

STATES IN GENERALIZED PROBABILISTIC MODELS: AN APPROACH BASED IN ALGEBRAIC GEOMETRY

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ABSTRACT. We present a characterization of states in generalized probabilistic models by appealing to a non-commutative version of geometric probability theory based on algebraic geometry techniques. Our theoretical framework allows for incorporation of invariant states in a natural way.

1. INTRODUCTION

States in physical theories must determine well defined probabilities for any testable empirical event. In classical theories, such as classical statistical mechanics, probabilistic states can be represented by measurable functions on a phase space. In this way, states of classical theories obey the axioms of Kolmogorov for probabilities. In Kolmogorov's setting, probabilities are considered as measures defined over a σ -algebra of measurable subsets of a given outcome set. Kolmogorov's achievement solved a problem posed in the year 1900 by the Mathematician David Hilbert [16, p. 454] (Hilbert's sixth problem) with regard to the axiomatization of probabilities. The ensuing solution was given in terms of measure theory, being suitable for a rigorous treatment of the problem.

Quantum mechanics showed that this correspondence between states and Kolmogorovian measures is not universal. Quantum states are represented by density operators [19, 20], which are very different mathematical objects. Indeed, a quantum state, defines a Kolmogorovian probability distribution for each empirical setup [39]. But if we mix events from different and incompatible setups, then Kolmogorov's rules for probability can be shown to fail [13]. Thus, the best we can do is to represent a quantum state as a *family* of Kolmogorovian measures pasted in a harmonic way [21]. Note that probabilities given by measurements on a single empirical setup are not enough: we must perform measurements of different incompatible observables in order to determine a quantum state completely (see for example [27]).

This can be expressed by using other words: while the set of states of a classical probabilistic theory forms a simplex, the sets of quantum states exhibit a more involved geometrical structure [6]. This distinction acquires a very specific mathematical expression: while classical probabilistic states can be described as measures over Boolean lattices, quantum states are described by measures over non-Boolean ones. This fact is related to the non-commutative character of observables. The algebras involved in models of standard quantum mechanics can differ from those used in quantum field theory [14], or in quantum statistical mechanics [3, 4]. While in non-relativist quantum mechanics we only have Type I factors, other factors can appear in a relativistic setting and in quantum statistical mechanics [17, 37, 3]. The algebras associated to classical systems are commutative [3]. The rigorous formulation of quantum theory can be considered as an extension of measure theory to the non-commutative setting [15]. States in quantum theories can be also

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characterized as non-Kolmogorovian probabilistic measures [37]. These formal developments can be considered as a continuation of the program outlined in Hilbert's sixth problem.

In many situations of interest, the task of the physicist is to find out which is the state of the system under study, given that certain constraints are specified. One of the most important constraints are given by symmetries, and symmetries are usually represented mathematically by the action of groups. Thus, the problem of determining measures which are invariant under the action of certain groups is an important one for physics. The problem of finding invariant states in operator algebras was studied in the mathematical physics literature, and is a hard one (see for example [3], Section 4.3).

In this work, we study this problem in the setting of general orthomodular lattices. Our results can be connected with those of the above mentioned algebraic approach in a natural way, because the algebraic structure of some of the operator algebras used in mathematical physics is in close relation with the properties of the lattices formed by their self-adjoint idempotent elements (i.e., projection operators) [22, 29, 30, 41, 31]. Applications of probabilities defined over non-Boolean structures are not restricted to physical theories (see for example [25] and [44]). The case of measures over bounded lattices is studied separately [28] and should not be confused with the study of states in probabilistic theories. Our contribution is twofold

- From the mathematical point of view, we provide an extension of the mathematical framework involved in the derivation of Groemer's integral theorem in a non-Boolean setting. This could be considered as a non-commutative generalization of geometric probability theory based on algebraic geometry techniques.
- We apply our extension to quantum theories represented by event structures which can be non-Boolean in general. This allows us, via our non-Boolean (or non-commutative) version of Groemer's theorem, to characterize all states which are invariant under the action of a general group G .

In order to find a suitable framework for incorporating conditions imposed by the action of groups, in [18] we proposed a reformulation of the MaxEnt problem in terms of a non-Boolean version of *geometric probability theory* [36, 26]. This rephrasing of the problem can be helpful for an axiomatic approach to the study of symmetries in the quantum setting, and thus, it can also be considered as a continuation of the program outlined by Hilbert's sixth problem. The present work is a natural follow up of the ideas presented in [18].

Our formalism could be helpful in contexts such as the ones appearing in the algebraic formulation of quantum statistical mechanics or quantum field theory. It is also suited for the application of the MaxEnt method to a wide range of models, because it is compatible with the specification of linear constraints (or more general ones) representing mean values [18].

Outline of the paper: In Section 2 we review elementary notions and start by the description of states in physical theories, and other mathematical preliminaries. We place emphasis on the fact that states of classical theories can be considered as measures over Boolean algebras, while states of quantum theories must appeal to measures over orthomodular lattices.

In Section 3 we discuss our approach for the particular case of complete Boolean atomistic lattices. This example is very important, because classical probabilistic models can be described by measures over σ -algebras which are Boolean lattices. The constructions and methods of this Section help to illuminate the extensions to the non-Boolean case of the rest of the paper. The content of this Section can be considered as a physical version of geometric probability theory.

In Section 4 we extend the construction of Section 3 to the more general case of additive orthogonal measures on orthocomplemented lattices. In this Section we apply our general

theoretical framework to the kind of measures which appear in physical theories of interest (such as classical statistical mechanics, non-relativistic quantum mechanics, algebraic quantum field theory and algebraic quantum statistical mechanics). In this way, we show how our theoretical framework is useful to characterize invariant states. We also discuss the case of Gleason's theorem [15, 9, 2].

The mathematical intuition behind our construction is that of extending geometric probability theory to the non-Boolean case, by appealing to algebraic geometry techniques. The physical intuition is that of considering physical states as invariant measures (see [18] for a previous version of this approach). The cases of interest for physics are given by orthomodular lattices and measures valued in the ring of real numbers. But we build our mathematical framework for the general case first, because in this way it is easier to understand the essential mathematical structures underlying our construction. A key feature of our theoretical framework is that we appeal to a mathematical technique based in algebraic geometry. Starting with an arbitrary orthocomplemented lattice \mathcal{L} , we build the abelian group $\mathbb{Z}^{\oplus \mathcal{L}}$ generated by \mathcal{L} , formed by all possible formal sums of its elements. By taking the ratio $M(\mathcal{L}) = \mathbb{Z}^{\oplus \mathcal{L}}/S$ with respect to a suitably chosen subgroup S , we obtain a commutative group which will be useful to characterize invariant measures¹. We show that a linear function from $M(\mathcal{L})$ is the same as a measure on \mathcal{L} . One of our main results is given by Theorem 4.9, which allows us to characterize, by means of Corollary 4.6, all invariant measures acting by automorphisms on the lattice. This means that, given an arbitrary lattice endowed with a set of measures (which could represent states of a given physical model), and an arbitrary group (which could be a group of automorphisms representing a physical symmetry), we obtain a canonical and constructive characterization of its invariant measures. We end Section 4 with some examples regarding positive measures and Gleason's theorem.

Finally, in Section 5, we draw some conclusions.

2. PRELIMINARIES

In this Section we introduce some elementary notions on lattice theory and states of physical theories as measures over lattices. The reader familiarized with lattice theory, may skip this.

2.1. Lattice Theory. A poset (\mathcal{L}, \leq) is called a *lattice* if every two elements x, y have a supremum $x \vee y$ and an infimum $x \wedge y$ [22]. We denote by $\mathbf{0}$ and $\mathbf{1}$ to the bottom and top elements, respectively, of a bounded lattice $(\mathcal{L}, \leq, \vee, \wedge)$. A bounded lattice $(\mathcal{L}, \leq, \vee, \wedge)$ is called *complemented* if for every $x \in \mathcal{L}$ there exists $y \in \mathcal{L}$ such that

$$x \vee y = \mathbf{1}, \quad x \wedge y = \mathbf{0}.$$

A morphism between bounded lattices $f : (\mathcal{L}, \leq, \vee, \wedge, \mathbf{0}, \mathbf{1}) \rightarrow (\mathcal{L}', \leq, \vee, \wedge, \mathbf{0}, \mathbf{1})$ is a poset function such that

$$f(x \wedge_{\mathcal{L}} y) = f(x) \wedge_{\mathcal{L}'} f(y), \quad f(x \vee_{\mathcal{L}} y) = f(x) \vee_{\mathcal{L}'} f(y), \quad f(\mathbf{0}) = \mathbf{0}, \quad f(\mathbf{1}) = \mathbf{1}.$$

An element z of a poset is an *atom* if the interval $\{\mathbf{0} \leq a \leq z\}$ consists on two elements,

$$\{\mathbf{0} \leq a \leq z\} = \{\mathbf{0}, z\}.$$

A lattice \mathcal{L} is called *atomic* if for every $x \in \mathcal{L} \setminus \mathbf{0}$, there exists an atom z such that $z \leq x$. It is said to be *atomistic* if every element can be written as a join of atoms. A lattice \mathcal{L} is called *distributive*

¹This construction is standard in commutative algebra. See for example the construction of the tensor product in [1, Prop.2.12].

if for every $x, y, z \in \mathcal{L}$,

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

A complemented distributive lattice is called a *Boolean* lattice. In a distributive lattice, complements are unique. A lattice in which every countable subset has a meet and a join is called a σ -lattice. The next Proposition is going to be useful in Section 3.

Proposition 2.1. *If z is an atom of a distributive lattice \mathcal{L} , then*

$$z \leq a \vee b \iff z \leq a \text{ or } z \leq b.$$

Proof. First, note that $z \wedge b$ and $z \wedge a$ are less than or equal to z , hence they must be equal to 0 or to z . If $z \wedge b = 0 = z \wedge a$, from the distributive law, $z \wedge (a \vee b) = 0$. But, from the equation $z \leq a \vee b$, we obtain $z = z \wedge (a \vee b)$, a contradiction. The other implication is obvious. \square

An *orthocomplementation* on a bounded lattice is a function that maps each element a to a complement a^\perp in such a way that the following axioms are satisfied:

- Complement: $a \vee a^\perp = 1$ and $a \wedge a^\perp = 0$.
- Involution: $(a^\perp)^\perp = a$.
- Order-reversing: if $a \leq b$ then $b^\perp \leq a^\perp$.

An *orthocomplemented* lattice is a bounded lattice which is equipped with an orthocomplementation. Orthocomplemented lattices, like Boolean algebras, satisfy de Morgan's laws:

$$(a \vee b)^\perp = a^\perp \wedge b^\perp, \quad (a \wedge b)^\perp = a^\perp \vee b^\perp.$$

A morphism between orthocomplemented lattices is a lattice map that preserves the orthocomplementation. We say that a group G acts on a orthocomplemented lattice if there is a group map from G to the group of automorphisms of the lattice.

An *orthomodular* lattice \mathcal{L} is an orthocomplemented lattice such that

$$a \leq b \implies a \vee (a^\perp \wedge b) = b, \quad \forall a, b \in \mathcal{L}.$$

We say that x is *orthogonal* to y , denoted $x \perp y$, if $y \leq x^\perp$.

2.2. States as measures. A general approach to physical theories, very useful for discussing foundational purposes, is based on von-Neumann algebras [17, 37]. Let $\mathcal{B}(H)$ be the algebra of bounded linear operators acting on a separable Hilbert space H . A *von Neumann algebra* is a $*$ -subalgebra of $\mathcal{B}(H)$ that contains the identity operator and that is closed in the weak operator topology. The *weak operator topology* is defined as the weakest topology such that the functional sending an operator A to the complex number $\langle A\phi, \psi \rangle$, is continuous for all vectors $\phi, \psi \in \mathcal{H}$. Due to the celebrated von Neumann's double commutant theorem [34], von Neumann algebras can be regarded as a $*$ -subalgebra \mathcal{W} , containing the identity operator and satisfying $\mathcal{W}'' = \mathcal{W}$, where given $S \subseteq \mathcal{B}(H)$, S' is defined as

$$S' = \{A \in \mathcal{B}(H) \mid AB - BA = 0 \forall B \in S\} \tag{1}$$

$\mathcal{B}(H)$ is a von Neumann algebra, but it is not the only one. For example, it is possible to consider the algebra of observables of a classical theory as a commutative von Neumann algebra. In algebraic relativistic quantum field theory, the algebras associated to space time regions will be generally the class of von Neumann algebras known as Type III factors [43]. A similar situation occurs in the algebraic approach to quantum statistical mechanics, where Type II and Type III factors may appear [37]. Thus, the approach based on von Neumann algebras is adequate for a very general family of physical models. A *state* $\nu : \mathcal{W} \rightarrow \mathbb{C}$ is defined as a continuous positive linear functional

such that $\nu(\mathbf{1}) = 1$ (see Reference [37]²). Positivity means that $\nu(A^*A) \geq 0$ for all $A \in \mathcal{W}$ or, equivalently, that $\nu(A) \geq 0$ for all $A \geq 0$, see [38, §2.1]. *Normal states* can be defined as those states satisfying the condition $\nu(\sup_\alpha(A_\alpha)) = \sup_\alpha \nu(A_\alpha)$ for any uniformly bounded increasing net A_α of positive elements of \mathcal{W} (equivalently, states satisfying $\nu(\sum_{i \in I} E_i) = \sum_{i \in I} \nu(E_i)$ for any countable and pairwise orthogonal family of projection operators³ $\{E_i\}_{i \in I}$). Normal states are real-valued for self-adjoint elements of \mathcal{W} (recall that A is self adjoint iff $A^* = A$) and they form a convex set. It can be shown that the set of projection operators associated to an arbitrary von Neumann algebra \mathcal{W} is an orthomodular lattice $\mathcal{L}(\mathcal{W})$ [34]. Thus, normal states of physical theories define probabilities on orthomodular lattices satisfying the following properties:

$$\nu : \mathcal{L} \rightarrow [0, 1], \quad (2a)$$

such that

$$\nu(\mathbf{1}) = 1, \quad (2b)$$

and, for a denumerable and pairwise orthogonal family of events $\{E_i\}_{i \in I}$,

$$\nu\left(\sum_{i \in I} E_i\right) = \sum_{i \in I} \nu(E_i). \quad (2c)$$

where \mathcal{L} is the lattice associated to a particular algebra. In standard quantum mechanics \mathcal{L} is the lattice of projection operators on a Hilbert space and Gleason's theorem ensures that it defines a density operator. In classical mechanics \mathcal{L} is the Boolean lattice of measurable subsets of phase space. In algebraic relativistic quantum field theory, each local region and a global state will define a lattice (associated to a Type III factor) satisfying the above axioms. Notice that Eqs. 2 make sense for mathematical objects more general than orthomodular lattices [13]. For example, one may use orthocomplemented lattices, or more generally, orthocomplemented posets (which are not necessarily lattices) [13]. Another important remark is that if, instead of condition 2c, we only impose additivity, we will obtain the more general notion of *additive probability*:

$$\nu : \mathcal{L} \rightarrow [0, 1], \quad (3a)$$

such that

$$\nu(\mathbf{1}) = 1, \quad (3b)$$

and, for orthogonal elements E_1 and E_2 ,

$$\nu(E_1 \vee E_2) = \nu(E_1) + \nu(E_2). \quad (3c)$$

In this work, we will refer to probabilities defined by Eqs. 2 and 3 as σ -*additive* and *additive*, respectively. In fact, we will change the codomain of ν in conditions (2a) and (3a) in order to work with *measures*. See [15, 8] for generalizations of Gleason's theorem.

It is important to notice that the Boolean vs. non-Boolean distinction is crucial here: while states of classical theories are measures over Boolean lattices (and thus, they obey Kolmogorov's axioms), states of non-classical theories cannot be equated to Kolmogorovian measures. Furthermore, all quantal theories "suffer" of contextuality: the Kochen-Specker theorem can be generalized to arbitrary von Neumann algebras [7]. This crucial distinction allows one to see that quantum theories define a new form of measure theory, much more general than the one used in Kolmogorov's axioms [37, 15]. In particular, as we show in our work, the whole theoretical framework must be adapted in order to deal with invariant measures representing physical symmetries.

²Throughout this paper we assume that \mathcal{W} is unital, i.e., that it possesses an identity element $\mathbf{1}$

³Here, a *projection operator* is understood as a self adjoint and idempotent element of the algebra. *Elementary tests* to be performed on the system involved, also known as *events*, are represented by projection operators (c.f. [34]).

A measure defined in an orthomodular lattice satisfying Eqns. 2 can be considered as a family of Kolmogorovian measures: if the lattice is non-Boolean, this family has only one member (the Boolean lattice itself). But if the lattice is not Boolean, it is possible to show that an orthomodular lattice can be considered as a pasting of maximal Boolean subalgebras [32], and a state will define a Kolmogorovian measure when restricted to one of them [21, 39].

The following Definition formalizes the idea of a state which possesses a symmetry represented by the action of a group:

Definition 2.2. Given an algebra \mathcal{W} and a group G , denote the action of G by $A \in \mathcal{W}$ as $\tau_g(A) \in \mathcal{W}$ for all $g \in G$. A state ν is said to be G -invariant if $\nu(\tau_g(A)) = \nu(A)$ for all $A \in \mathcal{W}$ and for all $g \in G$.

Notice that if G is represented by automorphisms in the lattice of projection operators, similar definitions can be made for states defined as measures in Eqns. 2 and 3.

Suppose now that the group G acts on an orthomodular lattice \mathcal{L} by automorphisms. Denote by α_g to the automorphism corresponding to $g \in G$. Thus, a measure ν satisfying Equations 2, is said to be *invariant* under the action of G if $\nu(\alpha_g x) = \nu(x)$, for all $x \in \mathcal{L}$ and all $g \in G$, see [40].

3. THE RING OF FUNCTIONS ON A COMPLETE BOOLEAN ATOMISTIC LATTICE

In this Section, we restrict ourselves to complete Boolean atomistic lattices in order to illustrate on some properties that will be extended later to the non-Boolean case. Let us start by recalling the description of a classical statistical system. States of a classical system will be described by points in a measurable phase space Γ . Observables are described by functions $f : \Gamma \rightarrow \mathbb{R}$ such that for every measurable set $E \subseteq \mathbb{R}$, $f^{-1}(E)$ is measurable in Γ , usually $\Gamma = \mathbb{R}^{2N}$ endowed with the Lebesgue measure. The appeal to measure theory allows one to build well defined probabilistic states: testable events will be defined as measurable subsets of Γ . As it is well known, the measurable subsets of Γ form a Boolean lattice (thus, states can be defined as in Eqns. 2, with \mathcal{L} equal to the lattice of measurable subsets of Γ).

Let us show how to assign a ring to a complete Boolean atomistic lattice. Let \mathcal{B} be a complete Boolean atomistic lattice and let $\mathcal{A}(\mathcal{B})$ be the set of its atoms. For every $x \in \mathcal{B}$, we can consider the *indicator function* of x , $I_x : \mathcal{A}(\mathcal{B}) \rightarrow \mathbb{R}$ as

$$I_x(z) = \begin{cases} 1 & \text{if } z \leq x, \\ 0 & \text{if not.} \end{cases} \quad (4)$$

Using the properties of \mathcal{B} as an atomistic lattice and Prop. 2.1 of Section 2, it is easy to see that the indicator functions satisfy for all $x, y, x_1, \dots, x_n \in \mathcal{B}$,

$$\begin{aligned} I_x I_y &= I_{x \wedge y} \\ I_{x \vee y} + I_{x \wedge y} &= I_x + I_y \\ I_{x_1 \vee \dots \vee x_n} &= 1 - (1 - I_{x_1}) \dots (1 - I_{x_n}) \end{aligned} \quad (5)$$

Let us call $Fun(\mathcal{A}(\mathcal{B}))$ the algebra of functions from $\mathcal{A}(\mathcal{B})$ to \mathbb{R} . The algebra operations are given by pointwise addition and multiplication. The real numbers act by scalar multiplication. Inside $Fun(\mathcal{A}(\mathcal{B}))$ we have the subalgebra $Sim(\mathcal{A}(\mathcal{B}))$ of simple functions generated by the indicators (i.e., all possible finite combinations of the form $\sum_{i=1}^n \alpha_i I_{x_i}$). Notice that we can assign to every $x \in \mathcal{B}$ the indicator function of its atoms,

$$\mathcal{B} \rightarrow Sim(\mathcal{A}(\mathcal{B})), \quad x \mapsto I_x := I_{\{z \in \mathcal{A}(\mathcal{B}) : z \leq x\}}.$$

Given that \mathcal{B} is atomistic, this assignment is injective and by Prop. 2.1, $I_{x \vee y} = I_{\{z \leq x\} \cup \{z \leq y\}}$. From the completeness of \mathcal{B} it follows that any indicator function I_A is equal to I_x , $x = \bigvee_{z \in A} z$.

Notice that a linear functional $\tilde{\nu}$ from $\text{Sim}(\mathcal{A}(\mathcal{B}))$ to \mathbb{R} must satisfy

$$\tilde{\nu}(I_{x \vee y}) + \tilde{\nu}(I_{x \wedge y}) = \tilde{\nu}(I_x) + \tilde{\nu}(I_y) \quad (6)$$

An important idea is at stake here. Given a Boolean lattice \mathcal{B} , we say that a function $\nu : \mathcal{B} \rightarrow \mathbb{R}$ is an *additive measure*⁴ (or just a *measure*) if given $x, y \in \mathcal{B}$,

$$\nu(x \vee y) + \nu(x \wedge y) = \nu(x) + \nu(y).$$

The measure ν is called *bounded* if it has a bounded image, *positive* if it takes positive values. It is called a *probability function* if its image is the interval $[0, 1]$ (and thus, it is a particular case of Eqs. 3, for \mathcal{L} a Boolean lattice). As it is well known, the above definition of measure can be derived from the following axiom,

$$\nu(x \vee y) = \nu(x) + \nu(y), \quad \forall x \perp y.$$

The advantage of this axiom is that it can be used to extend the definition of additive measure to any orthocomplemented lattice \mathcal{L} . If we instead impose the condition

$$\nu\left(\bigvee_{i=0}^{\infty} x_i\right) = \sum_{i=0}^{\infty} \nu(x_i), \quad x_i \perp x_j, \forall i \neq j,$$

we say that ν is a σ -*additive measure* on \mathcal{L} ⁵.

Let us go back to our previous constructions. We noticed that a linear functional $\tilde{\nu}$ from $\text{Sim}(\mathcal{A}(\mathcal{B}))$ to \mathbb{R} can be viewed (by restriction to \mathcal{B}) as a measure on the complete Boolean atomistic lattice \mathcal{B} . This connection is provided by Eq. 6. Indeed, given a function $\tilde{\nu}$ defined on $\text{Sim}(\mathcal{A}(\mathcal{B}))$, we obtain a measure on \mathcal{B} by defining $\nu(x) = \tilde{\nu}(I_x)$, for all $x \in \mathcal{B}$. Eq. 6 guarantees that ν satisfies the axioms of an additive measure. Analogously, a measure ν on \mathcal{B} determines a linear functional $\tilde{\nu}$ from $\text{Sim}(\mathcal{A}(\mathcal{B}))$ to \mathbb{R} by defining $\tilde{\nu}(I_x) = \nu(x)$. By Eqs. 5 we obtain the desired functional. Specifically, we proved a bijection,

$$\phi : \{\nu : \mathcal{B} \rightarrow \mathbb{R} : \nu(x \vee y) + \nu(x \wedge y) = \nu(x) + \nu(y)\} \rightarrow \{\tilde{\nu} : \text{Sim}(\mathcal{A}(\mathcal{B})) \rightarrow \mathbb{R} : \tilde{\nu} \text{ linear}\}.$$

If $\phi(\nu) = \phi(\nu')$, then $\phi(\nu)(I_x) = \phi(\nu')(I_x)$ for all $x \in \mathcal{B}$. Hence, $\nu = \nu'$. Given $\tilde{\nu}$, set $\nu = \tilde{\nu}|_{\mathcal{B}}$. Then, $\phi(\nu) = \tilde{\nu}$. This mathematical fact has a simple physical explanation (when probabilities are considered): the probabilities are determined by their values in elementary cases. The simpler functional is perhaps the evaluation map, $ev_z(I_x) = I_x(z)$, which corresponds to measuring the property z .

If a group G acts⁶ on \mathcal{B} , it follows that an invariant measure can be made equivalent to an invariant linear functional $\text{Sim}(\mathcal{A}(\mathcal{B})) \rightarrow \mathbb{R}$. Indeed, given an invariant measure $\nu : \mathcal{B} \rightarrow \mathbb{R}$, we define $\tilde{\nu} : \text{Sim}(\mathcal{A}(\mathcal{B})) \rightarrow \mathbb{R}$ as $\tilde{\nu}(I_x) = \nu(x)$. It is easy to prove that the action induces a linear action of G in $\text{Sim}(\mathcal{A}(\mathcal{B}))$ by $g \cdot I_x = I_{g \cdot x}$. Hence, $\tilde{\nu}$ is invariant if and only if ν is invariant.

⁴Compare with the general definition given by Eqs. 3. The main difference relies on the fact that probabilistic measures such as the ones defined by 3 are normalized (its range is restricted to the interval $[0, 1]$), while this last condition is not assumed for additive measures.

⁵Again, compare with the general Definition on arbitrary orthomodular lattices given by Eqs. 2.

⁶The action is defined as a group map $G \rightarrow \text{Aut}(\mathcal{B})$, where $\text{Aut}(\mathcal{B})$ is the set (group) of bijective morphisms of complete Boolean atomistic lattices, $\mathcal{B} \rightarrow \mathcal{B}$, that is, preserving $0, 1, \wedge, \vee$ and the atoms.

Let us characterize σ -additive measures on \mathcal{B} . Denote by $Fun^b(\mathcal{A}(\mathcal{B}))$ to the space of bounded functions with respect to the supremum norm $\|f\| = \sup_{\mathcal{A}(\mathcal{B})} |f(x)|$. It is a non-separable metric space and $\|I_x\| = 1$ for all $x \in \mathcal{B}$. Let us denote by $Sim(\mathcal{A}(\mathcal{B}))^*$ to the space of continuous linear functionals on $Sim(\mathcal{A}(\mathcal{B}))$.

Let ν be a σ -additive measure. Then, $\tilde{\nu}$ defines a linear functional $Sim(\mathcal{A}(\mathcal{B})) \rightarrow \mathbb{R}$. Let I_{x_n} be a convergent sequence of indicator functions such that $\lim_n I_{x_n} = I_x$. Then, there exists an orthogonal sequence $y_n \in \mathcal{B}$ such that $x_n = y_1 \vee \dots \vee y_n$. Then,

$$x_n = y_1 \vee \dots \vee y_n \implies \tilde{\nu}(I_{x_n}) = \tilde{\nu}(I_{y_1 \vee \dots \vee y_n}) = \sum_{i=1}^n \tilde{\nu}(I_{y_i}) = \sum_{i=1}^n \nu(y_i) \implies$$

$$\lim_n \tilde{\nu}(I_{x_n}) = \lim_n \sum_{i=1}^n \nu(y_i) = \nu(\lim_n \bigvee_{i=1}^n y_i) = \tilde{\nu}(\lim_n I_{x_n}) = \tilde{\nu}(I_x).$$

Analogously, if $\tilde{\nu}$ is linear continuous, take $\{y_i\}$ an orthogonal sequence, then

$$\nu(\bigvee_{i=1}^{\infty} y_i) = \tilde{\nu}(\lim_n I_{\bigvee_{i=1}^n y_i}) = \tilde{\nu}(\lim_n \sum_{i=1}^n I_{y_i}) = \lim_n \sum_{i=1}^n \tilde{\nu}(I_{y_i}) = \sum_{i=1}^{\infty} \nu(y_i).$$

Summing up, we have a bijection between σ -additive measures on \mathcal{B} and $Sim(\mathcal{A}(\mathcal{B}))^*$. Using the same arguments as before, there exists a bijection between σ -additive invariant measures on \mathcal{B} and invariant continuous linear functionals on $Sim(\mathcal{A}(\mathcal{B}))$.

3.1. Generating sets. It is the case that many lattices of interest in physics exhibit the property of being atomistic: each element can be expressed as the join of atoms. Examples of lattices with this property are points of phase space for a classical system and the set of one dimensional projections in standard quantum mechanics. This is a very useful property, but in the general case, lattices can be non-atomic (and thus, non-atomistic), for example, the continuous geometries studied by von Neumann [22]. In order to generalize the constructions presented in the previous Section, the following definitions will be useful:

Definition 3.1. Let \mathcal{B} be a Boolean lattice and let $B \subseteq \mathcal{B}$ be a subset closed under finite meets. We say that B is a

- *generating set* if every $x \in \mathcal{B}$ satisfies $x = b_1 \vee \dots \vee b_k$ for some $b_1, \dots, b_k \in B$.
- *σ -generating set* if every $x \in \mathcal{B}$ is equal to $x = \bigvee b_i$ for some denumerable subset $\{b_i\} \subseteq B$.

Before we continue, it is important to make the following remark. We can still characterize measures on a lattice (atomic or not) as measures on a manifold by using a suitably chosen generating set. This is the power of Groemer's integral theorem:

Theorem 3.2 (Groemer's integral theorem). *Let \mathcal{B} be a distributive σ -lattice and let B be a σ -generating set for \mathcal{B} . Let $\nu : B \rightarrow \mathbb{R}$ be a function satisfying $\nu(0) = 0$ and $\nu(b_1 \vee b_2) + \nu(b_1 \wedge b_2) = \nu(b_1) + \nu(b_2)$ for all $b_1, b_2, b_1 \vee b_2 \in B$. Then, the following statements are equivalent*

- (1) ν extends uniquely to a measure on \mathcal{B} .
- (2) ν satisfies the inclusion-exclusion identities

$$\nu(b_1 \vee \dots \vee b_k) = \sum_i \nu(b_i) - \sum_{i < j} \nu(b_i \wedge b_j) + \dots, \quad \forall b_1, \dots, b_k, b_1 \vee \dots \vee b_k \in B, k \geq 2.$$

Proof. See [35, Th.2.2.1] or [12, Th.1]. □

Although Groemer's Theorem yields a strong result, we want to extend it in two directions. The first one is to a situations in which a group is acting and the second is to orthomodular lattices. To illustrate these ideas, let us consider the following examples. The first one was extracted from [35].

Example 3.3. A polyconvex set in \mathbb{R}^n is a finite union of compact convex sets of dimension n . Union and intersections of polyconvex sets are polyconvex sets, hence the family of polyconvex sets is a non-atomic Boolean lattice. In this lattice, the set of parallelotopes $[0, x_1] \times \dots \times [0, x_n]$ combined with the group of Euclidean motions (translations and rotations) is a σ -generating set. Let us denote the Euclidean group as G and B to the set of parallelotopes of the form $[0, x_1] \times \dots \times [0, x_n]$, $B \cong \mathbb{R}^n$. Notice that the the *normalizer* subgroup of B , $N_G(B) = \{g \in G : gB = B\}$, is the group of permutations S_n ,

$$g \cdot [0, x_1] \times \dots \times [0, x_n] = [0, x_{g \cdot 1}] \times \dots \times [0, x_{g \cdot n}] \in B, \quad g \in S_n.$$

According to [35, Ch.5, Ch.4], there is a bijection between G -invariant measures and S_n -invariant functions $B = \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying the inclusion-exclusion identities.

In the next Subsection we extend the above constructions over orthomodular lattices (under the action of more general groups). The following example illustrates what happens in a non-distributive lattice:

Example 3.4. If \mathcal{L} is the (non-Boolean) lattice of subspaces of \mathbb{R}^n , the manifold of atoms is given by \mathbb{P}^{n-1} and the general linear group G acts on \mathcal{L} . Fix a line $\langle v \rangle$. We prove in the next Section that any G -invariant measure ν on \mathcal{L} is determined by assigning a real number $\alpha \in \mathbb{R}$ to the line $\langle v \rangle$. In fact, the value of ν over a k -dimensional subspace $L = \langle v_1, \dots, v_k \rangle$ is $k\alpha$,

$$\nu(L) = \nu(\langle v_1 \rangle) + \dots + \nu(\langle v_k \rangle) = k\alpha.$$

Then, invariant measures over this lattice are essentially the dimension functions.

4. INVARIANT MEASURES ON ARBITRARY ORTHOCOMPLEMENTED LATTICES.

Let us now extend the results of the previous Section to more general lattices. As seen in Section 2.2, the need of extending the approach to a more general setting is related to the fact that quantum theories are based on non-commutative algebras for which the event structures are non-distributive (and thus, non-Boolean). The category of orthomodular lattices is large enough to include many physical examples of interest [23].

4.1. Measures over orthocomplemented lattices. To understand the set of measures on a particular lattice in the light of our constructions, it is more convenient to use category theory. We introduce the notion of a measure with values in an abelian group A . Next, we show that this defines a functor (a type of mapping between categories) that is representable, i.e., there is a group *parameterizing*, the set of measures.

Definition 4.1. Let \mathcal{L} be an orthocomplemented lattice and let A be an abelian group.

- An *additive measure* on \mathcal{L} with values in A is a function $\nu : \mathcal{L} \rightarrow A$ satisfying

$$\nu(x \vee y) = \nu(x) + \nu(y), \quad \text{if } x \perp y.$$

- If A is complete, a σ -*additive measure* on \mathcal{L} with values in A is a function $\nu : \mathcal{L} \rightarrow A$ satisfying

$$\nu\left(\bigvee x_i\right) = \sum_{i=1}^{\infty} \nu(x_i), \quad \text{if } x_i \perp x_j, \forall i \neq j \in \mathbb{N}.$$

where the term in the right is a convergent series in A .

It is important to remark that in Eqs. 2 and 3 of Section 2.2, i) the abelian group is taken to be the set of real numbers, ii) the lattices are considered orthomodular, and iii) the measures are normalized to the interval $[0, 1]$ (this is the environment of interest for describing states in physical theories).

Let us denote by $\mathcal{M}(\mathcal{L}; A)$ the space of measures with values in A . If a group G acts by lattice automorphisms in \mathcal{L} , an *invariant measure* is a measure ν on \mathcal{L} such that $\nu(g \cdot x) = \nu(x)$ for all $x \in \mathcal{L}$ and $g \in G$. The space of invariant measures is denoted $\mathcal{M}(\mathcal{L}; A, G)$. By definition $\mathcal{M}(\mathcal{L}; A) = \mathcal{M}(\mathcal{L}; A, 0)$, where 0 is the trivial group. We always assume that the action of G in A is trivial.

Remark 4.2. In what follows, we are going to use some results and constructions from category theory and commutative algebra. First, let us denote by \mathbf{Ab} the category of abelian groups. We say that a functor $F : \mathbf{Ab} \rightarrow \mathbf{Ab}$ is *representable* if there exists $M \in \mathbf{Ab}$ such that $F(A) = \text{Hom}_{\mathbb{Z}}(M, A)$. Representable functors are an important tool nowadays in mathematics. This is due to the fact that they allow us to represent structures taken from an abstract (or not completely understood) category in terms of sets (or abelian groups) and functions. Usually, these notions are easier to handle, allowing for i) a better understanding of the original structures, and eventually, ii) to the application of known techniques to more abstract domains.

Secondly, we are going to use the \otimes -Hom adjunction that will be important in what follows. Let K_1, K_2 be two commutative rings and let M a K_1 -module, N a K_2 -module, and T a (K_1, K_2) -bimodule. Then, there exists an isomorphism of abelian groups

$$\text{Hom}_{K_1}(N \otimes_{K_2} T, M) \cong \text{Hom}_{K_2}(N, \text{Hom}_{K_1}(T, M)).$$

Third, given a set L we can construct the *free abelian group generated by L* , $\mathbb{Z}^{\oplus L}$. It is given by all finite formal sums of elements of L , $\{\sum_{i=1}^k l_i\}$. Another (equivalent) characterization is given by the set of all functions with finite support $L \rightarrow \mathbb{Z}$. The function associated to an element $l \in L$ is the indicator function $I_{\{l\}}$. We show below that the group $\mathbb{Z}^{\oplus L}$ has the following *universal property*: any function $\nu : L \rightarrow A$ extends uniquely to a \mathbb{Z} -linear map $\bar{\nu} : \mathbb{Z}^{\oplus L} \rightarrow A$:

$$\begin{array}{ccc} L & \xrightarrow{\nu} & A \\ \pi \downarrow & \circlearrowleft & \nearrow \\ \mathbb{Z}^{\oplus L} & & \exists! \bar{\nu} \end{array}$$

Finally, given an abelian group M where a group G acts, we can construct the module of invariants M^G and the module of coinvariants M_G ,

$$M^G = \{m \in M : g \cdot m = m, \forall g \in G\}, \quad M_G = M/R.$$

where R is the submodule generated by $\{g \cdot m - m : g \in G, m \in M\}$. The module M_G has the following universal property: for every linear map $\nu : M \rightarrow A$, where G acts trivially on A , there exists a unique factorization $\bar{\nu} : M_G \rightarrow A$,

$$\begin{array}{ccc} M & \xrightarrow{\nu} & A \\ \pi \downarrow & \circlearrowleft & \nearrow \\ M_G & & \exists! \bar{\nu} \end{array}$$

Now we apply all these notions to study different measures of interest on orthocomplemented lattices:

Proposition 4.3. *Let \mathcal{L} be an orthocomplemented lattice. Then, there exists an abelian group $M = M(\mathcal{L})$ representing the functor $\mathcal{M}(\mathcal{L}; -)$,*

$$\mathcal{M}(\mathcal{L}; -) = \text{Hom}_{\mathbb{Z}}(M, -)$$

This means that a measure in \mathcal{L} valued in A is equivalent to a \mathbb{Z} -linear map from $M(\mathcal{L})$ to A .

Proof. Let $\mathbb{Z}^{\oplus \mathcal{L}}$ be the free abelian group generated by \mathcal{L} . Let $S \subseteq \mathbb{Z}^{\oplus \mathcal{L}}$ be the subgroup generated by the elements

$$\{x \vee y - x - y : x \perp y\}.$$

where the elements of \mathcal{L} are embedded in $\mathbb{Z}^{\oplus \mathcal{L}}$ as usual (i.e., by using their characteristic functions). Let us see that $M = \mathbb{Z}^{\oplus \mathcal{L}}/S$ represents the functor $\mathcal{M}(\mathcal{L}; -)$.

First, assume that $x_i \perp x_j$ for every $1 \leq i < j \leq k$. Given that $x_i \leq x_1^\perp$ for $2 \leq i \leq k$, we have $x_2 \vee \dots \vee x_k \leq x_1^\perp$ and then, $x_1 \vee (x_2 \vee \dots \vee x_k) = x_1 + (x_2 \vee \dots \vee x_k)$ in M . By induction, we obtain that $x_1 \vee \dots \vee x_n = x_1 + \dots + x_n$ in M .

Second, any measure $\mathcal{L} \rightarrow A$ defines a function $\mathbb{Z}^{\oplus \mathcal{L}} \rightarrow A$ mapping S to zero. Analogously, any linear function $\nu : M \rightarrow A$ is a linear function from $\mathbb{Z}^{\oplus \mathcal{L}}$ such that $\nu(S) = 0$. Then, the restriction of ν to \mathcal{L} defines a measure on \mathcal{L} . \square

Corollary 4.4. *Let \mathcal{L} be an orthocomplemented lattice and assume that a group G acts by automorphism. Let A be an abelian group where G acts trivially.*

The map $\pi : \mathcal{L} \rightarrow M(\mathcal{L})$ is a measure and satisfies the following universal property: any measure with values in A factorizes as a \mathbb{Z} -linear map $M(\mathcal{L}) \rightarrow A$.

Also, the map $\pi_G : \mathcal{L} \rightarrow M_G$ is an invariant measure and satisfies the following universal property: any invariant measure with values in A factorizes as a \mathbb{Z} -linear map $M_G \rightarrow A$.

We can represent the properties of π and π_G by the following diagrams:

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\forall \nu} & A \\ \pi \downarrow & \circlearrowleft & \nearrow \\ M(\mathcal{L}) & & \exists! \bar{\nu} \end{array} \quad \begin{array}{ccc} \mathcal{L} & \xrightarrow{\forall \nu'} & A \\ \pi_G \downarrow & \circlearrowleft & \nearrow \\ M(\mathcal{L})_G & & \exists! \bar{\nu}' \end{array}$$

where ν (resp. ν') is a measure (resp. invariant measure) and $\bar{\nu}$, $\bar{\nu}'$ are linear maps. The commutativity means that $\nu = \bar{\nu}\pi$, $\nu' = \bar{\nu}'\pi_G$.

Proof. Clearly, π is a measure with values in M and π_G is an invariant measure with values in M_G . The result for π follows from Proposition 4.3.

Now recall that the functor $(-)_G$ is naturally isomorphic to $(-) \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ and the functor $(-)^G$ is naturally isomorphic to $\text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, -)$, [42, Lemma 6.1.1]. Then, using the \otimes -Hom adjunction and the equality $A = A^G$, we see that any map $\bar{\nu}' \in \text{Hom}_{\mathbb{Z}}(M, A)$ is in fact a G -linear map. Next, recall that any G -linear map $M \rightarrow A$ into a trivial G -module A factors uniquely over a map from M_G . Hence, $\bar{\nu}'$ factorizes uniquely over M_G .

Summing up, given $\nu' \in \mathcal{M}(\mathcal{L}; A, G)$ there exists a unique $\bar{\nu}' \in \text{Hom}_{\mathbb{Z}}(M_G, A)$ such that $\bar{\nu}'\pi = \nu'$. \square

Definition 4.5. We call π the *universal measure* and π_G the *universal invariant measure*. When no confusion arises, we write just x instead of $\pi(x)$ (or $\pi_G(x)$).

The following Corollaries summarize our constructions up to now:

Corollary 4.6. *We have the following characterizations,*

$$\mathcal{M}(\mathcal{L}; A) = \text{Hom}_{\mathbb{Z}}(M, A), \quad \mathcal{M}(\mathcal{L}; A, G) = \text{Hom}_{\mathbb{Z}}(M, A)^G.$$

Moreover, if K is a commutative ring and A is a K -module (ex. $K = \mathbb{R}$ and $A = \mathbb{R}^n$),

$$\mathcal{M}(\mathcal{L}; A) = \text{Hom}_K(M_K, A), \quad \mathcal{M}(\mathcal{L}; A, G) = \text{Hom}_K(M_K, A)^G,$$

where $M_K := M \otimes_{\mathbb{Z}} K$ is the extension of M from \mathbb{Z} to K . Also, the map $\pi_K : \mathcal{L} \rightarrow M_K$ is the universal K -linear measure.

Proof. Using the \otimes -Hom adjunction, we have

$$\begin{aligned} \text{Hom}_{\mathbb{Z}}(M_G, A) &= \text{Hom}_{\mathbb{Z}}(\mathbb{Z} \otimes_{\mathbb{Z}[G]} M, A) = \\ &= \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \text{Hom}_{\mathbb{Z}}(M, A)) = \text{Hom}_{\mathbb{Z}}(M, A)^G. \end{aligned}$$

Now, if A is a K -module, from the \otimes -Hom adjunction we have,

$$\text{Hom}_K(M_K, A) = \text{Hom}_K(M \otimes_{\mathbb{Z}} K, A) = \text{Hom}_{\mathbb{Z}}(M, \text{Hom}_K(K, A)) = \text{Hom}_{\mathbb{Z}}(M, A).$$

□

Corollary 4.7. *Let K be a commutative ring. Then, $\mathcal{M}(-; K)$ is a functor from the category of orthocomplemented lattices to the category of K -modules.*

Also, if we fix the orthocomplemented lattice, $\mathcal{M}(\mathcal{L}; -)$ is a functor from the category of K -modules to the category of K -modules.

Proof. Let $f : \mathcal{L} \rightarrow \mathcal{L}'$ be a lattice map between two orthocomplemented lattices. Pre-composing with f , we get a K -linear map, $f^* : \mathcal{M}(\mathcal{L}'; K) \rightarrow \mathcal{M}(\mathcal{L}; K)$.

Analogously, if $\phi : A \rightarrow A'$ is K -linear, $\phi_* : \mathcal{M}(\mathcal{L}; A) \rightarrow \mathcal{M}(\mathcal{L}; A')$ is also K -linear. □

Let us mention some simple applications of our result. For example, if we are interested in measures with values over \mathbb{C} , our results show that they are parameterized by complex linear functionals. Also, if we are interested in (for example) G -invariant real measures, we proved that they are parameterized by G -invariant real linear maps.

4.2. Generalization of Groemer's integral theorem. In this Subsection we extend Groemer's integral theorem to the more general setting of orthocomplemented lattices. For this purpose, we need to generalize the notion of *generating set* to arbitrary orthocomplemented lattices:

Definition 4.8. Let \mathcal{L} be an orthocomplemented lattice and assume that a group G acts on the lattice \mathcal{L} by automorphisms. Let $B \subseteq \mathcal{L}$ be a subset closed under finite meets. The group G may not act on B but we always has an action of its normalizer subgroup,

$$N_G(B) = \{g \in G : gB = B\}.$$

Notice that the inclusion $B \subseteq \mathcal{L}$ induces a set function (not necessarily injective) $\mu : B/N_G(B) \rightarrow \mathcal{L}/G$. We say that B is

- an *orthogonal generating set* if every $x \in \mathcal{L}$ is equal to $x = b_1 \vee \dots \vee b_k$ for some pairwise orthogonal $b_1, \dots, b_k \in B$.
- a *σ -orthogonal generating set* if every $x \in \mathcal{L}$ is equal to $x = \bigvee b_i$ for some pairwise orthogonal numerable subset $\{b_i\} \subseteq B$.
- an *orthogonal generating set for the action* if μ is injective and the set $G \cdot B$ is an orthogonal generating set for \mathcal{L} .
- a *σ -orthogonal generating set for the action* if μ is injective and the set $G \cdot B$ is a σ -orthogonal generating set for \mathcal{L} .

Now we are ready to construct our extension of Groemer's integral theorem to orthocomplemented lattices:

Theorem 4.9 (Non-Boolean Groemer's integral theorem). *Let \mathcal{L} be an orthocomplemented lattice where a group G acts. Let B be an orthogonal generating set for the action of G . Then, invariant measures on \mathcal{L} are in bijection with $N_G(B)$ -invariant functions on B , ν , such that*

$$\nu(b_1 \vee b_2) = \nu(b_1) + \nu(b_2), \quad \forall b_1, b_2, b_1 \vee b_2 \in B, \quad b_1 \perp b_2.$$

Proof. Let a) $S \subseteq \mathbb{Z}^{\oplus \mathcal{L}}$ be the subgroup generated by $\{x \vee y - x - y : x \perp y\}$ and b) $\overline{B} \subseteq \mathcal{L}/G$ be the image of $\mu : B/N_G(B) \rightarrow \mathcal{L}/G$. Recall that $M = M(\mathcal{L}) \cong \mathbb{Z}^{\oplus \mathcal{L}}/S$. By hypothesis, the following map is surjective,

$$\tilde{\mu} : \mathbb{Z}^{\overline{B}} \rightarrow M_G, \quad \overline{b} \mapsto [b].$$

Then, $\mathbb{Z}^{\overline{B}}/\ker(\tilde{\mu}) \cong M_G$, where

$$\ker(\tilde{\mu}) = \mathbb{Z}^{\overline{B}} \cap S_G = \langle \overline{b_1 \vee b_2} - \overline{b_1} - \overline{b_2} : \overline{b_1}, \overline{b_2}, \overline{b_1 \vee b_2} \in \overline{B}, \overline{b_1} \perp \overline{b_2} \rangle.$$

Finally, applying the functor $\text{Hom}_{\mathbb{Z}}(-, A)$, we obtain

$$\text{Hom}_{\mathbb{Z}}(M_G, A) \cong \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\overline{B}}/\ker(\tilde{\mu}), A).$$

The first space is isomorphic to $\mathcal{M}(\mathcal{L}; A, G)$. A map ν in the second space is a map in $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\overline{B}}, A)$, such that $\nu(\ker(\tilde{\mu})) = 0$. More specifically, it is the same as a function $\nu : \overline{B} \rightarrow A$ such that

$$\nu(\overline{b_1 \vee b_2}) = \nu(\overline{b_1}) + \nu(\overline{b_2}), \quad \overline{b_1}, \overline{b_2}, \overline{b_1 \vee b_2} \in \overline{B}, \quad \overline{b_1} \perp \overline{b_2}.$$

The result follows from $B/N_G(B) \cong \overline{B}$. □

Remark 4.10. A similar result holds for σ -additive measures. The steps are the following:

- (1) Define the set $\mathcal{M}^{\sigma}(\mathcal{L}; A)$ of σ -additive measures with values in a Banach vector space A over a complete field K (say $K = \mathbb{R}$).
- (2) Given that $M_K = M \otimes_{\mathbb{Z}} K$ is complete, see [1, Ch.10], we can define the set of continuous linear maps (i.e. bounded operators) from M_K to A , that is, $\nu \in \text{Cont}_K(M_K, A)$ if and only if

$$\nu\left(\bigvee_{i=1}^{\infty} x_i\right) = \sum_{i=1}^{\infty} \nu(x_i), \quad x_i \perp x_j, \forall i \neq j.$$

From the following commutative square, it follows that $\mathcal{M}^{\sigma}(\mathcal{L}; A) \cong \text{Cont}_K(M_K, A)$,

$$\begin{array}{ccc} \mathcal{M}(\mathcal{L}; A) & \xrightarrow{\sim} & \text{Hom}_K(M_K, A) \\ \uparrow & \circlearrowleft & \uparrow \\ \mathcal{M}^{\sigma}(\mathcal{L}; A) & \longrightarrow & \text{Cont}_K(M_K, A) \end{array}$$

- (3) Finally, using Theorem 4.9 and a density argument, it is easy to prove the result for σ -additive measures. Indeed, let B be a σ -orthogonal generating set for the action of G . The set $\mathcal{M}^{\sigma}(\mathcal{L}; A, G)$ is in bijection with functions $\nu : B/N_G(B) \rightarrow A$, such that

$$\nu\left(\bigvee_{i=1}^{\infty} b_i\right) = \sum_{i=1}^{\infty} \nu(b_i), \quad \bigvee_{i=1}^{\infty} b_i, b_j \in B, b_i \perp b_j \forall i \neq j.$$

Let us consider an example for what we have just proved. The above theorem (and the accompanying remark) has the following consequence: in order to define a measure on the orthomodular lattice of orthogonal projectors $\mathcal{L}(\mathcal{W})$ of a von Neumann algebra \mathcal{W} , it is sufficient to know an orthogonal generating set for the projectors in \mathcal{W} . For the case of Type I factors, this generating set is given by one dimensional projections. We will come back to this point later.

Remark 4.11. It is important to mention that, in general, it is a difficult task to prove that a set $B \subseteq \mathcal{L}$ is an orthogonal generating set for the action of a group G . But, in some cases, it is possible to check that the set $G \cdot B \subseteq \mathcal{L}$ is an orthogonal generating set. Using this hypothesis for B let us prove a weak version of the Non-Boolean Groemer's integral theorem:

Theorem 4.12 (Weak Non-Boolean Groemer's integral theorem). *The set $\mathcal{M}(\mathcal{L}; A, G)$ is included in the set of $N_G(B)$ -invariant functions $\nu : B \rightarrow A$ such that*

$$\nu(b_1 \vee b_2) = \nu(b_1) + \nu(b_2), \quad \forall b_1, b_2 \in B, b_1 \perp b_2.$$

Proof. By hypothesis, the multiplication map $\mu : \mathbb{Z}^{\oplus B} \otimes_{\mathbb{Z}[N_G(B)]} \mathbb{Z}[G] \rightarrow M$ is surjective. Applying $(-)_G$, the map $\tilde{\mu} : (\mathbb{Z}^{\oplus B})_{N_G(B)} \rightarrow M_G$ is also surjective. Notice that $\tilde{\mu}(\overline{R}) = 0$, where $R \subseteq \mathbb{Z}^{\oplus B}$, is generated by $b_1 \vee b_2 - b_1 - b_2$ for all $b_1, b_2 \in B, b_1 \perp b_2$ and $\overline{R} = R_{N_G(B)}$. Then,

$$(\mathbb{Z}^{\oplus B}/R)_{N_G(B)} \twoheadrightarrow M_G \implies \text{Hom}_{\mathbb{Z}}(M_G, A) \subseteq \text{Hom}_{\mathbb{Z}}((\mathbb{Z}^{\oplus B})_{N_G(B)}/\overline{R}, A).$$

□

4.3. Positive measures on orthocomplemented lattices. Now, we restrict ourselves to the study of positive measures on orthocomplemented lattices.

Definition 4.13. Let \mathcal{L} be an orthocomplemented lattice. A *positive measure* on \mathcal{L} is a measure taking positive values in \mathbb{R} . We use the following notation

- $\mathcal{M}_+(\mathcal{L})$ denotes the set of positive measures,
- $\mathcal{M}_+(\mathcal{L}; G)$ denotes the set of invariant positive measures,
- $\mathcal{M}_+^\sigma(\mathcal{L})$ denotes the set of σ -positive measures,
- $\mathcal{M}_+^\sigma(\mathcal{L}; G)$ denotes the set of invariant σ -positive measures.

We discuss next some examples of generating sets and their geometric properties. But first, let us give a geometric construction

Definition 4.14. Given a subset C of a real vector space V we define its *dual cone* as,

$$C^* = \{\ell : V \rightarrow \mathbb{R} : \ell(C) \geq 0\} = \{H \subseteq V : C \subseteq H^+\}.$$

From [5, §2.6.1], we know that the double dual cone of C is its convex hull and it determines the same dual cone.

Example 4.15. Geometrical properties of generating sets. Let B be an orthogonal generating set for \mathcal{L} . Let us denote by $M_{\mathbb{R}}$ the real vector space $M(\mathcal{L}) \otimes_{\mathbb{Z}} \mathbb{R}$. The kernel of a non-zero linear functional $M_{\mathbb{R}} \rightarrow \mathbb{R}$ is an hyperplane H inside $M_{\mathbb{R}}$. By appealing to the universal real measure we can identify every element $b \in B$ with an element of $M_{\mathbb{R}}$. Thus, we can write $B \subseteq M_{\mathbb{R}}$ instead of $\pi_{\mathbb{R}}(B) \subseteq M_{\mathbb{R}}$. Let us apply the *dual cone* construction to $B \subseteq M_{\mathbb{R}}$.

Using Theorem 4.9, it is easy to check that the dual cone of B is the set of positive measures

$$\mathcal{M}_+(\mathcal{L}) \cong B^* := \{H \subseteq M_{\mathbb{R}} : B \subseteq H^+\}.$$

Similarly, it follows

$$\begin{aligned}\mathcal{M}_+(\mathcal{L}; G) &\cong \{H \subseteq M_{\mathbb{R}} : B \subseteq H^+, H = H^G\} \\ \mathcal{M}_+^{\sigma}(\mathcal{L}) &\cong \{H \subseteq M_{\mathbb{R}} : B \subseteq H^+, H = \overline{H}\} \\ \mathcal{M}_+^{\sigma}(\mathcal{L}; G) &\cong \{H \subseteq M_{\mathbb{R}} : B \subseteq H^+, H = H^G = \overline{H}\}.\end{aligned}$$

Example 4.16. Hilbert space and standard quantum mechanics. Let \mathcal{H} be a separable Hilbert space and consider the orthomodular lattice of orthogonal projectors \mathcal{L} . Given that any closed subspace S possesses an orthonormal basis $\{v_i\}_{i=1}^{\infty}$, the family of projectors of range 1 (basically, one dimensional subspaces) is a σ -orthogonal generating set. Notice that any projector of rank 1 is given by $|v\rangle\langle v| \in \mathcal{B}(\mathcal{H})$ for some $v \in \mathcal{H}$. Thus, for an arbitrary separable Hilbert space, a σ -orthogonal generating set is given by $B = \{|v\rangle\langle v| : v \in \mathcal{H}, \|v\| = 1\}$. Notice that \mathcal{L} is defined inside $\mathcal{B}(\mathcal{H})$ (see Proposition 4.17 below). More generally, if \mathcal{W} is a von Neumann algebra, we still have a similar picture. Recall that the lattice associated to \mathcal{W} is the lattice $\mathcal{L}(\mathcal{W})$ of projectors in \mathcal{W} . Then, a σ -orthogonal generating set B will be some subset of projectors on \mathcal{W} . In the previous case, $\mathcal{W} = \mathcal{B}(\mathcal{H})$, the rank one projectors form a σ -orthogonal generating set. In more general situations, B will be a set of projectors (not necessarily of rank one, ex. Type III factor) which always exists.

Let us now turn to the following natural question [15, Ch. 5]:

Does any real measure on $\mathcal{L}(\mathcal{W})$ extend to a bounded linear functional on \mathcal{W} ?

The above question concerns the relationship between $\mathcal{M}(\mathcal{L}(\mathcal{W}); \mathbb{R})$ and $\text{Cont}_{\mathbb{C}}(\mathcal{W}; \mathbb{C})$. In order to describe this problem in our terms, let us prove the following fundamental fact:

Proposition 4.17. *Let \mathcal{W} be a von Neumann algebra on a separable Hilbert space and let $\mathcal{L}(\mathcal{W})$ be its orthomodular lattice of projectors. Then, the universal complex measure is injective*

$$\begin{array}{ccc} \mathcal{L}(\mathcal{W}) & \xrightarrow{i} & \mathcal{W} \\ \pi_{\mathbb{C}} \downarrow & \nearrow \bar{i} & \\ M_{\mathbb{C}} & & \end{array}$$

The map $\pi_{\mathbb{C}}$ is the universal complex measure and the map i is the inclusion $\mathcal{L}(\mathcal{W}) \subseteq \mathcal{W}$. Moreover, the image of \bar{i} is dense in \mathcal{W} .

Notice that \bar{i} is not necessarily injective.

Proof. The set $\mathcal{L}(\mathcal{W})$ of orthogonal projectors in \mathcal{W} forms an orthomodular lattice which can be viewed as a subset of \mathcal{W} (i.e., $\mathcal{L}(\mathcal{W}) \subseteq \mathcal{W}$). Due to the fact that any von Neumann algebra is a complex vector space with respect to the sum and multiplication by scalars, the inclusion map $i : \mathcal{L}(\mathcal{W}) \hookrightarrow \mathcal{W}$ satisfies the axioms of a measure with values in a complex vector space and then, from our previous results, this inclusion factorizes through $M_{\mathbb{C}}$. This means that there exists a unique map \bar{i} such that $i = \bar{i} \circ \pi_{\mathbb{C}}$. From the fact that i is an injection, it follows that $\pi_{\mathbb{C}}$ is also injective.

Now, let us recall a standard result of von Neumann algebras: the complex vector space spanned by $\mathcal{L}(\mathcal{W})$ is norm-dense in \mathcal{W} (see for example Corollary 0.4.9(b) and the comment that follows in [38]). Then, given that the space spanned by $\mathcal{L}(\mathcal{W})$ is contained in the image of \bar{i} , we can conclude that $\bar{i}(M_{\mathbb{C}})$ is also dense in \mathcal{W} . \square

Corollary 4.18. *Let \mathcal{W} be a von Neumann algebra over a separable Hilbert space such that \bar{i} is injective ($\mathcal{L}(\mathcal{W}) \subseteq M_{\mathbb{C}} \subseteq \mathcal{W}$). Then, there exists a bijection between σ -probabilities (resp. σ -positive measures) on $\mathcal{L}(\mathcal{W})$ and normal states (resp. positive linear functionals) on \mathcal{W} .*

Proof. As we mentioned in Section 2.2, states are defined as positive complex-valued continuous linear functionals $\mu : \mathcal{W} \rightarrow \mathbb{C}$. Moreover, the restriction of μ to $\mathcal{L}(\mathcal{W})$ defines a σ -probability.

Suppose that we have a σ -probability $\nu : \mathcal{L}(\mathcal{W}) \rightarrow [0, 1]$. Given that $[0, 1]$ is included in \mathbb{C} , we can assume that the codomain of ν is \mathbb{C} and it follows now that ν satisfies the axiom of a complex σ -measure. Then, from what we have proved above, there exists a unique extension to a continuous linear functional $\bar{\nu} : M_{\mathbb{C}} \rightarrow \mathbb{C}$. From the fact that $M_{\mathbb{C}}$ can be injected as a dense subset inside \mathcal{W} , we can extend $\bar{\nu}$ by continuity (in a unique way) to $\tilde{\nu} : \mathcal{W} \rightarrow \mathbb{C}$. Diagrammatically,

$$\begin{array}{ccccc}
 \mathcal{L}(\mathcal{W}) & \hookrightarrow & M_{\mathbb{C}} & \hookrightarrow & \mathcal{W} \\
 \downarrow \forall \nu & \searrow & \downarrow \exists! \bar{\nu} & \swarrow & \downarrow \exists! \tilde{\nu} \\
 [0, 1] & \hookrightarrow & \mathbb{C} & &
 \end{array}$$

It remains to check that $\tilde{\nu}$ is a normal state. Given that a positive operator A has a positive spectrum $sp(A)$ ([38, p.3]), we can approximate the identity over $sp(A)$ by simple functions with positive coefficients. Hence, any positive operator $A \in \mathcal{W}$ can be approximated by projectors in $\mathcal{L}(\mathcal{W})$. In particular, given a continuous linear map $\tilde{\nu} : \mathcal{W} \rightarrow \mathbb{C}$ such that $\tilde{\nu}(P) \geq 0$ for all $P \in \mathcal{L}(\mathcal{W})$, it follows that $\tilde{\nu}(A) \geq 0$ for all $A \geq 0$ (i.e. $\tilde{\nu}$ is positive). The normality condition follows from the σ -additivity condition on projection operators.

The bijection between σ -positive measures on $\mathcal{L}(\mathcal{W})$ and positive linear functional on \mathcal{W} is similar replacing $[0, 1]$ with \mathbb{R} . \square

Let us now use these constructions to rephrase Gleason's theorem in our terms. Gleason's theorem can be usually stated as follows (see [24, Th. 3.13] or [11]):

Theorem (Gleason). *Let \mathcal{H} be a Hilbert space with $\dim(\mathcal{H}) \geq 3$. Then the set of σ -positive measures ν on $\mathcal{L}(\mathcal{H})$ is in bijection with the set of positive operators T of the trace class such that*

$$\nu = \text{tr}(T-).$$

We can give a similar result appealing to Proposition 4.17 and its Corollary:

Theorem 4.19. *Let \mathcal{H} be a separable Hilbert space such that $\mathcal{L}(\mathcal{H}) \subseteq M_{\mathbb{C}} \subseteq \mathcal{B}(\mathcal{H})$. Then the set of σ -positive measures ν on $\mathcal{L}(\mathcal{H})$ is in bijection with the set of positive operators T of the trace class such that*

$$\nu = \text{tr}(T-).$$

Proof. According to the previous Corollary taking $\mathcal{W} = \mathcal{B}(\mathcal{H})$, a σ -positive measure is the same a positive linear functional $\nu : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{C}$. As it is well known, measures of this kind are given by $\text{tr}(T-)$ for some positive operator $T \in \mathcal{B}(\mathcal{H})$. Then, σ -positive measures on $\mathcal{L}(\mathcal{H})$ are in bijection with functionals $\text{tr}(T-)$, where T is some positive operator. Given that $I \in \mathcal{L}(\mathcal{H})$, by imposing $\nu(I) = 1$ (which is the normalization condition for any quantum state), we obtain that T is of the trace class. \square

5. CONCLUSIONS

In this paper we presented a characterization of states in orthocomplemented lattices. This was done by appealing to a non-commutative version of geometric probability theory based on algebraic geometry techniques. Our theoretical framework incorporates invariant states (under

groups acting by automorphisms) in a natural way. Our main result, given by Theorem 4.9, allows us to characterize states in orthomodular lattices and invariant states as well. Finally, we were able to recast Gleason's theorem in these terms.

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