CONVEX ENVELOPES ON TREES

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Abstract. We introduce two notions of convexity for an infinite regular tree. For these two notions we show that given a continuous boundary datum there exists a unique convex envelope on the tree and characterize the equation that this envelope satisfies. We also relate the equation with two versions of the Laplacian on the tree. Moreover, for a function defined on the tree, the convex envelope turns out to be the solution to the obstacle problem for this equation.

1. Introduction

In mathematics, a real-valued function defined on an interval is called convex if the line segment between any two points on the graph of the function lies above or on the graph. Equivalently, a function is convex if its epigraph (the set of points on or above the graph of the function) is a convex set. For a twice differentiable function of a single variable, if the second derivative is always greater than or equal to zero in the entire interval then the function is convex.

Convex functions play an important role in many areas of mathematics. They are especially important in the study of optimization problems where they are distinguished by a number of convenient properties. For instance, a (strictly) convex function has no more than one minimum. Even in infinite-dimensional spaces, under suitable additional hypotheses, convex functions continue to satisfy such properties and as a result, they are the most well-understood functionals in the calculus of variations. In probability theory, a convex function applied to the expected value of a random variable is always less than or equal to the expected value of the convex function of the random variable.

On the other hand, linear and nonlinear equations (coming mainly from mean value properties) on trees are models that are close (and related to) to linear and nonlinear PDEs in the unit ball of $\mathbb{R}^N$, therefore, it seems natural to look for convex functions on trees. This is our main goal in this paper. For other analytical issues on discrete structures (including graphs such as trees) we refer to [1, 3, 5, 6, 7, 10, 11, 14, 15, 16] and references therein.

Let us begin by making precise the well-known notion of convexity in $\mathbb{R}^N$. We fix a bounded smooth domain $\Omega \subset \mathbb{R}^N$. A function $u\colon \Omega \to \mathbb{R}$ is said to be a convex function if for any two points $x, y \in \Omega$ such that the segment $[x, y]$ is contained in Ω, it holds that

$$u(tx + (1-t)y) \le tu(x) + (1-t)u(y), \qquad \forall t \in (0,1).$$

With this definition one can define the convex envelope of a boundary datum $g\colon \partial\Omega \to \mathbb{R}$ as

$$u^*(x) \coloneqq \sup \left\{ u(x) \colon u \text{ is convex and verifies } u|_{\partial\Omega} \le g \right\}.$$

Here by $u|_{\partial\Omega} \leq g$, we understand

$$\limsup_{\Omega \ni x \to z} u(z) \leq g(z) \quad \forall z \in \partial\Omega.$$

This convex envelope turns out to be the largest solution to

$$\lambda_1(D^2u)(x) = 0 \qquad x \in \Omega,$$

(the equation has to be interpreted in viscosity sense) with

$$u(x) \leq g(x) \qquad x \in \partial\Omega.$$

Here $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ are the ordered eigenvalues of the Hessian matrix, D^2u. We refer to [4, 9, 12, 13]. Notice that the equation

$$\lambda_1(D^2u)(x) = 0$$

is equivalent to

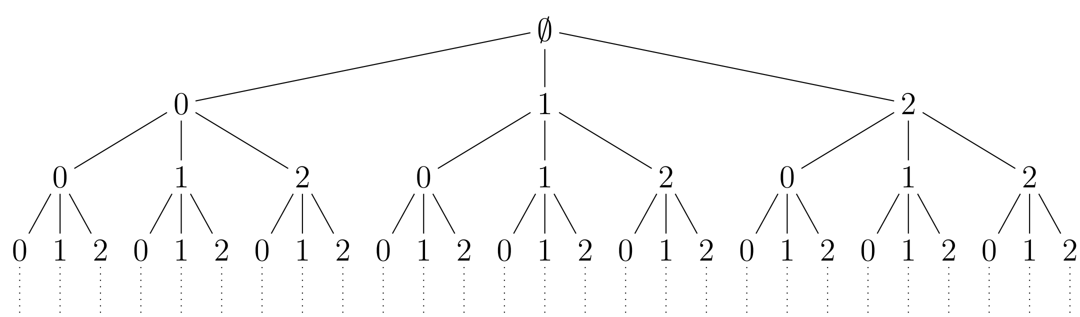
$$\min\left\{\langle D^2u(x)v, v\rangle : v \in \mathbb{R}^N \text{ such that } \|v\| = 1\right\} = 0.$$

This says that the equation that governs the convex envelope is just the minimum among all possible directions of the second derivative of the function at x equal to zero. Here we notice that we have existence of a continuous up to the boundary convex envelope for every continuous data if and only if the domain is strictly convex, see [4, 9, 12].

In this paper, our main goal is to develop similar ideas and concepts on a tree. When one wants to expend the notion of convexity to an ambient space beyond the Euclidean setting the two key ideas are to introduce what is a "segment" in our space and once this is done, to understand what is a "midpoint" in the segment. We introduce two different definitions of segments and midpoints and study the associated notion of convexity linked to each of them. In particular, for both definitions we are able to characterize the related equation that governs the convex envelope of a given boundary datum.

Closely related to our results is [2] where the authors considered convex functions on finite trees and showed that some relevant functions that are naturally related to spectral problems on the tree are convex.

Before starting with our main goal, we need to introduce our ambient space. Given $m \in \mathbb{N}_{\geq 2}$, a tree $\mathbb{T}_m$ with regular $m-$branching is an infinite graph that consists of the empty set $\emptyset$ and all finite sequences $(a_1, a_2, \ldots, a_k)$ with $k \in \mathbb{N}$, whose coordinates a_i are chosen from $\{0, 1, \ldots, m-1\}$.



A tree with 3−branching.

The elements in $\mathbb{T}_m$ are called vertices. Each vertex x has m successors, obtained by adding another coordinate. We will denote by

$$\mathcal{S}(x) := \{(x, i) \colon i \in \{0, 1, \ldots, m-1\}\}$$

the set of successors of the vertex x. If x is not the root then x has a only an immediate predecessor, which we will denote $\hat{x}$. The segment connecting a vertex x with $\hat{x}$ is called an edge and denoted by $(\hat{x}, x)$.

A vertex $x \in \mathbb{T}_m$ has level $k \in \mathbb{N}$ if $x = (a_1, a_2, \ldots, a_k)$. The level of x is denoted by $|x|$ and the set of all $k-$level vertices is denoted by $\mathbb{T}_m^k$. We say that the edge $e = (\hat{x}, x)$ has $k-$level if $x \in \mathbb{T}_m^k$.

A branch of $\mathbb{T}_m$ is an infinite sequence of vertices, where each one of them is followed by one of its immediate successors. The collection of all branches forms the boundary of $\mathbb{T}_m$, denoted by $\partial\mathbb{T}_m$. Observe that the mapping $\psi : \partial\mathbb{T}_m \to [0, 1]$ defined as

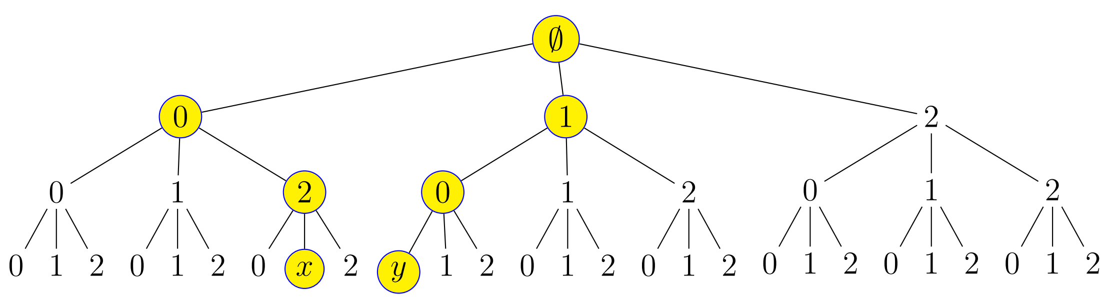
$$\psi(\pi) := \sum_{k=1}^{+\infty} \frac{a_k}{m^k}$$

is surjective, where $\pi = (a_1, \ldots, a_k, \ldots) \in \partial\mathbb{T}_m$ and $a_k \in \{0, 1, \ldots, m-1\}$ for all $k \in \mathbb{N}$. Whenever $x = (a_1, \ldots, a_k)$ is a vertex, we set

$$\psi(x) := \psi(a_1, \ldots, a_k, 0, \ldots, 0, \ldots).$$

At this point, we just have to recall that as we have mentioned, the definition of a convex function depends on what we understand by a segment and how to define the midpoint of the segment.

A path from a vertex x to a vertex y in $\mathbb{T}_m$ is a sequence of vertices $x_0, x_1, x_2, \ldots, x_k$ such that $x_0 = x$, $x_k = y_0$ and for any $i = 1, 2, \ldots, k$ we have that either $\hat{x}_{i-1} = x_i$ or $x_i \in \mathcal{S}(x_{i-1})$, that is, x_i and x_{i+1} are adjacent (connected) in the graph $\mathbb{T}_m$. A path is called a minimal path if and only if all the vertices on the path are different. Observe that for any $x, y \in \mathbb{T}_m$ there is a unique minimal path from x to y. This minimal path is denoted by $[x, y]$. This is our first idea of what is a segment of $\mathbb{T}_m$.



A path from a vertex x to a vertex y.

In a slight abuse of notation, we say that an edge e belongs to a path γ if there is a vertex $x \in \gamma$ such that $e = (\hat{x}, x)$. We now define the length of an edge e as follows:

$$\text{length}(e) := \frac{1}{m^k} \qquad \text{if } e \text{ has level } k.$$

With this length we can define the total length of a path γ as the sum of the lengths of the edges involved in γ, that is,

$$\text{lenght}(\gamma) := \sum_{e\in\gamma} \text{length}(e).$$

Now, with this notion of length of an edge and then of a path, let us consider the distance between nodes given by the length of the minimal path. That is, given two nodes x, y we let

$$d(x,y) := \text{lenght}([x,y]).$$

Remark that d is a genuine metric since $d(x,y) \geq 0$, $d(x,y) = 0$ iff $x = y$ and the triangular inequality holds (since the infimum of the lengths of the paths that joins x with y is less or equal than the infimum of the length of the paths that joins x with z plus the infimum of the length of the paths that joins z with y).

We are now ready to introduce our first notion of convex function. We say that a function $u : \mathbb{T}_m \mapsto \mathbb{R}$ is convex if for any $x, y, z \in \mathbb{T}_m$ with $z \in [x, y]$, it holds that

$$u(z) \leq \frac{d(y,z)}{d(x,y)} u(x) + \frac{d(x,z)}{d(x,y)} u(y).$$

Using this definition, we can look for the convex envelope of a function defined on $\partial\mathbb{T}_m$. Given a function $g\colon [0,1] \to \mathbb{R}$, we define the convex envelope of g on $\mathbb{T}_m$ as follows

$$u_g^*(x) := \sup\Big\{u(x)\colon u \in \mathcal{C}(g)\Big\}, \tag{1.1}$$

where

$$\mathcal{C}(g) := \Big\{u\colon \mathbb{T}_m \to \mathbb{R}\colon u \text{ is a convex function and } \limsup_{x\to\pi\in\partial\mathbb{T}_m} u(x) \leq g(\psi(\pi))\Big\}.$$

The convex envelope is unique (this follows easily since the maximum of two convex functions is also convex). Moreover, associated to this convex envelope we have a nonlinear equation that is verified on the whole tree.

Theorem 1.1. *The convex envelope of a continuous function $g\colon [0,1] \to \mathbb{R}$ is the largest solution to*

$$u(x) = \min\left\{ \min_{\substack{y,z\in\mathcal{S}(x)\\ y\neq z}} \frac{u(y)+u(z)}{2};\ \min_{y\in\mathcal{S}(x)} \frac{u(\hat{x}) + m\,u(y)}{m+1} \right\} \quad \text{on } \mathbb{T}_m, \tag{1.2}$$

that verifies

$$\limsup_{x\to\pi\in\partial\mathbb{T}_m} u(x) \leq g(\psi(\pi)). \tag{1.3}$$

Let us clarify that in the case $x = \emptyset$, relation (1.2) is understood as

$$u(x) = \min_{\substack{y,z\in\mathcal{S}(x)\\ y\neq z}} \frac{u(y)+u(z)}{2},$$

since $\emptyset$ does not have a predecessor because it is the root of $\mathbb{T}_m$.

Notice that (1.2) is a nonlinear mean value property at the nodes of the tree.

Once we have characterized the convex envelope by means of being the largest solution to (1.2) that is below the datum on $\partial\mathbb{T}_m$, we want to study the associated Dirichlet problem, that is, given a datum g on $\partial\mathbb{T}_m$ we want to solve the equation in $\mathbb{T}_m$ and find a solution that attains continuously the datum in the sense that

$$\lim_{x\to\pi\in\partial\mathbb{T}_m} u(x) = g(\psi(\pi)). \tag{1.4}$$

For this Dirichlet problem, we can show existence and uniqueness for continuous data.

Theorem 1.2. *Given a continuos boundary datum g, there is a unique solution to (1.2) that verifies (1.4).*

Therefore, from Theorems 1.1 and 1.2, we have that for every continuous datum on $\partial\mathbb{T}_m$ the unique convex envelope attains this datum with continuity, that is, (1.4) holds. Recall that in the Euclidean case we have to ask the domain Ω to be strictly convex for the validity of this continuity up to the boundary of the convex envelope.

Notice that the equation (1.2) can be rewritten as

$$0 = \min\left\{ \min_{\substack{y,z\in\mathcal{S}(x)\\ y\neq z}} \frac{u(y)+u(z)-2u(x)}{2}; \min_{y\in\mathcal{S}(x)} \frac{u(\hat{x})+mu(y)-(m+1)u(x)}{m+1}, \right\} \tag{1.5}$$

and hence we identify one possible analogous to the eigenvalues of the Hessian for the Euclidean case but in the case of the tree

$$\left\{\frac{u(x,i)+u(x,j)-2u(x)}{2}\right\}_{i<j} \quad \text{and} \quad \left\{\frac{u(\hat{x})+mu(y)-(m+1)u(x)}{m+1}\right\}_{y\in\mathcal{S}(x)}. \tag{1.6}$$

Recall that for the convex envelope in the Euclidean space the associated equation reads as

$$\min_{1\leq j\leq N} \lambda_j(D^2u) = 0,$$

and compare it with (1.5). Then, recalling that in the Euclidean setting when we add the eigenvalues of the Hessian we obtain the Laplacian, we obtain the following versions of a Laplacian on the tree adding the expressions found in (1.6),

$$u(x) = \frac{2}{(m+1)^2}u(\hat{x}) + \frac{m^2+2m-1}{(m+1)^2}\frac{1}{m}\sum_{y\in\mathcal{S}(x)} u(y).$$

Notice that this is a special case of the equations (given by mean value properties) that we previously studied in [8] showing existence and uniqueness for the Dirichlet problem.

Finally, we also study the convex envelope of a function defined on $\mathbb{T}_m$. That is, given a bounded function $f:\mathbb{T}_m\mapsto\mathbb{R}$ (not necessarily convex), we look for its convex envelope that is given by

$$u_f^*(x) \coloneqq \sup\left\{u(x)\colon u\in\mathcal{C}(f)\right\}, \tag{1.7}$$

where

$$\mathcal{C}(f) \coloneqq \Big\{u\colon \mathbb{T}_m\to\mathbb{R}\colon u \text{ is a convex function and } u(x)\leq f(x)\ \forall x\in\mathbb{T}_m\Big\}.$$

The convex envelope is unique (this follows easily since the maximum of two convex functions is also convex). Notice that when f attains a minimum this minimum coincides with the minimum of the convex envelope (and it is attained at the same vertices). This convex envelope turns out to be the solution to the obstacle problem for the equation (1.2). For the analogous property in the Euclidean setting, we refer to [13]. Recall that for the obstacle problem one important set is the coincidence set, that is given by the set of points where f and its convex envelope u_f^* coincide,

$$CS(f) = \left\{ x \in \mathbb{T}_m \, : \, f(x) = u_f^*(x) \right\}.$$

Theorem 1.3. *The convex envelope of a function $f\colon \mathbb{T}_m \to \mathbb{R}$ is the solution to the obstacle problem for the equation (1.2). That is, u_f^* is the largest solution to*

$$u(x) \leq \min \left\{ \min_{\substack{y,z \in \mathcal{S}(x)\\ y \neq z}} \frac{u(y)+u(z)}{2}; \min_{y \in \mathcal{S}(x)} \frac{u(\hat{x})+mu(y)}{m+1} \right\} \quad \text{on } \mathbb{T}_m,$$

that verifies

$$u(x) \leq f(x) \qquad \forall x \in \mathbb{T}_m.$$

In the coincidence set, the function f verifies an inequality

$$f(x) \leq \min \left\{ \min_{\substack{y,z \in \mathcal{S}(x)\\ y \neq z}} \frac{f(y)+f(z)}{2}; \min_{y \in \mathcal{S}(x)} \frac{f(\hat{x})+mf(y)}{m+1} \right\} \quad \text{on } CS(f),$$

while outside the coincidence set the convex envelope, u_f^*, is a solution of the equation, i.e., it holds

$$u_f^*(x) = \min \left\{ \min_{\substack{y,z \in \mathcal{S}(x)\\ y \neq z}} \frac{u_f^*(y)+u_f^*(z)}{2}; \min_{y \in \mathcal{S}(x)} \frac{u_f^*(\hat{x})+mu_f^*(y)}{m+1} \right\} \quad \text{on } \mathbb{T}_m \setminus CS(f).$$

On the other hand, in the arborescence (directed) tree, i.e., the tree defined as before but adding a direction to the edges in such a way that any two edges are not directed to the same vertex and the root is the unique vertex that has no edge directed into it), the Laplacian is defined as the mean value of the successors minus the value at the vertex, that is,

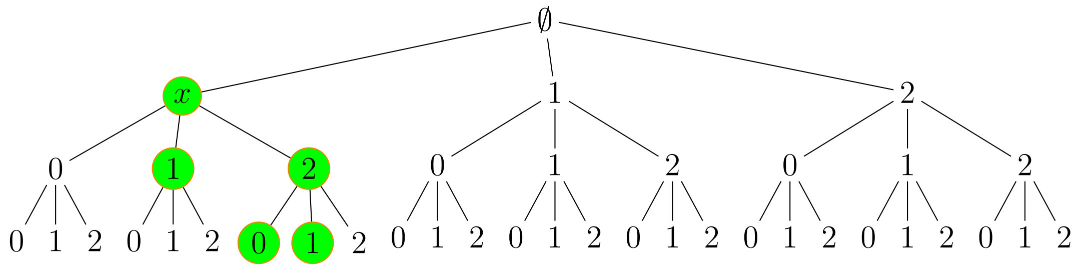
$$\Delta u(x) := \frac{1}{m} \sum_{y \in \mathcal{S}(x)} u(y) - u(x) \quad \forall x \in \mathbb{T}_m,$$

see [10]. Now, to obtain an interpretation of this Laplacian as the sum of eigenvalues of the Hessian as we did before, we just have to change the notion of convexity.

Now we need to introduce extra notations. Given $x \in \mathbb{T}_m$, $\mathbb{T}_2^x$ denotes the set of all subgraphs $\mathbb{B}$ that are formed from a finite subset of the vertices of $\mathbb{T}_m$ and such that $x \in \mathbb{B}$, $\mathcal{S}(x) \cap \mathbb{B}$ has two elements and for any $y \in \mathbb{B} \setminus \{x\}$,

- $|y| > |x|$;
- either $\mathcal{S}(y) \cap \mathbb{B} = \emptyset$ or $\mathcal{S}(y) \cap \mathbb{B}$ has exactly two elements.

We say that $y \in \mathbb{B}$ is an endpoint of $\mathbb{B}$ if $\mathcal{S}(y) \cap \mathbb{B} = \emptyset$. The set of all endpoints of $\mathbb{B}$ is denoted $\mathcal{E}(\mathbb{B})$. That is, $\mathbb{B}$ is just a finite binary subtree of $\mathbb{T}_m$.



An element of $\mathbb{T}_2^x$. Root: x. Endpoints: $(x,1)$, $(x,2,0)$, and $(x,2,1)$.

Our second notion of convexity is the following: a function $u:\mathbb{T}_m\to\mathbb{R}$ is called binary convex if for any $x\in\mathbb{T}_m$

$$u(x)\le \sum_{y\in\mathcal{E}(\mathbb{B})}\frac{1}{2^{|y|-|x|}}u(y)\quad \forall \mathbb{B}\in\mathbb{T}_2^x.$$

In this notion of convexity, a segment is $\mathbb{B}$, a finite binary subgraph of $\mathbb{T}_m$; a midpoint is the root of this subgraph $\mathbb{B}$ and the convexity property just says that the value of the function at the midpoint is less or equal than the mean value of the values at the endpoints.

Associated to this new version of convexity, we have a convex envelope of a bounded boundary datum g that is, defined as in (1.1), that is we define the binary convex envelope of g on $\mathbb{T}_m$ as follows

$$\tilde{u}_g(x):=\sup\{u(x)\colon u\in\mathcal{B}(g)\}$$

where

$$\mathcal{B}(g):=\left\{u\colon\mathbb{T}_m\to\mathbb{R}\colon \text{ is a binary convex function on } \mathbb{T}_m,\ \limsup_{x\to\pi\in\partial\mathbb{T}_m}u(x)\le g(\psi(\pi))\right\}.$$

For this notion of binary convex envelope, we also have an equation (simpler than with the previous notion).

Theorem 1.4. *The binary convex envelope of a bounded boundary datum g is the largest solution to*

$$u(x)=\min_{\substack{y,z\in\mathcal{S}(x)\\ y\neq z}}\frac{u(y)+u(z)}{2}\quad \text{on } \mathbb{T}_m, \tag{1.8}$$

that verifies (1.3)*.*

Again, when g is continuous we have a unique solution to the equation that attains the boundary datum continuously.

Theorem 1.5. *Given a continuous boundary datum g, there exists a unique* (1.8) *that verifies* (1.4)*.*

In this case, written (1.8) as

$$0=\min_{\substack{y,z\in\mathcal{S}(x)\\ y\neq z}}\left\{\frac{1}{2}u(y)+\frac{1}{2}u(z)-u(x)\right\},$$

we can identify the analogous to the eigenvalues of the Hessian that for this case are given by,

$$\left\{\frac{1}{2}u(x,i)+\frac{1}{2}u(x,j)-u(x)\right\}_{i<j}. \tag{1.9}$$

Then, adding the eigenvalues in (1.9), we obtain

$$0=\frac{1}{m}\sum_{y\in\mathcal{S}(x)}u(y)-u(x). \tag{1.10}$$

Notice that this is the usual Laplacian in the arborescence tree studied in [10].

For this notion of convexity we can also deal with the problem of the convex envelope of a function $f:\mathbb{T}_m\mapsto\mathbb{R}$ defined as in (1.7). In this case we also find that this convex envelope can be characterized as the solution to the obstacle problem for the associated equation, (1.8), and prove a theorem analogous to Theorem 1.3 for this case. Once we have proved Theorem 1.4, the proof of this result is similar to the one of Theorem 1.3 and hence we leave the statement and the proof to the reader.

To end this introduction, let us give a natural generalization of the notion of binary convexity. Given $k\in\{2,\dots,m-2\}$ and $x\in\mathbb{T}_m$, $\mathbb{T}_2^{xk}$ denotes the set of all subgraphs $\mathbb{B}$ that are formed from a finite subset of vertices in $\mathbb{T}_m$ and such that, $x\in\mathbb{B}$, $\mathcal{S}(x)\cap\mathbb{B}$ has exactly k elements and for any $y\in\mathbb{B}\setminus\{x\}$,

- $|y|>|x|$;
- either $\mathcal{S}(y)\cap\mathbb{B}=\emptyset$ or $\mathcal{S}(y)\cap\mathbb{B}$ has exactly k elements.

Observe that $\mathbb{T}_2^{x2}=\mathbb{T}_2^{x}$. As before we denote by $\mathcal{E}(\mathbb{B})$ (the set of endpoints) the set of points $y\in\mathbb{B}$ such that $\mathcal{S}(y)\cap\mathbb{B}=\emptyset$.

A function $u:\mathbb{T}_m\to\mathbb{R}$ is called $k-$convex if for any $x\in\mathbb{T}_m$

$$u(x)\leq\sum_{y\in\mathcal{E}(\mathbb{B})}\frac{1}{k^{|y|-|x|}}u(y)\quad\forall\mathbb{B}\in\mathbb{T}_2^{xk}.$$

As before, associated to this version of convexity, we have a convex envelope of a bounded boundary datum g that we will call the $k-$convex envelope of g. Following the proof of Theorems 1.4 and 1.5, we can show that the $k-$convex envelope of g is the largest solution to

$$u(x)=\min_{\substack{x_1,\dots,x_k\in\mathcal{S}(x)\\ x_i\neq x_j}}\frac{1}{k}\sum_{i=1}^{k}u(x_i)\quad\text{on }\mathbb{T}_m, \tag{1.11}$$

that verifies (1.3). Moreover, if g is a continuous function then the $k-$convex envelope of g is the a unique solution to (1.11) that verifies (1.4).

In this case, written (1.11) as

$$0=\min_{\substack{x_1,\dots,x_k\in\mathcal{S}(x)\\ x_i\neq x_j}}\left\{\frac{1}{k}\sum_{i=1}^{k}u(x_i)-u(x)\right\},$$

we identify the analogous to the eigenvalues of the Hessian,

$$\left\{ \frac{1}{k} \sum_{i=1}^{k} u(x, j_i) - u(x), \right\}_{j_i < j_{i+1}}. \tag{1.12}$$

Adding the eigenvalues in (1.12), we obtain again (1.10), the usual Laplacian on the arborescence tree.

Organization of the paper. In Section 2, we will prove general results for convex functions; In Section 3, we prove our main result for the convex envelope; In Sections 4 and 5, we extend the results of previous sections to the notion of binary convexity.

2. Convex functions

We begin this section showing a different characterization of convex functions.

Lemma 2.1. *A function u on the tree is convex if and only if u satisfies*

$$u(x) \leq \min \left\{ \min_{\substack{y,z \in \mathcal{S}(x) \\ y \neq z}} \frac{u(y) + u(z)}{2}; \min_{y \in \mathcal{S}(x)} \frac{u(\hat{x}) + m u(y)}{m+1} \right\} \qquad \forall x \in \mathbb{T}_m. \tag{2.13}$$

Proof. Let u be a convex function. Our goal is to show that (2.13) holds. Given x for any $y, z \in \mathcal{S}(x)$ with $y \neq z$ we have that

$$u(x) \leq \frac{1}{2} u(y) + \frac{1}{2} u(z)$$

due to the fact that u is a convex function (just take $x \in [y, z]$, $d(y, z) = \frac{2}{m^{|x|+1}}$, and $d(y, x) = d(z, x) = \frac{1}{m^{|x|+1}}$ in the definition of convexity). Then, we get

$$u(x) \leq \min_{\substack{y,z \in \mathcal{S}(x) \\ y \neq z}} \frac{u(y) + u(z)}{2}$$

for any $x \in \mathbb{T}_m$.

Now, given $x \in \mathbb{T}_m$ for any $y \in \mathcal{S}(x)$

$$u(x) \leq \frac{m}{m+1} u(y) + \frac{1}{m+1} u(\hat{x})$$

again due to the fact that u is a convex function (in this case, take $x \in [\hat{x}, y]$, $d(\hat{x}, y) = \frac{m+1}{m^{|x|+1}}$, $d(\hat{x}, x) = \frac{1}{m^{|x|}}$, and $d(y, x) = \frac{1}{m^{|x|+1}}$). Thus

$$u(x) \leq \min_{y \in \mathcal{S}(x)} \frac{u(\hat{x}) + m u(y)}{m+1}$$

for any $x \in \mathbb{T}_m$. Therefore, we have that if u is a convex function then u satisfies (2.13).

To see the converse, let u be a function defined on the tree that verifies (2.13) at every node and $x, y \in \mathbb{T}_m$. We begin by analyzing the case that $[x, y]$ is a "straight line", that is the vertices $x_0, \ldots, x_N$ of $[x, y]$ are such that $x_0 = x$, $x_N = y$, $\hat{x}_i = x_{i+1}$ for any $i \in \{0, \ldots, N-1\}$. More precisely, first we show that if $[x, y]$ is a "straight line" then

$$u(x_i) \leq \frac{d(x_i, x_0)}{d(x_N, x_0)} u(x_N) + \frac{d(x_i, x_N)}{d(x_N, x_0)} u(x_0) \quad \forall i \in \{0, \ldots, N\}. \tag{2.14}$$

We proceed by induction in N. When $N=2$, by (2.13), we have

$$u(x_1) \le \frac{u(x_2)+mu(x_0)}{m+1} = \frac{d(x_1,x_0)}{d(x_2,x_0)}u(x_2) + \frac{d(x_1,x_2)}{d(x_2,x_0)}u(x_0)$$

since $d(x_1,x_2)=\frac{1}{m^{|x_1|}}$, $d(x_1,x_0)=\frac{1}{m^{|x_0|}}=\frac{1}{m^{|x_1|+1}}$ and $d(x_2,x_0)=\frac{m+1}{m^{|x_1|+1}}$.

Suppose now that (2.14) is true for all straight line that have $N-1$ vertices, where $N>2$. Then,

$$u(x_1) \le \frac{d(x_1,x_{N-1})}{d(x_{N-1},x_0)}u(x_0) + \frac{d(x_1,x_0)}{d(x_{N-1},x_0)}u(x_{N-1})$$

and

$$u(x_{N-1}) \le \frac{d(x_1,x_{N-1})}{d(x_N,x_1)}u(x_N) + \frac{d(x_N,x_{N-1})}{d(x_N,x_1)}u(x_1).$$

Thus,

$$\begin{aligned} u(x_1) \le \frac{d(x_1,x_{N-1})}{d(x_{N-1},x_0)}u(x_0) &+ \frac{d(x_1,x_0)}{d(x_{N-1},x_0)}\frac{d(x_1,x_{N-1})}{d(x_N,x_1)}u(x_N) \\ &+ \frac{d(x_1,x_0)}{d(x_{N-1},x_0)}\frac{d(x_N,x_{N-1})}{d(x_N,x_1)}u(x_1). \end{aligned}$$

Therefore,

$$\begin{aligned} & d(x_1,x_{N-1})\left[d(x_1,x_N)u(x_0)+d(x_1,x_0)u(x_N)\right] \\ & \ge \left[d(x_{N-1},x_0)d(x_1,x_N) - d(x_1,x_0)d(x_{N-1},x_N)\right]u(x_1) \\ & \ge \left[\{d(x_N,x_0)-d(x_N,x_{N-1})\}\{d(x_N,x_0)-d(x_1,x_0)\} - d(x_1,x_0)d(x_{N-1},x_N)\right]u(x_1) \\ & \ge d(x_N,x_0)\left[d(x_N,x_0)-d(x_N,x_{N-1})-d(x_1,x_0)\right]u(x_1) \\ & \ge d(x_N,x_0)d(x_1,x_{N-1})u(x_1). \end{aligned}$$

Then

$$u(x_1) \le \frac{d(x_1,x_0)}{d(x_N,x_0)}u(x_N) + \frac{d(x_1,x_N)}{d(x_N,x_0)}u(x_N).$$

In similar manner, we get

$$u(x_{N-1}) \le \frac{d(x_{N-1},x_0)}{d(x_N,x_0)}u(x_N) + \frac{d(x_{N-1},x_N)}{d(x_N,x_0)}u(x_N). \tag{2.15}$$

If $z \in [x,y] \setminus \{x_0, x_1, x_{N-1}, x_N\}$, by the inductive hypothesis and (2.15), we have

$$
\begin{aligned}
u(z) &\leq \frac{d(z,x_{N-1})}{d(x_{N-1},x_0)}u(x_0) + \frac{d(z,x_0)}{d(x_{N-1},x_0)}u(x_{N-1}) \\
&\leq \frac{d(z,x_0)}{d(x_N,x_0)}u(x_N) + \frac{d(z,x_{N-1})d(x_N,x_0) + d(z,x_0)d(x_{N-1},x_N)}{d(x_{N-1},x_0)d(x_N,x_0)}u(x_0) \\
&\leq \frac{d(z,x_0)}{d(x_N,x_0)}u(x_N) \\
&\quad + \frac{\left[d(z,x_N) - d(x_{N-1},x_N)\right]d(x_N,x_0) + \left[d(x_N,x_0) - d(z,x_N)\right]d(x_{N-1},x_N)}{d(x_{N-1},x_0)d(x_N,x_0)}u(x_0) \\
&\leq \frac{d(z,x_0)}{d(x_N,x_0)}u(x_N) + d(z,x_N)\frac{d(x_N,x_0) - d(x_{N-1},x_N)}{d(x_{N-1},x_0)d(x_N,x_0)}u(x_0) \\
&\leq \frac{d(z,x_0)}{d(x_N,x_0)}u(x_N) + \frac{d(z,x_N)}{d(x_N,x_0)}u(x_0),
\end{aligned}
$$

showing that indeed (2.14) holds when $[x,y]$ is a straight line.

When $[x,y]$ is not a straight line, there is a $z \in [x,y]$ such that $[x,z]$ and $[z,y]$ are straight lines. Observe that $[x,y] = [x,z] \cup [z,y]$ and $\mathcal{S}(z) \cap [x,y] = \{w_1, w_2\}$. We can assume that $w_1 \in [x,z]$ and $w_2 \in [z,y]$.

Thus, from (2.14), we have

$$
\begin{aligned}
2u(z) &\leq u(w_1) + u(w_2) \\
&\leq \left[\frac{d(w_1,x)}{d(x,z)} + \frac{d(w_2,y)}{d(y,z)}\right]u(z) + \frac{d(w_1,z)}{d(x,z)}u(x) + \frac{d(w_2,z)}{d(y,z)}u(y).
\end{aligned}
$$

Then,

$$
\frac{2d(x,z)d(y,z) - d(w_1,x)d(z,y) - d(w_2,y)d(z,x)}{d(z,x)d(z,y)}u(z) \leq \frac{d(w_1,z)}{d(x,z)}u(x) + \frac{d(w_2,z)}{d(y,z)}u(y).
$$

Since $d(w_1,z) = d(w_2,z)$, we get

$$
\begin{aligned}
&d(w_1,z)\left[d(y,z)u(x) + d(x,z)u(y)\right] \\
&\qquad \geq \left[2d(x,z)d(y,z) - d(w_1,x)d(z,y) - d(w_2,y)d(z,x)\right]u(z) \\
&\qquad \geq \left\{2d(x,z)d(y,z) - \left[d(x,z) - d(w_1,z)\right]d(z,y) - \left[d(y,z) - d(w_2,z)\right]d(z,x)\right\}u(z) \\
&\qquad \geq d(w_1,z)\left[d(z,y) + d(z,x)\right]u(z) \\
&\qquad \geq d(w_1,z)d(x,y)u(z).
\end{aligned}
$$

Therefore, we obtain

$$
u(z) \leq \frac{d(x,z)}{d(x,y)}u(y) + \frac{d(y,z)}{d(x,y)}u(x).
$$

If $w \in [x,y] \setminus \{x,z,y\}$ then $w \in [x,z]$ or $w \in [z,y]$. In the case that $w \in [x,z]$, since $[x,z]$ is a straight line we have

$$\begin{aligned}
u(w) &\leq \frac{d(x,w)}{d(x,z)}u(z) + \frac{d(z,w)}{d(x,z)}u(x) \\
&\leq \frac{d(x,w)}{d(x,y)}u(y) + \left[\frac{d(x,w)d(y,z)}{d(x,z)d(x,y)} + \frac{d(z,w)}{d(x,z)}\right]u(x) \\
&\leq \frac{d(x,w)}{d(x,y)}u(y) + \frac{d(x,w)d(y,z) + d(z,w)d(x,y)}{d(x,z)d(x,y)}u(x) \\
&\leq \frac{d(x,w)}{d(x,y)}u(y) + \frac{\left[d(x,y) - d(y,w)\right]d(y,z) + d(z,w)d(x,y)}{d(x,z)d(x,y)}u(x) \\
&\leq \frac{d(x,w)}{d(x,y)}u(y) + \frac{\left[d(y,z) + d(z,w)\right]d(x,y) - d(y,w)d(y,z)}{d(x,z)d(x,y)}u(x) \\
&\leq \frac{d(x,w)}{d(x,y)}u(y) + \frac{d(y,w)\left[d(x,y) - d(y,z)\right]}{d(x,z)d(x,y)}u(x) \\
&\leq \frac{d(x,w)}{d(x,y)}u(y) + \frac{d(y,w)}{d(x,y)}u(x).
\end{aligned}$$

In the case that $w \in [z,y]$ the proof is similar.

Therefore, we conclude that a function u that verifies (2.13) is a convex function in $\mathbb{T}_m$. □

Our second result show that the sum of convex function is also a convex function.

Corollary 2.2. *Let $u, v\colon \mathbb{T}_m \to \mathbb{R}$ be convex functions. Then $u+v$ is a convex function.*

Proof. Since u, v are convex function, by Lemma 2.1, for any $x \in \mathbb{T}_m$ we have that

$$\begin{aligned}
u(x) + v(x) \leq \min &\left\{ \min_{\substack{y,z\in\mathcal{S}(x)\\ y\neq z}} \frac{u(y)+u(z)}{2}; \min_{y\in\mathcal{S}(x)} \frac{u(\hat{x}) + mu(y)}{m+1} \right\} \\
&+ \min\left\{ \min_{\substack{y,z\in\mathcal{S}(x)\\ y\neq z}} \frac{v(y)+v(z)}{2}; \min_{y\in\mathcal{S}(x)} \frac{v(\hat{x}) + mv(y)}{m+1} \right\} \\
\leq \min &\left\{ \min_{\substack{y,z\in\mathcal{S}(x)\\ y\neq z}} \frac{u(y)+u(z)}{2} + \frac{v(y)+v(z)}{2}; \min_{y\in\mathcal{S}(x)} \frac{u(\hat{x}) + mu(y)}{m+1} + \frac{v(\hat{x}) + mv(y)}{m+1} \right\}.
\end{aligned}$$

Therefore, by Lemma 2.1, $u+v$ is a convex function. □

It is immediate to check that the constant function $u = 1$ is a convex function such that

$$\lim_{x\to\pi} u(x) = \chi_{[0,1]}(\psi(\pi)) \quad \forall \pi \in \partial\mathbb{T}_m.$$

We now show that for any $x_0 \in \mathbb{T}_m \setminus \{\emptyset\}$ there is a convex function u such that

$$\limsup_{x\to\pi} u(x) \le \chi_{I_{x_0}}(\psi(\pi)) \quad \pi \in \partial\mathbb{T}_m.$$

Here I_{x_0} is the interval associated to the vertex x_0 of length $\frac{1}{m^{|x_0|}}$ given by

$$I_{x_0} := \left[\psi(x_0), \psi(x_0) + \frac{1}{m^{|x_0|}}\right].$$

Observe that for $x_0 \in \mathbb{T}_m$, $I_{x_0} \cap \partial\mathbb{T}_m$ is the subset of $\partial\mathbb{T}_m$ consisting of all branches that pass through x_0.

To find such a convex function we introduce the following set: given $x_0 \in \mathbb{T}_m$, let us consider

$$\mathbb{T}_m^{x_0} := \{x \in \mathbb{T}_m \colon |x| \ge |x_0|, I_x \subset I_{x_0}\}.$$

Lemma 2.3. *Let $x_0 \in \mathbb{T}_m \setminus \{\emptyset\}$. Then the function $u_{x_0} \colon \mathbb{T}_m \to \mathbb{R}$*

$$u_{x_0}(x) := \frac{m-1}{m}\begin{cases} 0 & \text{if } x \notin \mathbb{T}_m^{x_0}, \\ \displaystyle\sum_{i=0}^{|x|-|x_0|} \frac{1}{m^i} & \text{if } x \in \mathbb{T}_m^{x_0}, \end{cases}$$

is a convex function such that

$$\limsup_{x\to\pi} u_{x_0}(x) \le \chi_{I_{x_0}}(\psi(\pi)) \quad \forall \pi \in \partial\mathbb{T}_m.$$

Moreover,

$$u_{x_0}(x) = \min\left\{ \min_{\substack{y,z\in\mathcal{S}(x)\\ y\neq z}} \frac{u_{x_0}(y)+u_{x_0}(z)}{2}; \min_{y\in\mathcal{S}(x)} \frac{u_{x_0}(\hat{x}) + m u_{x_0}(y)}{m+1} \right\} \quad \forall x \in \mathbb{T}_m,$$

and for any $\pi \in \mathbb{T}_m$ such that $\psi(\pi)$ is not one of the two endpoints of I_{x_0} we have

$$\lim_{x\to\pi} u_{x_0}(x) = \chi_{I_{x_0}}(\psi(\pi)).$$

Proof. Let us start by showing that the function u_{x_0} is convex. By Lemma (2.1), it is enough to show that u_{x_0} satisfies (2.13). If $x \in \mathbb{T}_m \setminus \mathbb{T}_m^{x_0}$ then there exist $y, z \in \mathcal{S}(x)$ such that $y \neq z$, $u_{x_0}(y) = u_{x_0}(z) = 0$. So, we have

$$\min\left\{ \min_{\substack{y,z\in\mathcal{S}(x)\\ y\neq z}} \frac{u_{x_0}(y)+u_{x_0}(z)}{2}; \min_{y\in\mathcal{S}(x)} \frac{u_{x_0}(\hat{x}) + m u_{x_0}(y)}{m+1} \right\} = 0 = u_{x_0}(x). \tag{2.16}$$

If $x = x_0$, then $u_{x_0}(\hat{x}_0) = 0$ and $u_{x_0}(y) = \frac{m-1}{m}(1+\frac{1}{m})$ for any $y \in \mathcal{S}(x_0)$. Therefore

$$\begin{aligned} &\min\left\{ \min_{\substack{y,z\in\mathcal{S}(x_0)\\ y\neq z}} \frac{u_{x_0}(y)+u_{x_0}(z)}{2}; \min_{y\in\mathcal{S}(x_0)} \frac{u_{x_0}(\hat{x}) + m u_{x_0}(y)}{m+1} \right\} \\ &= \frac{m-1}{m}\min\left\{1+\frac{1}{m}; 1\right\} \\ &= \frac{m-1}{m} \\ &= u_{x_0}(x_0). \end{aligned} \tag{2.17}$$

Now, suppose that $x \in \mathbb{T}_m \setminus \{x_0\}$, and so,

$$u_{x_0}(\hat{x}) = \frac{m-1}{m} \sum_{i=0}^{|x|-1-|x_0|} \frac{1}{m^i}$$

and

$$u_{x_0}(y) = \frac{m-1}{m} \sum_{i=0}^{|x|+1-|x_0|} \frac{1}{m^i}.$$

Hence, we obtain

$$\begin{aligned} \min \left\{ \min_{\substack{y,z \in \mathcal{S}(x_0) \\ y \neq z}} \frac{u_{x_0}(y) + u_{x_0}(z)}{2}; \min_{y \in \mathcal{S}(x_0)} \frac{u_{x_0}(\hat{x}) + m u_{x_0}(y)}{m+1} \right\} \\ = \frac{m-1}{m} \min \left\{ \sum_{i=0}^{|x|+1-|x_0|} \frac{1}{m^i}; \sum_{i=0}^{|x|-|x_0|} \frac{1}{m^i} \right\} \\ = \frac{m-1}{m} \sum_{i=0}^{|x|-|x_0|} \frac{1}{m^i} \\ = u_{x_0}(x). \end{aligned} \tag{2.18}$$

Therefore, by (2.16), (2.17) and (2.18) we get that

$$u_{x_0}(x) = \min \left\{ \min_{\substack{y,z \in \mathcal{S}(x) \\ y \neq z}} \frac{u_{x_0}(y) + u_{x_0}(z)}{2}; \min_{y \in \mathcal{S}(x)} \frac{u_{x_0}(\hat{x}) + m u_{x_0}(y)}{m+1} \right\} \quad \forall x \in \mathbb{T}_m.$$

Thus, u_{x_0} is a convex function.

Finally, we have to show that

$$\limsup_{x \to \pi} u_{x_0}(x) \leq \chi_{I_{x_0}}(\psi(\pi)) \quad \forall \pi \in \partial \mathbb{T}_m.$$

Case 1. If $\pi \in \partial\mathbb{T}_m$, $\psi(\pi) \in I_{x_0}$ and $\psi(\pi)$ is not an endpoint of I_{x_0} then for any sequence $\{x_k\}_{k\in\mathbb{N}}$ in $\mathbb{T}_m$ such that $\pi = (x_1, \dots, x_k, \dots)$, there is $k_0 \in \mathbb{N}$ such that $x_k \in \mathbb{T}_m^{x_0}$ for all $k \geq k_0$. Then

$$u_{x_0}(x_k) = \frac{m-1}{m} \sum_{i=0}^{|x_k|-|x_0|} \frac{1}{m^i} \quad \forall k \leq k_0.$$

Thus, as $k \to \infty$ we have

$$u_{x_0}(x_k) \to 1 = \chi_{I_{x_0}}(\psi(\pi)).$$

Case 2. Similarly, if $\pi \in \partial\mathbb{T}_m$, $\psi(\pi) \notin I_{x_0}$ then for any sequence $\{x_k\}_{k\in\mathbb{N}}$ on $\mathbb{T}_m$ such that $\pi = (x_1, \dots, x_k, \dots)$, we get

$$u_{x_0}(x_k) \to 0 = \chi_{I_{x_0}}(\psi(\pi))$$

as $k \to \infty$.

Case 3. Finally suppose that $\pi \in \partial\mathbb{T}_m$, $\psi(\pi) \in I_{x_0}$ and $\psi(\pi)$ is an endpoint of I_{x_0}.

In the case that $\psi(\pi) = 0$ or $\psi(\pi) = 1$ for any sequence $\{x_k\}_{k\in\mathbb{N}}$ on $\mathbb{T}_m$ such that $\pi = (x_1, \dots, x_k, \dots)$, there is $k_0 \in \mathbb{N}$ such that $x_k \in \mathbb{T}_m^{x_0}$ for all $k \geq k_0$. Therefore

$$u_{x_0}(x_k) \to 1 = \chi_{I_{x_0}}(\psi(\pi))$$

as $k \to \infty$.

In the case that $\psi(\pi) \notin \{0, 1\}$ there exists sequences $\{x_k\}_{k\in\mathbb{N}}$, $\{y_k\}_{k\in\mathbb{N}}$ on $\mathbb{T}_m$ and $k_0 \in \mathbb{N}$ such that $\pi = (x_1, \dots, x_k, \dots)$, $\pi = (y_1, \dots, y_k, \dots)$, for all $k \geq k_0$ we get $x_k \in \mathbb{T}_m^{x_0}$ and $y_k \in \mathbb{T}_m^{x_0}$. Therefore,

$$\begin{aligned} u_{x_0}(x_k) &\to 1 = \chi_{I_{x_0}}(\psi(\pi)), \\ u_{x_0}(y_k) &\to 0 \leq \chi_{I_{x_0}}(\psi(\pi)). \end{aligned}$$

This fact, together with the previous cases 1 and 2, completes the proof. □

3. Convex envelopes

In this section we deal with convex functions on the tree. Let us start by showing that the convex envelop u_g^* of function $g\colon [0,1] \to \mathbb{R}$, defined in (1.1), is a convex function.

Lemma 3.1. *For any function $g\colon [0,1] \to \mathbb{R}$, the convex envelop u_g^* is a convex function.*

Proof. This follows easily from the fact that the supremum of convex functions is also convex. Given $g\colon [0,1] \to \mathbb{R}$, for every function $u \in \mathcal{C}(g)$ it holds that

$$u(z) \leq \frac{d(y,z)}{d(x,y)} u(x) + \frac{d(x,z)}{d(x,y)} u(y) \leq \frac{d(y,z)}{d(x,y)} u_g^*(x) + \frac{d(x,z)}{d(x,y)} u_g^*(y)$$

for any $x, y, z \in \mathbb{T}_m$ with $z \in [x, y]$. Hence we get

$$u_g^*(z) \leq \frac{d(y,z)}{d(x,y)} u_g^*(x) + \frac{d(x,z)}{d(x,y)} u_g^*(y)$$

for any $x, y, z \in \mathbb{T}_m$ with $z \in [x, y]$. Thus u_g^* is a convex function. □

Our second aim is to show that if g is a continuous function then

$$\lim_{x\to\pi\in\partial\mathbb{T}_m} u_g^*(x) = g(\psi(\pi)) \quad \forall \pi \in \partial\mathbb{T}_m. \tag{3.1}$$

To prove this property, we need to show a comparison principle.

Lemma 3.2. *Let u and v satisfy*

$$u(x) \geq \min\left\{ \min_{\substack{y,z\in\mathcal{S}(x_0)\\ y\neq z}} \frac{u(y)+u(z)}{2}; \min_{y\in\mathcal{S}(x_0)} \frac{u(\hat{x}_0)+mu(y)}{m+1} \right\} \quad \forall x \in \mathbb{T}_m, \tag{3.2}$$

$$v(x) \leq \min\left\{ \min_{\substack{y,z\in\mathcal{S}(x_0)\\ y\neq z}} \frac{v(y)+v(z)}{2}; \min_{y\in\mathcal{S}(x_0)} \frac{v(\hat{x}_0)+mv(y)}{m+1} \right\} \quad \forall x \in \mathbb{T}_m, \tag{3.3}$$

with

$$\lim_{x\to\pi\in\partial\mathbb{T}_m} u(x) \geq \lim_{x\to\pi\in\partial\mathbb{T}_m} v(x),$$

for every $\pi \in \partial\mathbb{T}_m$. Then,

$$u(x) \geq v(x) \qquad \forall x \in \mathbb{T}_m.$$

Proof. Adding a positive constant c to u, we may assume that

$$\lim_{x\to\pi\in\partial\mathbb{T}_m} u(x) > \lim_{x\to z\in\partial\mathbb{T}_m} v(x). \tag{3.4}$$

Our goal is to show that in this case we have

$$u(x) \geq v(x) \quad \forall x \in \mathbb{T}_m$$

(and then we obtain the result just by letting $c \to 0$).

We argue by contradiction and assume that

$$M = \max_{x\in\mathbb{T}_m}(v(x) - u(x)) > 0.$$

Notice that the maximum is attained thanks to (3.4). Also thanks to (3.4), we have that M is attained only in a finite set of nodes. Let x be one of such nodes. From (3.2) and (3.3) we obtain

$$M = v(x) - u(x) \leq \min\left\{ \min_{\substack{y,z\in\mathcal{S}(x_0)\\ y\neq z}} \frac{v(y)+v(z)}{2}; \min_{y\in\mathcal{S}(x_0)} \frac{v(\hat{x}_0)+mv(y)}{m+1} \right\}$$
$$- \min\left\{ \min_{\substack{y,z\in\mathcal{S}(x_0)\\ y\neq z}} \frac{u(y)+u(z)}{2}; \min_{y\in\mathcal{S}(x_0)} \frac{u(\hat{x}_0)+mu(y)}{m+1} \right\}.$$

From this inequality, using that

$$M \geq \left(\frac{v(y)+v(z)}{2}\right) - \left(\frac{u(y)+u(z)}{2}\right), \qquad \forall y, z \in \mathcal{S}(x_0)\, y \neq z,$$

and

$$M \geq \left(\frac{v(\hat{x}_0)+mv(y)}{m+1}\right) - \left(\frac{u(\hat{x}_0)+mu(y)}{m+1}\right), \qquad \forall y \in \mathcal{S}(x_0),$$

we get

$$M \leq \min\left\{ \min_{\substack{y,z\in\mathcal{S}(x_0)\\ y\neq z}} \frac{v(y)+v(z)}{2}; \min_{y\in\mathcal{S}(x_0)} \frac{v(\hat{x}_0)+mv(y)}{m+1} \right\}$$
$$- \min\left\{ \min_{\substack{y,z\in\mathcal{S}(x_0)\\ y\neq z}} \frac{u(y)+u(z)}{2}; \min_{y\in\mathcal{S}(x_0)} \frac{u(\hat{x}_0)+mu(y)}{m+1} \right\} \leq M.$$

Hence, we obtain that there are two nodes x_1 and x_2 connected with x (one of them can be the predecessor) such that

$$v(x_1) - u(x_1) = M, \qquad \text{and} \qquad v(x_2) - u(x_2) = M.$$

Since this happens for every x in the set of maximums of $v - u$ and this set is finite, we obtain a contradiction that shows that

$$u(x) \geq v(x),$$

and proves the result. □

Now we will prove (3.1).

Theorem 3.3. *Let $g\colon [0,1] \to \mathbb{R}$ be a continuous function. Then*

$$\lim_{x \to \pi \in \partial \mathbb{T}_m} u_g^*(x) = g(\psi(\pi))$$

for any $\pi \in \partial \mathbb{T}_m$.

Proof. Let us start by observing that, for any constant c, $u \in \mathcal{C}(g)$ if only if $u+c \in \mathcal{C}(g+c)$. Therefore, without loss of generality, we may assume that g is a nonnegative function.

Let $\pi_0 = (y_1, \dots, y_k, \dots) \in \partial \mathbb{T}_m$. For any $n \in \mathbb{T}_m$, there exist $z_n \in \mathbb{T}_m^n$ and k_0 such that $\psi(y_k) \in I_{z_n}$ for all $k \geq k_0$. Now taking $c = \min\{g(t)\colon t \in I_{z_n}\}$ and $w_n = c u_{z_n}$ where u_{z_n} is given by Lemma 2.3, we have that w_n is a convex function such that

$$\lim_{x \to \pi} w_n(x) \leq g(\psi(\pi)), \qquad \forall \pi \in \partial \mathbb{T}_m.$$

Here, we are using that $g \geq 0$. Then, $w_n \in \mathcal{C}(g)$, and therefore $w_n(x) \leq u_g^*(x)$ for any $x \in \mathbb{T}_m$. In particular, $w_n(y_k) \leq u_g^*(y_k)$ for any k. Therefore,

$$\min\{g(t)\colon t \in I_{z_n}\} = \lim_{k \to \infty} w_n(y_k) \leq \liminf_{k \to \infty} u_g^*(y_k).$$

Taking the limit as $n \to \infty$, we have

$$g(\psi(\pi_0)) \leq \liminf_{k \to \infty} u^*(y_k)$$

since g is a continuous function.

Moreover, taking

$$w^n(x) = a(1 - u_{z_n}) + b u_{z_n} = a + (b-a) w_n$$

where $a = 2\max\{g(t)\colon t \in [0,1]\}$ and $b = \max\{g(t)\colon t \in I_{z_n}\}$, we have that

$$\begin{aligned}
w^n(x) &= a + (b-a) w_n(x) \\
&= a + (b-a) \min\left\{ \min_{\substack{y,z \in \mathcal{S}(x_0) \\ y \neq z}} \frac{w_n(y) + w_n(z)}{2}; \min_{y \in \mathcal{S}(x_0)} \frac{w_n(\hat{x}_0) + m w_n(y)}{m+1} \right\} \\
&= \max\left\{ \max_{\substack{y,z \in \mathcal{S}(x_0) \\ y \neq z}} a + \frac{(b-a)(w_n(y) + w_n(z))}{2}; \max_{y \in \mathcal{S}(x_0)} a + \frac{(b-a)(w_n(\hat{x}_0) + m w_n(y))}{m+1} \right\} \\
&= \max\left\{ \max_{\substack{y,z \in \mathcal{S}(x_0) \\ y \neq z}} \frac{w^n(y) + w^n(z)}{2}; \max_{y \in \mathcal{S}(x_0)} \frac{w^n(\hat{x}) + m w^n(y)}{m+1} \right\} \\
&\geq \min\left\{ \min_{\substack{y,z \in \mathcal{S}(x_0) \\ y \neq z}} \frac{w^n(y) + w^n(z)}{2}; \min_{y \in \mathcal{S}(x_0)} \frac{w^n(\hat{x}) + m w^n(y)}{m+1} \right\}
\end{aligned}$$

for any $x \in \mathbb{T}_m$ and

$$g(\psi(\pi)) \leq \liminf_{z \to \pi} w^n(x_k), \qquad \forall \pi \in \partial \mathbb{T}_m.$$

Thus, by Lemma 3.2, for any $u \in \mathcal{C}(g)$ we have that $u(x) \leq w^n(x)$ for any $x \in \mathbb{T}_m$. Therefore $u_g^*(x) \leq w^n(x)$ for any $x \in \mathbb{T}_m$. In particular, $u_g^*(y_k) \leq w^n(y_k)$ for any k. Then

$$\limsup_{k \to \infty} u_g^*(y_k) \leq \lim w_n(y_k) = \max\{g(t)\colon t \in I_{z_n}\}.$$

Again, taking the limit as $n \to \infty$, we have

$$\limsup_{k\to\infty} u_g^*(y_k) \le g(\psi(\pi_0)).$$

Therefore, we conclude that

$$\lim_{k\to\infty} u^*(y_k) = g(\psi(\pi_0)).$$

As $\pi_0 \in \partial\mathbb{T}_m$ was arbitrary, we conclude

$$\lim_{x\to\pi_0} u^*(x) = g(\psi(\pi_0))$$

for any $\pi_0 \in \partial\mathbb{T}_m$. □

Now our next goal is to find the equation that u_g^* verifies on $\mathbb{T}_m$.

Theorem 3.4. *Let $g\colon [0,1] \to \mathbb{R}$ be a continuous functions. The convex envelope u_g^* is characterized as the largest solution to*

$$u(x) = \min\left\{ \min_{\substack{y,z\in\mathcal{S}(x)\\ y\neq z}} \frac{u(y)+u(z)}{2} ; \min_{y\in\mathcal{S}(x)} \frac{u(\hat{x})+mu(y)}{m+1} \right\} \quad \text{on } \mathbb{T}_m, \tag{3.5}$$

that verifies

$$\limsup_{x\to\pi\in\partial\mathbb{T}_m} u(x) \le g(\psi(\pi)).$$

Proof. Given $g\colon [0,1] \to \mathbb{R}$, by Lemmas 3.1 and 2.1 we get that u_g^* verifies (2.13).

Now, to see that we have an equality, we argue by contradiction. Assume that at some node $x_0 \in \mathbb{T}_m$, we have

$$u_g^*(x_0) < \min\left\{ \min_{\substack{y,z\in\mathcal{S}(x_0)\\ y\neq z}} \frac{u_g^*(y)+u_g^*(z)}{2} ; \min_{y\in\mathcal{S}(x_0)} \frac{u_g^*(\hat{x}_0)+mu_g^*(y)}{m+1} \right\}$$

Taking $\delta > 0$ such that

$$u_g^*(x_0) + \delta < \min\left\{ \min_{\substack{y,z\in\mathcal{S}(x)\\ y\neq z}} \frac{u_g^*(y)+u_g^*(z)}{2} ; \min_{y\in\mathcal{S}(x)} \frac{u_g^*(\hat{x})+mu_g^*(y)}{m+1} \right\}$$

and consider

$$v(x) = \begin{cases} u_g^*(x) & \text{if } x \neq x_0, \\ u_g^*(x_0) + \delta & \text{if } x = x_0. \end{cases}$$

Observe that v verifies (2.13). Thus, by Lemma 2.1, v is convex. In addition, we have that $v \in \mathbb{C}(g)$. Therefore

$$v(x) \le u_g^*(x) \quad \forall x \in \mathbb{T}_m,$$

leading to a contradiction. This proves that u_g^* is a solution to (3.5).

Finally, to see that u_g^* is the largest solution to (3.5) that verifies

$$\limsup_{x\to\pi\in\partial\mathbb{T}_m} u_g^*(x) \le g(\psi(\pi)),$$

it is enough to define

$$\overline{u}(x) = \sup\left\{u(x)\colon u \text{ verifies } (3.5) \text{ and } \limsup_{x\to\pi\in\partial\mathbb{T}_m} u(x) \le g(\psi(\pi))\right\}.$$

This function $\overline{u}$ trivially verifies

$$\overline{u}(x) \ge u_g^*(x) \quad x \in \mathbb{T}_m,$$

just notice that u_g^* belongs to the set defining $\overline{u}$.

On the other hand, since $\overline{u}$ is a solution to (3.5), by Lemma 2.1, we have that $\overline{u}$ is convex and therefore $\overline{u} \in \mathcal{C}(g)$. Then

$$\overline{u}(x) \le u_g^*(x) \quad \forall x \in \mathbb{T}_m.$$

We conclude that

$$u_g^*(x) = \overline{u}(x) = \sup\left\{v(x)\colon u \text{ verifies } (3.5) \text{ and } \limsup_{x\to\pi\in\partial\mathbb{T}_m} u(x) \le g(\psi(\pi))\right\}.$$

□

Observe that by Lemma 3.2 and Theorem 3.4, for any continuous function $g\colon [0,1] \mapsto \mathbb{R}$, the equation defining the convex envelope has a unique solution that attains the datum g continuously.

Theorem 3.5. *Let $g\colon [0,1] \mapsto \mathbb{R}$ be a continuous function. There exists a unique solution to* (3.5) *such that*

$$\lim_{x\to\pi\in\partial\mathbb{T}_m} u(x) = g(\psi(\pi)).$$

for any $\pi \in \mathbb{T}_m$.

To end this section we prove Theorem 1.3 that deals with the convex envelope of a function $f:\mathbb{T}_m \mapsto \mathbb{R}$ given by (1.7).

Theorem 3.6. *The convex envelope of a function $f\colon \mathbb{T}_m \to \mathbb{R}$ is the solution to the obstacle problem for the equation* (1.2)*.*

Proof. Let us call v^* to the largest solution to

$$u(x) \le \min\left\{\min_{\substack{y,z\in\mathcal{S}(x)\\ y\neq z}} \frac{u(y)+u(z)}{2};\ \min_{y\in\mathcal{S}(x)} \frac{u(\hat{x})+mu(y)}{m+1}\right\} \quad \text{on } \mathbb{T}_m, \tag{3.6}$$

that verifies

$$u(x) \le f(x) \qquad \forall x \in \mathbb{T}_m.$$

We have to prove that the convex enevolpe of f, u_f^*, verifies

$$u_f^*(x) = v^*(x), \qquad \forall x \in \mathbb{T}_m.$$

Since u_f^* is convex from Lemma 2.1 we obtain that it is a solution to (3.6) that verifies $u_f^* \le f$ on $\mathbb{T}_m$ and then we obtain

$$u_f^*(x) \le v^*(x), \qquad \forall x \in \mathbb{T}_m.$$

We have also from Lemma 2.1 that v^* being a solution to (3.6) is a convex function and it verifies $v^* \leq f$ on $\mathbb{T}_m$. Hence,

$$v^*(x) \leq u_f^*(x), \qquad \forall x \in \mathbb{T}_m.$$

We conclude that

$$u_f^*(x) = v^*(x), \qquad \forall x \in \mathbb{T}_m.$$

In the coincidence set, the function f verifies an inequality. From the fact that u_f^* is convex and smaller than f we obtain for $x \in CS(f)$,

$$\begin{aligned} f(x) &= u_f^*(x) \\ &\leq \min\left\{ \min_{\substack{y,z\in\mathcal{S}(x)\\ y\neq z}} \frac{u_f^*(y)+u_f^*(z)}{2}; \min_{y\in\mathcal{S}(x)} \frac{u_f^*(\hat{x})+m\,u_f^*(y)}{m+1} \right\} \\ &\leq \min\left\{ \min_{\substack{y,z\in\mathcal{S}(x)\\ y\neq z}} \frac{f(y)+f(z)}{2}; \min_{y\in\mathcal{S}(x)} \frac{f(\hat{x})+mf(y)}{m+1} \right\}. \end{aligned}$$

Finally, outside of the coincidence set the convex envelope, u_f^*, is a solution to the equation. In fact, arguing by contradiction, assume that for some $x_0 \notin CS(f)$ it holds

$$u_f^*(x_0) < \min\left\{ \min_{\substack{y,z\in\mathcal{S}(x_0)\\ y\neq z}} \frac{u_f^*(y)+u_f^*(z)}{2}; \min_{y\in\mathcal{S}(x_0)} \frac{u_f^*(\hat{x_0})+mu_f^*(y)}{m+1} \right\}. \tag{3.7}$$

Then, since $x_0 \notin CS(f)$ and we have a strict inequality in (3.7) there exists $\delta > 0$ such that the function

$$v(x) = \begin{cases} u_f^*(x) & \text{if } x \neq x_0, \\ u_f^*(x_0)+\delta & \text{if } x = x_0 \end{cases}$$

is convex and still verifies $v \leq f$ on $\mathbb{T}_m$ contradicting the maximality of the convex envelope u_f^*. This contradiction shows that we have an equality in (3.7). □

4. Binary convex functions

As in Section 2, we begin showing a different characterization of binary convex functions.

Lemma 4.1. *A function u on the tree is binary convex if and only if u satisfies*

$$u(x) \leq \min_{\substack{y,z\in\mathcal{S}(x)\\ y\neq z}} \frac{u(y)+u(z)}{2} \qquad \forall x \in \mathbb{T}_m. \tag{4.8}$$

Proof. Let us start the proof observing that if $x \in \mathbb{T}_m$, $y, z \in \mathcal{S}(x)$ and $y \neq z$ then $[y, z] \in \mathbb{T}_2^x$ and $\mathcal{E}([y, z]) = \{y, z\}$. Therefore if u is a binary convex function, $x \in \mathbb{T}_m$ and $y, z \in \mathcal{S}(x)$ are such taht $y \neq z$ then

$$u(x) \leq \frac{u(y)+u(z)}{2}.$$

Thus, u satisfies (4.8) in $\mathbb{T}_m$.

Now assume that u satisfies (4.8). Our aim is to prove that u is a binary convex function, that is, we aim to show that

$$u(x) \leq \sum_{y\in\mathcal{E}(\mathbb{B})} \frac{u(y)}{2^{|y|-|x|}} \quad \forall \mathbb{B} \in \mathbb{T}_2^x. \tag{4.9}$$

Fix $x \in \mathbb{T}_m$. Given $\mathbb{B} \in \mathbb{T}_2^x$, we define

$$|\mathbb{B}| \coloneqq \max\left\{|z| - |x| \colon z \in \mathcal{E}(\mathbb{B})\right\} \in \mathbb{N}$$

and

$$\mathbb{T}_2^{xn} \coloneqq \left\{\mathbb{B} \in \mathbb{T}_2^x \colon |\mathbb{B}| = n\right\} \subset \mathbb{T}_2^x.$$

The proof of (4.9) runs by induction in n. Observe that in the case $|\mathbb{B}| = 1$ there exist $y, z \in \mathcal{S}(x)$ such that $\mathbb{B} = [y, z]$ and obviously $\mathcal{E}(\mathbb{B}) = \{y, z\}$. Then, since u satisfies (4.8), we get

$$u(x) \leq \frac{u(y) + u(z)}{2} = \sum_{y\in\mathcal{E}(\mathbb{B})} \frac{u(y)}{2^{|y|-|x|}}.$$

That is (4.9) holds for any $\mathbb{B} \in \mathbb{T}_2^{x1}$.

Now we assume that (4.9) holds for any $\mathbb{B} \in \mathbb{T}_2^{xn}$, and we will show that it also holds for any $\mathbb{B} \in \mathbb{T}_2^{x(n+1}$.

If $\mathbb{B} \in \mathbb{T}_2^{x(n+1)}$ then $\mathbb{B}' = \mathbb{B} \setminus \{y \in \mathcal{E}(\mathbb{B}) \colon |y| - |x| = n + 1\} \in \mathbb{T}_2^{xn}$. Then, by the inductive hypothesis, we get

$$u(x) \leq \sum_{y\in\mathcal{E}(\mathbb{B}')} \frac{u(y)}{2^{|y|-|x|}}. \tag{4.10}$$

On the other hand, for any $y \in \mathcal{E}(\mathbb{B})$ we have that $y \in \mathcal{E}(\mathbb{B}')$ or there are $w \in \mathcal{E}(\mathbb{B}')$ and $z \in \mathcal{E}(\mathbb{B}) \setminus \{y\}$ such that $y, z \in \mathcal{S}(w)$. Thus, since u satisfies (4.8), from (4.10), we have that

$$u(x) \leq \sum_{y\in\mathcal{E}(\mathbb{B})} \frac{u(y)}{2^{|y|-|x|}}.$$

Finally, since x is arbitrary, we conclude that u is a binary convex function. □

Remark 4.2. Now, by Lemmas 2.1 and 4.1, it is easy to check that a convex function is also a binary convex function.

Proceeding as in the proof of Corollary 2.2 we can prove the following result.

Corollary 4.3. *Let $u, v \colon \mathbb{T}_m \to \mathbb{R}$ be binary convex functions. Then $u + v$ is a binary convex function.*

Now, we obtain the following result, whose proof is similar to Lemma 2.3.

Lemma 4.4. *Let $x_0 \in \mathbb{T}_m \setminus \{\emptyset\}$. Then the function $u_{x_0} \colon \mathbb{T}_m \to \mathbb{R}$ defined by*

$$u_{x_0}(x) \coloneqq \begin{cases} 0 & \text{if } x \notin \mathbb{T}_m^{x_0}, \\ 1 & \text{if } x \in \mathbb{T}_m^{x_0}, \end{cases}$$

is a binary convex function such that

$$\limsup_{x\to\pi} u_{x_0}(x) \le \mathcal{X}_{I_{x_0}}(\psi(\pi)) \quad \forall \pi \in \partial\mathbb{T}_m.$$

Moreover,

$$u_{x_0}(x) = \min_{\substack{y,z\in\mathcal{S}(x)\\ y\neq z}} \frac{u_{x_0}(y)+u_{x_0}(z)}{2} \quad \forall x \in \mathbb{T}_m,$$

and for any $\pi \in \mathbb{T}_m$ such that $\psi(\pi)$ is not an endpoint of I_{x_0} we have

$$\lim_{x\to\pi} u_{x_0}(x) = \mathcal{X}_{I_{x_0}}(\psi(\pi)).$$

5. Binary Convex envelopes

Proceeding as in the proof of Lemma 3.1 we can show that the binary convex envelop is a binary convex function.

Lemma 5.1. *For any function $g\colon [0,1]\to\mathbb{R}$, the binary convex envelop $\tilde{u}_g$ is a binary convex function.*

In a similar way to Section 3, we will show that if g is a continuous function, then

$$\lim_{x\to\pi\in\partial\mathbb{T}_m} \tilde{u}_g(x) = g(\psi(\pi)) \quad \forall \pi \in \partial\mathbb{T}_m. \tag{5.11}$$

As before, to prove this claim we need a comparison principle.

Lemma 5.2. *Let u and v satisfy*

$$u(x) \ge \min_{\substack{y,z\in\mathcal{S}(x_0)\\ y\neq z}} \frac{u(y)+u(z)}{2} \quad \forall x \in \mathbb{T}_m, \tag{5.12}$$

$$v(x) \le \min_{\substack{y,z\in\mathcal{S}(x_0)\\ y\neq z}} \frac{v(y)+v(z)}{2}, \quad \forall x \in \mathbb{T}_m, \tag{5.13}$$

with

$$\lim_{x\to\pi\in\partial\mathbb{T}_m} u(x) \ge \lim_{x\to\pi\in\partial\mathbb{T}_m} v(x),$$

for every $\pi\in\partial\mathbb{T}_m$. Then,

$$u(x) \ge v(x) \qquad \forall x \in \mathbb{T}_m.$$

Proof. Adding a positive constant c to u, we may assume that

$$\lim_{x\to\pi\in\partial\mathbb{T}_m} u(x) > \lim_{x\to z\in\partial\mathbb{T}_m} v(x). \tag{5.14}$$

We argue by contradiction, so, assume that

$$M = \max_{x\in\mathbb{T}_m}(v(x)-u(x)) > 0.$$

Notice that the maximum is attained thanks to (5.14). Also by (5.14), we have that M is attained only in a finite set of vertices. Let x be one of such vertices. From (5.12) and (5.13) we obtain

$$M = v(x)-u(x) \le \min_{\substack{y,z\in\mathcal{S}(x_0)\\ y\neq z}} \frac{v(y)+v(z)}{2} - \min_{\substack{y,z\in\mathcal{S}(x_0)\\ y\neq z}} \frac{u(y)+u(z)}{2}.$$

Now, using that

$$M \geq \left(\frac{v(y)+v(z)}{2}\right) - \left(\frac{u(y)+u(z)}{2}\right), \qquad \forall y,z \in \mathcal{S}(x_0)\, y \neq z,$$

we get

$$M \leq \min_{\substack{y,z\in\mathcal{S}(x_0)\\ y\neq z}} \frac{v(y)+v(z)}{2} - \min_{\substack{y,z\in\mathcal{S}(x_0)\\ y\neq z}} \frac{u(y)+u(z)}{2} \leq M.$$

Then, there exist two nodes $x_1, x_2 \in \mathcal{S}(x)$ such that

$$v(x_1) - u(x_1) = M, \qquad \text{and} \qquad v(x_2) - u(x_2) = M.$$

Since this happens for every x in the set of maximums of $v - u$ and this set is finite, we obtain a contradiction that shows that

$$u(x) \geq v(x)$$

and proves the result. □

Now we will show (5.11).

Theorem 5.3. *Let $g\colon [0,1] \to \mathbb{R}$ be a continuous function. Then, for any $\pi \in \partial\mathbb{T}_m$*

$$\lim_{x\to\pi\in\partial\mathbb{T}_m} \tilde{u}_g(x) = g(\psi(\pi)).$$

Proof. Let us start by observing that, for any constant c, $u \in \mathcal{B}(g)$ if only if $u + c \in \mathcal{B}(g + c)$. Therefore, without loss of generality, we may assume that g is nonnegative.

Consequently, by Remark 4.2 and Theorem 3.3, we have

$$\liminf_{x\to\pi\in\partial\mathbb{T}_m} \tilde{u}_g(x) \geq \lim_{x\to\pi\in\partial\mathbb{T}_m} u_g^*(x) = g(\psi(\pi))$$

for any $\pi \in \partial\mathbb{T}_m$.

To complete the proof, we proceed as in the end of the proof of Theorem 3.3, using Lemmas 4.4 and 5.2 instead of Lemmas 2.3 and 3.2. □

Finally, with analogous arguments of the Section 3, we get the following results.

Theorem 5.4. *Let $g\colon [0,1] \to \mathbb{R}$ be a continuous functions. The binary convex envelope $\tilde{u}_g$ is characterized as the largest solution to*

$$u(x) = \min_{\substack{y,z\in\mathcal{S}(x)\\ y\neq z}} \frac{u(y)+u(z)}{2} \quad \text{on } \mathbb{T}_m, \tag{5.15}$$

that verifies

$$\limsup_{x\to\pi\in\partial\mathbb{T}_m} u(x) \leq g(\psi(\pi)).$$

Theorem 5.5. *Let $g\colon [0,1] \mapsto \mathbb{R}$ be a continuous function. Then, there exists a unique solution to (5.15) such that for any $\pi \in \mathbb{T}_m$*

$$\lim_{x\to\pi\in\partial\mathbb{T}_m} u(x) = g(\psi(\pi)).$$

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