On frontal operators in Hilbert algebras

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Abstract

In this article, we introduce and study a family of compatible functions in Hilbert algebras which in the case of Heyting algebras agree with the frontal operators given by Esakia (2006, *J. Appl. Non-Class. Log.*, 16, 349–366). Moreover, we give a representation theory, based on previous works by Cabrer, Celani and Montangie, for Hilbert algebras with a frontal operator and for Hilbert algebras with some particular frontal operators.

Keywords: Hilbert algebras, compatible functions, modal operators, frontal operators, Hilbert duality.

1 Introduction and basic results

Hilbert algebras represent the algebraic counterpart of the implicative fragment of intuitionistic propositional logic. These algebras were introduced in early 1950s by Henkin for some investigations of implication in intuitionistic and other non-classical logics ([20], p. 16). In the 1960s, they were studied especially by Horn and Diego in [11, 15]. For the general development of Hilbert algebras, the notion of deductive system (also called implicative filter) plays an important role. For example, it is known that the set of all deductive systems of a Hilbert algebra is a distributive lattice order isomorphic to the lattice of congruences (see [17]). In this work, we characterize compatible functions in Hilbert algebras. We introduce and study a family of compatible functions, which are particular cases of the modal operators considered by Celani and Montangie in [8], and which also generalize the frontal operators given by Esakia in [12]. We further develop some relevant examples that were studied by Caicedo and Cignoli in [4] for the case of Heyting algebras (see also [10]).

In Section 2, we give our characterization of compatible functions in Hilbert algebras. We use it to show that the variety of Hilbert algebras is not locally affine complete, and expansive modal maps are compatible. In Section 3, we define and study frontal operators in Hilbert algebras, a particular case of expansive modal maps. Moreover, we introduce three examples of frontal operators (S, γ and G) that generalize the operators explored in [4, 10]. In particular, we show that S is a new implicit connective in the sense of [4, 5], G is a new implicit connective over Hilbert algebras with (explicit) bottom and γ is also an implicit connective over Hilbert algebras with (explicit) bottom. Finally, we use the irreducible deductive systems of a Hilbert algebra to prove that G is a frontal operator. In Section 4, we give a representation theory for Hilbert algebras with a frontal operator and determine explicitly the representation for Hilbert algebras with some particular frontal operators, based on previous works by Cabrer, Celani and Montangie ([3, 8, 9]).

We start with some definitions and preliminary results.

Definition 1
A *Hilbert algebra* is an algebra $\langle H, \rightarrow, 1\rangle$ of type $(2,0)$ which satisfies the following conditions:

(a) $a \rightarrow (b \rightarrow a) = 1$.

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(b) $a \to (b \to c) = (a \to b) \to (a \to c)$.
(c) If $a \to b = b \to a = 1$, then $a = b$.

It is well known that the class of Hilbert algebras forms a variety. The following equations are an equational basis for this variety:

1. $a \to a = 1$,
2. $1 \to a = a$,
3. $a \to (b \to c) = (a \to b) \to (a \to c)$,
4. $(a \to b) \to ((b \to a) \to a)) = (b \to a) \to ((a \to b) \to b))$.

In every Hilbert algebra we have the partial order

$$a \leq b \text{ if and only if } a \to b = 1,$$

which is called *natural order*. Relative to the natural order on H, 1 is the greatest element. If H has a first element, 0, H is called *bounded*; in this case, for $x \in H$ we denote $\neg x = x \to 0$. In what follows, we say that a Hilbert algebra is bounded if the bottom is in the language of the Hilbert algebra.

Now we shall give some basic results about Hilbert algebras (see e.g. [2]).

Lemma 2
Let H be a Hilbert algebra and $a, b, c \in H$. Then we have the following conditions:

1. $a \leq b \to a$;
2. $a \leq (a \to b) \to b$;
3. $a \to 1 = 1$;
4. $a \to (b \to c) = b \to (a \to c)$;
5. $((a \to b) \to b) \to b = a \to b$;
6. $a \to b \leq (b \to c) \to (a \to c)$;
7. If $a \leq b$, then $c \to a \leq c \to b$ and $b \to c \leq a \to c$;
8. $(a \to b) \to (b \to a) = b \to a$;
9. $((a \to b) \to a) \to b = a \to b$.

The following example will be useful in this work:

Example 3
In any poset H with top element 1, it is possible to define the following binary operation:

$$a \to b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$$

The structure $\langle H, \to, 1 \rangle$ is a Hilbert algebra.

Definition 4
A subset D of a Hilbert algebra is a *deductive system* if

(d1) $1 \in D$.
(d2) If $a \in D$ and $a \to b \in D$ then $b \in D$.

Let H be a Hilbert algebra. If $C \subseteq H$ we write $\langle C \rangle$ for the deductive system generated by C. It follows from Lemma 2.3 of [1] that

$$\langle C \rangle = \{a \in H : a_1 \to (a_2 \to (a_3 \ldots\ldots (a_n \to a) \ldots\ldots) \text{ for some } a_1, \ldots, a_n \in C\}.$$

In particular, we have that

$$\langle\{a,b\}\rangle = \{c\in H : a\to(b\to c)=1\} = \{c\in H : a\leq b\to c\}. \tag{1}$$

A proper deductive system D of a Hilbert algebra H is *irreducible* if for any deductive systems D_1 and D_2 such that $D=D_1\cap D_2$, it follows that $D=D_1$ or $D=D_2$. The set of all irreducible deductive systems of H is denoted $X(H)$. For the following two lemmas see [3].

Lemma 5
Let H be a Hilbert algebra and D a proper deductive system of H. The following conditions are equivalent:

1. $D\in X(H)$.
2. If $a,b\notin D$, there exists $c\notin D$ such that $a,b\leq c$.
3. If $a,b\notin D$, there exists $c\notin D$ such that $a\to c, b\to c\in D$.

Lemma 6
Let H be a Hilbert algebra. Then

1. For all $a,b\in H$, if $a\nleq b$ then there exists $P\in X(H)$ such that $a\in P$ and $b\notin P$.
2. If $P\in X(H)$ then $a\to b\notin P$ if and only if there exists $Q\in X(H)$ such that $P\subseteq Q$, $a\in Q$ and $b\notin Q$.

2 Compatible functions

Compatibility of functions is a classical topic in Universal Algebra. In [4] compatible functions were studied in Heyting algebras, following basically the characterization of compatible functions by means of the relationship between congruences and filters. In this section, we study compatible functions in Hilbert algebras using the link between congruences and deductive systems ([17]). More precisely, we establish a characterization for compatible functions in Hilbert algebras. We also prove that the variety of Hilbert algebras is not locally affine complete, and we show that expansive modal operators are compatible.

If θ is a congruence of H and $a,b\in H$, we write a/θ for the equivalence class of a and $\theta(a,b)$ for the congruence generated by (a,b).

The following remark is part of the folklore of Hilbert algebras:

Remark 1
There exists an order isomorphism between the lattice of congruences of H and the lattice of deductive systems of H. The isomorphism is established via the assignments $\theta\to 1/\theta$ and $D\to \theta_D=\{(a,b)\in H\times H : a\to b\in D \text{ and } b\to a\in D\}$.

Lemma 2
Let H be a Hilbert algebra, θ a congruence of H and $a,b\in H$.

(a) Let $c,d\in H$. Then $c\theta d$ if and only if $(c\to d,1)\in\theta$ and $(d\to c,1)\in\theta$.
(b) Let $c,d\in H$. Then $(c,d)\in\theta(a,b)$ if and only if $a\to b\leq(b\to a)\to(c\to d)$ and $a\to b\leq(b\to a)\to(d\to c)$.

Proof. By Remark 1 we obtain the item (a). An alternative proof can be obtained using the equation $(a\to b)\to((b\to a)\to a))=(b\to a)\to((a\to b)\to b))$. Now we prove the item (b). By Remark 1 again we have that $1/\theta(a,b)=\langle\{a\to b, b\to a\}\rangle$. Using the equality (1) and the item (a) we obtain that $(c,d)\in\theta(a,b)$ if and only if $(c\to d,1)\in\theta(a,b)$ and $(d\to c,1)\in\theta(a,b)$ if and only if $a\to b\leq (b\to a)\to(c\to d)$ and $a\to b\leq(b\to a)\to(d\to c)$. ■

Now we recall the following:

DEFINITION 3
Let H be an algebra and let $f : H^n \to H$ be a function.

1. We say that f is *compatible with a congruence θ of H* if $(a_i, b_i) \in \theta$ for $i = 1, \ldots, n$ implies $(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in \theta$.
2. We say that f is a *compatible function* of H provided it is compatible with all the congruences of H.

The simplest examples of compatible functions in an algebra H are the polynomial functions. The notion of polynomial used here is simply that from universal algebra ([16]). It is known that a function $f : H \to H$ is compatible if and only if $(f(a), f(b)) \in \theta(a, b)$ for every $a, b \in H$.

Taking into account Lemma 2 we obtain the following

PROPOSITION 4
Let H be a Hilbert algebra and $f : H \to H$ a function.
The following conditions are equivalent:

1. f is compatible.
2. For every $a, b \in H$ we have that $a \to b \leq (b \to a) \to (fa \to fb)$.

Let A be an algebra, $f : A^k \to A$ a function and $a = (a_1, \ldots, a_k) \in A^k$. For $i = 1, \ldots, k$, define unary functions $f_i^a : A \to A$ by $f_i^a(x) := f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_k)$.

Then, we have the following characterization for the compatibility of a k-ary function f:

LEMMA 5
Let A be an algebra and $f : A^k \to A$ a function.

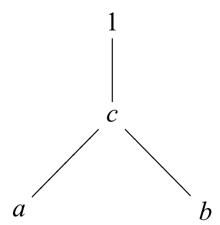
The following conditions are equivalent:

(a) f is compatible.
(b) For every $a \in A^k$ and every $i = 1, \ldots, k$, the functions $f_i^a : A \to A$ are compatible.

Proposition 4 together with Lemma 5 allow us to characterize compatible k-ary functions on a Hilbert algebra.

We apply Proposition 4 to give the following example of a compatible function that is not a polynomial function:

EXAMPLE 6
Consider the following poset:



This poset, with the implication given in the Example 3 from the Introduction, is a Hilbert algebra H. It follows from Proposition 4 that the function $f : H \to H$ given by $fx = x \vee a$ is compatible. We shall see, through straightforward computations based on the complexity (number of connectives) of a polynomial, that any function $P:H\to H$ such that $Pa=a$ and $Pb=Pc=c$ is not a polynomial. Note that the case of complexity 0 and 1 follows from direct computations. Suppose that the property holds for polynomials of complexity less than or equal to n, and let P be a polynomial of complexity $n+1$. Then there exist Q and R polynomials of complexity less than or equal to n such that $Px=Qx\to Rx$. We will see that $Pa\neq a$ or $Pb\neq c$ or $Pc\neq c$. Suppose that $Pa=a$ and $Pb=Pc=c$. Hence we obtain that $Qb=Qc=1$ and $Rb=Rc=c$, so $Ra\neq a$. But $Pa=a=Qa\to Ra$, so $Ra=a$, which is a contradiction. Then, $Pa\neq a$ or $Pb\neq c$ or $Pc\neq c$. Therefore, f is a compatible function which is not a polynomial.

An algebra H is affine complete if any compatible function of H is given by a polynomial of H. It is locally affine complete provided that any compatible function is given by a polynomial on each finite subset of H. The variety of Boolean algebras is affine complete ([14]). The variety of Heyting algebras is not affine complete but it is locally affine complete ([4]). It follows from Example 6 the following

Corollary 7
The variety of Hilbert algebras is not locally affine complete.

We finish this section showing that certain modal operators in Hilbert algebras are compatible. A Hilbert algebra with a modal operator ([8]) is a pair $\langle H,\Box\rangle$, where H is a Hilbert algebra and $\Box$ is a semi-homomorphism defined on H, i.e. $\Box$ is a map such that satisfies the following conditions:

1. $\Box 1=1$,
2. $\Box(a\to b)\leq \Box a\to\Box b$, for all $a,b\in H$.

Lemma 8
Let H be a Hilbert algebra and $\Box:H\to H$ a semi-homomorphism which satisfies the additional inequality $a\leq\Box a$ for every $a\in H$. Then $\Box$ is a compatible function.

Proof. Let $a,b\in H$. Then $a\to b\leq\Box(a\to b)\leq\Box a\to\Box b$. As

$$a\to b\leq\Box a\to\Box b\leq(b\to a)\to(\Box a\to\Box b),$$

the conclusion follows from Proposition 4. ∎

3 Frontal operators

In this section, we introduce and study frontal operators in Hilbert algebras as a generalization of the frontal operators in Heyting algebras given by Esakia in [12] (see also [19]). Moreover, we build up three examples of frontal operators (S, γ and G) that generalize the operators studied in [4, 10].

As expressed in the openning of [4], if we consider intuitionistic and intermediate propositional calculi as logics with truth values in Heyting algebras, it is natural to consider new connectives for these logics as operations in the algebras. For example, the modalized Heyting calculus *mHC* was considered in [12], which consists of an augmentation of the Heyting propositional calculus by a modal operator. The algebraic models of *mHC* are Heyting algebras with a unary operator subject to additional identities. These identities must be the algebraic counterpart of the axioms that the modal operator satisfies on the logic.

Let $\langle H,\wedge,\vee,\to,0,1\rangle$ be a Heyting algebra. The map $\tau:H\to H$ is a *frontal operator* if it satisfies the following conditions for every $a,b\in H$:

(f1): $\tau(a\wedge b)=\tau a\wedge\tau b$,
(f2): $a\leq\tau a$,
(f3): $\tau a\leq b\vee(b\to a)$.

Equivalently, τ is a frontal operator if and only if it satisfies **(f2)**, **(f3)** and the inequality $\tau(a\to b)\leq\tau a\to\tau b$. The main motivation to study frontal operators in Heyting algebras stemmed from topological semantics in which τ is interpreted as the co-derivative operator. Frontal operators were also generalized in residuated lattices and in weak Heyting algebras ([7, 10]).

LEMMA 1
Let H be a Heyting algebra and $\tau:H\to H$ a map. Then τ satisfies **(f3)** if and only if it satisfies $\tau a\leq((b\to a)\to b)\to b$ for every $a,b\in H$.

PROOF. Suppose that τ satisfies **(f3)**. It follows from $b\leq((b\to a)\to b)\to b$ and $b\to a\leq((b\to a)\to b)\to b$ that $\tau a\leq b\vee(b\to a)\leq(b\to a)\to b)\to b$. Conversely, suppose that τ satisfies $\tau a\leq((c\to a)\to c)\to c$. Put $c=b\vee(b\to a)$. As $c\to a=(b\to a)\wedge((b\to a)\to a)=(b\to a)\wedge a=a$, we have that $(c\to a)\to c=a\to c=1$. Therefore, $\tau a\leq 1\to c=c=b\vee(b\to a)$. ∎

Inspirated by the previous lemma we give the following

DEFINITION 2
Let H be a Hilbert algebra and let $\tau:H\to H$ be a map. We say that τ is a *frontal operator* if it satisfies the following conditions for every $a,b\in H$:

(i1): $\tau(a\to b)\leq\tau a\to\tau b$,
(i2): $a\leq\tau a$,
(i3): $\tau a\leq((b\to a)\to b)\to b$.

Note that a frontal operator in a Hilbert algebra is a semi-homomorphism such that satisfies (**i2**) and (**i3**). In particular, it follows from Lemma 8 of the previous Section, that frontal operators are compatible. Moreover, in every Hilbert algebra the identity map is a frontal operator.

Let H be a Hilbert algebra. We define the function $S:H\to H$ through the inequality (**i3**) and the additional equation

(S): $Sa\to a\leq Sa$.

It is immediate that a function which satisfies the previous conditions is necessarily unique. This map will be called *successor*, and it is a generalization of the unary operation introduced on Heyting algebras in [18] (see also [4, 10]).

LEMMA 3
Let H be a Hilbert algebra and $a,b\in H$. Then $b\to a\leq b$ if and only if $a\leq b$ and $b\to a=a$.

PROOF. Suppose that $b\to a\leq b$. As $a\leq b\to a$ we have that $a\leq b$. Besides, taking into account the item 9. of Lemma 2 of the Introduction we obtain that $b\to a=((b\to a)\to b)\to a=1\to a=a$. The converse is immediate. ∎

COROLLARY 4
Let H be a Hilbert algebra. The successor function is characterized by (**i2**), (**i3**) and the equation $Sa\to a=a$.

PROPOSITION 5
Let H be a Hilbert algebra.

(a) S is characterized by $Sa = min\{b \in H : b \to a \leq b\}$.
(b) The successor function is a frontal operator.

PROOF. (a) Let S be the successor function. Straightforward computations show that $Sa = min\{b \in H : b \to a \leq b\}$. Conversely, let S be the map given by $Sa = min\{b \in H : b \to a \leq b\}$. As $Sa \in \{b \in H : b \to a \leq b\}$ we obtain the inequality $Sa \to a \leq Sa$. Let $c = ((b \to a) \to b) \to b$. By properties of Hilbert algebras we have that $b \leq c$ and $b \to a \leq c$, so $c \to a \leq b \to a \leq c$. Hence, $S(a) \leq c = ((b \to a) \to b) \to b$.

(b) We only need to prove the condition (**i1**). We have that

$$\begin{array}{rcl} S(a \to b) & \leq & ((Sb \to (a \to b)) \to Sb) \to Sb \\ & = & ((Sb \to a) \to (Sb \to b)) \to Sb) \to Sb \\ & = & ((Sb \to a) \to b) \to Sb) \to Sb. \end{array}$$

Then,

$$S(a \to b) \leq ((Sb \to a) \to b) \to Sb) \to Sb. \tag{2}$$

As $b \leq Sb$ we have that $(Sb \to a) \to b \leq (Sb \to a) \to Sb$. Thus,

$$((Sb \to a) \to Sb) \to Sb \leq ((Sb \to a) \to b) \to Sb. \tag{3}$$

Using inequality (3) we obtain that

$$Sa \leq ((Sb \to a) \to Sb) \to Sb \leq ((Sb \to a) \to b) \to Sb,$$

so we deduce the inequality

$$((Sb \to a) \to b) \to Sb) \to Sb \leq S(a) \to S(b). \tag{4}$$

Therefore, it follows from (2) and (4) that $S(a \to b) \leq Sa \to Sb$. ∎

In [4] it was defined a frontal operator called γ, and in [10] it was proved that γ is characterized by $\gamma a =$ min $\{b : \neg b \vee a \leq b\}$. Inspirated in this fact, we define the following unary map in bounded Hilbert algebras:

$$\gamma a = \text{min } \{b \in H : \neg b \leq b \text{ and } a \leq b\}.$$

PROPOSITION 6
γ is a frontal operator.

PROOF. For every $a \in H$, put $\gamma_a = \{b \in H : \neg b \leq b$ and $a \leq b\}$. It is immediate that γ satisfies (**i2**). To prove the condition (**i3**), let $a, b \in H$. Straightforward computations show that the element defined as $c = ((b \to a) \to b) \to b$ is such that $a \leq c$. Besides $c \to 0 \leq c \to a \leq c$ (see proof of Proposition 5). Thus, $c \in \gamma_a$, i.e. $\gamma a \leq c = ((b \to a) \to b) \to b$.

Finally we will prove the condition (**i1**). Let $a, b \in H$. We will see that $\gamma a \to \gamma b \in \gamma_{a \to b}$. First note that

$$\begin{array}{rcl} (a \to b) \to (\gamma a \to \gamma b) & = & ((a \to b) \to \gamma a) \to (b \to (\gamma a \to \gamma b)) \\ & = & ((a \to b) \to \gamma a) \to (\gamma a \to (\gamma b \to \gamma b)) \\ & = & 1, \end{array}$$

so $a \to b \leq \gamma a \to \gamma b$. Now we will see that $\neg(\gamma a \to \gamma b) \leq \gamma a \to \gamma b$. To prove it, we make the following computation:

$$\begin{array}{rcl}
\neg(\gamma a \to \gamma b) \to (\gamma a \to \gamma b) & = & \gamma a \to (\neg(\gamma a \to \gamma b) \to \gamma b)) \\
& = & (\gamma a \to \neg(\gamma a \to \gamma b)) \to (\gamma a \to \gamma b) \\
& = & ((\gamma a \to (\gamma a \to \gamma b)) \to \neg\gamma a) \to (\gamma a \to \gamma b) \\
& = & ((\gamma a \to \gamma b) \to \neg\gamma a) \to (\gamma a \to \gamma b) \\
& = & (\gamma a \to \neg\gamma b) \to (\gamma a \to \gamma b) \\
& = & \gamma a \to (\neg\gamma b \to \gamma b) \\
& = & \gamma a \to 1 \\
& = & 1.
\end{array}$$

Then, $\neg(\gamma a \to \gamma b) \leq \gamma a \to \gamma b$. Hence, $\gamma a \to \gamma b \in \gamma_{a \to b}$, that is, $\gamma(a \to b) \leq \gamma a \to \gamma b$. Therefore, we conclude that γ is a frontal operator. ∎

Proposition 7
Let H be a bounded Hilbert algebra. The map γ is characterized by the following inequalities:

1. $a \leq \gamma a$,
2. $\neg\gamma a \leq \gamma a$,
3. $\gamma a \leq ((a \to b) \to ((\neg b \to b) \to b))$.

Proof. It is immediate that if we have a unary operation that satisfies the inequalities given in the proposition, then this operation is given by $\gamma a = \min\{b \in H : \neg b \leq b \text{ and } a \leq b\}$. Conversely, suppose that there exists the unary operation given by $\gamma a = \min\{b \in H : \neg b \leq b \text{ and } a \leq b\}$. It is immediate that $a \leq \gamma a$ and $\neg\gamma a \leq \gamma a$. Let $c = ((a \to b) \to ((\neg b \to b) \to b))$. We shall prove that $\neg c \leq c$. Suppose that $\neg c \not\leq c$. In what follows we shall use Lemma 6 of the Introduction. We have that there exists $P \in X(H)$ such that $\neg c \in P$ and $c \notin P$, so there exists $Q \in X(H)$ such that $P \subseteq Q$, $a \to b \in Q$ and $(\neg b \to b) \to b \notin Q$. Thus, there is $Z \in X(H)$ such that $Q \subseteq Z$, $\neg b \to b \in Z$ and $b \notin Z$. In particular, $\neg b \notin Z$, so there is $W \in X(H)$ such that $Z \subseteq W$ and $b \in W$. Taking into account that $b \leq c$ and $P \subseteq W$, we obtain that $c \in W$. Besides we have that $\neg c \in W$. Hence, we have that $0 \in W$, which is a contradiction. Thus, $\neg c \leq c$. Straightforward computations show that $a \leq c$. Therefore, $\gamma a \leq c$. ∎

Let H be a Heyting algebra with γ. This operation is a polynomial in H because $\gamma x = x \vee \gamma 0$. It is natural to ask whether γ is a polynomial function in the Hilbert reduct of a bounded Hilbert algebra H.

Remark 8
Let H be the Hilbert reduct of the Heyting chain $\{0, a, 1\}$, with $0 < a < 1$. Straightforward computations based on the complexity of a polynomial proves that any polynomial $P : H \to H$ is such that $P0 \neq a$ or $Pa \neq a$. As $\gamma 0 = \gamma a = a$, we obtain that γ is not a polynomial in H. Note we have obtained an alternative example to show that the variety of Hilbert algebras is not locally affine complete.

Let H be a bounded Hilbert algebra. We define the function G through the inequalities (**i2**), (**i3**) and the following additional inequalities:

(i4): $Ga \leq \neg\neg a$,
(i5): $Ga \to a \leq \neg\neg a \to a$.

We will prove that if this function exists then it is necessarily unique. This map generalizes the *Gabbay's function* given in [13] (see also [4, 10]). It follows from (**i4**) and (**i5**) that $Ga \to a = \neg\neg a \to a$.

LEMMA 9
Let H be a bounded Hilbert algebra and $a, b \in H$. Then

(a) $a \leq \neg\neg a$ and $\neg a = \neg\neg\neg a$.
(b) $b \to a \leq \neg\neg a \to b$ if and only if $b \to a \leq \neg\neg a \to a$ and $a \leq b$.
(c) $\neg\neg a \to \neg\neg b = a \to \neg\neg b$.
(d) $\neg\neg(a \to b) \leq \neg\neg a \to \neg\neg b$.

PROOF. (a). Let $a \in H$. Then $a \to \neg\neg a = a \to (\neg a \to 0) = \neg a \to (a \to 0) = \neg a \to \neg a = 1$, so $a \leq \neg\neg a$. Using the previous property we obtain that $\neg\neg\neg a \leq \neg a \leq \neg\neg\neg a$, so $\neg a = \neg\neg\neg a$.

(b). Now let $a, b \in H$. Suppose that $b \to a \leq \neg\neg a \to b$. As $a \leq \neg\neg a$ and $a \leq b \to a \leq \neg\neg a \to b$ we obtain that $a \leq b$. Besides $1 = (b \to a) \to (\neg\neg a \to b) = \neg\neg a \to ((b \to a) \to b)$, so $\neg\neg a \leq (b \to a) \to b$. Thus, $\neg\neg a \to a \geq ((b \to a) \to b) \to a = b \to a$. Hence, $a \leq b$ and $b \to a \leq \neg\neg a \to a$. Conversely, suppose that $a \leq b$ and $b \to a \leq \neg\neg a \to a$. So, $b \to a \leq \neg\neg a \to a \leq \neg\neg a \to b$. Therefore, $b \to a \leq \neg\neg a \to b$.

(c). Let $a, b \in H$. Then

$$\begin{aligned}
\neg\neg(a \to b) &= \neg\neg a \to (\neg b \to 0) \\
&= \neg b \to \neg\neg\neg a \\
&= \neg b \to \neg a \\
&= \neg b \to (a \to 0) \\
&= a \to \neg\neg b.
\end{aligned}$$

(d). Finally we have that

$$\begin{aligned}
a &\leq (a \to b) \to b \\
&\leq \neg(a \to b) \to \neg b \\
&\leq \neg\neg(a \to b) \to \neg\neg b,
\end{aligned}$$

so $a \leq \neg\neg(a \to b) \to \neg\neg b$. Thus, $\neg\neg(a \to b) \leq a \to \neg\neg b = \neg\neg a \to \neg\neg b$. ∎

PROPOSITION 10
Let H be a bounded Hilbert algebra. The Gabbay's function is characterized by $Ga = min\{b \in H : b \to a \leq \neg\neg a \to b\}$.

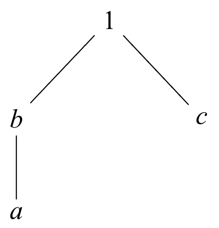
PROOF. For every $a \in H$, we define $G_a = \{b \in H : b \to a \leq \neg\neg a \to b\}$. Suppose that there exists the Gabbay's function G, and let $a \in H$. By **(i2)**, **(i5)** and Lemma 9 we have $Ga \in G_a$. Let $b \in G_a$, i.e. $b \to a \leq \neg\neg a \to b$. It follows from **(i3)** and the previous inequality that $Ga \leq ((b \to a) \to b) \to b \leq ((\neg\neg a \to b) \to b) \to b = \neg\neg a \to b$. By **(i4)** we have that $Ga \leq \neg\neg a$, so $Ga \leq b$. Then, $Ga = min\, G_a$. Conversely, suppose that $Ga = min\, G_a$. By Lemma 9 we obtain **(i2)** and **(i5)**. As $\neg\neg a \in G_a$ we have that $Ga \leq \neg\neg a$, i.e. the condition **(i4)**. Let $b \in H$, and let $c = ((b \to a) \to b) \to b$. We have that $a \leq c$ and $c \to a \leq c$ (see proof of Proposition 5). Taking into account Lemma 3 we have that $a \leq c$ and $c \to a = a \leq \neg\neg a \to a$. It follows from Lemma 9 that $c \in G_a$, so we conclude the condition **(i3)**. ∎

REMARK 11
In Proposition 18 we shall prove that G satisfies **(i1)**.

In what follows, we consider posets with the implication given in Example 3 in the Introduction.

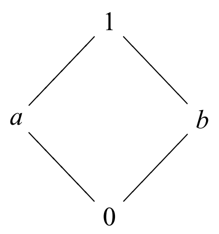
EXAMPLE 12
Consider the Hilbert algebra given in Example 6 of Section 2. We have that $Sa = Sb = c$ and $Sc = S1 = 1$.

EXAMPLE 13
Consider the following poset:



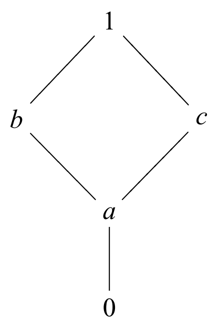
We have that $Sa = b$ and $Sb = Sc = S1 = 1$.

EXAMPLE 14
Consider the following poset:



There is no $S0$ because $\{y : y \to 0 \leq y\} = \{a, b, 1\}$, and there is no $\gamma 0$ because $\{y : y \to 0 \leq y$ and $0 \leq y\} = \{y : y \to 0 \leq y\}$. However, there exists G, with $G0 = 0$ and $Ga = Gb = G1 = 1$.

EXAMPLE 15
Consider the following poset:



There is no Sa because $\{y : y \to a \leq y\} = \{b, c, 1\}$, and there is no Ga because $\{y : y \to a \leq \neg\neg a \to y\} = \{y : y \to a \leq y\}$. However, there exists γ, with $\gamma 0 = a$, $\gamma a = a$, $\gamma b = b$, $\gamma c = c$ and $\gamma 1 = 1$.

In what follows, we shall give some results to prove that G satisfies the inequality (**i1**).

LEMMA 16
Let H be a Hilbert algebra and $f : H \to H$ a map that satisfies (**i3**). Let $P \in X(H)$ and $a, b \in H$ such that $fa \in P$ and $b \notin P$. Then $b \to a \in P$.

Proof. Suppose that $b\to a\notin P$. As $b\notin P$, by Lemma 5 of the Introduction, we have that there exists $c\notin P$ such that $b\to c, (b\to a)\to c\in P$. We have that

$$b\to c\leq (c\to a)\to(b\to a)\leq((b\to a)\to c)\to((c\to a)\to c),$$

so we obtain that $(b\to a)\to c\leq(b\to c)\to((c\to a)\to c)$. As $(b\to a)\to c\in P$ we have that $(b\to c)\to((c\to a)\to c)\in P$. It follows from that $b\to c\in P$ the fact that $(c\to a)\to c\in P$. Thus, by (**i3**) we have that $fa\leq((c\to a)\to c)\to c$. So we conclude that $c\in P$, which is a contradiction. Therefore, $b\to a\in P$. ∎

Let X be a set. If $U\subseteq X$, we write U^c for the complement of U relative to X. If $\langle X,\leq\rangle$ is a poset and $U\subseteq X$, we write U_M to indicate the set of maximal elements of U.

Let H be a Hilbert algebra and $a\in H$. We define the set

$$\varphi(a)=\{P\in X(H): a\in P\}.$$

Lemma 17
Let H be a Hilbert algebra with G, and let $a\in H$. Then $\varphi(Ga)=\varphi(a)\cup(\varphi(\neg\neg a)\cap(\varphi(a)^c)_M)$.

Proof. Let $P\in\varphi(Ga)$, i.e. $Ga\in P$. Suppose that $a\notin P$. As $Ga\leq\neg\neg a$ we have that $\neg\neg a\in P$. Let $Q\in X(H)$ such that $P\subseteq Q$ and $a\notin Q$. Suppose that there exists $b\in Q$ such that $b\notin P$. Taking into account Lemma 16 we obtain that $b\to a\in P\subseteq Q$. As $b\in Q$ we obtain that $a\in Q$, which is absurd. Hence, we obtain that $\varphi(Ga)\subseteq\varphi(a)\cup(\varphi(\neg\neg a)\cap(\varphi(a)^c)_M)$.

Conversely, let $P\in\varphi(a)\cup(\varphi(\neg\neg a)\cap(\varphi(a)^c)_M)$. If $a\in P$ we have that $Ga\in P$ because $a\leq Ga$. Let $P\in\varphi(\neg\neg a)\cap(\varphi(a)^c)_M$. Suppose that $Ga\notin P$. Recall that $Ga\to a=\neg\neg a\to a$. If $\neg\neg a\to a\in P$, as $\neg\neg a\in P$ we have that $a\in P$, which is imposible. Thus, $Ga\to a\notin P$. Taking into account Lemma 6 of the Introduction we have that there exists $Q\in X(H)$ such that $P\subseteq Q$, $Ga\in Q$ and $a\notin Q$. But $P\in(\varphi(a)^c)_M$, so $P=Q$. Then $Ga\in Q$, which is a contradiction. Therefore, we conclude that $\varphi(Ga)=\varphi(a)\cup(\varphi(\neg\neg a)\cap(\varphi(a)^c)_M)$. ∎

Proposition 18
G satisfies the inequality (**i1**).

Proof. Suppose that there exist a,b such that $G(a\to b)\nleq Ga\to Gb$. It follows from Lemma 6 of the Introduction that there exists $P\in X(H)$ such that $G(a\to b)\in P$ and $Ga\to Gb\notin P$. Taking again into account Lemma 6 we obtain that there exists $Q\in X(H)$ such that $Ga\in Q$, $Gb\notin Q$ and $P\subseteq Q$. So, by Lemma 17 we have that $P\in\varphi(G(a\to b))=\varphi(a\to b)\cup(\varphi(\neg\neg(a\to b))\cap(\varphi(a\to b)^c)_M)$, $Q\in\varphi(Ga)=\varphi(a)\cup(\varphi(\neg\neg a)\cap(\varphi(a)^c)_M)$ and $Q\notin\varphi(Gb)=\varphi(b)\cup(\varphi(\neg\neg b)\cap(\varphi(b)^c)_M)$. In what follows, we shall consider the cases $a\to b\in P$ and $a\to b\notin P$ to obtain a contradiction.

Case $a\to b\in P$. If $a\in Q$, as $a\to b\in P\subseteq Q$ we have that $b\in Q$, which is a contradiction. Thus, $Q\in\varphi(\neg\neg a)\cap(\varphi(a)^c)_M$. Suppose that $\neg\neg b\notin Q$. If $\neg\neg a\to\neg\neg b\in Q$, as $\neg\neg a\in Q$ we obtain that $\neg\neg b\in Q$, which is a contradiction. Hence, $\neg\neg a\to\neg\neg b\notin Q$, so $\neg\neg a\to\neg\neg b\notin P$. Using that $a\to b\leq\neg\neg(a\to b)\leq\neg\neg a\to\neg\neg b$ (Lemma 9) we have that $a\to b\notin P$, which is a contradiction again. Hence, we have that $Q\notin(\varphi(b)^c)_M$ because $Q\notin\varphi(Gb)$. Then there exists $Z\in X(H)$ such that $b\notin Z$ and $Q\subset Z$. If $a\in Z$, as $a\to b\in Z$ we have that $b\in Z$, which is absurd. So $a\notin Z$. Thus, we conclude that $Q=Z$ because $Q\in(\varphi(a)^c)_M$, which is impossible.

Case $a\to b\notin P$. We have that $P\in\varphi(\neg\neg(a\to b))\cap(\varphi(a\to b)^c)_M$. Suppose that $a\in Q$. If $a\to b\in Q$ we have that $b\in Q$, which is not possible. Thus, $a\to b\notin Q$. As $P\subseteq Q$ and $P\in(\varphi(a\to b)^c)_M$ we obtain that $P=Q$. Then $\neg\neg(a\to b)\in P=Q$. As $\neg\neg(a\to b)\leq\neg\neg a\to\neg\neg b$ and $\neg\neg a\in P=Q$ we have that

$\neg\neg b \in Q$. It implies that $Q \notin (\varphi(b)^c)_M$. Hence, there exists $Z \in X(H)$ such that $b \notin Z$ and $Q \subset Z$. We have that $a \to b \notin Z$ (if $a \to b \in Z$ we have that $b \in Z$ because $a \in Z$, which is a contradiction). As $Q \in (\varphi(a \to b)^c)_M$ we obtain that $Q = Z$, which is absurd. In consequence, we obtain that $a \notin Q$. Suppose that $\neg\neg b \notin Q$, so by Lemma 6 in the Introduction, there exists $Z \in X(H)$ such that $\neg b \in Z$ and $Q \subseteq Z$. Then $\neg\neg b \notin Z$. Besides $\neg\neg a \in Z$. Using that $\neg\neg(a \to b) \in P \subseteq Q \subseteq Z$ and that $\neg\neg(a \to b) \leq \neg\neg a \to \neg\neg b$, we obtain that $\neg\neg b \in Z$, which is a contradiction. Then we have necessarily that $Q \notin (\varphi(b)^c)_M$, so there exists $Z \in X(H)$ such that $b \notin Z$ and $Q \subset Z$. If $a \to b \notin Z$ then $P = Q = Z$, which is a contradiction. So $a \to b \in Z$. If $a \in Z$ then $b \in Z$, which is impossible. Thus, $a \notin Z$ and in consequence $Q = Z$, which is a contradiction again.

Therefore, we conclude that $G(a \to b) \leq Ga \to Gb$. ∎

COROLLARY 19
G is a frontal operator.

4 Representation theory

In [8] a dual equivalence for the category of Hilbert algebras with modal operators is given. As frontal operators are modal, in this section we give an explicit description of this equivalence for the category of Hilbert algebras with frontal operators, and a detailed description for the particular frontal operators S, γ and G. We start with some basic results about the dual equivalence for the category of Hilbert algebras with modal operators. We recommend the reader to have the refer [2, 3, 9] while reading this section.

Let us consider a poset $\langle X, \leq \rangle$. A subset $U \subseteq X$ is said to be increasing (decreasing) if for all $x, y \in X$ such that $x \in U$ ($y \in U$) and $x \leq y$, we have $y \in U$ ($x \in U$). For each $Y \subseteq X$, the increasing (decreasing) set generated by Y is $[Y) = \{x \in X : \text{there is } y \in Y \text{ such that } y \leq x\}$ ($(Y] = \{x \in X : \text{there is } y \in Y \text{ such that } x \leq y\}$. If $Y = \{y\}$, then we will write $[y)$ and $(y]$ instead of $[\{y\})$ and $(\{y\}]$, respectively. It is well known that the set of upsets of $\langle X, \leq \rangle$ is a Hilbert algebra, where the implication $\Rightarrow$ is given by

$$U \Rightarrow V = (U \cap V^c]^c = \{x \in X : [x) \cap U \subseteq V\}. \tag{5}$$

Moreover, the set of upsets of $\langle X, \leq \rangle$ is a Heyting algebra.

Consider a pair $\langle X, \mathcal{K} \rangle$, where X is a set, $\emptyset \neq \mathcal{K} \subseteq P(X)$ and $P(X)$ is the set of all subsets of X. We define a relation $\leq_{\mathcal{K}} \subseteq X \times X$ by $x \leq_{\mathcal{K}} y$ if and only if for every $W \in \mathcal{K}$, if $x \notin W$ then $y \notin W$. It is immediate that $\leq_{\mathcal{K}}$ is a reflexive and a transitive relation. Define the operators *sat* and *cl* on $P(X)$ as follows. For each $Y \subseteq X$, let $sat(Y) = \bigcap\{W : Y \subseteq W \text{ and } W \in \mathcal{K}\}$ and $cl(Y) = \bigcap\{X - W : Y \cap W = \emptyset \text{ and } W \in \mathcal{K}\}$. If $\mathcal{K}$ is a basis of a topology T defined on X, then $\leq_{\mathcal{K}}$ is the *specialization dual order of* X, $sat(Y)$ is the *saturation* of Y, and $cl(Y)$ is the closure of Y. We note that the relation $\leq_{\mathcal{K}}$ can be defined in terms of the operator *cl* as follows: $x \leq_{\mathcal{K}} y$ if and only if $y \in cl(\{x\})$. The relation $\leq_{\mathcal{K}}$ is a partial order when X is T_0. In this case, $cl(Y) = [Y)$, $sat(Y) = (Y]$, and every open (resp. closed) subset is a decreasing (resp. increasing) subset respect to $\leq_{\mathcal{K}}$.

Let X be a topological space. An arbitrary non-empty subset Y of X is irreducible if for any closed subsets Z and W such that $Y \subseteq Z \cup W$, we have that $Y \subseteq Z$ or $Y \subseteq W$. A topological space X is sober if, for every irreducible closed set Y, there exists a unique $x \in X$ such that $cl(\{x\}) = Y$. Notice that a sober space is automatically T_0. A topological space X with a base $\mathcal{K}$ will be denoted by $\langle X, T_{\mathcal{K}} \rangle$ or simply $\langle X, \mathcal{K} \rangle$. From now on, for every topological space $\langle X, \mathcal{K} \rangle$ we shall write $\leq$ in place of $\leq_{\mathcal{K}}$.

Definition 1
(Definition 3.4 of [9]) An H-space is a topological space $\langle X, \mathcal{K} \rangle$ such that:

(H1) $\mathcal{K}$ is a base of open and compact subsets for the topology $T_{\mathcal{K}}$ on X.
(H2) For every $A, B \in \mathcal{K}$, $sat(A \cap B^c) \in \mathcal{K}$.
(H3) $\langle X, \mathcal{K} \rangle$ is sober.

If $\left\langle X, T_{\mathcal{K}} \right\rangle$ is an H-space, then $\mathbf{D}(X) = \langle D(X), \Rightarrow \rangle$ is a Hilbert algebra, where $D(X) = \{U \subseteq X : U^c \in \mathcal{K}\}$ and $\Rightarrow$ is the binary map given in (5). This algebra is called the dual Hilbert algebra of the H-space $\left\langle X, T_{\mathcal{K}} \right\rangle$. If H is a Hilbert algebra then $\mathbf{X}(H) = \langle X(H), \mathcal{K}_H \rangle$ is an H-space, where $\mathcal{K}_H = \{\varphi(h)^c : h \in H\}$. Moreover, we have that $\leq_{\mathcal{K}_H} = \subseteq$. We also write φ for the isomorphism of Hilbert algebras from H to $\mathbf{D}(\mathbf{X}(H))$ given by $\varphi(a) = \{P \in X(H) : a \in P\}$. If $\left\langle X, T_{\mathcal{K}} \right\rangle$ is an H-space, then the map $\epsilon : X \to X(D(X))$ given by $\epsilon(x) = \{U \in D(X) : x \in U\}$ is well defined and it is an homeomorphism between the topological spaces X and $\mathbf{X}(\mathbf{D}(X))$(see [3, Theorem 2.2]).

If X, Y are sets and $R \subseteq X \times Y$, we define $R(x) = \{y \in Y : (x,y) \in R\}$. If $U \subseteq Y$, we define $R^{-1}(U) = \{x \in X : R(x) \subseteq U\}$.

Definition 2
Let $\mathbf{X}_1 = \langle X_1, \mathcal{K}_1 \rangle$ and $\mathbf{X}_2 = \langle X_2, \mathcal{K}_2 \rangle$ be two H-spaces. Let us consider a relation $R \subseteq X_1 \times X_2$. We say that R is an H-relation from $\mathbf{X}_1$ into $\mathbf{X}_2$ if it satisfies the following properties:

(HR1) $R^{-1}(U) \in \mathcal{K}_1$, for every $U \in \mathcal{K}_2$.
(HR2) $R(x)$ is a closed subset of $\mathbf{X}_2$, for all $x \in X_1$.

We say that R is an H-functional relation if it satisfies the following additional condition:

(HF) If $(x,y) \in R$ there exists $z \in X_1$ such that $x \leq z$ and $R(z) = [y)$.

If $\langle X, \mathcal{K} \rangle$ is an H-space, the relation $\epsilon^* \subseteq X \times X(D(X))$ given by $(x, P) \in \epsilon^*$ if and only if $\epsilon(x) \subseteq P$ is an H-functional relation which is an isomorphism in the category whose objects are H-spaces and whose morphisms are H-functional relations. If R is an H-functional relation from $\langle X_1, \mathcal{K}_1 \rangle$ into $\langle X_2, \mathcal{K}_2 \rangle$, then the map h_R from $\mathbf{D}(X_2)$ into $\mathbf{D}(X_1)$ given by $h_R(U) = \{x \in X_1 : R(x) \subseteq U\}$ is an homomorphism of Hilbert algebras. If $h : A \to B$ is a function between Hilbert algebras, we define the relation $R_h \subseteq X(B) \times X(A)$ by $(P, Q) \in R_h$ if and only if $h^{-1}(P) \subseteq Q$. If h is an homomorphism of Hilbert algebras, then R_h is an H-functional relation.

It follows from [9] and [3, Theorem 3.7] as below.

Theorem 3
There exists a dual equivalence between the category of Hilbert algebras and the category whose objects are H-spaces and whose morphisms are H-functional relations.

A Hilbert algebra with a modal operator $\Box$, or $H\Box$-algebra for short, is a pair $\langle A, \Box \rangle$ where A is a Hilbert algebra and $\Box$ is a semi-homomorphism defined on A. We write H$\Box$ for the variety of $H\Box$-algebras. Let $A, B \in$ H$\Box$. A map $h : A \to B$ is a $\Box$-homomorphism if h is an homomorphism such that $h(\Box a) = \Box(h(a))$, for all $a \in A$. We also denote by H$\Box$ to the category of $H\Box$-algebras with $\Box$-homomorphisms.

Let X be a set and Q a binary relation defined on X. For each $U \subseteq X$ consider the set $\Box_Q(U) = \{x \in X : Q(x) \subseteq U\}$. A triple $\langle X, \mathcal{K}, Q \rangle$ is an $H\Box$-space if $\langle X, \mathcal{K} \rangle$ is an H-space and $Q \subseteq X \times X$ is an H-relation. Let $\langle X_1, \mathcal{K}_1, Q_1 \rangle$ and $\langle X_2, \mathcal{K}_2, Q_2 \rangle$ be two $H\Box$-spaces and $R \subseteq X_1 \times X_2$ be an H-functional relation. We say that R is an $H\Box$-functional relation if $Q_1 \circ R = R \circ Q_2$, where $\circ$ is the composition of relations.

If A is a $H\Box$-algebra, then $\langle X(A), \mathcal{K}_{X(A)}, R_{\Box}\rangle$ is an an $H\Box$-space. If $h: A \to B$ is a $\Box$-homomorphism, then R_h is an $H\Box$-functional relation. Let $A \in \mathrm{H}\Box$. The map $\varphi: A \to \mathbf{D}(\mathbf{X}(A))$ is an $\Box$-homomorphism, i.e. $\varphi(\Box a) = \Box_{R_{\Box}}(\varphi(a))$ for every $a \in A$. If $\langle X, \mathcal{K}, Q\rangle$ is an $H\Box$-space, then $\langle D(X), \Box_Q\rangle$ is a $H\Box$-algebra. Let $\langle X_1, \mathcal{K}_1, Q_1\rangle$ and $\langle X_2, \mathcal{K}_2, Q_2\rangle$ be two $H\Box$-spaces and $R \subseteq X_1 \times X_2$ be an $H\Box$-functional relation. Then h_R is a morphism of $\mathrm{H}\Box$. If $\langle X, \mathcal{K}, Q\rangle$ is an $H\Box$-space, then the relation ϵ^* is a morphism of $H\Box$-spaces.

The following theorem follows from results of [8]:

Theorem 4
There exists a dual equivalence between $\mathrm{H}\Box$ and the category whose objects are $H\Box$-spaces and whose morphisms are $H\Box$-functional relations. Moreover, there exists a dual equivalence between the full subcategory of $\mathrm{H}\Box$ whose objects satisfies the inequality $a \leq \Box a$ and the full subcategory of $H\Box$-spaces with $H\Box$-functional relations whose objects $\langle X, \mathcal{K}, Q\rangle$ satisfies the additional condition $Q \subseteq \leq$.

Proposition 5
Let H be a Hilbert algebra and $f: H \to H$ a function. Then f satisfies (**i3**) if and only if $< \subseteq R_f$, where $<$ is the strict inclusion in $X(H)$.

Proof. Suppose that τ satisfies (**i3**). Let $P, Q \in X(H)$ such that $P \subset Q$. Then there exists $b \in Q$ such that $b \notin P$. Let $a \in f^{-1}(P)$, i.e. $fa \in P$. By Lemma 16 of previous Section, we have that $b \to a \in P \subset Q$. As $b \in Q$ we obtain that $a \in Q$. Hence, $< \subseteq R_f$. Conversely, suppose that $< \subseteq R_f$. Suppose that there exist $a, b \in H$ such that $fa \nleq ((b \to a) \to b) \to b$. Hence, by Lemma 6 of the Introduction, there exists $P \in X(H)$ such that $fa \in P$ and $((b \to a) \to b) \to b \notin P$, so using this lemma again we have that there exists $Q \in X(H)$ such that $P \subseteq Q$, $(b \to a) \to b \in Q$ and $b \notin Q$. As $b \notin Q$ we have that $b \to a \notin Q$, so there exists $Z \in X(H)$ such that $Q \subseteq Z$, $b \in Z$ and $a \notin Z$. As $b \in Z$ and $b \leq (b \to a) \to b) \to b$ we have that $(b \to a) \to b) \to b \in Z$. Using that $(b \to a) \to b) \to b \notin P$, we obtain that $P \subset Z$. Thus, it follows from the hypothesis that $f^{-1}(P) \subseteq Z$. So $a \in Z$ because $fa \in P$, which is a contradiction. Therefore, f satisfies (**i3**). ∎

Proposition 6
Let $\langle X, \mathcal{K}, Q\rangle$ be a $H\Box$-space.
If $< \subseteq Q$ then $\Box_Q$ satisfies (**i3**).

Proof. Let $< \subseteq Q$. It follows from Theorem 4 that $< \subseteq R_{\Box_Q}$. Hence, taking into account Proposition 5 we obtain that $\Box_Q$ satisfies the condition (**i3**), which was our aim. ∎

We write $F\mathcal{H}$ for the category whose objects are algebras $\langle H, \tau\rangle$, where H is a Hilbert algebra and τ is a frontal operator on H. The morphisms are homomorphisms of Hilbert algebras that commute with the frontal operator. Note that a frontal operator τ on a Hilbert algebra H can be seen as an semi-homomorphism that satisfies the additional conditions (**i2**) and (**i3**).

Theorem 7
There exists a dual equivalence between $F\mathcal{H}$ and the full subcategory of $H\Box$-spaces with $H\Box$-functional relations whose objects $\langle X, \mathcal{K}, Q\rangle$ satisfies the additional condition $< \subseteq Q \subseteq \leq$.

In what follows, we establish a dual equivalence for the category of Hilbert algebras with successor.

Lemma 8
If H is a Hilbert algebra with successor, then $\varphi(Sa) = \varphi(a) \cup (\varphi(a)^c)_M$ for every $a \in A$.

PROOF. Let $P\in\varphi(a)\cup(\varphi(a)^c)_M$. If $P\in\varphi(a)$, as $a\leq Sa$ we have that $P\in\varphi(Sa)$. If $P\in(\varphi(a)^c)_M$ we have that $a\notin P$. Using that $a=Sa\to a$ by Lemma 6 in the Introduction, we obtain that there is $Q\in X(H)$ such that $P\subseteq Q$, $Sa\in Q$ and $a\notin Q$. Thus, $P=Q$ and then $P\in\varphi(Sa)$. Hence, $\varphi(a)\cup(\varphi(a)^c)_M\subseteq\varphi(Sa)$. Conversely, let $P\in\varphi(Sa)$, that is, $Sa\in P$. Suppose that $a\notin P$, and let $Q\in X(H)$ such that $P\subseteq Q$ and $a\notin Q$. Let $b\in Q$. We will show that $b\in P$. Suppose that $b\notin P$. Taking into account Lemma 16 we have that $b\to a\in P\subseteq Q$. But $b\in Q$, so $a\in Q$, which is a contradiction. Thus, $b\in Q$. Hence, $P=Q$. Therefore, we conclude that $\varphi(Sa)=\varphi(a)\cup(\varphi(a)^c)_M$. ∎

We say that an H-space $\langle X,\mathcal{K}\rangle$ is an SH-space if for every $U\in D(X)$, $U\cup(U^c)_M\in D(X)$.

COROLLARY 9
If H is a Hilbert algebra with successor then $\mathbf{X}(H)$ is an SH-space.

LEMMA 10
Let H be a Hilbert algebra, $a,b\in H$ and $P\in X(H)$. The following conditions are not simultaneously verified: $(b\to a)\to b\in P, b\notin P$ and $P\in(\varphi(a)^c)_M$.

PROOF. Suppose that the assertion is not true. In particular, $b\to a\notin P$. It follows from Lemma 6 of the Introduction, that there exists $Q\in X(H)$ such that $P\subseteq Q$, $b\in Q$ and $a\notin Q$. But $P\in(\varphi(a)^c)_M$, so $P=Q$, which is a contradiction because $b\notin P$ and $b\in Q$. ∎

COROLLARY 11
If $\langle X,\mathcal{K}\rangle$ is an SH-space, then $\mathbf{D}(X)$ is a Hilbert algebra with successor.

PROOF. Let $U\in D(X)$. We will prove that $TU:=U\cup(U^c)_M$ is the successor function. It is immediate that $U\subseteq TU$. We will show that $TU\subseteq((V\Rightarrow U)\to V)\Rightarrow V$. Suppose that there is $x\in TU$ such that $x\notin((V\Rightarrow U)\Rightarrow V)\Rightarrow V$. Thus, there is y such that $x\leq y$, $y\in(V\Rightarrow U)\to V$ and $y\notin V$. Suppose that $y\in U$, so $y\in V\Rightarrow U$. As $y\in(V\Rightarrow U)\to V$ we have that $y\in V$, which is a contradiction. Thus, $y\notin U$. Then, $x\notin U$ and $x=y$. Hence, we obtain that $x\in(V\Rightarrow U)\Rightarrow V$, $x\notin V$ and $x\in(U^c)_M$, which is a contradiction by Lemma 10. Finally, we will prove that $TU\Rightarrow U\subseteq U$. First note that $(TU\Rightarrow U)^c=((U^c)_M]$. Let $x\notin U$. Taking into account Lemma 7 of [11] we have that there is $y\in(U^c)_M$ such that $x\leq y$. Therefore, $TU\Rightarrow U\subseteq U$. ∎

A moment's reflection shows the following two remarks.

REMARK 12
Let H,M be Hilbert algebras with successor and $f:H\to M$ a morphism of Hilbert algebras.

(a) If f is an isomorphism of Hilbert algebras, then $S(fh)=f(Sh)$ for every $h\in H$.
(b) If f commutes with the successor function, then $h_{R_f}(S(\varphi h))=S(h_{R_f}(\varphi h))$ for every $h\in H$.

REMARK 13
Let $\langle X,\mathcal{K}\rangle$ and $\langle Y,\mathcal{K}\rangle$ be SH-spaces, and let $R\subseteq X\times Y$ be an isomorphism of H-spaces. Then h_R is an isomorphism of Hilbert algebras and in consequence $h_R(SU)=S(h_R(U))$ for every $U\in D(X)$. Moreover, as $\epsilon^*\subseteq X\times X(D(X))$ is an isomorphism of H-spaces, we have that h_{ϵ^*} is such that $h_{\epsilon^*}(SU)=S(h_{\epsilon^*}U)$.

Taking into account the previous results and Theorem 3, we obtain the following:

THEOREM 14
There exists a dual equivalence between the category of Hilbert algebras with successor and the category whose objects are SH-spaces and whose morphisms are morphisms of H-spaces $R\subseteq X\times Y$ such that $h_R(SU)=Sh_R(U)$ for every $U\in D(X)$.

In what follows, we establish a dual equivalence for the category of bounded Hilbert algebras with gamma.

Lemma 15
If H is a bounded Hilbert algebra with γ, then $\varphi(\gamma a)=\varphi(a)\cup(X(H))_M$ for every $a\in A$.

Proof. Let $P\in\varphi(a)\cup(\mathbf{X}(H))_M$. If $a\in P$ then $P\in\varphi(a)$ because $a\le\gamma a$. Suppose that $a\notin P$ and that $\varphi a\notin P$. As $\neg\gamma a\le\gamma a$ we obtain that $\neg\gamma a\notin P$. Hence, by Lemma 6 of the Introduction we have that there exists $Q\in X(H)$ such that $P\subseteq Q$ and $\gamma a\in Q$. Taking into account that $P\in(X(H))_M$ and $P\subseteq Q$ we obtain that $P=Q$. Then $\gamma a\in P$. Hence, $\varphi(a)\cup(X(H))_M\subseteq\varphi(\gamma a)$.

Conversely, let $P\in\varphi(\gamma a)$, and suppose that $a\notin P$. First we will prove that $\gamma 0\in P$. To show it, suppose that $\gamma 0\notin P$. Thus, by Lemma 5 in the Introduction we have that there is $c\notin P$ such that $a\le c$ and $\gamma 0\le c$. Then $\neg c\le\neg\gamma 0\le\gamma 0\le c$, so $\gamma a\le c$. As $c\notin P$ we obtain that $\gamma a\notin P$, which is a contradiction. Then we have that $\gamma 0\in P$. Let $Q\in X(H)$ such that $P\subseteq Q$, and suppose that there exists $b\in Q$ such that $b\notin P$. Thus, by Lemma 16 of Section 3 we have that $\neg b\in P$. So, $\neg b\in Q$. Using that $b\in Q$ we obtain that $0\in Q$, which is a contradiction. Therefore, $\varphi(\gamma a)\subseteq\varphi(a)\cup(X(H))_M$. ∎

Remark 16
A Hilbert algebra H has bottom if and only if $X(H)\in\mathcal{K}_H$. Equivalently, a Hilbert algebra H has bottom if and only if $\emptyset\in D(X(H))$.

We say that an H-space $\langle X,\mathcal{K}\rangle$ is a γH-space if $\emptyset\in D(X)$ and if $U\cup X_M\in D(X)$ for every $U\in D(X)$.

Lemma 17
If $\langle X,\mathcal{K}\rangle$ is a γH-space, then $D(X)$ is a bounded Hilbert algebra with γ.

Proof. Recall that if H is a bounded Hilbert algebra, for every $a\in H$ we have defined $\gamma_a=\{b\in H:\neg b\le b \text{ and } a\le b\}$.

Let $U\in D(X)$, and let $TU=U\cup X_M$. It is obvious that $U\subseteq TU$. To show that $\neg TU=\emptyset$, suppose that there exists $x\in X$ such that $[x)\cup TU=\emptyset$. In particular, $x\notin U$. Taking into account Lemma 7 of [11] we obtain that there exists $y\in X_M$ such that $x\le y$, so $[x)\cap X_M\neq\emptyset$, which is a contradiction. Hence, $\neg TU=\emptyset$ and in consequence $TU\in\gamma_U$. Let $V\in\gamma_U$, i.e. $\neg V\subseteq V$ and $U\subseteq V$. Now let us see that $TU\subseteq V$. Let $x\in TU$. If $x\in U$ then $x\in V$ because $U\subseteq V$. Suppose that $x\notin U$, so $x\in X_M$. Suppose that $x\notin V$. Thus, $x\notin\neg V$ because $\neg V\subseteq V$. Then we obtain that $[x)\cap V\neq\emptyset$, i.e. there exists $y\in X$ such that $x\le y$ and $y\in V$. Taking into account that $x\in X_M$, we have that $x=y$. Then $x\in V$, which is a contradiction. Therefore, $TU\subseteq V$. ∎

Similar results to those given in Remarks 12 and 13 allow us to prove the following

Theorem 18
There exists a dual equivalence between the category of bounded Hilbert algebras with γ and the category whose objects are γH-spaces and whose morphisms are morphisms of H-spaces $R\subseteq X\times Y$ such that $h_R(\gamma U)=\gamma(h_R U)$ for every $U\in D(X)$.

Finally we establish a dual equivalence for the category of bounded Hilbert algebras with the Gabbay's function.

We say that an H-space $\langle X,\mathcal{K}\rangle$ is a GH-space if $\emptyset\in D(X)$ and if for every $U\in D(X)$ we have that $U\cup((\neg\neg U)\cap(U^c)_M)\in D(X)$.

Lemma 19
If $\langle X,\mathcal{K}\rangle$ is a GH-space, then $\mathbf{D}(X)$ is a bounded Hilbert algebra with the Gabbay's function.

Proof. Let $U \in D(X)$. We will prove that $TU := U \cup ((\neg\neg U) \cap (U^c)_M)$ is the Gabbay's function.

It is immediate that $U \subseteq TU$. Let $V \in D(X)$. We shall prove that $TU \subseteq ((V \Rightarrow U) \Rightarrow V) \Rightarrow UV$. Suppose that there exists $x \in X$ such that $x \in TU$ and $x \notin ((V \Rightarrow U) \Rightarrow V) \Rightarrow V$. In particular, there exists $y \in X$ such that $x \leq y$, $y \in (V \rightarrow U) \rightarrow V$ and $y \notin V$. If $y \in U$ then $y \in V \Rightarrow U$. Then $y \in V$, which is a contradiction. Thus $y \notin U$, so $x \notin U$ and in consequence $x = y$. Hence, $x \in (V \Rightarrow U) \Rightarrow V$, $x \notin V$ and $x \in (U^c)_M$, which is a contradiction by Lemma 10. Then we obtain that $TU \subseteq ((V \Rightarrow U) \Rightarrow V) \Rightarrow V$.

To prove that $TU \subseteq \neg\neg U$, suppose that there exists $x \in TU$ such that $x \notin \neg\neg U$. Thus, we have that $[x) \cap \neg U \neq \emptyset$. Taking into account we have that there exists $y \in X$ such that $x \leq y$ and $[y) \cap U = \emptyset$. If $x \in U$ then $y \in U$, which is a contradiction. Hence we have that $x \in \neg\neg U$, which is a contradiction. Then, $TU \subseteq \neg\neg U$.

Finally we will prove that $TU \Rightarrow U \subseteq \neg\neg U \Rightarrow U$. Let $x \in TU \Rightarrow U$, i.e. $[x) \cap TU \subseteq U$. We need to prove that $[x) \cap \neg\neg U \subseteq U$. Suppose that there exists $y \in [x) \cap \neg\neg U$ such that $y \notin U$. By Lemma 7 of [11] we have that there exists $z \in (U^c)_M$ such that $y \leq z$. Then we have that $z \in [x) \cap TU \subseteq U$, which is a contradiction. Therefore, we obtain that $TU \Rightarrow U \subseteq \neg\neg U \Rightarrow U$. ∎

Similar results to those given in Remarks 12 and 13 (together with Lemma 17 of Section 3) allow us to prove the following:

Theorem 20
There exists a dual equivalence between the category of bounded Hilbert algebras with G and the category whose objects are GH- spaces and whose morphisms are morphisms of H-spaces $R \subseteq X \times Y$ such that $h_R(GU) = G(h_R U)$ for every $U \in D(X)$.

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