g(x(t))]\Delta t < 0 \text{ if } x(0) = \alpha_j, \; \left\|x^\Delta\right\|_\infty < a. \tag{12}$$

Then, for any continuous T-periodic function $p(t)$ with mean value zero, problem (1) has at least n different T-periodic solutions.

Proof: Using the same argument as in the proof of Theorem 3.2, we obtain that if $x \in C_T^1(\mathbb{T}, \mathbb{R})$ is a solution of (5) with $\lambda \in [0, 1]$, then $|x^\Delta(t)| < a$ and $x(0) \neq \alpha_j$, for any $j = 0, \ldots, n$. Therefore, problem (5) has no solution in $\partial\Omega_j$ for all $j = 0, \ldots, n - 1$, where

$$\Omega_j := \left\{x \in C_T^1(\mathbb{T}, \mathbb{R}) / x(0) \in (\alpha_j, \alpha_{j+1}), \; \left\|x^\Delta\right\|_\infty < a\right\}.$$

From the homotopy invariance of the Leray-Schauder degree, we obtain

$$\begin{aligned} deg_{LS}(I - M(1,\cdot), \Omega_j, 0) &= deg_{LS}(I - M(0,\cdot), \Omega_j, 0) = \\ &= deg_{LS}(I - (P + QN_f), \Omega_j, 0) = \\ &= deg_B\left(I - (P + QN_f)\Big|_{\overline{\Omega_j \cap \mathbb{R}}}, \Omega_j \cap \mathbb{R}, 0\right) = \\ &= deg_B(g, \Omega_j \cap \mathbb{R}, 0) = \\ &= deg_B(g, (\alpha_j, \alpha_{j+1}), 0). \end{aligned}$$

Moreover, observe that employing condition (12) for $x \equiv \alpha_j$ it is verified that $(-1)^j g(\alpha_j) < 0$ which, in turn, implies that $deg_B(g, (\alpha_j, \alpha_{j+1}), 0) \neq 0$. We conclude that the operator $M(1, \cdot) = M_f$ has a fixed point $x_j \in \Omega_j$. Finally, observe that $x_j(0) \in (\alpha_j, \alpha_{j+1})$ hence all the solutions are different. □

Remark 4.2 *Condition (12) may be replaced by*

$$(-1)^j \int_0^T [h(x(t))x^\Delta(t) + g(x(t))]\Delta t > 0 \text{ if } x(0) = \alpha_j, \; \left\|x^\Delta\right\|_\infty < a.$$

Similarly to Corollary 3.4, condition (12) can be obtained from appropriate explicit assumptions on g and h, provided that the set $L(S)$ of right-scattered points has only finitely many limit points.

Corollary 4.3 *Assume that $L(S)$ is finite and that there exists a strictly increasing sequence $\{\alpha_j\}_{j=0}^n$ such that*

1. *h is nonincreasing and $g(s) > 0$ for $s \in (\alpha_j - \frac{aT}{2}, \alpha_j + \frac{aT}{2})$ and j odd.*

2. *h is nondecreasing and $g(s) < 0$ for $s \in (\alpha_j - \frac{aT}{2}, \alpha_j + a\frac{aT}{2})$ and j even.*

Then, for any continuous T-periodic function $p(t)$ with mean value zero, problem (3) has at least n different T-periodic solutions.

Proof: From the previous proof and the reasoning of Corollary 3.4, it suffices to verify that if $x \in C^1_T(\mathbb{T}, \mathbb{R})$ is a solution of (5) with $\lambda \in [0,1]$ and $x(0) = \alpha_j$, then $x(t) \in (\alpha_j - \frac{aT}{2}, \alpha_j + \frac{aT}{2})$ for all t. To this end, observe that if $|x(t) - \alpha_j| \geq \frac{aT}{2}$ for some $t \in (0, T)_{\mathbb{T}}$, then

$$a\frac{T}{2} \leq |x(t) - \alpha_j| \leq \int_0^t |x^{\Delta}(s)|\Delta s < at,$$

whence $t > \frac{T}{2}$. Due to the periodicity, we also deduce that $T - t > \frac{T}{2}$, a contradiction. □

Remark 4.4 *In particular, the conditions in the previous theorem imply that $\alpha_{j+1} - \alpha_j \geq aT$ for $j = 0, 1, \ldots, n-1$.*

Also, in the spirit of Corollary 3.7 we obtain:

Corollary 4.5 *Assume there exists a strictly increasing sequence $\{\alpha_j\}_{j=0}^n$ and constants γ_j such that*

1. *$g(x) > a|h(x) + \gamma_j|$ for $s \in (\alpha_j - \frac{aT}{2}, \alpha_j + \frac{aT}{2})$ and j odd.*

2. *$g(x) < -a|h(x) + \gamma_j|$ for $s \in (\alpha_j - \frac{aT}{2}, \alpha_j + \frac{aT}{2})$ and j even.*

Then, for any continuous T-periodic function $p(t)$ with mean value zero, problem (3) at least n different T-periodic solutions.

Remark 4.6 *In particular, suppose that g has slow oscillations, that is, there exists a sequence of zeros $x_j \nearrow +\infty$ such that $(-1)^j g(x) > 0$ for $x \in (x_j, x_{j+1})$, with $x_{j+1} - x_j > aT$, then the problem has infinitely many solutions, provided for example that $a|h(x)| < |g(x)|$ in $(\alpha_j - \frac{aT}{2}, \alpha_j + \frac{aT}{2})$ for all j, where $\alpha_j = \frac{x_j + x_{j+1}}{2}$.*

5 Examples

In order to illustrate the above results, we consider some examples.

Example 5.1 *Let us consider the equation*

$$\left(\frac{x^{\Delta}(t)}{\sqrt{1 - x^{\Delta}(t)^2}}\right)^{\Delta} + e^{-x^2(t)}x^{\Delta}(t) + \arctan(x(t)) = \sin(4\pi t) \quad t \in \mathbb{T} \tag{13}$$

where $\mathbb{T}$ is a 1/2-periodic time scale with

$$[0, 1/2]_{\mathbb{T}} = [0, 1/8] \cup \{3/16\} \cup \{1/4\} \cup [5/16, 3/8] \cup [7/16, 1/2]\,.$$

By Corollary 3.4 or Corollary 3.7, we deduce that the problem (13) has at least one 1/2-periodic solution.

Example 5.2 *Let $h : \mathbb{R} \to \mathbb{R}$ be continuous. Let us study the existence of a 2π-periodic solution to the following problem*

$$\left(\frac{x^\Delta(t)}{\sqrt{1-\frac{x^\Delta(t)^2}{c^2}}}\right)^\Delta + h(x(t))x^\Delta(t) + x^3(t-r) = \cos(t), \quad t \in \mathbb{R}, \tag{14}$$

where $c > 0$ and $r \geq 0$. Using Corollaries 3.4 and 3.7, it follows that problem (14) has at least one 2π-periodic solution if for example one of the following assumptions is verified:

1. *$S = \emptyset$.*
2. *There exists $R > 0$ such that*
$$h(y) \leq h(x) \qquad \text{for } y \geq x \geq R \;\text{ or }\; y \leq x \leq -R.$$
3. *$\limsup_{x\to\pm\infty} \left|\frac{h(x)}{x^3}\right| < 1$*

Example 5.3 *Let us consider the relativistic pendulum equation on time scales*

$$\left(\frac{x^\Delta(t)}{\sqrt{1-\frac{x^\Delta(t)^2}{c^2}}}\right)^\Delta + h(x(t))x^\Delta(t) + \sin(x(t)) = p(t), \quad t \in \mathbb{T}, \tag{15}$$

where $h, p : \mathbb{R} \to \mathbb{R}$ are continuous functions and p is T-periodic with mean value zero. If $cT \leq \pi$, then problem (15) has infinitely many T-periodic solutions under one of the following assumptions:

1. *$S = \emptyset$.*
2. *$L(S)$ is finite and $(-1)^{j+1}h$ is nondecreasing in $(\alpha_j - \frac{cT}{2}, \alpha_j + \frac{cT}{2})$, where $\alpha_j = (2j+1)\frac{\pi}{2}$ for $j \in \mathbb{Z}$.*
3. *$c\,|h(x) + \gamma_j| < |\sin(x)|$ for $j \in \mathbb{Z}$ and some constants γ_j.*

Clearly the latter condition is satisfied when h is constant although, in this case, the solutions are not necessarily different in geometric sense (see [19]). It is worth observing that the restriction $cT \leq \pi$, which comes from Remark 4.4, improves the one in the original work by Torres, but it is slightly worse than the one obtained in [20] which, as mentioned in the introduction, reads $cT < 2\sqrt{3} = 3.46\ldots$ However, the method in [20] involves a change of variables that cannot be extended to a general time scale. The better bound given in [2] is easily obtained in the continuous case, due to the Sobolev inequality

$$\|x - \overline{x}\|_\infty^2 \leq \frac{T}{12}\|x'\|_{L^2}^2,$$

valid for T-periodic functions. Indeed, it suffices to observe that, if we replace P by Q in the definition of the operator M in (6) then our main theorem is also *valid, changing $x(0)$ by $\overline{x}$ in condition (12) and the definition of Ω. Thus, any possible solution of (5) satisfying for example $\overline{x} = \frac{\pi}{2}$ verifies $|x(t) - \frac{\pi}{2}| \leq \frac{cT}{2\sqrt{3}}$ for all t. If $cT \leq \sqrt{3}\pi$, then $x(t) \in [0, \pi]$ for all t and*

$$0 = \int_0^T \sin(x(t))\, dt > 0,$$

a contradiction. For a general time scale, the argument is essentially the same and yields the condition $s(\mathbb{T})c\sqrt{T} \leq \frac{\pi}{2}$, where $s(\mathbb{T})$ is the constant of the corresponding Sobolev inequality. We recall that, in the continuous case, the obtention of the value $s(\mathbb{R}) = \frac{T}{12}$ relies on the Fourier series expansion for periodic functions (see e.g. [15]), which should be adapted accordingly to the general context. For example, a rapid computation shows, for arbitrary $\mathbb{T}$, that $s(\mathbb{T}) \leq \frac{T}{4}$ which, applied to this case, retrieves the condition $cT \leq \pi$.

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