EIGENVALUES AND MINIMIZERS FOR A NON-STANDARD GROWTH
NON-LOCAL OPERATOR

ARIEL M. SALORT

Abstract. In this article we study eigenvalues and minimizers of a fractional non-standard growth
problem. We prove several properties on this quantities and their corresponding eigenfunctions.

Contents

1. Introduction 1
2. Preliminary results 4
2.1. Young functions 4
2.2. Fractional Orlicz-Sobolev spaces 5
2.3. The fractional $g$–Laplace operator 6
3. Some useful results on fractional Orlicz-Sobolev spaces 6
3.1. Poincaré’s inequalities 6
3.2. A strong maximum principle for continuous solutions 8
4. The eigenvalue problem 9
4.1. The first eigenvalue 9
5. The minimization problem 12
5.1. Continuity with respect to $\rho$ 13
6. Further properties 15
6.1. Nodal domains 15
6.2. Behaviour of $\alpha_{1,\mu}$ as $s \to 1$ 17
Acknowledgements 17
References 17

1. Introduction

In the last years the eigenvalue problem associated with the $p$–Laplacian operator

$$
\begin{cases}
-\Delta_p u := -\text{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

(1.1)

has received a huge attention, where $\Omega \subset \mathbb{R}^n$ is an open and bounded set and $p > 1$.

Properties on the spectrum of (1.1) and its principal eigenvalue

$$
\lambda_1 := \inf\{\|\nabla u\|_{L^p(\Omega)} : \|u\|_{L^p(\Omega)} = 1\}
$$

have been widely studied and generalized, and a vast bibliography is available. We refer for instance the pioneering works of Anane [2], Allegreto and Huang [1], Lindqvist [18], Anane and Tsouli [3] and references to them. The generalization to homogeneous monotone operators of the form $-\text{div}(a(x, \nabla u))$ has been dealt for instance by Kawohl et al. [13] and Fernández Bonder et al. [9]. The extension to operators involving behaviors more general than powers was treated by several authors: the eigenvalue problem related with the $g$–Laplacean defined as $\Delta_g u = \text{div}(g(|\nabla u|)\nabla u)$, where $g$ is a positive nondecreasing function, was studied by Gossez and Mansevich in [11], García-Huidobro et al. in [10] and Mustonen and Tienari in [20], for instance. In the same spirit, in [19] Montenegro studies a related minimization problem.

2010 Mathematics Subject Classification. 46E30, 35R11, 45G05.

Key words and phrases. Fractional order Sobolev spaces, nonlocal eigenvalues, $g$–laplace operator, nonlocal Hardy inequalities.
Eigenvalue problems have been also treated in nonlocal settings. In [22] Servadai and Valdonoci as well as Kwaśnicki in [16] study the spectrum of different non-local linear operators. In [17] Lindqvist and Lindgren define and study properties of the first eigenvalue of the fractional p–Laplacian
\[ (-\Delta_p)^s u(x) := 2\text{p.v.} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{n+sp}} \, dy \]
where \( s \in (0, 1) \) and \( p > 1 \).

The main aim of this manuscript is to study eigenvalues and minimizers involving the non-local non-linear non-homogeneous operator
\[ (-\Delta_p)^s u := 2\text{p.v.} \int_{\mathbb{R}^n} g(|Du|) \frac{D_x u}{|Du|} \frac{dy}{|x-y|^{n+sp}}, \]
defined in [7], where the \( s \)-Hölder quotient is defined as
\[ D_x u(x, y) = \frac{u(x) - u(y)}{|x-y|^s} \]
Here p.v. stands for in principal value, \( s \in (0, 1) \) is a fractional parameter and \( g \) is a positive non-decreasing function such that \( g = G' \), being \( G \) an function belonging to the so-called Young class (see Section 2 for details) satisfying the growth condition
\[ 1 < p^- \leq \frac{tg(t)}{G(t)} \leq p^+ < \infty \quad \forall t > 0 \]
for some constants \( p^\pm \).

Given an open and bounded set \( \Omega \subset \mathbb{R}^n \) and \( \lambda \in \mathbb{R} \) we consider the problem
\[ \begin{align*}
(-\Delta_p)^s u = \lambda g(|u|) |u| & \quad \text{in } \Omega, \\
u = 0 & \quad \text{on } \mathbb{R}^n \setminus \Omega.
\end{align*} \tag{1.2} \]

In this context we say that \( \lambda \) is an eigenvalue of (1.2) with eigenfunction \( u \) belonging to the fractional Orlicz-Sobolev space \( W^{s,G}_0(\Omega) \setminus \{0\} \) (see Section 2 for details) provided that
\[ \langle (-\Delta_p)^s u, v \rangle := \int_{\mathbb{R}^n} g(|D_x u|) \frac{D_x u}{|D_x u|} D_x v \, d\mu = \lambda \int_\Omega g(|u|) \frac{u}{|u|} v \, dx \tag{1.3} \]
holds for all \( v \in W^{s,G}_0(\Omega) \), where we have denoted the measure \( d\mu(x,y) = \frac{dy}{|x-y|^s} \).

The spectrum \( \Sigma \) is defined as the set
\[ \Sigma := \{ \lambda \in \mathbb{R} : \text{there exists } u \in W_{0}^{s,G}(\Omega) \text{ nontrivial solution to (1.3)} \}. \]
Problem (1.2) is the the Euler-Lagrange equation corresponding to the minimization problem
\[ \alpha_{1,\mu} = \inf_{u \in M_\mu} \frac{F(u)}{G(u)} \quad \text{with} \quad M_\mu = \{ u \in W^{s,G}_0(\Omega) : G(u) = \mu \}, \tag{1.4} \]
where functionals \( F, G : W^{s,G}_0(\Omega) \to \mathbb{R} \) are defined by
\[ F(u) = \int_{\mathbb{R}^n \times \mathbb{R}^n} G(|D_x u|) \, d\mu, \quad G(u) = \int_\Omega G(|u|) \, dx. \tag{1.5} \]

By means of the direct method of the calculus of variations, in Proposition 5.1 it is proved that for each election of \( \mu > 0 \), the minimization problem (1.4) is attained for a function \( u_{1,\mu} \in W^{s,G}_0(\Omega) \). Moreover, since \( F \) and \( G \) are Fréchet differentiable due to Proposition 4.1, by the Lagrange multipliers method, Theorem 4.2 states that there exists a number \( \lambda_{1,\mu} \in \mathbb{R} \) being an eigenvalue of (1.2) with associated eigenfunction \( u_{1,\mu} \), i.e.
\[ \langle (-\Delta_p)^s u_{1,\mu}, v \rangle = \lambda_{1,\mu} \int_\Omega g(|u_{1,\mu}|) \frac{u_{1,\mu}}{|u_{1,\mu}|} v \, dx \quad \forall v \in W^{s,G}_0(\Omega). \tag{1.6} \]
In contrast with $p$–Laplacian type problems, $\alpha_{1,\mu}$ may differ from $\lambda_{1,\mu}$, although both quantities are comparable: in Corollary 5.3 it is proved that there are constants $c_1, c_2 > 0$ independent on $\mu$ such that
\[
0 < c_1 \alpha_{1,\mu} \leq \lambda_{1,\mu} \leq c_2 \alpha_{1,\mu}.
\]
We remark that the number $\alpha_{1,\mu}$ can be seen as the best Poincaré’s constant in $W^{s,G}_0(\Omega)$, and, in general, it is not an eigenvalue. Moreover, the eigenvalue $\lambda_{1,\mu}$ in general does not admit a variational characterization.

Due to the possible lack of homogeneity of (1.2), the numbers $\alpha_{1,\mu}$ and $\lambda_{1,\mu}$ strongly depend on the energy level $\mu$. Therefore, we can consider the less quantities over all possible choices of $\mu$. We define
\[
\lambda_1 = \inf \{ \lambda_{1,\mu} : \mu > 0 \}, \quad \alpha_1 = \inf \{ \lambda_{1,\mu} : \mu > 0 \}.
\]
Since in Proposition 4.5 we prove that $\Sigma$ is a closed set, it is derived in Corollary 4.7 that $\lambda_1$ is in fact an eigenvalue of (1.2). Furthermore, an inequality of the type (1.7) still being true between $\alpha_1$ and $\lambda_1$.

With regard to higher eigenvalues $\lambda$ with continuous sign changing eigenfunction $u$, in Proposition 6.2 it is stated the following relation
\[
p^+\lambda(\Omega) > \lambda_1(\Omega^+), \quad p^+\lambda(\Omega) > \lambda_1(\Omega^-),
\]
where $\Omega^+$ and $\Omega^-$ denote the subset of $\Omega$ where $u > 0$ and $u < 0$, respectively.

An important property the eigenfunctions $u_{1,\mu}$ of $\lambda_{1,\mu}$ is established in Theorem 4.4: it is one-signed in $\Omega$ whenever it is a continuous function.

In an analogous way, one could multiply the right side of (1.2) by a weight function $\rho$. Hence, given a function $\rho$ satisfying
\[
0 < \rho_- \leq \rho(x) \leq \rho_+ < \infty \quad \forall x \in \mathbb{R}^n
\]
for certain constant $\rho_{\pm}$, and $\mu > 0$, one can consider the corresponding quantity $\alpha_{1,\mu}(\rho)$ defined as
\[
\alpha_{1,\mu}(\rho) = \inf_{u \in M_\mu} \frac{\mathcal{F}(u)}{\int_\Omega \rho G(|u|) \, dx}.
\]
In Theorem 5.4 we prove that $\alpha_{1,\mu}(\rho)$ is continuous with respect to $\rho$. Namely, if $\mu > 0$ is fixed and \{\rho_{\varepsilon}\}_{\varepsilon > 0} is a sequence of functions satisfying (1.9) such that $\rho_{\varepsilon} \rightharpoonup \rho_0$ weakly* in $L^\infty(\Omega)$, then it holds that
\[
\lim_{\varepsilon \to 0} \alpha_{1,\mu}(\rho_{\varepsilon}) = \alpha_{1,\mu}(\rho_0).
\]
Moreover, when the family \{\rho_{\varepsilon}\}_{\varepsilon > 0} is $Q$–periodic, being $Q$ the unit cube in $\mathbb{R}^n$, then the rate of the convergence can be estimated. Indeed, in this case $\rho_0 = \frac{1}{Q} \rho$ and in Theorem 5.6 we prove that
\[
|\alpha_{1,\mu}(\rho_{\varepsilon}) - \alpha_{1,\mu}(\rho_0)| \leq C \varepsilon^{sp^+}(\alpha_{1,\mu})^2
\]
where $C$ is a constant independent of $\varepsilon$ and $\mu$.

Finally, in Proposition 6.3, through a $\Gamma$–convergence argument we prove that
\[
\lim_{s \uparrow 1} \alpha_{1,\mu,s} = \alpha_{1,\mu,1} := \inf \left\{ \frac{\Phi_{\tilde{G}}(\nabla u)}{\Phi_G(u)} : u \in W^{1,\tilde{G}}_0(\Omega), \Phi_{\tilde{G}}(u) = \mu \right\}
\]
where we have stressed the dependence on $s$ in $\alpha_{1,\mu,s}$, and here $\tilde{G}$ is a suitable limit Young function explicitly given in terms of $G$. Observe that $\alpha_{1,\mu,1}$ is a minimizer of the well-known local operator $\tilde{g}$–Laplacian, being $\tilde{g} = \tilde{G}'$.

The paper of organized as follows: in Section 2 we introduce the class of Young functions and some useful properties on them as well as the fractional Orlicz-Sobolev spaces. In Section 3 we prove some Poincaré’s type inequalities and maximum principles. Section 4 is devoted to study the eigenvalue problem (1.2) whilst Section 5 is dedicated to treat the corresponding minimizers. Finally, in Section 6 some further results are provided.
2. Preliminary results

In this section we introduce the classes of Young function and fractional Orlicz-Sobolev functions as well as the fractional $g$-Laplacian.

2.1. Young functions. We say that a function $G : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the Young class if it admits the integral formulation $G(t) = \int_0^t g(s)\,ds$, where the right continuous function $g$ defined on $[0, \infty)$ has the following properties:

\begin{align*}
(g_1) & \quad g(0) = 0, \quad g(t) > 0 \text{ for } t > 0, \\
(g_2) & \quad g \text{ is nondecreasing on } (0, \infty), \\
(g_3) & \quad \lim_{t \to \infty} g(t) = \infty.
\end{align*}

From these properties it is easy to see that a Young function $G$ is continuous, nonnegative, strictly increasing and convex on $[0, \infty)$. Without loss of generality $G$ can be normalized such that $G(1) = 1$.

The complementary Young function $G^*$ of a Young function $G$ is defined as

$$G^*(t) = \sup\{tw - G(w) : w > 0\}.$$ 

From this definition the following Young-type inequality holds

$$st \leq G(s) + G^*(t) \quad \text{for all } s, t \geq 0. \quad (2.1)$$

Moreover, it is not hard to see that $G^*$ can be written in terms of the inverse of $g$ as

$$G^*(t) = \int_0^t g^{-1}(s)\,ds, \quad (2.2)$$

see [21, Theorem 2.6.8],

The following growth condition on the Young function $G$ will be assumed

$$1 < p^- \leq \frac{tg(t)}{G(t)} \leq p^+ < \infty \quad \forall t > 0 \quad \text{(L)}$$

where $p^\pm$ are fixed numbers.

The following properties are well-known in the theory of Young function. We refer, for instance, to the books [15] and [21] for an introduction to Young functions and Orlicz spaces, and the proof of these results. See also [8].

Lemma 2.1. Let $G$ be a Young function satisfying (L) and $s, t \geq 0$. Then

\begin{align*}
(G_1) & \quad \min\{s^{p^-}, s^{p^+}\}G(t) \leq G(st) \leq \max\{s^{p^-}, s^{p^+}\}G(t), \\
(G_2) & \quad G(s + t) \leq C(G(s) + G(t)) \quad \text{with } C := 2^{p^+}, \\
(G_3) & \quad G \text{ is Lipschitz continuous: } |G(s) - G(t)| \leq |g(s)||s - t|. \\
\end{align*}

Condition $(G_2)$ is known as the $\Delta_2$ condition or doubling condition and, as it is showed in [15, Theorem 3.4.4], it is equivalent to the right side inequality in (L).

It is easy to see that condition (L) implies that

$$\left(p^+\right)' \leq \frac{t(G^*)'(t)}{G(t)} \leq \left(p^-\right)' \quad \forall t > 0, \quad (G_3')$$

from where it follows that $G^*$ also satisfies the $\Delta_2$ condition.

Lemma 2.2. Let $G$ be a Young function satisfying (L) and $s, t \geq 0$. Then

\begin{align*}
(G_1') & \quad \min\{s^{(p^-)'}, s^{(p^+)'}\}G^*(t) \leq G^*(st) \leq \max\{s^{(p^-)'}, s^{(p^+)'}\}G^*(t), \\
\end{align*}

where $(p^\pm)' = \frac{p^\pm}{p^\pm - 1}$. 


Since $g^{-1}$ is increasing, from (2.2) and (L) it is immediate the following relation.

**Lemma 2.3.** Let $G$ be an Young function satisfying (L) such that $g = G'$ and denote by $G^*$ its complementary function. Then

$$G^*(g(t)) \leq p^+ G(t)$$

holds for any $t \geq 0$.

**Example 2.4.** The family of Young functions includes the following examples.

1. **Powers.** If $g(t) = t^{p-1}$, $p > 1$ then $G(t) = \frac{t^p}{p}$, and $p^\pm = p \pm 1$.
2. **Powers \times logarithms.** Given $b, c > 0$ if $g(t) = t^\log(b + ct)$ then
   $$G(t) = \frac{1}{4c^2} \left( c(t - ct) - 2(b^2 - c^2) \log(b + ct) \right)$$
   and $p^- = 2, p^+ = 3$. In general, if $a, b, c > 0$ and $g(t) = t^\log(b + ct)$ then
   $$G(t) = \frac{t^{1+a}}{(1+a)^2} \left( 2F_1(1 + a, 1, 2 + a, -\frac{at}{b}) + (1 + a) \log(b + ct) - 1 \right)$$
   with $p^- = 1 + a, p^+ = 2 + a$, where $2F_1$ is a hyper-geometric function.
3. **Different powers behavior.** An important example is the family of functions $G$ allowing different power behavior near 0 and infinity. The function $G$ can be considered such that
   $$g \in C^1([0, \infty)), \quad g(t) = c_1 t^{p_1} \text{ for } t \leq s \quad \text{and} \quad g(t) = c_2 t^{p_2} + d \text{ for } t \geq s.$$
   In this case $p^- = 1 + \min\{a_1, a_2\}$ and $p^+ = 1 + \max\{a_1, a_2\}$.
4. **Linear combinations.** If $g_1$ and $g_2$ satisfy (L) then $a_1 g_1 + a_2 g_2$ also satisfies (L) when $a_1, a_2 \geq 0$.
5. **Products.** If $g_1$ and $g_2$ satisfy (L) with constants $p_i^\pm, i = 1, 2$, then $g_1 g_2$ also satisfies (L) with constants $p^- = p_1^- + p_2^- - 1$ and $p^+ = p_1^+ + p_2^+ - 1$.
6. **Compositions.** If $g_1$ and $g_2$ satisfy (L) with constants $p_i^\pm, i = 1, 2$, then $g_1 \circ g_2$ also satisfies (L) with constants $p^- = 1 + (p_1^- - 1)(p_2^- - 1)$ and $p^+ = 1 + (p_1^+ - 1)(p_2^+ - 1)$.

### 2.2. Fractional Orlicz-Sobolev spaces

Given a Young function $G$, a fractional parameter $s \in (0, 1)$ and an open and bounded set $\Omega \subseteq \mathbb{R}^n$, we consider the following spaces:

$$L^G(\Omega) := \{ u : \mathbb{R}^n \to \mathbb{R} \text{ Lebesgue measurable, such that } \Phi_G(u) < \infty \},$$

$$W^{s,G}(\Omega) := \{ u \in L^G(\Omega) \text{ such that } \Phi_{s,G}(u) < \infty \},$$

where the modulars $\Phi_G$ and $\Phi_{s,G}$ are defined as

$$\Phi_G(u) := \int_{\Omega} G(|u(x)|) \, dx, \quad \Phi_{s,G}(u) := \iint_{\mathbb{R}^n \times \mathbb{R}^n} G(|D_s u(x,y)|) \, d\mu,$$

with the $s$–Hölder quotient defined as

$$D_s u(x,y) = \frac{u(x) - u(y)}{|x-y|^s},$$

and $d\mu(x,y) := \frac{dx \, dy}{|x-y|^n}$. These spaces are endowed with the so-called **Luxemburg norms**

$$\|u\|_G := \inf \left\{ \lambda > 0 : \Phi_G \left( \frac{u}{\lambda} \right) \leq 1 \right\}, \quad \|u\|_{s,G} := \|u\|_G + [u]_{s,G},$$

where the $(s,G)$-Gagliardo semi-norm is defined as

$$[u]_{s,G} := \inf \left\{ \lambda > 0 : \Phi_{s,G} \left( \frac{u}{\lambda} \right) \leq 1 \right\}.$$

We also consider the following space

$$W_0^{s,G}(\Omega) := \{ u \in W^{s,G}(\mathbb{R}^n) : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \}.$$

Observe that the following inclusions hold

$$W_0^{s,G}(\Omega) \subset W^{s,G}(\mathbb{R}^n) \subset L^G(\mathbb{R}^n).$$
Hereafter, Ω will always stand for a bounded open set in \( \mathbb{R}^n \) whose diameter is denoted as
\[
d = \text{diam}(\Omega) = \sup \{|x - y| : x, y \in \Omega\}.
\]

We finish this section recalling some useful results on fractional Orlicz-Sobolev spaces.

**Proposition 2.5** ([7], Proposition 2.10). Let \( s \in (0, 1) \) and \( G \) a Young function satisfying (L). Then \( W^{s,G}(\mathbb{R}^n) \) is a reflexive and separable Banach space. Moreover, \( C_\infty^\alpha(\mathbb{R}^n) \) is dense in \( W^{s,G}(\mathbb{R}^n) \).

A variant of the well-known Fréchet-Kolmogorov compactness theorem gives the compactness of the inclusion of \( W^{s,G} \) into \( L^G \).

**Proposition 2.6** ([7], Theorem 3.1). Let \( s \in (0, 1) \) and \( G \) a Young function satisfying (L). Then \( W^{s,G}(\Omega) \in L^G(\Omega) \).

Another useful result regarding strong convergence is the following.

**Proposition 2.7** ([21], Theorem 12). Let \( \{u_n\}_{n \in \mathbb{N}} \) be a sequence in \( L^G \) and \( u \in L^G \). If \( G^* \) satisfies the \( \Delta_2 \) condition, \( \Phi_G(u_n) \to \Phi_G(u) \) and \( u_n \to u \) a.e., then \( u_n \to u \) in the \( L^G \) norm.

Finally we recall that fractional Orlicz-Sobolev spaces are embedded into the usual fractional Sobolev spaces as well.

**Proposition 2.8.** [8, Corollary 2.10] Given \( 0 < t < s < 1 \) and a Young function \( G \) satisfying (L), for any \( q \) such that \( 1 \leq q < p^- \) it holds that \( W^{s,p^-}(\Omega) \subset W^{t,q}(\Omega) \) with continuous inclusion.

As a consequence, since \( W^{0,p^-}(\Omega) \) is continuously embedded into \( C^{0,\alpha}(\Omega) \) for \( \alpha = s - \frac{n}{p^-} > 0 \), see [6, Section 8], we can characterize continuous functions in fractional Orlicz-Sobolev spaces.

**Corollary 2.9.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded and open set and \( s \in (0, 1) \). If \( G \) is a Young function satisfying (L) such that \( s p^- > n \), then \( W^{s,G}(\Omega) \subset C^{0,\alpha}(\Omega) \) with \( \alpha = s - \frac{n}{p^-} \).

### 2.3. The fractional \( g \)-Laplacian operator.

Let \( G \) be a Young function and \( s \in (0, 1) \) be a parameter. The fractional \( g \)-Laplace operator is defined as
\[
(\Delta_g)^s u := 2\text{p.v.} \int_{\mathbb{R}^n} g(|D_s u|) \frac{D_s u}{|D_s u|} \frac{dy}{|x - y|^{n+s}},
\]
where \( \text{p.v.} \) stands for in principal value and \( g = G' \). This operator can be seen as the gradient of the modular \( \Phi_{s,G}(u) \) and is well defined between \( W^{s,G}(\mathbb{R}^n) \) and its dual space \( W^{-s,G'}(\mathbb{R}^n) \). In fact, in [7, Theorem 6.12] the following representation formula is provided
\[
\langle (\Delta_g)^s u, v \rangle = \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} g(|D_s u|) \frac{D_s u}{|D_s u|} D_s v d\mu,
\]
for any \( v \in W^{s,G}(\mathbb{R}^n) \).

### 3. Some useful results on fractional Orlicz-Sobolev spaces

In this section we probe two Poincaré’s inequalities and a maximum principle in the context of nonlocal Orlicz-Sobolev spaces.

#### 3.1. Poincaré’s inequalities.

We start this section proving a modular inequality for small cubes. We will denote \( (u)_Q \) the average of \( u \) on \( Q \).

**Lemma 3.1.** Let \( Q \) be the unit cube in \( \mathbb{R}^n \), \( n \geq 1 \) and let \( G \) be a Young function. Then, for every \( u \in W^{s,G}(Q_\varepsilon) \) we have that
\[
\int_{Q_\varepsilon} G(|u - (u)_Q|) dx \leq c \varepsilon^{sp^+} \int_{Q_\varepsilon} G(|D_s u|) d\mu
\]
where \( 0 < \varepsilon \leq 1 \), \( Q_\varepsilon = \varepsilon Q \) and \( c \) is a constant depending only on \( n \).
Proof. Given \( u \in W^{s,G}(Q_\varepsilon) \), by using Jensen’s inequality it follows that
\[
\int_{Q_\varepsilon} G(|u - (u)_{Q_\varepsilon}|) dx = \int_{Q_\varepsilon} G \left( \int_{Q_\varepsilon} (u(x) - u(y)) dy \right) dx \\
\leq \int_{Q_\varepsilon} \int_{Q_\varepsilon} G(|u(x) - u(y)|) dy dx \\
\leq \varepsilon^{sp^+} \int_{Q_\varepsilon} \int_{Q_\varepsilon} G \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) dx dy
\]
from where the result follows. \( \square \)

The following Poincaré’s inequality for modulars in \( W^{s,G}_0(\Omega) \) gives as a consequence that \( \cdot \) is an equivalent norm in \( W^{s,G}_0(\Omega) \).

Proposition 3.2. Let \( \Omega \subset \mathbb{R}^n \) be open and bounded and let \( G \) be a Young function satisfying \((L)\). Then for \( s \in (0,1) \) it holds that
\[
\Phi_G(u) \leq \Phi_{s,G}(C_p d^s u)
\]
for all \( u \in W^{s,G}_0(\Omega) \), where \( C_p = \left( \frac{sp^+}{n\omega_n} \right)^{1/p} \) with \( \omega_n \) standing for the volume of the unit ball in \( \mathbb{R}^n \).

Proof. Let \( C_p \) be a positive constant to determine. Given \( x \in \Omega \), observe that when \( |x - y| \geq d \), then \( y \not\in \Omega \). Hence, by using \((G_1)\) we get
\[
\Phi_{s,G}(C_p d^s u) \geq \int_{\Omega} \int_{|x-y| \geq d} G \left( \frac{d^s}{|x-y|^s} C_p |u(x)| \right) dy dx \\
\geq C_p^{-d^s} \int_{|x| \geq d} \left( \int_{|z| \geq d} \frac{dz}{|x-y|^{n+sp^+}} \right) G(|u(x)|) dx
\]
since, without loss of generality we can assume that \( C_p \geq 1 \).

Now, by using polar coordinates we have that
\[
\int_{|z| \geq d} \frac{dz}{|x-y|^{n+sp^+}} = \frac{n\omega_n}{sp^+ d^{-sp^+}}
\]
and the result follows choosing properly the constant \( C_p \). \( \square \)

As a direct implication we obtain an inequality for norms.

Corollary 3.3. Under the same assumptions than in Theorem 3.2, it holds that
\[
\|u\|_G \leq C_p d^s [u]_{s,G}
\]
for every \( s \in (0,1) \) and \( u \in W^{s,G}_0(\Omega) \).

Proof. Given \( u \in W^{s,G}_0(\Omega) \), applying Theorem 3.2 to the function \( u/C_p d^s [u]_{s,G} \), we get
\[
\Phi_G \left( \frac{u}{C_p d^s [u]_{s,G}} \right) \leq \Phi_{s,G} \left( \frac{u}{[u]_{s,G}} \right) = 1
\]
by definition of the Luxemburg’s norm. Consequently,
\[
\|u\|_G = \inf \{ \lambda : \Phi_G \left( \frac{u}{\lambda} \right) \leq 1 \} \leq C_p d^s [u]_{s,G}
\]
as desired. \( \square \)

Condition \((G_1)\) on Proposition 3.2 gives the following inequality.
Corollary 3.4. Under the same assumptions than in Theorem 3.2, it holds that
\[ \Phi_G(u) \leq C \max\{d^{sp^+}, d^{sp^-}\} \Phi_{x,G}(u) \]
for all \( s \in (0, 1) \) and \( u \in W_0^{s,G}(\Omega) \), where \( C = C(s, a, p^\pm) \).

3.2. A strong maximum principle for continuous solutions. In order to define the main result in this paragraph it is convenient to define the notion of weak and viscosity solutions in our settings. Given an open and bounded set \( \Omega \subset \mathbb{R}^n \) and \( f \in L^{G^*}(\Omega) \), consider the following Dirichlet equation
\[ \begin{cases} (-\Delta_g)^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \]

We say that \( u \in W_0^{s,G}(\Omega) \) is a \textit{weak sub-solution (super-solution)} to (3.1) if
\[ \langle (-\Delta_g)^s u, v \rangle \leq (\geq) \int_\Omega f(v) \quad \text{for all non-negative } v \in W_0^{s,G}(\Omega). \]
If \( u \) is simultaneously a weak super- and sub-solution, then we say that \( u \) is a \textit{weak solution} to (3.1).

We say that an upper (lower) semi-continuous function \( u \) such that \( u \leq 0 \) (\( u \geq 0 \)) in \( \mathbb{R}^n \setminus \Omega \) is a \textit{viscosity sub-solution (super-solution)} to (3.1) if whenever \( x_0 \in \Omega \) and \( \varphi \in C^1_c(\mathbb{R}^n) \) are such that
\[ (i) \quad \varphi(x_0) = u(x_0), \quad (ii) \quad u(x) \leq (\geq) \varphi(x) \quad \text{for } x \neq x_0 \]
then \((-\Delta_g)^s \varphi(x_0) \leq (\geq) f(x_0)\).

Finally, a continuous function \( u \) is a \textit{viscosity solution} to (3.1) if it is a viscosity super-solution and a viscosity sub-solution.

Remark 3.5. Since \((-\Delta_g)^s(\varphi + C) = (-\Delta_g)^s \varphi\), the previous definitions are equivalent if the function \( \varphi(x) + C \) (or \( \psi(x) - C \)) touches \( u \) from below (from above, respectively) at \( x_0 \).

Furthermore, in the previous definitions we may assume that the test function touches \( u \) strictly. Indeed, for a test function \( \varphi \) touching \( u \) from below, consider the function \( h(x) = \varphi(x) - \eta(x) \), where \( \eta \in C^\infty(\mathbb{R}^n) \) satisfies \( \eta(x_0) = 0 \) and \( \eta(x) > 0 \) for \( x \neq x_0 \). Notice that \( h \) touches \( u \) strictly. Moreover, since the function \( g \) is increasing it holds that \((-\Delta_g)^s h(x_0) \leq (-\Delta_g)^s \varphi(x_0)\). For further details about general theory of viscosity solutions we refer, for instance, to the classical monographs [4, 12].

The theory of viscosity solutions is based on a point-wise testing: by [8, Lemma 2.17], \((-\Delta_g)^s\) is well defined point-wisely for any test function \( \varphi \in C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) and for every \( x \in \mathbb{R}^n \) provided that
\[ (1 - s)p^- > 1. \]
Moreover, in light of Corollary 2.9, weak solutions are continuous when
\[ sp^- > n. \]
Therefore, in order to deal with viscosity solutions coming from continuous weak solutions, we will impose a lower bound for the growth of the Young function \( G \) satisfying (L), namely,
\[ (S) \quad p^- > \frac{n}{s(1 - s)}. \]
Under this assumptions, weak and viscosity solutions can be related.

Proposition 3.6. [8, Lemma 3.7] Let \( \Omega \subset \mathbb{R}^n \) be open and bounded and let \( G \) be a Young function satisfying (L) and (S). Then, a weak solution \( u \in W_0^{s,G}(\Omega) \) of (3.1) is a viscosity solution of (3.1).

We state the following weak maximum principle.

Proposition 3.7. Let \( \Omega \subset \mathbb{R}^n \) be open and bounded and let \( G \) be a Young function. Then, if \( f \geq 0 \) in \( \Omega \), then a weak solution \( u \in W_0^{s,G}(\Omega) \) of \((-\Delta_g)^s u = 0\) satisfies \( u \geq 0 \) in \( \Omega \).
Proof. Let \( u^+ := \max\{u, 0\} \) and \( u^- := \max\{-u, 0\} \) be the positive and negative parts of \( u \), respectively. Testing with \( u^- \in W_0^{s,G}(\Omega) \) we have
\[
0 \leq \int_{\Omega} f u^- = \int_{\mathbb{R}^n} g(|D_s u|) \frac{|D_s u|}{|D_s u|} D_s u^- \, d\mu
\]
\[
= \int_{\mathbb{R}^n} D_s u D_s u^- \left( \int_0^1 g((1-t)|D_s u|) \, dt \right) \, d\mu.
\]
Observe that
\[
D_s u(x, y) D_s u^-(x, y) = -\frac{u(x)u^-(y) + u(y)w^-(x)}{|x-y|^s} - (D_s u^-(x, y))^2
\]
\[
\leq -(D_s u^-(x, y))^2.
\]
Since \( g' \geq 0 \) and \( g'(t) = 0 \) \( if \) and only if \( t = 0 \), from the last two relations we get that \( u^- \equiv 0 \) in \( \Omega \). \( \square \)

Moreover, the following strong maximum principle for continuous functions holds.

Proposition 3.8. Let \( \Omega \in \mathbb{R}^n \) be open and bounded and assume that \( G \) fulfills \((L)\) and \((S)\). If \( u \in W_0^{s,G}(\Omega) \) is weak super-solution of \((-\Delta_g)^s u = 0 \) in \( \Omega \), then either \( u > 0 \) in \( \Omega \) or \( u \equiv 0 \).

Proof. Let \( u \in W_0^{s,G}(\Omega) \) be a weak super-solution of \((-\Delta_g)^s u = 0 \) in \( \Omega \). In light of condition \((S)\), from Proposition 3.6 we have that \( u \) is also a viscosity super-solution of the same equation.

Let \( x_0 \in \Omega \) be a point where \( u(x_0) = 0 \). By definition of viscosity super-solution, for any test function \( \varphi \in C_c^1(\mathbb{R}^n) \) such that
\[
0 = u(x_0) = \varphi(x_0), \quad \varphi(x) < u(x) \text{ if } x \neq x_0
\]
it holds that
\[
0 \geq -(-\Delta_g)^s \varphi(x_0) = 2p.v. \int_{\mathbb{R}^n} \frac{|\varphi(y)|}{|x_0 - y|^s} \frac{\varphi(y)}{|\varphi(y)|} \frac{d\mu}{|x_0 - y|^{n+s}}.
\]
If \( \varphi \geq 0 \) then it follows that \( \varphi \equiv 0 \), from where \( u \equiv 0 \) in \( \mathbb{R}^n \).

If \( u \neq 0 \), by using the continuity of \( u \), we can select a test function \( \varphi \) such that \( 0 \leq \varphi \leq u \) which is positive at some point. Consequently \( u \equiv 0 \) or \( u > 0 \) in \( \Omega \). \( \square \)

4. The eigenvalue problem

4.1. The first eigenvalue. In this section we prove the existence of the eigenvalue \( \lambda_{1,\mu} \) for each \( \mu > 0 \) according to definition \((1.6)\), as well as some properties on it and its eigenfunction.

Proposition 4.1. Let \( G \) be a Young function satisfying \((L)\). Then, the functional \( \mathcal{F}, \mathcal{G} : W_0^{s,G}(\Omega) \to \mathbb{R} \)
defined in \((1.5)\) are class \( C^1 \) and their Fréchet derivatives \( \mathcal{F}', \mathcal{G}' : W_0^{s,G}(\Omega) \to (W_0^{s,G}(\Omega))^* \) satisfy
\[
\langle \mathcal{F}'(u), v \rangle = \langle (-\Delta_g)^s u, v \rangle, \quad \langle \mathcal{G}'(u), v \rangle = \int_{\Omega} g(|u|) \frac{u}{|u|} v \, dx
\]
for \( u, v \in W_0^{s,G}(\Omega) \).

Proof. For \( u, v \in W_0^{s,G}(\Omega) \) and \( t > 0 \) we compute
\[
\frac{\mathcal{F}(u + tv) - \mathcal{F}(v)}{t} = \int_{\mathbb{R}^n} \left( \frac{1}{t} \int_{|D_s u|}^{[D_s u + tD_s v]} g(s) \, ds \right) \, d\mu.
\]
As \( t \to 0 \), \( D_s u + tD_s v \to D_s u \) almost everywhere. Now, since \( g \) is increasing, for \( t \) small we get
\[
\left| \frac{1}{t} \int_{|D_s u|}^{[D_s u + D_s v]} g(s) \, ds \right| \leq g(|D_s u|) \leq g(|D_s u| + |D_s v|) |D_s v|.
\]
We claim that \( g(|D_s w|) \in L^{G^*}(\mathbb{R}^{2n}, d\mu) \) for all \( w \in W^{r, G}_0(\Omega) \). Indeed, by using (2.2), (L) and the fact that \( g^{-1} \) is increasing we obtain that

\[
\int_{\mathbb{R}^n \times \mathbb{R}^n} G^* (|g(|D_s w|)|) d\mu = \int_{\mathbb{R}^n \times \mathbb{R}^n} \left( \int_0^{g(|D_s w|)} g^{-1}(s) \, ds \right) d\mu \\
\leq \int_{\mathbb{R}^n \times \mathbb{R}^n} g^{-1}(g(|D_s w|)) g(|D_s w|) \, d\mu \\
\leq p^+ \int_{\mathbb{R}^n \times \mathbb{R}^n} G(|D_s w|) \, d\mu \\
= p^+ \Phi_s, G(w).
\]

Then, \( g(|D_s u| + |D_s v|) \in L^{G^*}(\mathbb{R}^{2n}, d\mu) \), and using (2.1), we get that

\[
\int_{\mathbb{R}^n \times \mathbb{R}^n} g(|D_s u| + |D_s v|) D_s v \, d\mu < \infty.
\]

Thus, by the dominated convergence theorem,

\[
\langle \mathcal{F}'(u), v \rangle = \lim_{t \to 0} \frac{\mathcal{F}(u + tv) - \mathcal{F}(u)}{t} = \frac{d}{dt} \mathcal{F}(u + tv) \bigg|_{t=0} \\
= \int_{\mathbb{R}^n \times \mathbb{R}^n} g(|D_s u|) \frac{D_s u}{|D_s u|} D_s v \, d\mu \\
= \langle (-\Delta_g)^s u, v \rangle.
\]

Now, let us see that \( \mathcal{F}' \) is continuous. Let \( \{u_j\}_{j \in \mathbb{N}} \subset W^{r, G}_0(\Omega) \) be a such that \( u_j \to u \) and observe that

\[
|\langle \mathcal{F}'(u_j) - \mathcal{F}'(u), v \rangle| = \left| \int_{\mathbb{R}^n \times \mathbb{R}^n} \left( g(|D_s u|) \frac{D_s u}{|D_s u|} - g(|D_s u_j|) \frac{D_s u_j}{|D_s u_j|} \right) D_s v \, d\mu \right|
\]

then, by Egoroff’s Theorem, there exists a positive sequence \( \delta_j \to 0 \) such that

\[
\sup_{\|v\|_{G} \leq 1} \int_{\mathbb{R}^n \times \mathbb{R}^n} \left( g(|D_s u|) \frac{D_s u}{|D_s u|} - g(|D_s u_j|) \frac{D_s u_j}{|D_s u_j|} \right) D_s v \, d\mu \\
\leq \left\| g(|D_s u|) \frac{D_s u}{|D_s u|} - g(|D_s u_j|) \frac{D_s u_j}{|D_s u_j|} \right\|_{L^{G^*}(\mathbb{R}^{2n}, d\mu)} + \delta_k,
\]

where we have used the Hölder’s inequality for Orlicz spaces (see [15, Theorem 3.3.8]). Now, since \( G^* \) satisfies \( (G_1^*) \), by Proposition 2.7 we get

\[
\|g(D_s u) - g(D_s u_k)\|_{L^{G^*}(\mathbb{R}^{2n}, d\mu)} \to 0,
\]

and therefore \( \|\mathcal{F}'(u_n) - \mathcal{F}'(u)\|_{(W^{r, G}_0(\Omega))'} \to 0 \) as required.

A similar reasoning allow us to claim that \( \mathcal{G} \in C^1 \) and

\[
\lim_{t \to 0} \frac{\mathcal{G}(u + tv) - \mathcal{G}(v)}{t} = \frac{d}{dt} \Phi_{G}(u + tv) \bigg|_{t=0} = \int_{\Omega} g(|u|) \frac{u}{|u|} v
\]

and the proof concludes. \( \square \)

As a consequence, we get the eigenvalue existence.

**Theorem 4.2.** Let \( G \) be a Young function satisfying (L). Then, for every \( \mu > 0 \) there exists a positive eigenvalue \( \lambda_{1, \mu} \) of (1.2) with non-negative eigenfunction \( u_{1, \mu} \in W^{r, G}_0(\Omega) \) such that \( \mathcal{G}(u_{1, \mu}) = \mu \).
Moreover, \( \lambda_{1, \mu} \) is bounded by below independently of \( \mu \).
Proof. Given a fixed value of $\mu > 0$, in light of Proposition 5.1 there exists a function $u_{1,\mu} \in W_0^{s,G}(\Omega)$ attaining the minimum in (1.4). In view of Proposition 4.1, from the Lagrange multiplier rule there exists $\lambda_{1,\mu}$ such that the constraint $G(u_{1,\mu}) = \mu$ is satisfied and

$$\langle (-\Delta_g)^s u_{1,\mu}, v \rangle = \lambda_{1,\mu} \int_{\Omega} g(|u_{1,\mu}|) \frac{|u_{1,\mu}|}{|u_{1,\mu}|} v \quad \forall v \in W_0^{s,G}(\Omega).$$

Choosing $v = u_{1,\mu}$ in the last expression, we obtain that $\lambda_{1,\mu} > 0$.

By definition, $u_{1,\mu}$ realizes the infimum in the expression of $\alpha_{1,\mu}$ defined in (1.4). Since the functionals $F$ and $G$ are invariant by replacing $u_{1,\mu}$ with $|u_{1,\mu}|$ we may assume that $u_{1,\mu}$ is one-signed in $\Omega$.

Finally, from (L) and Proposition 3.2 we get

$$\lambda_{1,\mu} = \langle (-\Delta_g)^s u_{1,\mu}, u_{1,\mu} \rangle \geq \frac{p^+}{p^+} \lambda_{1,\mu} \geq C \min\{d^{sp^+}_-, d^{-sp^-}_-\} > 0$$

where $C$ depends only on $s$, $n$ and $p^\pm$. \hfill $\square$

Corollary 4.3. Given a Young function satisfying (L), the quantity $\lambda_1$ defined in (1.8) is strictly positive. More precisely,

$$\lambda_1 \geq C \min\{d^{sp^+}_-, d^{-sp^-}_-\} > 0$$

where $C$ depends only on $s$, $n$ and $p^\pm$.

Theorem 4.2 asserts that an eigenfunction of $\lambda_{1,\mu}$ is non-negative in $\Omega$. The following result claims that in fact, it is positive in $\Omega$ whenever it is a continuous function.

Theorem 4.4. Let $\Omega$ be open and bounded and let $G$ be a Young function satisfying (L) and (S). Then any eigenfunction of $\lambda_{1,\mu}$, $\mu > 0$, has constant sign in $\Omega$.

Proof. Fixed $\mu > 0$, let $(\lambda_{1,\mu}, v_{1,\mu})$ be an eigenpair of (1.2), i.e., $v_{1,\mu} \in W_0^{s,G}(\Omega)$ is such that

$$\langle (-\Delta_g)^s u_{1,\mu}, v \rangle = \lambda_{1,\mu} \int_{\Omega} g(|u_{1,\mu}|) v \quad \forall v \in W_0^{s,G}(\Omega).$$

Since Theorem 4.2 gives that $\lambda_{1,\mu} > 0$ and $u_{1,\mu}$ is non-negative, we get

$$\langle (-\Delta_g)^s u_{1,\mu}, v \rangle \geq 0 \quad \forall v \in W_0^{s,G}(\Omega),$$

i.e., $u_{1,\mu}$ is a weak super-solution of $(-\Delta_g)^s u = 0$. Therefore, since $u_{1,\mu}$ is not non-trivial the result follows in light of Proposition 3.8. \hfill $\square$

Finally, we prove that $\Sigma$ is closed, from where we deduce that $\lambda_1$ is an eigenvalue of (1.2) as well.

Proposition 4.5. The spectrum of (1.2) is closed.

Proof. Let $\lambda_j \in \Sigma$ be such that $\lambda_j \to \lambda$ and let $u_j \in W_0^{s,G}(\Omega)$ be an eigenfunction associated to $\lambda_j$, i.e.,

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} g(|D_su_j|) \frac{D_su_{ij}}{|D_su_{ij}|} D_s v d\mu = \lambda_j \int_{\Omega} g(|u_j|) \frac{u_j}{|u_j|} v \quad \forall v \in W_0^{s,G}(\Omega).$$

By Proposition 2.6, up to a subsequence, there exists $u \in W_0^{s,G}(\Omega)$ such that

$$u_j \to u \quad \text{weakly in } W_0^{s,G}(\Omega),$$

$$u_j \to u \quad \text{strongly in } L^G(\Omega)$$

$$u_j \to u \quad \text{a.e. in } \mathbb{R}^n$$

as a consequence,

$$G(D_su_j) \to G(D_su) \quad \text{a.e. in } \Omega.$$

Observe that from Lemma 2.3 and (L) we have that

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} G^*(g(|D_s u_j|)) d\mu \leq p^+ \Phi, G(u_j) \leq \frac{p^+}{p} \lambda_j \int_{\Omega} g(|u_j|) |u_j| \leq \frac{(p^+)^2}{p} \lambda \Phi(u).$$
for \( j \) big enough. So, we can assume that \( g(|D_s u_j|) \frac{D_s u_j}{|D_s u_j|} \rightarrow \eta \) weakly in \( L^G(\mathbb{R}^n, d\mu) \). Again, from (4.2) we get
\[
g(|D_s u_j|) \frac{D_s u_j}{|D_s u_j|} \rightarrow g(|D_s u|) \frac{D_s u}{|D_s u|} \quad \text{a.e. in } \Omega
\]
and hence, taking limit as \( j \rightarrow \infty \) in (4.1) we can assume that \( \eta = g(|D_s u|) \frac{D_s u}{|D_s u|} \) a.e., consequently
\[
\int_{\mathbb{R}^n} g(|D_s u|) \frac{D_s u}{|D_s u|} D_s v \, d\mu = \lambda \int_{\Omega} g(|u|) \frac{|u|}{|u|} v 
\quad \text{for all } v \in W_0^{s,G}(\Omega)
\]
from where the proof concludes. \( \square \)

Remark 4.6. In contrast with the local case, the point-wise convergence of the \( s \)-Hölder quotients \( D_s u_j \) simplifies considerably the proof, not being necessarily the deal with the monotonicity of the operator. See [9] for details.

**Corollary 4.7.** The number \( \lambda_1 \) defined in (1.8) is an eigenvalue of (1.2).

## 5. The minimization problem

In this section we study the minimization problem (1.4) related to the Euler-Lagrange equation (1.2).

**Proposition 5.1.** Let \( G \) be a Young function satisfying (L). Then, the minimization problem (1.4) has a solution \( \alpha_{1, \mu} \) for each \( \mu > 0 \).

**Proof.** Let \( \{u_j\}_{j \in \mathbb{N}} \subset M_\mu \) be a minimizing sequence for \( \alpha_{1, \mu} \), i.e., \( \Phi_{s,G}(u_j) = \mu \) and
\[
\Phi_{s,G}(u_j) \rightarrow \mu \alpha_{1, \mu} \quad \text{as } j \rightarrow \infty.
\]
Let us see that \( \|u_j\|_{s,G} \) is bounded independently of \( j \). If \( \|u_j\|_{s,G} \leq 1 \) there is nothing to prove. Assume that \( \|u_j\|_{s,G} \geq 1 + \varepsilon \) for some \( \varepsilon > 0 \), then by using (G1) we obtain that
\[
\Phi_{s,G}(u_j) \geq \Phi_{s,G}\left( \frac{(1 + \varepsilon)u_j}{\|u_j\|_{s,G}} \right) \frac{\|u_j\|_{s,G}}{1 + \varepsilon} \left( \frac{\|u_j\|_{s,G}}{1 + \varepsilon} \right)^{p^-} = \Phi_{s,G}\left( \frac{u_j}{\|u_j\|_{s,G}} \right) \frac{\|u_j\|_{s,G}}{1 + \varepsilon} \left( \frac{\|u_j\|_{s,G}}{1 + \varepsilon} \right)^{p^-},
\]
where the last equality follows from the definition of the Luxemburg norm. Hence, when \( \|u_j\|_{s,G} > 1 \) the sequence \( \{u_j\}_{j \in \mathbb{N}} \subset M_\mu \) is uniformly bounded for \( j \) large enough:
\[
\|u_j\|_{s,G} \leq (\Phi_{s,G}(u_j))^{\frac{1}{p^-}} < (\mu \alpha_{1, \mu})^{\frac{1}{p^-}}.
\]
Then, by Proposition 2.6, up to a subsequence, there exists \( u \in W_0^{s,G}(\Omega) \) such that
\[
u_j \rightarrow u \quad \text{weakly in } W_0^{s,G}(\Omega),
\]
\[
u_j \rightarrow u \quad \text{strongly in } L^G(\Omega) \quad \text{and a.e. in } \Omega,
\]
from where \( G(u) = \mu \) and then \( u \in M_\mu \).

Now, since the application \( u \mapsto \Phi_{s,G}(u) \) is lower semi-continuous due to the convexity of the modular, by the Fatou’s lemma we get
\[
\Phi_{s,G}(u) \leq \liminf_{j \rightarrow \infty} \Phi_{s,G}(u_j) = \mu \alpha_{1, \mu}.
\]
Since by definition \( \mu \alpha_{1, \mu} \leq \Phi_{s,G}(u) \), the result follows. \( \square \)

As a consequence of the Poincaré’s inequality, namely, Corollary 3.4, we obtain the following.

**Proposition 5.2.** The number \( \alpha_{1, \mu} \) is strictly positive. Moreover,
\[
\alpha_{1, \mu} \geq C \min\{d^{-sp^-}, d^{-sp^+}\} > 0
\]
where \( C = C(n, s, p^\pm) \).
As a direct consequence (L) and Proposition 5.2 we get the following result.

**Corollary 5.3.** The quantities \( \lambda_1 \) and \( \alpha_1 \) defined in (1.8) are strictly positive and comparable. More precisely,

\[
C \min \{d^{-sp^\cdot}, d^{-sp^+}\} \leq \frac{p^-}{p^+} \alpha_1 \leq \frac{p^+}{p^-} \alpha_1
\]

where \( C = C(n, s, p^\pm) \).

5.1. **Continuity with respect to \( \rho \).** In this subsection we prove continuity of the numbers \( \alpha_{1,\mu}(\rho) \) defined in (1.10) with respect to \( \rho \), and in the case of periodic weights we obtain estimates on the rate of convergence.

Without any additional assumption on the weight functions we prove the following.

**Theorem 5.4.** Let \( \{\rho_\varepsilon\}_{\varepsilon > 0} \) be a sequence of functions satisfying (1.9) such that \( \rho_\varepsilon \to \rho_0 \) weakly* in \( L^\infty(\Omega) \). Let \( \alpha_{1,\mu}(\rho_\varepsilon) \) and \( \alpha_{1,\mu}(\rho_0) \) be the numbers defined in (1.10). Then

\[
\lim_{\varepsilon \to 0} \alpha_{1,\mu}(\rho_\varepsilon) = \alpha_{1,\mu}(\rho_0).
\]

The proof is based in the following convergence result.

**Lemma 5.5.** Let \( \Omega \subset \mathbb{R}^n \) be an open and bounded domain and \( G \) a Young function. Let \( \{\rho_\varepsilon\}_{\varepsilon > 0} \) be a sequence of functions satisfying (1.9) such that \( \rho_\varepsilon \to \rho_0 \) weakly* in \( L^\infty(\Omega) \). Then

\[
\lim_{\varepsilon \to 0} \int_{\Omega} (\rho_\varepsilon - \rho_0) G(u) = 0
\]

for every \( u \in W^{s,G}(\Omega) \), \( 0 < s < 1 \).

**Proof.** The weak* convergence of \( \{\rho_\varepsilon\}_{\varepsilon > 0} \) in \( L^\infty(\Omega) \) says that \( \int_{\Omega} \rho_\varepsilon \varphi \to \int_{\Omega} \rho_0 \varphi \) for all \( \varphi \in L^1(\Omega) \). In particular, since \( u \in W^{s,G}(\Omega) \), we have that \( G(u) \in L^1(\Omega) \) and the result is proved. \( \square \)

**Proof of Theorem 5.4.** Let \( v_\varepsilon \in W^{s,G}(\Omega) \) be a minimizer of \( \alpha_{1,\mu}(\rho_\varepsilon) \). Since \( v \) is admissible in the characterization of \( \alpha_{1,\mu}(\rho_0) \) we have that

\[
\alpha_{1,\mu}(\rho_0) \leq \frac{\mathcal{F}(v_\varepsilon)}{\int_{\Omega} \rho_\varepsilon G(|v_\varepsilon|) \, dx} \leq \frac{\int_{\Omega} \rho_\varepsilon G(|v_\varepsilon|) \, dx}{\int_{\Omega} \rho_0 G(|v_\varepsilon|) \, dx} = \alpha_{1,\mu}(\rho_\varepsilon) \frac{I_{\Omega} \rho_\varepsilon G(|v_\varepsilon|) \, dx}{\int_{\Omega} \rho_0 G(|v_\varepsilon|) \, dx}
\]

From Lemma 5.5 we get

\[
\frac{\int_{\Omega} \rho_\varepsilon G(|v_\varepsilon|) \, dx}{\int_{\Omega} \rho_0 G(|v_\varepsilon|) \, dx} = 1 + o(1)
\]

from where we obtain that

\[
(5.1) \quad \alpha_{1,\mu}(\rho_0) - \alpha_{1,\mu}(\rho_\varepsilon) \leq o(1) \alpha_{1,\mu}(\rho_\varepsilon).
\]

Interchanging the roles of \( \alpha_{1,\mu}(\rho_0) \) and \( \alpha_{1,\mu}(\rho_\varepsilon) \), similarly can be obtained that

\[
(5.2) \quad \alpha_{1,\mu}(\rho_\varepsilon) - \alpha_{1,\mu}(\rho_0) \leq o(1) \alpha_{1,\mu}(\rho_0).
\]

From (5.1), (5.2) and (1.9) we obtain that

\[
|\alpha_{1,\mu}(\rho_\varepsilon) - \alpha_{1,\mu}(\rho_0)| \leq o(1) \max\{\alpha_{1,\mu}(\rho_\varepsilon), \alpha_{1,\mu}(\rho_0)\} \leq o(1) \frac{\alpha_{1,\mu}}{\rho_-}
\]

and the proof concludes. \( \square \)

When the family \( \{\rho_\varepsilon\}_{\varepsilon > 0} \) is defined in terms of a \( Q \)-periodic function \( \rho \) satisfying (1.9) as \( \rho_\varepsilon(x) = \rho(\varepsilon x) \) for any \( x \in \mathbb{R}^n \), being \( Q \) the unit cube in \( \mathbb{R}^n \), it is well-known that \( \rho_\varepsilon \to \overline{\rho} := \int_Q \rho \) weakly* in \( L^\infty \) as \( \varepsilon \to 0 \). In this case, more information about the convergence of the minimizers defined in (1.10) can be obtained.
Theorem 5.6. Let $\alpha_{1,\mu}(\bar{\rho})$ and $\alpha_{1,\mu}(\rho_s)$ be the numbers defined in (1.10). Then, there exists a positive constant $C = C(p^+, \rho_+, s, n, \Omega)$ such that
\[
|\alpha_{1,\mu}(\rho_s) - \alpha_{1,\mu}(\bar{\rho})| \leq C\varepsilon^{sp^+}(\alpha_{1,\mu})^2.
\]

The proof of Theorem 5.6 is based on the following key lemma.

Lemma 5.7. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $G$ a Young function satisfying (L) and denote by $Q$ the unit cube in $\mathbb{R}^n$. Let \(\{\rho_\varepsilon\}_{\varepsilon > 0}\) be a sequence defined as $\rho_\varepsilon(x) = \rho(\frac{x}{\varepsilon})$ in terms of a $Q$-periodic function $\rho$ satisfying (1.9) such that $\bar{\rho} = 0$. Then, there exists $C = C(\rho_+, p^+, n, d) > 0$ such that
\[
\left|\int_{\Omega} \rho_\varepsilon G(|v|)\right| \leq C\varepsilon^{sp^+}\Phi_{s,G}(v)
\]
holds for every $v \in W^{s,G}_{0}(\Omega)$ with $s \in (0, 1)$.

Proof. Denote by $I_\varepsilon$ the set of all $z \in Z^n$ such that $Q_{z,\varepsilon} \cap \Omega \neq \emptyset$, $Q_{z,\varepsilon} := \varepsilon(z + Q)$. Given $v \in W^{s,G}_{0}(\Omega)$ we consider the function $\bar{v}_\varepsilon$ given by the formula
\[
\bar{v}_\varepsilon(x) = \frac{1}{\varepsilon^n} \int_{Q_{z,\varepsilon}} v(y) \, dy
\]
for $x \in Q_{z,\varepsilon}$. We denote by $\Omega_1 = \bigcup_{z \in I_\varepsilon} Q_{z,\varepsilon} \supset \Omega$. Thus, we can write
\begin{align}
\left|\int_{\Omega} \rho_\varepsilon G(|v|)\right| &= \left|\int_{\Omega_1} \rho_\varepsilon(G(|v|) - G(\bar{v}_\varepsilon)) + \int_{\Omega_1} \rho_\varepsilon(G(\bar{v}_\varepsilon))\right| \\
&\leq \int_{\Omega_1} \rho_\varepsilon(G(|v|) - G(\bar{v}_\varepsilon)) + \int_{\Omega_1} \rho_\varepsilon(G(\bar{v}_\varepsilon)) := (i) + (ii).
\end{align}

We can split (i) as follows
\begin{align}
(i) &= \int_{I_1} \rho_\varepsilon(G(|v|) - G(\bar{v}_\varepsilon)) + \int_{I_2} \rho_\varepsilon(G(\bar{v}_\varepsilon)) - G(|v|))
\end{align}
where $I_1 = \{x \in \Omega_1 : G(|v|) - G(\bar{v}_\varepsilon) \geq 0\}$ and $I_2 = \{x \in \Omega_1 : G(|v|) - G(\bar{v}_\varepsilon) < 0\}$.

Observe that from $(G_3)$, Young’s inequality (2.1) and Lemma 2.3 we get
\[
G(|v|) - G(\bar{v}_\varepsilon) \leq g(|v|)|v - \bar{v}_\varepsilon|
\leq G^*(g(|v|)) + G(|v - \bar{v}_\varepsilon|)
\leq p^+ G(|v|) + G(|v - \bar{v}_\varepsilon|)
\]
and similarly,
\[
G(\bar{v}_\varepsilon) - G(|v|) \leq p^+ G(|\bar{v}_\varepsilon|) + G(|v - \bar{v}_\varepsilon|).
\]

So, in light of (5.4) and (1.9) we have that
\[
(i) \leq p^+ \left( \int_{I_1} \rho_\varepsilon G(|v|) + \int_{I_2} G(|v - \bar{v}_\varepsilon|) + \int_{I_2} G(|\bar{v}_\varepsilon|) + \int_{I_2} G(|v - \bar{v}_\varepsilon|) \right).
\]
Adding and subtracting $\bar{v}_\varepsilon$ in the first integral and using the $\Delta_2$, from the last inequality we get
\[
(i) \leq p^+ \left( C \int_{I_1} \rho_\varepsilon G(|v - \bar{v}_\varepsilon|) + \int_{I_1} \rho_\varepsilon G(|\bar{v}_\varepsilon|) + \int_{I_1} G(|v - \bar{v}_\varepsilon|) + \int_{I_2} \rho_\varepsilon G(|\bar{v}_\varepsilon|) + \int_{I_2} G(|v - \bar{v}_\varepsilon|) \right)
\]
and since the integrands are positive we can enlarge the domain of integration to obtain
\begin{align}
(i) &\leq p^+ \left( (2 + \rho_+ C) \int_{\Omega_1} G(|v - \bar{v}_\varepsilon|) + (1 + C) \int_{\Omega_1} \rho_\varepsilon G(|\bar{v}_\varepsilon|) \right).
\end{align}
Now, by using Lemma 3.1 we have
\[
\int_{\Omega_t} G(|v - \tilde{v}_\varepsilon|) = \sum_{z \in I^+, \alpha} \int_{\mathcal{Q}_{z,\varepsilon}} G(|v - \tilde{v}_\varepsilon|) dx
\leq c \varepsilon^{s_p^+} \sum_{z \in I^+, \alpha} \int_{\mathcal{Q}_{z,\varepsilon} \times \mathcal{Q}_{z,\varepsilon}} G(|D_x v|) d\mu
\leq c \varepsilon^{s_p^+} \int_{\mathcal{Q}_{z,\varepsilon} \times \mathcal{Q}_{z,\varepsilon}} G(|D_x v|) d\mu
\leq c \varepsilon^{s_p^+} \int_{\mathcal{Q}_{z,\varepsilon} \times \mathcal{Q}_{z,\varepsilon}} G(|D_x v|) d\mu
\leq c \varepsilon^{s_p^+} \int_{\mathcal{Q}_{z,\varepsilon} \times \mathcal{Q}_{z,\varepsilon}} G(|D_x v|) d\mu.
\] (5.6)

Finally, since \( \bar{\rho} = 0 \) and since \( \rho \) is \( Q \)-periodic, we get
\[
\int_{\Omega_t} \rho \varepsilon G(|\tilde{v}_\varepsilon|) = \sum_{z \in I^+} G(|\tilde{v}_\varepsilon|) \int_{\mathcal{Q}_{z,\varepsilon}} \rho \varepsilon = 0.
\] (5.7)

Therefore, combining (5.3), (5.5), (5.6) and (5.7) we find that \( (ii) = 0 \) and
\[
\left| \int_{\Omega} \rho \varepsilon G(|v|) \right| \leq c \varepsilon^{s_p^+} (2 + \rho_s C) \varepsilon^{s_p^+} \Phi_{s,G}(v)
\]
and the proof finishes. \( \square \)

**Proof of Theorem 5.6.** The proof runs similarly to those of Theorem 5.4 by using Lemma 5.7 instead of Lemma 5.5. Indeed, let \( v_\varepsilon \in W^{s,G}_0(\Omega) \) be a minimizer of \( \alpha_{1,\mu}(\rho_\varepsilon) \). Since \( v \) is admissible in the characterization of \( \alpha_{1,\mu}(\bar{\rho}) \) we have that
\[
\alpha_{1,\mu}(\bar{\rho}) \leq \frac{\mathcal{F}(v_\varepsilon)}{\int_{\Omega_t} \rho \varepsilon G(|v_\varepsilon|) dx } \int_{\Omega_t} \rho \varepsilon G(|v_\varepsilon|) dx
= \alpha_{1,\mu}(\rho_\varepsilon) \int_{\Omega_t} \rho \varepsilon G(|v_\varepsilon|) dx \int_{\Omega_t} \rho \varepsilon G(|v_\varepsilon|) dx
\]
From Lemma 5.5 and (1.9) we get
\[
\frac{\int_{\Omega_t} \rho \varepsilon G(|v_\varepsilon|) dx}{\int_{\Omega_t} \rho \varepsilon G(|v_\varepsilon|) dx } = 1 + C \varepsilon^{s_p^+} \Phi_{s,G}(v_\varepsilon) \int_{\Omega_t} \rho \varepsilon G(|v_\varepsilon|) dx \leq 1 + C \varepsilon^{s_p^+} \frac{\rho}{\int_{\Omega_t} \rho \varepsilon G(|v_\varepsilon|) dx } \Phi_{s,G}(v_\varepsilon)
\]
from where we obtain that
\[
(5.8) \quad \alpha_{1,\mu}(\bar{\rho}) - \alpha_{1,\mu}(\rho_\varepsilon) \leq C \varepsilon^{s_p^+} (\alpha_{1,\mu}(\rho_\varepsilon))^2.
\]
Interchanging the roles of \( \alpha_{1,\mu}(\bar{\rho}) \) and \( \alpha_{1,\mu}(\rho_\varepsilon) \), similarly can be obtained that
\[
(5.9) \quad \alpha_{1,\mu}(\rho_\varepsilon) - \alpha_{1,\mu}(\bar{\rho}) \leq C \varepsilon^{s_p^+} (\alpha_{1,\mu}(\bar{\rho}))^2.
\]
From (5.8), (5.9) and (1.9) we obtain that
\[
|\alpha_{1,\mu}(\rho_\varepsilon) - \alpha_{1,\mu}(\bar{\rho})| \leq C \varepsilon^{s_p^+} \max(\alpha_{1,\mu}(\rho_\varepsilon), (\alpha_{1,\mu}(\bar{\rho}))^2) \leq C \varepsilon^{s_p^+} \frac{(\alpha_{1,\mu})^2}{(\rho_\varepsilon)^2},
\]
which concludes the proof. \( \square \)

6. **Further properties**

6.1. **Nodal domains.** In this subsection we need more regularity on the Young function \( G \), namely, we assume the renowned *Lieberman’s condition*
\[
(L') \quad p^- - 1 \leq \frac{t g(t)}{g(t)} \leq p^+ - 1 \quad \forall t > 0
\]
for certain constant \( 1 < p^- < p^+ < \infty \).
Observe that condition (L') on G implies (L).

The following auxiliary lemma is useful for our next result.

**Lemma 6.1.** Let $G$ be a Young function satisfying (L') such that $g = G'.$ Then, the function $h(t) = \frac{1}{t} g\left(\frac{1}{t}\right)$ is increasing for any fixed $c > 0.$

**Proof.** Since $h'(t) = g'\left(\frac{1}{t}\right) \frac{1}{t^2} - g\left(\frac{1}{t}\right) \frac{1}{t^2},$ $h$ is increasing if $\frac{g'(t/c)}{g(t/c)} \geq \frac{1}{t}$ which is guaranteed by (L'). □

**Proposition 6.2.** Let $\lambda_1(\Omega)$ be an eigenvalue of (1.2) with $G$ satisfying (L) with continuous sign-changing eigenfunction $u.$ Then

$$p^+ \lambda(\Omega) > \lambda_1(\Omega^+), \quad p^+ \lambda(\Omega) > \lambda_1(\Omega^-)$$

holds for the open sets $\Omega^+ = \{u > 0\}$ and $\Omega^- = \{u < 0\}.$

Moreover, if $\Omega$ has its diameter comparable with its Lebesgue measure, then

$$\lambda(\Omega) \geq C(n, s, p^+) |\Omega^-|^{-\gamma}$$

for some constant $\gamma > 0$ depending on $p^+, s$ and $n.$

**Proof.** We decompose $u = u^+ - u^-$ where $u^\pm = \max\{\pm u, 0\}$ denote the positive and negative part of $u,$ respectively. Observe that

$$(u(x) - u(y))(u^+(x) - u^-(y)) = |u^+(x) - u^+(y)|^2 + u^+(x)u^-(y) + u^+(y)u^-(x).$$

Hence, choosing $u^+ \in W_0^{s,G}(\Omega)$ as a test in (1.3), from the identity above we find that

$$\lambda \int_{\Omega} g(|u|) u^+ dx = \int_{\mathbb{R}^n \times \mathbb{R}^n} g(|D_s u|) \frac{|u(x) - u(y)| u^+(x) - u^+(y)}{|u(x) - u(y)| |x - y|^s} d\mu$$

$$= \int_{\mathbb{R}^n \times \mathbb{R}^n} g(|D_s u|) \frac{|u^+(x) - u^+(y)|^2}{|u(x) - u(y)| |x - y|^s} dx dy$$

$$+ 2 \int_{\mathbb{R}^n \times \mathbb{R}^n} g(|D_s u|) \frac{u^+(x)u^-(y)}{|u(x) - u(y)| |x - y|^{n+s}} dx dy := I + II.$$ 

Now, since

$$|u(x) - u(y)|^2 = |u^+(x) - u^+(y)|^2 + |u^-(x) - u^-(y)|^2 + 2u^+(x)u^-(y) + 2u^+(y)u^-(x),$$

we get that $|u(x) - u(y)| \geq |u^+(x) - u^+(y)|.$ Therefore, from Lemma 6.1 we get

$$I \geq \int_{\mathbb{R}^n \times \mathbb{R}^n} g(|D_s u^+|) |D_s u^+| d\mu := I'.$$

Moreover, the identity above also gives that $|u(x) - u(y)| \geq \sqrt{2u^+(x)u^-(y)},$ from where

$$II \geq \int_{\mathbb{R}^n \times \mathbb{R}^n} g\left(\frac{\sqrt{2u^+(x)u^-(y)}}{|x - y|^s}\right) \frac{u^+(x)u^-(y)}{|u(x) - u(y)| |x - y|^{n+s}} dx dy := II' > 0,$$

since $g$ is increasing. Consequently, we find that

$$\lambda(\Omega) \int_{\Omega^+} g(u^+) u^+ dx \geq I' + II' > I'.$$

and since $u^+ \in W_0^{s,G}(\Omega^+)$ the last expression yields $\lambda(\Omega) > \lambda_1(\Omega^+).$ Finally, if $d$ and $|\Omega|$ are comparable, from Corollary 5.3 we conclude that

$$\lambda(\Omega) > \lambda_1(\Omega^+) \geq C(n, s, p^+) |\Omega^-|^{-\gamma}$$

for some suitable positive number $\gamma.$ The proof for $\Omega^-$ is analogous. □
6.2. Behaviour of $\alpha_{1,\mu}$ as $s \to 1$. As a direct implication of the $\Gamma-$convergence of modulars stated in [7], the behavior of the Poincaré constant (1.4) as $s \to 1^+$ can be characterized. For definitions and an introduction to the $\Gamma-$convergence theory, see for instance [5].

In this paragraph it will be convenient to emphasize the dependence on $s$ in $\alpha_{1,\mu}$ and in the set $M_{\mu}$. Given an open and bounded set $\Omega \subset \mathbb{R}^n$, a parameter $s \in (0,1)$ and a Young function $G$ satisfying (L), we consider the fractional minimizer and the limit minimizer defined as

$$(6.1) \quad \alpha_{1,\mu,s} = \inf_{u \in M_{\mu,s}} \frac{\Phi_{s,G}(u)}{\Phi_G(u)}, \quad \alpha_{1,\mu,1} = \inf_{u \in M_{\mu}} \frac{\Phi_{G}(\|\nabla u\|)}{\Phi_G(u)},$$

where for $\mu > 0$ we consider the sets

$M_{\mu,s} = \{ u \in W_0^G(\Omega) : \Phi_G(u) = \mu \}, \quad \tilde{M}_{\mu,s} = \{ u \in W_0^{1,G}(\Omega) : \Phi_{\tilde{G}}(u) = \mu \}$

and the limit Young function $\tilde{G}$ is defined as follows

$$\tilde{G}(t) := \lim_{s \downarrow 1-} (1-s) \int_{0}^{1} \int_{S^{n-1}} G(t|z_n| r^{1-s}) dS_z \frac{dr}{r},$$

see [7, Proposition 2.16] for details.

In this context we define the energy functionals $\mathcal{J}_s, \mathcal{J} : L^G(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by

$$\mathcal{J}_s(u) = \begin{cases} (1-s) \mu \frac{\Phi_{s,G}(u)}{\Phi_G(u)} & \text{if } u \in M_{\mu,s}, \\ +\infty & \text{otherwise}, \end{cases} \quad \mathcal{J}(u) = \begin{cases} \mu \Phi_{\tilde{G}}(\|\nabla u\|) & \text{if } u \in \tilde{M}_{\mu} \\ +\infty & \text{otherwise}. \end{cases}$$

Proposition 6.3.

$$\lim_{s \to 1^+} (1-s) \alpha_{1,\mu,s} = \alpha_{1,\mu,1},$$

Proof. By [7, Theorem 6.5] the functional $\mathcal{J}_s$ $\Gamma-$converges to $\mathcal{J}$ as $s \to 1^+$. The main feature of the $\Gamma-$convergence is that it implies the convergence of minima (see [5, Theorem 7.4]):

$$\lim_{s \to 1^+} \min_{L^G(\Omega)} \mathcal{J}_s(u) = \min_{L^{G}(\Omega)} \mathcal{J}(u) = \min_{L^{\tilde{G}}(\Omega)} \mathcal{J}(u),$$

where the last equality follows since $G$ and $\tilde{G}$ define the same Orlicz space in light of [7, Proposition 2.16]. \hfill \square

Acknowledgements

This paper is partially supported by grants UBACyT 20020130100283BA, CONICET PIP 11220150100032CO and ANPCyT PICT 2012-0153. The author is member of CONICET.

References


Departamento de Matemática, FCEyN - Universidad de Buenos Aires and IMAS - CONICET.
Ciudad Universitaria, Pabellón I (1428) Av. Cantillo s/n.
Buenos Aires, Argentina.

E-mail address, A.M. Salort: asalort@dm.uba.ar

URL: http://mate.dm.uba.ar/~asalort