Retrofit of Multiproduct Batch Plants Through Generalized Disjunctive Programming

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Abstract—The retrofit problem for multiproduct batch plants deals with the modification of the original structure of the plant to meet new production conditions such as the introduction of new products, a new supply pattern, etc. For this problem, a disjunctive model is presented, which takes into consideration every usual alternative about the configuration of units and storage tanks. A disjunction is generated for that purpose. The selection of one of those alternatives sets up all the corresponding constraints, which are: operation time, unit size and cost, etc. Finally, the performance of this approach is analyzed through the resolution of a set of examples. © 2003 Elsevier Ltd. All rights reserved.

Keywords—Generalized disjunctive programming, MINLP, Retrofit, Multiproduct batch plants.

NOMENCLATURE

- $B_{ij}$: batch size of Product $i$ at Stage $j$ [kg]
- $c_j$: parameter in the variable part of the cost of units in Stage $j$ [U.S.A. dollars]
- $CT_{j}$: parameter in the variable part of the cost of storage tanks in Position $j$ [U.S.A. dollars]
- $CC_{jk}$: purchasing cost of the Unit $k$ in Stage $j$ [U.S.A. dollars]
- $CCT_{jk}$: purchasing cost of the new storage tank $kt$ in Position $j$ [U.S.A. dollars]
- $CE_{j}$: batch units cost at Stage $j$ [U.S.A. dollars]
- $CET_{j}$: storage tanks cost at Position $j$ [U.S.A. dollars]
- $CV_{jk}$: selling value of the Unit $k$ in Stage $j$ [U.S.A. dollars]
- $CVT_{jk}$: selling value of the storage tank $kt$ in Position $j$ [U.S.A. dollars]
- $H$: time horizon [hr]
- $HT_{j}$: number of terms in the disjunction corresponding to Stage $j$
- $H_j$: number of terms in the disjunction corresponding to Position $j$ for storage tanks
- $K_j$: fixed purchasing cost part of units in Stage $j$ [U.S.A. dollars]
- $KT_{j}$: fixed purchasing cost part of storage tanks in Position $j$ [U.S.A. dollars]
- $M_h$: number of groups in option $h$

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1. INTRODUCTION

Multiproduct batch plants manufacture a number of products sharing the same equipment. Various products are made following the same processing sequence. This equipment exploitation and the possibility of elaborating products with a high added value have focused the interest on this kind of plants [1].

The plant configuration implies the determination of the number of units to be used at each stage and their sizes. On this basis, two problems arise. First, the design problem assuming that the plant does not exist and it must be built to meet certain demands [2-4]. The second is the case of the retrofit problem, where the plant already exists, but production of new products, variations in the product demands, etc., forces a modification of the original plant configuration in order to satisfy this new scenarios. The changes include the allocation of new units; the sale of units that will not be used anymore, and a different equipment arrangement for the new production needs.

In the first work on this area [5], the objective of the proposed model was to maximize the plant incomes taking into account the price and maximum demand estimated for each product and the cost of the units to be added. The main weakness of that work consisted that all units were structured in the same way for all products. Fletcher, Hall and Johns [6] overcame this drawback allowing new units to be configured in different ways. Yoo, Lee, Ryu and Lee [7] presented a quite complex model that enables all plant units, either new or old, to be considered in different alternatives. Even useless units can be sold. Van den Heever and Grossmann [8]
presented a disjunctive model considering a multi-period horizon, although it did not allow all the options of the previous work. Finally, Montagna, Vecchietti and Iribarren [9] presented a general retrofit model that included every possible options for units configuration. In addition, this model considers the allocation of intermediate storage tanks for the first time.

In all cases, problems are modeled in the form of mixed integer nonlinear programs (MINLP). There is a set of decisions that must be posed by means of integer variables: the number of units in a stage, the intermediate storage tank allocation, etc. In general terms, previous works have been developed using conventional models.

This work presents a general model that has been developed using the generalized disjunctive programming methodology. Therefore, it involves both all-possible options for (either existing or new) units configuration, and the allocation of intermediate storage tanks. Old units that are not taken into account in the new plant configuration are sold. The disjunctive programming performance is analyzed according to previous approaches.

2. MULTIPRODUCT BATCH PLANTS

A batch plant is composed by a set of \( N \) stages. At each stage \( j \) \((j = 1, \ldots, N)\) there is a set of \( N_j \) units. A number of \( P \) products are made in the plant. For each product \( i \) \((i = 1, \ldots, P)\), the quantity to be produced \( Q_i \) should be determined, which is limited by the required demand for that product \( Q_i^U \). Production of Product \( i \) at Stage \( j \) requires a time period \( T_{ij} \).

At each batch stage \( j \), \( N_j \) units can be arranged in different ways so as to improve the plant productivity. In general terms, given the scale economy in the units cost, it is convenient to use a larger unit rather than two separate units. However, using several units at one stage increase the plant productivity. If the units operate out-of-phase, then the operation time \( T_{ij} \) is divided by the number of units at that stage. Successive batches of Product \( i \) use the different units and leave the stage with a smaller difference among them than when considering the original operation time. If Stage \( j \) has the longest operation time, the remaining stages must operate at its rate. Processing in that stage must be finished before considering the batch processing of the following stage. It is common to have idle times between batch stages. By incorporating out-phased duplicated units in the stage with the longest operation time, the time interval between stages can be reduced. This reduces the idle times of the remaining stages, and thus, increases their processing capacity. This is the reason for the use of out-of-phase duplicated units.

In-phase arrangement is another option for the operation of duplicated units. In this case, in-phase units operate together as if they were a major unit. When entering the stage, the batch is divided among all units in the group. This option is usually used where the required batch size exceeds the highest possible capacity for the unit size.

In models where the units can be structured in different ways for each product, they may operate out-of-phase for a given product because of the stage with the largest processing time. However, for other product, the same units can be arranged in-phase so as to increase the batch size to be processed, and improve the processing capacity of the stage.

Both configurations can be used together. Figure 1 shows Stage \( j \) with four units, where \( V_{jk} \) is the volume of Unit \( k \) at Stage \( j \). The overlapped units operate as a group, i.e., in-phase. Units 1 and 3 form Group 1 and operate in-phase, just like Units 2 and 4, which form Group 2. Both groups operate out-of-phase.

Another option for improving productivity in a batch plant is the allocation of intermediate storage tank between two stages. In this case, the production process is divided into two uncoupled subprocesses. Both subprocesses must have the same productivity in order to avoid material accumulation in the intermediate storage tank. In this case, there is a different batch size in each subprocess allowing the adjustment of unit sizes.
3. RETROFIT OF MULTIPRODUCTO BATCH PLANTS

As it was pointed out in the abstract section, the retrofit problem deals with a production plant, which was designed to meet certain specifications and those specifications have changed. So, there is an attempt to adjust its configuration to meet these new constraints. In the following section, we describe the basic constraints that model this problem by a conventional methodology. The model is general, allowing all possible configuration options for units and storage tanks. In addition, there are no constraints in the way old and new units can be grouped out.

In a multiproduct batch plant, all products follow the same operation sequence through the \(N\) stages conforming the plant. At each stage, there is a set of \(N_j\) units. Among these units, there are \(N_j^{\text{OLD}}\) existing units in the plant and \(N_j^{\text{NEW}}\) units that can be incorporated, so that

\[
N_j = N_j^{\text{OLD}} + N_j^{\text{NEW}}, \quad j = 1, \ldots, N. \tag{1}
\]

The units at each stage can be configured in a different way for each Product \(i\) produced in the plant. \(B_{ij}\) is the batch to be processed at Stage \(j\) for Product \(i\). When \(B_{ij}\) enters a group of units, i.e., configured in-phase, \(B_{ij}\) is divided among the units of the group. Thus, the sum of the volumes of each unit in the group must be enough to process the batch. This can be expressed by the following constraint (for Product \(i\), Stage \(j\)):

\[
\sum_{k \in g} V_{jk} \geq S_{ij} B_{ij}, \quad i = 1, \ldots, P; \quad j = 1, \ldots, N; \quad g \in h. \tag{2}
\]

\(S_{ij}\) is a size factor given by the production recipe; \(g\) is a group of units that are part of the configuration option \(h\). A configuration option \(h\) identifies a unit order corresponding to that group. Figure 1 corresponds to a configuration option for a stage with four available units, which have been grouped forming two groups. For example, if there are three available units in one stage, there are seven possible groups: \(\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\), where units between braces corresponds to a group. These groups may be arranged in 14 possible configuration options. These options are

\[
\begin{align*}
    h_1 &= \{1\}, \quad h_2 = \{2\}, \quad h_3 = \{3\}, \quad h_4 = \{1,2\}, \quad h_5 = \{1,3\}, \quad h_6 = \{2,3\}, \\
    h_7 &= \{1,2,3\}, \quad h_8 = \{1\} \cup \{2\}, \quad h_9 = \{1\} + \{3\}, \quad h_{10} = \{2\} + \{3\}, \quad h_{11} = \{1,2\} + \{3\}, \\
    h_{12} &= \{1\} + \{2,3\}, \quad h_{13} = \{1,3\} + \{2\}, \quad h_{14} = \{1\} + \{2\} + \{3\}.
\end{align*}
\]

Figure 1. Group existence in the Stage \(j\).
In this case, there are seven options with a single group \((h_1 \text{ to } h_7)\), six options with two groups \((h_8 \text{ to } h_{13})\) and one option with three groups \((h_{14})\). With more units, the number of groups increases, and therefore, the options. However, this number can be reduced if the designer considers that there are options, which are neither admitted nor optimal \textit{a priori}, and therefore, better efficiency is achieved in resolution.

Any new unit that is incorporated into the stage has a purchasing cost \(CC_{jk}\) expressed by

\[
CC_{jk} = \begin{cases} 
K_j + c_j V_{jk}, & \text{if Unit } k \text{ is used,} \\
0, & \text{otherwise,}
\end{cases} \quad j = 1, \ldots, N; \quad k = N_j^{OLD} + 1, \ldots, N_j. \tag{3}
\]

\(K_j\) is a fixed purchasing cost regardless of the unit size, whereas \(c_j\) is a parameter that corresponds to the variable part of the cost, proportional to the unit size.

In the same way, if Unit \(k\) existing in the plant is sold because it shall not be used in the plant, its selling value \(CV_{jk}\) can be formulated as

\[
CV_{jk} = \begin{cases} 
R_{jk}, & \text{if Unit } k \text{ is not used,} \\
0, & \text{otherwise,}
\end{cases} \quad j = 1, \ldots, N; \quad k = 1, \ldots, N_j^{OLD}. \tag{4}
\]

\(R_{jk}\) is a predetermined value that depends on the size and state of the existing equipment of the plant.

Let \(M_h\) be the number of groups included in option \(h\). Let \(TL_i\) be the limiting cycle time of Product \(i\), i.e., the time consumed between two consecutive batches of \(i\). This time coincides with the longest processing time. As was mentioned before, if there are groups out-of-phase, the time consumed between two successive batches is reduced. Then, the limiting cycle time for Product \(i\), for a certain units configuration \(h\) at Stage \(j\) is given by the following expression:

\[
TL_i \geq \frac{T_{ij}}{M_h}, \quad i = 1, \ldots, P; \quad j = 1, \ldots, N. \tag{5}
\]

For each product the productivity \(Pr_i\) is given by

\[
Pr_i = \frac{B_{is}}{TL_{is}}, \quad i = 1, \ldots, P; \quad \forall s = \text{subprocess}, \tag{6}
\]

where \(B_{is}\) and \(TL_{is}\) are the batch size and cycle time for Product \(i\) in subprocess \(s\). Subprocesses arise when an intermediate storage tank is allocated. This productivity is kept constant in all subprocesses of the plant so as to avoid material accumulation in the intermediate storage tanks. Taking into account that intermediate storage tank allocation is not known previously to the resolution, in order to simplify the model, this expression is substituted in (5), which results in

\[
B_{ij} \geq \frac{T_{ij}}{M_h} Pr_i, \quad i = 1, \ldots, P; \quad j = 1, \ldots, N. \tag{7}
\]

The intermediate storage tanks can be allocated between two consecutive batch stages. Position \(j\) is located between Stage \(j\) and Stage \(j + 1\). There is \(N - 1\) possible positions. At each position, there is a set of pre-existing tanks \(NT_j\). Only one new tank can be assigned at each position. It does not make sense to add a greater number of units, since there are no configuration options; intermediate storage tanks always operate as a single group.

If there is no intermediate storage tank between two batch stages, then the batch sizes used in those stages should be the same, i.e.,

\[
B_{ij} = B_{i,j+1}, \quad i = 1, \ldots, P; \quad j = 1, \ldots, N - 1. \tag{8}
\]
On the other hand, if there are intermediate storage tanks at Position $j$, batches at Stage $j$, and Stage $j + 1$ may be different and the following expression must be met:

$$\frac{1}{\theta} \leq \frac{B_{ij}}{B_{i,j+1}} \leq \theta, \quad i = 1, \ldots, P; \quad j = 1, \ldots, N - 1,$$

where $\theta$ is the maximal admitted relationship between consecutive batch sizes.

Storage tanks volume at Position $j$ must meet the following condition:

$$\sum_{k=1}^{N_{T,j}+1} V_{T,j,kt} \geq S_{T_{ij}}(B_{ij} + B_{i,j+1}), \quad i = 1, \ldots, P; \quad j = 1, \ldots, N - 1.$$

$S_{T_{ij}}$ is the tank size factor for Product $i$ at Position $j$. $V_{T,j,kt}$ is the tank volume size at Position $j$. If a new tank will be added, it corresponds to the $N_{T,j} + 1$ unit. This expression for sizing tanks is derived from [10].

For storage tanks cost, there are similar expressions to those for batch units. There is a purchasing cost $CCT_{j,kt}$ for the new unit, and a selling cost $CVT_{j,kt}$ for the units that already exists but will not be used, which are given by the following expressions:

$$CCT_{j,NT_{j}+1} = \begin{cases} KT_{j} + c_{t}V_{T,j,NT_{j}+1}, & \text{if unit } N_{T_{j}} + 1 \text{ is used,} \\ 0, & \text{otherwise,} \end{cases} \quad j = 1, \ldots, N - 1,$$

$$CVT_{j,kt} = \begin{cases} RT_{j,kt}, & \text{is unit } k \text{ is not used anymore,} \\ 0, & \text{otherwise,} \end{cases} \quad j = 1, \ldots, N - 1; \quad kt = 1, \ldots, N_{T_{j}}.$$

$K_{T_{j}}, c_{t}$ are parameters to evaluate the tank cost and $R_{T_{j},kt}$ corresponds to the selling value of unit $kt$ at Position $j$.

Let $N_{ij}$ be the number of batches of Product $i$ at Stage $j$. The total production of $i$, $Q_{i}$, should not exceed the quantity processed at each stage. Then

$$Q_{i} \leq N_{ij}B_{ij}, \quad i = 1, \ldots, P; \quad j = 1, \ldots, N.$$  

This number of batches $N_{ij}$ should be an integer variable. However, just like in previous works, it is assumed this variable can be considered as a continuous variable, due to its magnitude on long time horizons.

All products must be produced in a time horizon $H$, which is expressed by the following constraint:

$$\sum_{i} Q_{i} / P_{t_{i}} \leq H.$$  

The retrofit model posed above allows different unit configurations in the stages for each product. Each option has an associated cost that is given by the value of the units included in that option. If the unit already exists, no cost is charged. If the unit must be introduced into the plant, it generates an extra cost depending on the unit size. If an existing unit will not be used for any product, it can be sold, generating an income for the business. In the case of storage tanks, the work is done in a similar way, except for the fact that there are no configuration options, since tanks always work in-phase in a single group. Each of the previous works in this area counts on a set of discrete variables to model the different configuration options.

The objective considers the maximization of the benefits generated by the plant. It takes into account the incomes coming from the product sales and the unit sales that already exist in the plant and will not longer used after the retrofit. The expenditures come from acquisition of new units. The objective function is formulated as

$$\text{Max } z = \sum_{i=1}^{P} p_{i}Q_{i} + \sum_{j=1}^{N} N_{j}^{\text{OLD}} \sum_{k=1}^{N_{T_{j}}} \sum_{j=1}^{N-1} \sum_{kt=1}^{N_{T_{j}}} CV_{j,k} + \sum_{j=1}^{N-1} \sum_{kt=1}^{N_{T_{j}}} CVT_{j,k} - \sum_{j=1}^{N} \sum_{k=N_{T_{j}}^{\text{OLD}}+1}^{N_{T_{j}}} CC_{j,k} - \sum_{j=1}^{N-1} \sum_{kt=1}^{N_{T_{j}}} CCT_{j,NT_{j}+1},$$

where $p_{i}$ is the total benefit per unit of Product $i$. 
4. FORMULATING DISCRETE DECISIONS BY MEANS OF DISJUNCTIONS

A generalized disjunctive programming model has the following formulation [11]:

\[
\begin{align*}
\text{Min} & \quad z = \sum_{k \in K} c_k + f(x), \\
\text{s.t.} & \quad r(x) \leq 0, \\
& \quad \bigvee_{j \in J_k} \begin{bmatrix}
Y_{jk} \\
g_{jk}(x) \leq 0 \\
c_k = \gamma_{jk}
\end{bmatrix}, \quad k \in K, \\
\Omega(Y) = \text{True}, \\
& \quad x \geq 0, \quad c_k \geq 0, \quad Y_{jk} \in \{\text{true, false}\}.
\end{align*}
\]

In this model, \(x \in \mathbb{R}^n\) is the continuous variables vector and \(Y_{jk}\) are Boolean variables. \(c_k \in \mathbb{R}\) are continuous variables and \(\gamma_{jk}\) are values corresponding to the cost of the alternative. \(f: \mathbb{R}^n \to \mathbb{R}\) is the term of the objective function that depends on variables \(x\) and \(r: \mathbb{R}^n \to \mathbb{R}\) are a general set of constraints that do not depend on disjunctions. This general model assumes that \(f(x)\) and \(r(x)\) are convex functions.

A disjunction is formed by an operator OR and a set of terms. In each disjunction term, there is a Boolean variable \(Y_{jk}\), a set of convex constraints \(g_{jk}: \mathbb{R}^n \to \mathbb{R}\) and a cost variable \(c_k\). If the Boolean variable \(Y_{jk}\) is true, then conditions \(g_{jk}(x) \leq 0\) and \(c_k = \gamma_{jk}\) must be met. Otherwise, if \(Y_{jk}\) is false, the corresponding constraints are overlooked. Each disjunction term is assumed to give rise to a feasible and nonempty region. Finally, \(\Omega(Y) = \text{True}\) is a set of logic constraints generated by using the set of Boolean variables \(Y\).

In this work, a set of disjunctions are generated to pose the discrete decisions of the retrofit problem to evaluate the advantages of this approach. Two main groups of disjunctions must be considered. The first one is used to define how the units will be grouped in a batch stage. In a similar way, the other disjunction is posed to determine the tanks to be used at each Position \(j\).

One decision that must be made at each batch stage and for each product is the arrangement of the different equipment units belonging to the stage. Moreover, the size of the units to be incorporated must be determined. In order to represent these options, the following disjunction is formulated:

\[
\begin{align*}
& \quad \bigvee_{h \in H_j} \begin{bmatrix}
\sum_{k \in g} V_{jk} \geq S_{ij} B_{ij}, \quad \forall g \in h \\
B_{ij} \geq \frac{T_{ij}}{M_h} P_{ri} \\
\text{CEQ}_{ij} = \sum_{k=N_{LOD+1,j}}^{N_{LOD,j}} \sum_{k \in h} \gamma_{jk} C_{N_{LOD,j}} \\
& \quad \sum_{k=N_{LOD+1,j}}^{N_{LOD,j}} \sum_{k \in h} \gamma_{jk}
\end{bmatrix}, \quad i = 1, \ldots, P, \quad j = 1, \ldots, N. \quad (17)
\end{align*}
\]

Equation (17) represents a set of disjunctions for each Stage \(j\) and for each Product \(i\), of \(H_j\) terms, where each term \(h \in H_j\) corresponds to each possible equipment configuration that are admitted in the model. In case we have at most three pieces of equipment per stage, the disjunction contains 14 terms according to the groups that can be configured as it was previously explained.

Boolean variable \(Y_{ijk}\) serves as an indicator of which disjunction term is true. Only one term must be true for Product \(i\) at Stage \(j\), and all constraints included in that term must be met: the volume of Units \(k\) at Stage \(j\), equation (2), the limiting cycle time equation (7), and the equipment cost equation of that option, equations (3) and (4).

Another discrete decision that must be taken into account in the model, is whether to place new intermediate storage tanks between stages, and/or whether to sell the existing ones in case
they will no longer be used in the plant. For this case, a set of \( N - 1 \) disjunctions is defined, where \( N \) is the number of stages of the batch plant, in the following form:

\[
\forall \; h_t \in H_{T_j} = \begin{bmatrix}
\frac{1}{\theta} \leq \frac{B_{ij}}{B_{i,j+1}} \leq \theta \\
\sum_{k_t \in H_{T_j}} V_{T,j,kt} \geq S_{T_ij}(B_{ij} + B_{i,j+1}) \\
C_{EQT}_{ij} = \sum_{k_t \in H_{T_j}} C_{C,KT,j,kt} - \sum_{k_t \in H_{T_j}} C_{C,VT,j,kt}
\end{bmatrix}, \tag{18}
\]

\( i = 1, \ldots, P; \; j = 1, \ldots, N - 1. \)

Disjunction corresponding to Position \( j \) has \( H_{T_j} \) terms. This quantity is a function of the number of tanks (\( N_{T_j} \)) that exist in that position. Only one intermediate tank is allowed to be added. For example, if already exists a tank at Position \( j (N_{T_j} = 1) \), there are \( H_{T_j} = 4 \) options for that position and, thus disjunction terms, according to the following detail:

1. two tanks (the old one plus the new one being added),
2. the current tank,
3. the new tank (the old one is sold),
4. none (the old tank is sold).

All terms have the same form, as shown by equation (18), except for the term corresponding to option without tanks, which presents slight differences. Boolean variable \( Y_{T,j,ht} \) serves as an indicator if the disjunction term is true or false. Again, only one term must be true and all constraints included in the term must be met.

In (18), the first constraint in the term refers to the relationship between batches before and after Position \( j \), equation (9). In the term corresponding to the option without storage tanks, \( 0 - 1 \) is taken, equation (8). The equation for tank sizing must take into account the existing tanks in that location \( H_{T_j} \), equation (10). Again, in the option without tanks, this constraint is not included. The cost equation of the option must take into account both the sale of tanks no longer used and the additional cost caused by adding a new one, equations (11) and (12).

Working with disjunctions (17) and (18) the objective function is the following:

\[
\text{Max } z = \sum_{i=1}^{P} p_i Q_i - \sum_{j=1}^{N} C_{E_j} - \sum_{j=1}^{N-1} C_{E_{T_j}}, \tag{19}
\]

where \( C_{E_j} \) is the batch unit costs at Stage \( j \), which results from the following expression:

\[
C_{E_j} \geq C_{E Q,j}, \quad i = 1, \ldots, P; \; j = 1, \ldots, N, \tag{20}
\]

and \( C_{E_{T_j}} \) is the tanks cost in Position \( j \), which results from the following expression:

\[
C_{E_{T_j}} \geq C_{E Q,T_{ij}}, \quad i = 1, \ldots, P; \; j = 1, \ldots, N - 1. \tag{21}
\]

In addition, constraints (13) and (14) must be considered, since they are also part of the group of common constraints \( r(x) \) defined in (16). Among logic constraints \( \Omega \) defined in (16), the following constraint is derived, which assures that all products use the same units

\[
\sum_{g \in H} Y_{i,j,h} = \sum_{g \in H} Y_{ii,j,h}, \quad j = 1, \ldots, N, \; \forall k, i, \; ii = 1, \ldots, P, \; i \neq ii. \tag{22}
\]
To solve the problem, disjunctions are transformed into mixed-integer constraints applying convex-hull relaxation proposed by [12] for linear models. Also, bearing in mind that constraints (13) and (14) presents nonconvex terms, the same transformations used by [5] are introduced. Also, the first term in the objective function is concave. To overcome this problem, the same authors have proved that the negative exponential functions in that term can be approximated by a system of piecewise linear under estimators. This approximation overestimates the objective function so it can be employed to find the global solution of this model.

Therefore, a mixed-integer nonlinear problem (MINLP) is derived, which can be solved using any known algorithm for this kind of problems [13]. In our case, the MINLP has been solved by using the OA/ER/AP algorithm by [14], then completed by [15], and implemented in DICOPT⁺ solver of GAMS mathematical modelling system.

5. EXAMPLES

In all solved examples, the introduction of intermediate tanks relies on unit size and unit cost factors, which was not included in previous retrofit problems. These factors values have been appropriately selected to show the potential of the application. In general terms, for all the examples the solution depends on the relationship between the revenues from the product sales and the cost of the units to be incorporated.

EXAMPLE 1. This example is interesting because its resolution allows the consideration of several options included in the proposed model. Table 1 shows the data added to Example 5 presented in [7], whereas Table 2 shows the results reached by using the previous formulation [7] and those obtained with the approach proposed in this article. In Table 2, u2 means Unit 2. Units between square brackets form a group.

Table 1. Data added to Example 5 by [7] for Example 1 of this work.

<table>
<thead>
<tr>
<th>ST_{ij}</th>
<th>Product A</th>
<th>Product B</th>
<th>KT_j</th>
<th>c_{ij}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Position 1</td>
<td>4.</td>
<td>1.</td>
<td>1000</td>
<td>1.</td>
</tr>
</tbody>
</table>

Table 2. Results corresponding to Example 1.

<table>
<thead>
<tr>
<th>Q_j/1000</th>
<th>Reference [7]</th>
<th>Our Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Product A</td>
<td>Product B</td>
<td>Product A</td>
</tr>
<tr>
<td>2,000</td>
<td>4,000</td>
<td>2,000</td>
</tr>
<tr>
<td>New Units</td>
<td>New Units</td>
<td></td>
</tr>
<tr>
<td>Stage 1</td>
<td>1,000</td>
<td></td>
</tr>
<tr>
<td>[u2, u3]</td>
<td>[u2]</td>
<td></td>
</tr>
<tr>
<td>Stage 2</td>
<td>[u2]</td>
<td></td>
</tr>
<tr>
<td>Storage Tank Position 1</td>
<td>3,600</td>
<td></td>
</tr>
<tr>
<td>[u2]</td>
<td>[u3]</td>
<td></td>
</tr>
<tr>
<td>Sold Units</td>
<td>Sold Units</td>
<td></td>
</tr>
<tr>
<td>Stage 1</td>
<td>u1</td>
<td></td>
</tr>
<tr>
<td>Stage 2</td>
<td>u1, u3</td>
<td></td>
</tr>
<tr>
<td>Profit ($)</td>
<td>752,000</td>
<td>750,400</td>
</tr>
</tbody>
</table>

In this example and in both formulations, there are old units that are sold, as it is noted in Figures 2 and 3 that show the final structure of the plant for both formulations. Each unit includes its number and size. Units in grey are those that have been added to the plant. Figure 2 shows different configurations for both products. For Product A, both units at Stage 1 operate in-phase and for Product B, they operate out-of-phase. Unit 3 at Stage 1, which is grey-coloured in the figure, is added after the plant retrofit. Figure 3 corresponds to the solution using the
A new tank is incorporated between Stages 1 and 2 and it is used only for Product B. In a previous formulation considering storage tanks where discrete decisions were posed by algebraic constraints [9], tanks had to be used for both products, reaching a solution of $758,200. By formulating the disjunction using (18), it is possible to reach solutions as the one proposed model.
shown in Figure 3, in which not all products use the tank. This allows achieving a slight increase in the objective function.

Note that although Figures 2 and 3 correspond to solutions with a slight difference in the achieved optimal values, their structures are quite different.

This simple example shows some advantages of this approach for retrofit. A better optimal solution is found when intermediate storage tanks are admitted. This is a logical result if we take into account that usually this kind of units are cheaper than operation units. At the same time, a broader alternatives set is available. In this example, the original plant was too big to process the new production demands. Therefore, the units had to be adjusted to the new requirements, selling useless units. In the solution proposed in [7], three units are sold. The new unit in Stage 1 is used in-phase for Product A so as to increase the batch size, while for Product B it operates out-of-phase to reduce the cycle time. However, for Product B, the batch size is reduced because of the new unit is smaller than the previous one and it limits the batch size to be processed. In the solution with the proposed approach, also three units are sold. Since no new unit is added the productivity of the Product A is worse than previous solution. However, for Product B, the storage tank decouples the process, allowing different batch sizes and cycle times in both subprocesses determined by the tank. Thus, in the subprocess composed by Stage 1, a greater batch size can be processed, permitting a superior performance for Product B. This important improvement overcomes the poor performance reached by Product A without a new unit. In synthesis, the availability of more options for the retrofit allows to attain better solutions.

| Table 3. Data for Example 2. |

<table>
<thead>
<tr>
<th>Product / T_{ij}</th>
<th>Stage 1</th>
<th>Stage 2</th>
<th>Stage 3</th>
<th>Stage 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>6.3822</td>
<td>4.7393</td>
<td>8.3353</td>
<td>3.9443</td>
</tr>
<tr>
<td>B</td>
<td>6.7936</td>
<td>6.4175</td>
<td>6.4750</td>
<td>5.4392</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Product / S_{ij}</th>
<th>Stage 1</th>
<th>Stage 2</th>
<th>Stage 3</th>
<th>Stage 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>7.913</td>
<td>2.0615</td>
<td>5.2208</td>
<td>4.9223</td>
</tr>
<tr>
<td>B</td>
<td>0.7891</td>
<td>0.2671</td>
<td>0.2744</td>
<td>3.3951</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Product / S_{ij}</th>
<th>Stage 1</th>
<th>Stage 2</th>
<th>Stage 3</th>
<th>Stage 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>-</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( V_{ij}^{A} )</th>
<th>Stage 1</th>
<th>Stage 2</th>
<th>Stage 3</th>
<th>Stage 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>4.000 (1)</td>
<td>4.000 (1)</td>
<td>3.000 (2)</td>
<td>3.000 (1)</td>
</tr>
<tr>
<td>B</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( V_{ij}^{B} )</th>
<th>Stage 1</th>
<th>Stage 2</th>
<th>Stage 3</th>
<th>Stage 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>4.000</td>
<td>4.000</td>
<td>3.000</td>
<td>3.000</td>
</tr>
</tbody>
</table>

New Units

<table>
<thead>
<tr>
<th>( K_{ij} )</th>
<th>Stage 1</th>
<th>Stage 2</th>
<th>Stage 3</th>
<th>Stage 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>60,000</td>
<td>140,000</td>
<td>180,000</td>
<td>40,000</td>
</tr>
<tr>
<td>B</td>
<td>45.00</td>
<td>120.00</td>
<td>150.00</td>
<td>40.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( c_{ij} )</th>
<th>Stage 1</th>
<th>Stage 2</th>
<th>Stage 3</th>
<th>Stage 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2.000</td>
<td>2.000</td>
<td>2.000</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>2.000</td>
<td>2.000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Product / ( f_{ij} )</th>
<th>Stage 1</th>
<th>Stage 2</th>
<th>Stage 3</th>
<th>Stage 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1.1</td>
<td>500,000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>1.5</td>
<td>400,000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**EXAMPLE 2.** Table 3 presents the problem data. It deals with a four-stage plant, where there is one unit in each stage, except for the third one that has two units. Two products are elaborated. Table 4 shows the results for this case. In the solution without intermediate storage tanks, no unit has been added (Figure 4). The cost of the new units is so expensive that no unit has been incorporated. Production of A (which has the lower benefit) is reduced and its demand is not fulfilled.
Table 4. Results for Example 2.

<table>
<thead>
<tr>
<th>Solution without intermediate storage tanks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Product</td>
</tr>
<tr>
<td>Qi (kg)</td>
</tr>
<tr>
<td>Profit ($)</td>
</tr>
<tr>
<td>Stage</td>
</tr>
<tr>
<td>Vol. New Units</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Solution considering intermediate storage tanks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Product</td>
</tr>
<tr>
<td>Qi (kg)</td>
</tr>
<tr>
<td>Profit ($)</td>
</tr>
<tr>
<td>Stage</td>
</tr>
<tr>
<td>Vol. New Units</td>
</tr>
<tr>
<td>Position</td>
</tr>
<tr>
<td>Vol. Storage Tanks</td>
</tr>
</tbody>
</table>

Figure 4. Optimal solution of Example 2 without intermediate storage tanks.

When intermediate storage is allowed, all demands are satisfied (Figure 5). We have one unit in parallel at Stage 1 operating in different ways for each product: in-phase for Product A and out-of-phase for Product B. An intermediate storage tank is allocated between Stages 2 and 3. In this way, production capacity is increased at Stage 1 for Product A operating both units in-phase and reducing the limiting cycle time for Product B. The storage tank uncouples the original process into two subprocesses reducing the batch sizes for Stages 3 and 4 and allowing a higher productivity for both products. The added equipment cost is $227,900, that is justified by the increase of $294,800 in the value of Production A. Allocating a tank between Stages 2 and 3 improves the total benefits by 8%. In this particular example, allowance of intermediate storage tanks had a really important impact.

6. COMPUTATIONAL EFFICIENCY

Table 5 shows information on the different solved examples. The five examples proposed in [7] are included.
Figure 5. Optimal solution with the model proposed with disjunctions.

Table 5. Computational performance using generalized disjunctive programming with convex hull relaxation.

<table>
<thead>
<tr>
<th>Example</th>
<th>Number of Binary Variables</th>
<th>Total Number of Variables</th>
<th>Total Number of Constraints</th>
<th>CPU Time for this Proposal [sec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 1</td>
<td>60</td>
<td>155</td>
<td>234</td>
<td>3</td>
</tr>
<tr>
<td>Example 2</td>
<td>128</td>
<td>361</td>
<td>620</td>
<td>33</td>
</tr>
<tr>
<td>Example 3</td>
<td>120</td>
<td>295</td>
<td>468</td>
<td>15</td>
</tr>
<tr>
<td>Example 4</td>
<td>134</td>
<td>305</td>
<td>452</td>
<td>28</td>
</tr>
<tr>
<td>Example 5</td>
<td>208</td>
<td>455</td>
<td>687</td>
<td>28</td>
</tr>
</tbody>
</table>


<table>
<thead>
<tr>
<th>Example</th>
<th>Number of Binary Variables</th>
<th>Total Number of Variables</th>
<th>Total Number of Constraints</th>
<th>CPU Time for Reference [9] Formulation [sec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 1</td>
<td>37</td>
<td>118</td>
<td>287</td>
<td>2</td>
</tr>
<tr>
<td>Example 2</td>
<td>87</td>
<td>294</td>
<td>768</td>
<td>198</td>
</tr>
<tr>
<td>Example 3</td>
<td>73</td>
<td>216</td>
<td>535</td>
<td>21</td>
</tr>
<tr>
<td>Example 4</td>
<td>51</td>
<td>148</td>
<td>340</td>
<td>134</td>
</tr>
<tr>
<td>Example 5</td>
<td>101</td>
<td>260</td>
<td>585</td>
<td>155</td>
</tr>
</tbody>
</table>

The first and second columns correspond to the number of discrete variables and total variables of the model before performing the piece-wise linear approximation of the objective function. The following column corresponds to the total number of constraints of the model. Finally, the last column corresponds to the resolution time of the present formulation, i.e., using generalized disjunctive programming. All times correspond to the CPU time on the same equipment with a Pentium Celeron processor of 650 MHz, and using the same solvers.

The same examples were solved using the model proposed in [9], which contemplates intermediate storage tanks using conventional formulation. Table 6 shows the same information than
Table 5, but obtained with the conventional approach used in [9]. The same solvers and computer were employed.

Most of the examples show a decrease in the computation time required to solve the MINLP generated from the disjunctive formulation. One possible explanation for this behaviour is that the convex hull relaxation of the disjunctive set proposed gives a tighter feasible region than the conventional MINLP, as was shown in [16]. This tighter feasible region speeds up the problem solution, despite of the increase on the number of variables and restrictions of the problem. However, it cannot be conclusive neither generalized to any problem.

7. CONCLUSIONS

A generalized disjunctive model has been presented to solve the retrofit problem for multiproduct batch plants. This model takes into account all the currently allowed options. Disjunctions have been posed for the following discrete decisions.

(a) The different structures in which the \( N_j \) units can be arranged at each batch Stage \( j \) and for each Product \( i \), taking into account both: existing and new units. Useless units can be sold.

(b) Allocation of additional intermediate storage tanks among the different batch stages. Only one tank is allowed to be added to the existing ones. No longer-used tanks can be also sold.

In order to solve the model, disjunctions were transformed into mixed-integer constraints by applying their relaxation by means of the convex-hull.

By this formulation, advantages can be gained at the model level, since the latter is written in a more natural way, thus being much more clear and understandable than the ones previously presented [7,9]. Besides the greater easiness for modelling and understanding the problem, it is noted that the resolution time of the disjunctive model is shorter than the mixed-integer model (MINLP). On the other hand, the allocation of intermediate storage tanks, which is a nonavailable option in most of the previous articles, allows reaching better solutions with lower costs in the examples shown, even though it depends on the cost parameters assigned to the units. In the same way, the option of using intermediate storage tanks just for some products allows reaching better solutions as in the case of Example 1.

REFERENCES


