COLUMN SYMMETRIC POLYNOMIALS

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Résumé. Nous étudions l'algèbre des polynômes en une m x n matrice de variables sur un anneau contenant les rationnels, sujette à la condition que le produit de deux variables appartenant à une même colonne est nul. Nous montrons que la sous-algèbre des polynômes invariants sous l'action des n! permutations des colonnes est un quotient de l'algèbre des polynômes en m variables; l'application quotient envoie la i-ème variable en la somme des entrées de la i-ème ligne de la matrice. Une application en géométrie différentielle synthétique est esquissée.

Abstract. We study the polynomial algebra (over a ring containing the rationals) in an m by n matrix of variables, and subject to the relation that says that the product of any two variables in the same column is zero. We show that the sub-algebra of polynomials, which are invariant under the n! permutations of the columns, is a quotient of the polynomial algebra in m variables; the quotient map sends the i'th variable to the sum of the entries in the i'th row of the matrix. An application in synthetic differential geometry is sketched.

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Introduction

Let A be a commutative ring. It is classical how symmetric polynomials in $A[x_1, \ldots, x_n]$ are uniquely expressible as polynomials in the n elementary symmetric polynomials, cf. e.g. [4] §29. For instance for $n = 2$, the two elementary polynomials are $\sigma_1 := x_1 + x_2$ and $\sigma_2 := x_1 x_2$; and the symmetric polynomial $x_1^2 + x_2^2$ may be expressed as $\sigma_1^2 - 2\sigma_2$:

$$x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1 x_2.$$

Modulo the ideal I generated by x_1^2 and x_2^2, we therefore also have

$$x_1 x_2 = \frac{1}{2}(x_1 + x_2)^2,$$

provided $\frac{1}{2}$ exists in the base ring A.

In fact, we have more generally that if A contains the ring $\mathbb{Q}$ of rationals, then, modulo I, any symmetric polynomial in $A[x_1, \ldots, x_n]$ may be uniquely expressed as a polynomium in the single symmetric polynomium $x_1 + \ldots + x_n$, where I is the ideal generated by the x_i^2s. This is a well known and important fact, called "the symmetric functions property" in [2] Exercise I.3.3.

It is a result in this direction we intend to generalize from dimension 1 to dimension m. We are considering the polynomial ring in $m \times n$ variables $x_{i,j}$; the kind of symmetry we consider is not with respect to all the mn variables; we consider these variables organized in an $m \times n$ matrix, and we only consider invariance under the $n!$ permutations of the n columns. The result refers to what we can assert, modulo the ideal I generated by the degree 2 monomials $\{x_{ij} x_{i'j}\}_{j=1,\ldots,n,\ i=1,\ldots,m,\ i'=1,\ldots,m}$.

The result asserts that any polynomial, invariant under the $n!$ permutations of the columns can, modulo I, be expressed uniquely as a polymonial in the m "row-sums", $\{s_i = x_{i,1} + x_{i,2} + \ldots + x_{i,n}\}_{i=1,\ldots,m}$. The classical "symmetric functions property" is the special case where $m = 1$.

An application of this Theorem concerns formal exactness of closed differential 1-forms is sketched in Section 3 below.

Throughout A will be a commutative ring. It is assumed to contain $\mathbb{Q}$. All the A-modules which we consider are free. Therefore, we use terminology from linear algebra, as if A were a field.

1. Polynomials in a matrix of variables

1.1 The free commutative monoid

The free commutative monoid $M(X)$ on a set X is in a natural way a graded monoid. We call its elements *monomials* in X, we call X the set of *variables*; we write the monoid structure multiplicatively. We shall give an explicit presentation of $M(X)$.[1]

Let k be a positive integer; we let $[k]$ denote the set $[k] = \{1, 2, \ldots, k\}$. Then a monomial ω of degree k may be explicitly presented by a function $f : [k] \rightarrow X$; we write the monomium thus presented $\omega_f := x_{f(1)}x_{f(2)} \ldots x_{f(k)}$. Since the variables commute, it follows that two functions f and $f' : [k] \rightarrow X$ present the same monomium iff they differ by a permutation $\varepsilon : [k] \rightarrow [k]$ of $[k]$, i.e. $f' = f \circ \varepsilon$.

Later on in the proof of Proposition 1.5, we shall need a finer notation: We denote by $\|f\|$ the set of all functions $f \circ \varepsilon$ for $\varepsilon \in S_k$ (where S_k is the group of permutations of $[k]$). Thus $\|f\|$ is the orbit of f under the right action (by precomposition) of S_k. The monomials are actually indexed by these orbits, we have a well defined monomium $\omega_{\|f\|}$, and $\omega_{\|f\|} = \omega_{\|f'\|} \iff \|f\| = \|f'\|$.

1.2 The polynomial ring in a matrix of variables

If A is any commutative ring, the polynomial ring $A[X]$ with coefficients in A in a set X of indeterminates is the free commutative A-algebra on the set X. It may be constructed by a two-stage process: first, construct the free commutative monoid $M(X)$ on X, and then construct the free A-module on the set $M(X)$. It inherits its multiplication from that of $M(X)$. It is a *graded* A-algebra, with the degree-k part being the linear submodule with basis the monomials of degree k.

We shall be interested in some further structure which the algebra $A[X]$ has, in the case where the set X is given as a product set $[m] \times [n]$. We think of this X as the set of $m \times n$ matrices (m rows, n columns) with entries $x_{i,j}$ ($i \in [m], j \in [n]$), and write $A[M^{m\times n}] := A[[m] \times [n]] = A[x_{1,1}, \ldots, x_{m,n}]$.

[1] An equivalent description is that $M(X)$ is the set of finite *multi-subsets* of X.

A function $[k] \longrightarrow [m] \times [n]$ is given by a pair (f, g), where $f : [k] \to [m]$ and $g : [k] \to [n]$. The monomium presented by such function we denote $\omega_{(f,g)}$, or just $\omega_{f,g}$. Thus

$$\omega_{f,g} = \prod_{l \in [k]} x_{f(l),g(l)} \;=\; x_{f(1)g(1)} x_{f(2)g(2)} \;\cdots\; x_{f(k)g(k)} \tag{1}$$

Clearly, when g is monic, then so is any other g', for any other presentation (f', g') of the same monomium. Therefore, the following notion is well defined.

Definition 1.1. *The monomial $\omega_{f,g}$ is* admissible *if $g : [k] \to [n]$ is monic. A polynomium $\in A[M^{m\times n}]$ is called* admissible *if it is a linear combination of admissible monomials.*

So a monomium in the $x_{i,j}$'s is admissible if it does not contain two factors from any of the columns, like $x_{i,j} \cdot x_{i',j}$. In particular, it does not contain any squared factor $x_{i,j}^2$. Clearly, admissible polynomials are of degree $\leq n$.

If ω is not admissible, it is called inadmissible. If ω is inadmissible, then so is $\omega \cdot \theta$ for any monomium θ. It follows that the linear subspace of $A[M^{m\times n}]$ generated by the inadmissible monomials is an ideal $I \subseteq A[M^{m\times n}]$. The quotient algebra $A[M^{m\times n}]/I$ may be identified with the linear subspace (not a subalgebra) $A_a[M^{m\times n}] \subseteq A[M^{m\times n}]$ generated by the admissible monomials, with the projection morphism $A[M^{m\times n}] \longrightarrow A_a[M^{m\times n}]$ being the map which discards all terms containing an inadmissible factor. The algebra structure of $A_a[M^{m\times n}]$ is thus given by the multiplication table $\{x_{i,j} \cdot x_{i',j} = 0\}_{i\in[m],\, i'\in[m],\, j\in[n]}$, and no other relations.[2] The algebra $A_a[M^{m\times n}]$ inherits a grading from that of $A[M^{m\times n}]$. Note that in $A_a[M^{m\times n}]$ all non-zero elements are of degree $\leq n$.

Among the polynomials in $A[M^{m\times n}]$ we have the m "row-sums" s_i for $i = 1, \ldots, m$ (the sum of the entries in the ith row); they are all admissible:

$$s_i := \sum_{j \in [n]} x_{i,j} \;=\; x_{i,1} + x_{i,2}, \;\ldots\; x_{i,n}. \tag{2}$$

[2] $A_a[M^{m\times n}]$ is an example of what sometimes is called a *Weil-algebra* over A; in particular, it is finite-dimensional as an A-module. Likewise, the algebra $A_{\leq n}[y_1, \ldots, y_m]$ to be considered below, is a Weil-algebra.

Consider any map $f : [k] \to [m]$. By the distributive law, i.e. by multiplying out the product, we have the second equality sign in

$$\prod_{l\in[k]} s_{f(l)} = \prod_{l\in[k]} \sum_{j\in[n]} x_{f(l)j} = \sum_{[k]\xrightarrow{g}[n]} \prod_{l\in[k]} x_{f(l)g(l)},$$

where g ranges over the set of all maps $[k] \to [n]$. The admissible terms here are those where g is injective, so modulo I, equivalently, discarding inadmissible terms,

$$\prod_{l\in[k]} s_{f(l)} \;=\; \sum_{[k]\overset{g}{\hookrightarrow}[n]} \prod_{l\in[k]} x_{f(l),g(l)} \qquad \text{in the algebra } A_a[M^{m\times n}]. \tag{3}$$

where g now ranges over the set of monic maps $[k] \hookrightarrow [n]$.

1.3 Column symmetric polynomials

Let σ be a permutation $\sigma : [n] \to [n]$, i.e. $\sigma \in S_n$. One may permute the n columns of the matrix X of variables $x_{i,j}$ by σ. More explicitly, σ permutes the monomials by the recipe:

$$\sigma \cdot \omega_{f,g} := \omega_{f,\sigma\circ g}\,. \tag{4}$$

This is well defined with respect to different presentations of the same monomial. Thus, the set of monomials carry a left action by S_n. If $g : [k] \to [n]$ is injective, then so is $\sigma \circ g$, for any permutation $\sigma : [n] \to [n]$, hence the subset of admissible monomials is stable under the action. The action clearly extends to an action on the polynomial algebras $A[M^{m\times n}]$ and $A_a[M^{m\times n}]$. Note that the subspace inclusion as well as the quotient morphism preserve the action.

The polynomials which are invariant under the action of S_n, we call *column symmetric*. These elements form subalgebras of $A[M^{m\times n}]$ and of $A_a[M^{m\times n}]$; they deserve the notation $\mathrm{sym}(A[M^{m\times n}])$ and $\mathrm{sym}(A_a[M^{m\times n}])$, respectively.

In the sequel we study the structure of the elements of the algebra

$$\mathrm{sym}(A_a[M^{m\times n}]) \subseteq A_a[M^{m\times n}].$$

This is where we need that the ring A contains $\mathbb{Q}$.

If a finite group S acts on an algebra C over a commutative ring A, the elements in C invariant under the action of S form a subalgebra $\mathrm{sym}_S(C)$ of S-*symmetric* or S-*invariant* elements. If A contains the field of rational numbers $\mathbb{Q}$ as a subring, we further have that the subalgebra $\mathrm{sym}_S(C) \subseteq C$, seen just as a linear subspace, is a retract, with retraction the symmetrization operator sym given, for $a \in C$, by

$$\mathrm{sym}(a) := p^{-1} \cdot \sum_{\sigma \in S} \sigma \cdot a, \tag{5}$$

where p is the cardinality of S. And we have

$$a \; is \; invariant \iff a = \mathrm{sym}(a).$$

Proposition 1.2. *Any two admissible monomials $\omega_{f,g}$, $\omega_{f,g'}$ with the same $f : [k] \to [m]$ are in the same orbit of the action by S_n. It follows that $\mathrm{sym}(\omega_{f,g}) = \mathrm{sym}(\omega_{f,g'})$, see (5).*

Proof. Recall that if g and $g' : [k] \to [n]$ are monic, then we may find a permutation $[n] \xrightarrow{\tau} [n]$ with $\tau \circ g = g'$. There are in fact $(n-k)!$ such permutations. With such τ, we have $\tau \cdot \omega_{f,g} = \omega_{f,g'}$. It follows that $\mathrm{sym}(\omega_{f,g})$ and $\mathrm{sym}(\omega_{f,g'})$ have the same terms but in different order. □

The row-sum polynomials s_i, see (2), are clearly column-symmetric, and the product $\prod_{l \in [k]} s_{f(l)}$, as a k-fold product of homogeneous degree 1 polynomials, is a homogeneous degree k polynomial, and likewise column symmetric.

Proposition 1.3. *For any admissible monomium $\omega_{f,g}$ of degree k, we have (discarding inadmissible terms)*

$$\mathrm{sym}(\omega_{f,g}) = \frac{(n-k)!}{n!} \prod_{l \in [k]} s_{f(l)}.$$

Proof. Any $\sigma \in S_n$ defines, by restriction to the subset $[k] \subseteq [n]$, a monic map $g : [k] \hookrightarrow [n]$. Conversely, any monic $[k] \hookrightarrow [n]$ extends to a permutation $\sigma : [n] \to [n]$ in $(n-k)!$ different ways, by simple combinatorics.

Let $C(g) \subseteq S_n$ be the set of such extensions of g. These subsets of S_n are clearly disjoint. So we have

$$S_n = \coprod_{[k] \overset{g}{\hookrightarrow} [n]} C(g).$$

Therefore, we may rewrite $\sum_{\sigma \in S_n} \sigma \cdot \omega_{f,g}$ as follows

$$\sum_{[k] \overset{g}{\hookrightarrow} [n]} \sum_{\sigma \in C(g)} \prod_{l \in [k]} x_{f(l)\sigma(l)} = \sum_{[k] \overset{g}{\hookrightarrow} [n]} \sum_{\sigma \in C(g)} \prod_{l \in [k]} x_{f(l)g(l)}$$

since for each g and each $\sigma \in C(g)$, $\sigma(l) = g(l)$, for $l \in [k] \subseteq [n]$. Therefore, for a given g, the terms in the summation over $C(g)$ are equal, and there are $(n-k)!$ of them, so the equation continues

$$= \sum_{[k] \overset{g}{\hookrightarrow} [n]} (n-k)! \prod_{l \in [k]} x_{f(l)g(l)} = (n-k)! \sum_{[k] \overset{g}{\hookrightarrow} [n]} \prod_{l \in [k]} x_{f(l)g(l)} \, ,$$

and this expression equals $(n-k)! \prod_{l \in [k]} s_{f(l)}$ by equation (3). Dividing by $n!$ now gives the desired equation. □

From the Proposition, we may deduce (recall that $\mathbb{Q}$ is a subring of A)

Proposition 1.4. *Every column symmetric admissible polynomial can be expressed in $A_a[M^{m \times n}]$ as a polynomial in the s_i's. (This expression can be interpreted as an expression, modulo the ideal I of inadmissibles, in the polynomial ring $A[M^{m \times n}]$.)*

Proof. Any admissible polynomial $h \in A_a[M^{m \times n}]$ is a linear combination of admissible monomials, and sym is linear; by Proposition 1.3 sym of an admissible monomium is a polynomial in the s_i's. Therefore also $\mathrm{sym}(h)$ is so. If h is furthermore column symmetric, $h = \mathrm{sym}(h)$, then h itself is expressed as a polynomial of the s_i's, $h = G(s_1, \ldots, s_m)$ for some polynomium $G \in A([m]) = A[y_1, \ldots, y_m]$. □

We shall formulate the results so far and some of its consequences in the category $\mathcal{A}$ of commutative A-algebras.

Consider the algebra $A[y_1, \ldots, y_m]$. Since it is the free algebra in the generators y_i, and $s_i \in \mathrm{sym}(A[M^{m\times n}])$, there is a unique algebra map (preserving degree)

$$A[y_1, \ldots, y_m] \xrightarrow{S} \mathrm{sym}(A[M^{m\times n}]) \subseteq A[M^{m\times n}]), \tag{6}$$

namely the one which sends $y_i \in A[y_1, \ldots, y_m]$ to s_i.

Let J be the ideal in $A[y_1, \ldots, y_m]$ generated by the monomials of degree $n+1$. The quotient algebra $A[y_1, \ldots, y_m]/J$ may be identified with the linear subspace (not a subalgebra) $A_{\leq n}[y_1, \ldots, y_m] \subseteq A[y_1, \ldots, y_m]$ of polynomials of degree less or equal to n , the algebra structure given by the multiplication table $\{y_{f(1)}\, y_{f(2)}\, \cdots\, y_{f(n+1)} = 0\}_{f:[n+1]\to[m]}$, and no other relations.

It follows immediately from the respective multiplication tables (alternatively since S sends the ideal J into the ideal I) that we have an algebra map:

$$A_{\leq n}[y_1, \ldots, y_m] \xrightarrow{s} \mathrm{sym}\, A_a[M^{m\times n}] \tag{7}$$

making the diagram below commutative:

$$\begin{array}{ccc} A[y_1, \ldots, y_m] & \xrightarrow{S} & \mathrm{sym}(A[M^{m\times n}]) \\ \downarrow & & \downarrow \\ A_{\leq n}[y_1, \ldots, y_m] & \xrightarrow{s} & \mathrm{sym}(A_a[M^{m\times n}]) \end{array} \tag{8}$$

The vertical maps are quotient maps which discard terms of degree $> n$, respectively inadmissible terms. Thus the map s discards the inadmissible terms from the values of S.

Proposition 1.5. *The algebra map s in (7) is injective.*

Proof. (We refer to the last paragraph in Subsection 1.1 for the notation $\|f\|$ for the orbit of f under precomposition with permutations.) Clearly the monomials $\omega_{\|f\|}$ of degree $\leq n$ make up a vector basis of $A_{\leq n}[y_1, \ldots, y_m]$. We may define an equivalence relation $\sim$ on the set of monomials of degree k in $A_a[M^{m\times n}]$, namely $\omega_{f,g} \sim \omega_{f',g'}$ iff $\|f\| = \|f'\|$. We let $B_{\|f\|}$ be the equivalence class defined by $\|f\|$. It follows that $A_{\leq n}[M^{m\times n}]$ is a direct sum of the subspaces $V_{\|f\|}$ spanned by the $B_{\|f\|}$. We show that $s(\omega_{\|f\|})$ lies in $V_{\|f\|}$; recall equation (3) and note that for any g, $\omega_{f,g} \in B_{\|f\|}$:

$$s(\omega_{\|f\|}) = \prod_{l\in[k]} s_{f(l)} = \sum_{[k]\overset{g}{\hookrightarrow}[n]} \prod_{l\in[k]} x_{f(l)g(l)} = \sum_{[k]\overset{g}{\hookrightarrow}[n]} \omega_{f,g}.$$

Thus the map s sends a linear base into a set of lineary independent vectors, and its injectivity follows. □

Remark. A similar argument proves that also the map S in (6) is injective.

The surjectivity of the map s is a reformulation of Proposition 1.4. Thus, combining Propositions 1.4 and 1.5, we have our main result:

Theorem 1.6. *The algebra map s in (7) is an isomorphism.*

We shall paraphrase this in geometric terms:

2. Geometric interpretation

2.1 The category of A-algebras and its dual

The following Section only is a reminder, to fix notation etc. As above, $\mathcal{A}$ denotes the category of commutative A-algebras (here just called algebras.)

The dual category $\mathcal{A}^{op}$ is essentially the category of affine schemes over A. The objects, viewed in this category, we here just call *spaces*, and the maps in it, we call *functions*. If $A \in \mathcal{A}$, we denote $\overline{A} \in \mathcal{A}^{op}$ the correponding space, and similarly for maps.

A main object in $\mathcal{A}$ is the polynomial ring $A[x]$ in one variable; as a space it is denoted R,

$$R := \overline{A[x]}.$$

Because $A[x]$ is the free algebra in one generator x, there is, for any algebra B, a 1-1 correspondence between the set of elements of B and the set of algebra maps $A[x] \to B$, with dual notation, with the set of functions $\overline{B} \to \overline{A[x]} = R$. Thus, we have the basic fact:

elements of an algebra B correspond to R-valued functions on the space $\overline{B}$.

Since $\overline{A[x_1,\ldots,x_n]}$ is a coproduct in $\mathcal{A}$ of n copies of $A[x]$, it follows that $\overline{A[x_1,\ldots,x_n]} = R^n$, the "$n$-dimensional vector space over R", product of n copies of R. Therefore, the elements of $A[x_1,\ldots,x_n]$ correspond to functions $R^n \to R$, explaining in tautological terms the relationship between *polynomials* in n variables and *functions* $R^n \to R$; all functions $R^n \to R$ in $\mathcal{A}^{op}$ are polynomial.

Any ideal I in an an algebra B gives quotient map $B \twoheadrightarrow B/I$, and hence in $\mathcal{A}^{op}$ defines a monic function $\overline{B/I} \hookrightarrow \overline{B}$.

It is convenient to give names to some standard spaces thus defined. The space corresponding to $A[y_1,\ldots,y_m]/J$, where $J \subset A[y_1,\ldots,y_m]$ is the ideal generated by monomials of degree $n+1$, is denoted $D_n(m) \subset R^m$,

$$D_n(m) = \overline{A_{\leq n}[y_1,\ldots,y_m]},$$

and deserves the name "the nth infinitesimal neighbourhood of $0 \in R^m$". In the standard description of finite limits with internal variables we have:

$$D_n(m) = \{(x_1,\ \ldots . x_m) \in R^m \mid \forall\, f : [n{+}1] \to [m]\ \ x_{f(1)}\ \ldots\ x_{f(n+1)} = 0\}.$$

Likewise with the ideal $I \subseteq A[M^{m\times n}]$ described in Section 1.2. In this case we have

$$D_1(m)^n = \overline{A_a[M^{m\times n}]};$$

this follows since $A[M^{m\times n}]/I$ is the coproduct in $\mathcal{A}$ of n copies of $A[x_1,\ldots,x_m]/J$, where J now is the ideal generated by monomials of degree 2.

With internal variables we have the description:

$$D_1(m)^n = \{(x_{1,1},\ \ldots\ x_{m,n}) \in R^{m\times n} \mid x_{i,j}\, x_{i',j} = 0\}$$

(where i and i' range over $[m]$ and j over $[n]$), which is easily understood by the isomorphism $R^{m\times n} = (R^m)^n$.

2.2 Orbit space

Let B be an algebra, and let S be a finite group acting on B. The subalgebra $\mathrm{sym}_S(B) \subseteq B$ of *invariant* or *symmetric* elements may be described in categorical terms, in the category $\mathcal{A}$, as the joint equalizer of the automorphisms of the form $B \xrightarrow{\sigma} B$ over all the $\sigma, \sigma' \ldots \in S$,

$$\mathrm{sym}_S(B) \hookrightarrow B \underset{\sigma}{\overset{\sigma'}{\underset{\rightarrow}{\overset{\rightarrow}{\vdots}}}} B\ .$$

In the category $\mathcal{A}^{op}$, this becomes a joint coequalizer, thus the orbit object of the action of S,

$$\overline{B}/S \twoheadleftarrow \overline{B} \underset{\overline{\sigma}}{\overset{\overline{\sigma'}}{\underset{\leftarrow}{\overset{\leftarrow}{\vdots}}}} \overline{B}\ .$$

The isomorphism s in the Theorem 1.6, see diagram (8), is displayed in the following commutative diagram:

$$\begin{array}{ccccccc}
A[y_1, \ldots, y_m] & \xrightarrow{S} & \mathrm{sym}(A[M^{m\times n}]) & \hookrightarrow & A[M^{m\times n}] & \underset{\sigma}{\overset{\sigma'}{\underset{\rightarrow}{\overset{\rightarrow}{\vdots}}}} & A[M^{m\times n}] \\
\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\
A_{\leq n}[y_1, \ldots, y_m] & \underset{\cong}{\xrightarrow{s}} & \mathrm{sym}(A_a[M^{m\times n}]) & \hookrightarrow & A_a[M^{m\times n}] & \underset{\sigma}{\overset{\sigma'}{\underset{\rightarrow}{\overset{\rightarrow}{\vdots}}}} & A_a[M^{m\times n}]
\end{array}$$

By a tautological rewriting, this diagram becomes

$$\begin{array}{ccccccc}
R^m & \xleftarrow{\overline{S}} & (R^m)^n/S_n & \twoheadleftarrow & (R^m)^n & \underset{\overline{\sigma}}{\overset{\overline{\sigma'}}{\underset{\leftarrow}{\overset{\leftarrow}{\vdots}}}} & (R^m)^n \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
D_n(m) & \underset{\cong}{\xleftarrow{\overline{s}}} & D_1(m)^n/S_n & \twoheadleftarrow & D_1(m)^n & \underset{\overline{\sigma}}{\overset{\overline{\sigma'}}{\underset{\leftarrow}{\overset{\leftarrow}{\vdots}}}} & D_1(m)^n
\end{array} \qquad (9)$$

The composite map $(R^m)^n \to R^m$ in the diagram is, in synthetic terms: "take an n tuple of vectors in R^m, and add them up". It is symmetric in the n arguments; and it restricts to a map

$$\mathrm{sum} : D_1(m)^n \to D_n(m).$$

Theorem 1.6 then can be expressed as follows:

Theorem 2.1. *The addition map* sum : $D_1(m)^n \to D_n(m)$ *is the quotient map of* $D(m)^n$ *under permutations of the* n *factors, i.e. is universal among* S_n*-symmetric maps out of* $D_1(m)^n$.

The special case where $m = 1$ was called “the symmetric functions property” in the early days of synthetic differential geometry (see e.g. Exercise I.4.4 in [2]); in this form, it was used (see e.g. [2] Exercise I.8.3 and I.8.4, or [1] Proposition 3.4) to establish the Formal Integration for vector fields: extending a vector field $D \times M \longrightarrow M$ into a “formal flow” $D_\infty \times M \longrightarrow M$.

Remark. It is not hard to prove that the constructions and results so far can be presented in a coordinate-free way, i.e. referring to an abstract m-dimensional vector space V over R, rather than to R^m, thus replacing e.g. the subspace $D_n(m) \subseteq R^m$ by a subspace $D_n(V)$; see e.g. [3] 1.2 for the definition of this subobject.

3. Primitives for closed differential 1-forms

The following Section is sketchy, and is included to give an indication of the kind of motivation from synthetic differential geometry that lead to the algebraic result stated in Theorem 1.6 or Theorem 2.1. Therefore, we do not attempt to give the reasoning fully explained, or in its full generality (e.g. replacing the space R^m by an abstract vector space $V \cong R^m$, or even by an arbitrary manifold). Also, some of the structure involved, like the ring structure on R (= the co-ring structure on $A[x]$), we shall assume known. Details may be found in [3], and the references therein.

Two points x and y in R^m are called *first order neighbours* if $y - x \in D_1(m)$. In this case, we write $x \sim y$. The relation $\sim$ is symmetric and reflexive, but not transitive. A differential 1-form ω on R^m may synthetically be described as an R-valued function ω defined on pairs of 1st order neighbour points x, y in R^m, with $\omega(x, x) = 0$ for all x. It is *closed* if for any three points x, y, z with $x \sim y$, $y \sim z$ and $x \sim z$, we have $\omega(x, y) + \omega(y, z) = \omega(x, z)$. Now, in R^m, the data of a 1-form ω may be encoded by giving a function $\Omega(-;-) : R^m \times R^m \to R$, linear in the argument after the semicolon, and such that

$$\omega(x, y) = \Omega(x; y - x), \quad \text{for } x \sim y.$$

Closedness of ω implies that the bilinear $d\Omega(x; -, -) : R^m \times R^m \to R$ is symmetric (see Proposition 2.2.7 in [3]). Hence, by the symmetric functions property (for the given m, and for $n = 2$), or by simple polarization, we get that the bilinear form $d\Omega(x; -, -)$ only depends on the sum of the two arguments. From this, it is easy to conclude (essentially by the Taylor expansion in the proof of the quoted Proposition) that $\omega(x, y) + \omega(y, z)$ is independent of y, even without assuming that $x \sim z$.

If $f : R^m \to R$ is a function, we get a closed 1-form df on M by $df(x, y) := f(y) - f(x)$. If $\omega = df$, we say that f is a *primitive* of ω. We may attempt to find a primitive f of a given closed 1-form ω, in a neighbourhood of the form $x_0 + D_n(m)$, where $x_0 \in R^m$. For a chain $x_0 \sim x_1 \sim \ldots \sim x_n$ (with each $x_i \sim x_{i+1}$), we want to define $f(x_n)$ by the sum

$$\omega(x_0, x_1) + \omega(x_1, x_2) + \ldots + \omega(x_{n-1}, x_n); \quad (10)$$

is this "definition" of $f(x_n)$ independent of the "interpolating points" $x_1, \ldots, \ldots, x_{n-1}$? We may write $x_{i+1} = x_i + d_{i+1}$ with $d_i \in D(V)$ $(i = 0, \ldots, n-1)$. In this case, the first question is whether the proposed value of $f(x_0 + d_1 + \ldots + d_n)$ is independent the individual d_is $(i < n)$ and only depends on their sum. By the symmetric functions property, this will follow if the sum is independent of the order in which we take the increments d_i. But this independence follows because closedness of ω implies $\omega(x, x + d) + \omega(x + d, x + d + d') = \omega(x, x + d') + \omega(x + d', x + d + d')$, thus two consecutive summands in the proposed chain of d_is may be interchanged; and such transpositions generate the whole of S_n. So Theorem 2.1 allows us to define $f : x_0 + D_n(m) \to R$ by the formula (10).

It is then easy to conclude that $f(y) - f(x) = \omega(x, y)$ for any $y \sim x$, for any x in the "formal neighbourhood of x_0" (meaning the set of points which can be reached by a chain $x_0 \sim x_1 \sim \ldots \sim x$, starting in x_0. So f is a primitive of ω on this formal neighbourhood.

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