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WEIGHTED BEST LOCAL APPROXIMATION IN ORLICZ SPACE

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Abstract. We present a common fixed point theorem for generalized asymptotically nonexpansive and noncommuting mappings in normed linear spaces.

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1 Introduction

The best multipoint local approximations of a given data has been studied by several authors. In [1] Beatson and Chui obtained results for the uniform norm when $n=2$. In [5] Marano studied this problem in L^p, where the same speed at each point was assumed. This problem was also considered by Chui et al. in [2], and in this case, they introduced the concept of balanced point in L^p, where a different speed of convergence is considered at each point and where this fact also depends on p. In [3] Favier studied the best local approximation by polynomials with general norms in Orlicz spaces. In this paper we study the best local approximations in Orlicz spaces with a generalized concept of balanced points which depends also in this case, on the function ϕ.

We now introduce some notation. Let X be a bounded open set in $\mathbb{R}$ and $f : X \longrightarrow \mathbb{R}$ be a sufficiently smooth function. We consider a finite measure space $(X, \mathcal{A}, m)$, where m is the Lebesgue measure and denote $\mathcal{M} = \mathcal{M}(X, \mathcal{M}, m)$ the system of all equivalence classes of measurable real valued functions.

For each convex function $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with $\phi(x) = 0$ if and only if $x = 0$ define

$$L^{\phi}(X) = \{f \in \mathcal{M} : \int_X \phi(\alpha|f(x)|)dx < \infty, \text{ for some } \alpha > 0\}.$$

This class of function L^ϕ is called the Orlicz space determined by ϕ. The spaces L^ϕ can be endowed with the following norm

$$\|f\|_\phi := inf\{\lambda > 0 : \int_X \phi(\frac{|f(x)|}{\lambda})\mathrm{d}x \leq 1\},$$

called the Luxemburg norm of L^ϕ. The spaces $(L^\phi, \|.\|_\phi)$ is a Banach spaces (see e.g. [4]).

In this paper we will use the following condition. We say the function ϕ satisfies the Δ_2 condition if there exists a constant $M > 0$ such that $\phi(2x) \leq M\phi(x)$ for $x \geq 0$, and we say ϕ satisfies Δ' condition if there exists a constant $C > 0$ such that $\phi(xy) \leq C\phi(x)\phi(y)$ for $x, y \geq 0$. There is some property about these conditions, for example, Δ' condition imply Δ_2 condition, furthermore it is well known that if ϕ satisfies Δ_2 condition then L^ϕ space can be defined as

$$L^\phi(X) = \left\{ f \ : \ \int_X \phi(\alpha|f(x)|)\mathrm{d}x < \infty, \quad \text{for any} \quad \alpha > 0 \right\}.$$

We assume in the sequel that the convex function ϕ satisfies Δ' condition. For a detailed study of Orlicz spaces the reader is referred to [4].

Given n real points $\{x_1, \cdots, x_n\}$ we define for $\delta > 0$ a net of measu-rable sets

$$V_k = V_k(\delta), \qquad k = 1, \cdots, n,$$

where $V_k(\delta) = x_k + \varepsilon_k(\delta)\ A_k(\delta)$ and $A_k = A_k(\delta)$ a measurable set with measure 1 and $\varepsilon_k = \varepsilon_k(\delta) \searrow 0$ as $\delta \longrightarrow 0$. We point out that the sets A_k are uniformly bounded for all $\delta > 0$.

For each $\delta > 0$ the function f will be approximated on the set $V = V(\delta) = \bigcup V_k$ by a function from a subspace S_N of L^ϕ. Denote $g_\delta \in S_N$ so that

$$\int_V \phi(|f(x) - g_\delta(x)|)\mathrm{d}x \leq \int_V \phi(|f(x) - h(x)|)\mathrm{d}x$$

for all $h \in S_N$. Such a function g_δ is called the best $\phi-$approximation of f from S_N.

Given f, S_N, ϕ and the $n-$tuples $< x_k >:=< x_1, \cdots, x_n >$ and $< V_k >:=< V_1, \cdots, V_n >$, if we consider a net of best $\phi-$approximation functions $\{g_\delta\}$ and it has a limit in S_N as $\delta \longrightarrow 0$, then this limit is called the best local $\phi-$approximation of f from S_N. Under certain conditions the best local $\phi-$approximation can be obtained by Hermite interpolation. It can be calculated explicitly without having to find elements of the net $\{g_\delta\}$. The result will be presented in sections 3.

Now we make an assumption on the $n-$tuple $< \varepsilon_k >$ which will guarantee that the terms of the form $\phi(\varepsilon_k^{\alpha})\varepsilon_k$ can be compared with each other as functions of δ. We will assume a weaker condition than those in [2] for L^P case. However, in our case it depends also on ϕ. Namely, for any $\alpha, \beta \geq 0$ and any j,k such that $1 \leq j,k \leq n$, we assume that either $\phi(\varepsilon_j^{\beta})\ \varepsilon_j = O(\phi(\varepsilon_k^{\alpha})\ \varepsilon_k)$ or $\phi(\varepsilon_k^{\alpha})\ \varepsilon_k = O(\phi(\varepsilon_j^{\beta})\ \varepsilon_j)$ or both.

Given an $n-$tuple of functions $< \phi(\varepsilon_k^{\alpha_k})\ \varepsilon_k >$ where $\alpha_k 0$ is a positive real number, $\phi(\varepsilon_j^{\alpha_j})\ \varepsilon_j$ is said to be maximal if for all k, $1 \leq k \leq n$, $\phi(\varepsilon_k^{\alpha_k})\ \varepsilon_k = O(\phi(\varepsilon_j^{\alpha_j})\ \varepsilon_j)$. We denote it by $\max\{\phi(\varepsilon_k^{\alpha_k})\varepsilon_k\}$.

Remark 1. The assumption over the $n-$tuple $< \varepsilon_k >$ imply the existence of $\max\{\phi(\varepsilon_k^{\alpha_k})\varepsilon_k\}$ for any $n-$tuple $< \alpha_k >$ of positive real numbers.

We assume that f and S_N lie in $PC^m(X)$, where $PC^m(X)$ is the class of functions with $m-1$ continuous derivatives and with piecewise continuous m^{th} derivative. The space S_N is assumed to be fully interpolating at the set $< x_k >$, that is if $S_N \in PC^m(X)$ and $i_1, \cdots, i_n$ are nonnegative integers with $i_k \leq m$ and $\sum\limits_{k=1}^{n} i_k = N$ then there is a unique $g \in S_N$ such that $g^{(j)}(x_k) = a_{j,k}$, $0 \leq j \leq i_k - 1$, $1 \leq k \leq n$, where the $a_{j,k}$ are an arbitrary set of real numbers.

2 Preliminary Results

In this section we prove two Lemmas, which will be used to obtain the main results and which follow the way used in [2] for L^p case. The following Lemma provides an order of the error $\int_V \phi(|f-g|)dx$ when $g \in S_N$ and satisfies $f^{(j)}(x_k) = g^{(j)}(x_k)$, $0 \leq j \leq i_k - 1$, $1 \leq k \leq n$.

Lemma 2.1. *Let $i_1, \cdots, i_n$ be positive integers. Suppose $h \in PC^m(X)$, where $m = \max\{i_k\}$, and that $h^{(j)}(x_k) = 0$, $0 \leq j \leq i_k - 1$, $1 \leq k \leq n$. Then*

$$\int_V \phi(|h|)\mathrm{d}x \ = \ O(\max\{\phi(\varepsilon_k^{i_k})\varepsilon_k\}).$$

Proof. Approximating h by the Taylor polynomial about x_k, we have

$$h(x) = \sum_{j=0}^{i_k-1} \frac{h^{(j)}(x_k)}{j!}\ (x-x_k)^j\ + R_k(x),$$

where $R_k(x) = \dfrac{h^{(i_k)}(\xi)}{i_k!}(x-x_k)^{i_k}$, ξ is between x and x_k. It means $R_k(x) = O((x-x_k)^{i_k})$. Since $h^{(j)}(x_k) = 0$, $0 \leq j \leq i_k - 1$, we have $h(x) = O((x-x_k)^{i_k})$, $x \in V_k$. Thus, by the Δ_2 condition and setting $x - x_k = \varepsilon_k y$, we obtain

$$\int_{V_k} \phi(|h(x)|)\ \mathrm{d}x \ \leq M \int_{V_k} \phi(|x-x_k|^{i_k})\ \mathrm{d}x \leq M \int_{A_k} \varepsilon_k\ \phi(\varepsilon_k^{i_k}\ |y|^{i_k})\ \mathrm{d}y\ \leq M' \varepsilon_k \phi(\varepsilon_k^{i_k}),$$

since A_k are uniformly bounded for all $\delta > 0$. Finally

$$\int_V \phi(|h|)\mathrm{d}x \leq\ M \sum_{k=1}^{n} \phi(\varepsilon_k^{i_k})\varepsilon_k \leq M'\ \max\{\phi(\varepsilon_k^{i_k})\varepsilon_k\},$$

or

$$\int_V \phi(|h|)\mathrm{d}x\ =\ O(\max\{\phi(\varepsilon_k^{i_k})\varepsilon_k\}),$$

as required.

We now cite the Lemma 3 from [2] which will be used in the sequel.

Lemma 2.2. *Let Λ be a family of uniformly bounded measurable subsets of the real line with measure* 1. *Let $P(x) = c_0 + c_1 x + \cdots + c_m x^m$ be an arbitrary polynomial of degree m. Then there exists a constant M (depending on m) such that for all $P(x)$ and all $A \in \Lambda$,*

$$|c_k| \leq M\ \|P(x)\|_{L_p(A)},$$

$1 \leq p \leq \infty,\ \ 0 \leq k \leq m.$

As a consequence of this Lemma we obtain a similar result in L^ϕ.

Corollary 2.3. *Let Λ be a family of uniformly bounded measurable subsets of the real line with measure* 1. *Let $P(x) = c_0 + c_1 x + \cdots + c_m x^m$ be an arbitrary polynomial of degree m. Then there exists a constant M (depending on m) such that for all $P(x)$ and all $A \in \Lambda$,*

$$\phi(|c_k|) \leq M \int_A \phi(|P(x)|)\mathrm{d}x,$$

for $0 \leq k \leq m$.

Proof. In the following, the constant M can be different in each occurrence. We know there is a constant M such that for all $P(x)$ and all $A \in \Lambda$, $|c_k| \leq M \int_A |P(x)|\mathrm{d}x$, $0 \leq k \leq m$, then

$$\phi(|c_k|) \leq M'\phi(M)\phi\left(\int_A |P(x)|\mathrm{d}x\right),$$

since ϕ is an increasing function which satisfies the Δ' condition. Now, using the Jensen's inequality we obtain

$$\phi(|c_k|) \leq M \int_A \phi(|P(x)|)\mathrm{d}x, \qquad 0 \leq k \leq m,$$

for all $A \in \Lambda$. This completes the proof.

3 $\Phi-$Balanced Neighborhood in L^{ϕ}

The following definition generalizes a concept given in [2] for the L^p space.

Definition 3.1. Given an $n-$tuple $< \varepsilon_k >$; an $n-$tuple $< i_k >$ of nonnegative integers is said to be $\phi-$balanced if for each j such that $i_j > 0$,

$$\phi\left(\frac{1}{\varepsilon_j^{i_j-1}}\right)\max\left\{\frac{\phi(\varepsilon_k^{i_k})\varepsilon_k}{\varepsilon_j}\right\} = o(1).$$

If $< i_k >$ is $\phi-$balanced, then $\sum_{k=1}^{n} i_k$ is said to be a $\phi-$balanced integer.

The $n-$tuple $< V_k >$ is said to be $\phi-$balanced neighborhoods if the dimension N of the space S_N is a $\phi-$balanced integer.

Remark 2. If $\phi(x) = x^p$, $1 \le p < \infty$ the definition of $\phi-$balanced is equivalent to those considered by Chui et al. in [2].

Example 3.2. Let be $\phi(x) = x^3(1+|\ln x|)$ with $\phi(0) = 0$ a function that satisfies Δ' condition and $< \varepsilon_1, \varepsilon_2 > = < \delta, e^{-1/\delta} >$, then each integer N is a $\phi-$balanced integer.

Remark 3. For each $\phi-$balanced integer there corresponds exactly one $\phi-$balanced $< i_k >$. In fact, since ϕ satisfies the Δ' condition and we suppose there is $< i_k > \neq < i'_k >$ with $\sum_{k=1}^{n} i_k = \sum_{k=1}^{n} i'_k$ and $< i_k >$ $\phi-$balanced, then there exist j, l with $i_j < i'_j$ and $i'_l < i_l$, such that

$$\phi\left(\frac{1}{\varepsilon_j^{i'_j-1}}\right)\frac{\phi(\varepsilon_l^{i'_l})\varepsilon_l}{\varepsilon_j} \ge \phi\left(\frac{1}{\varepsilon_j^{i_j}}\right)\frac{\phi(\varepsilon_l^{i_l-1})\varepsilon_l}{\varepsilon_j} \ge M\,\frac{1}{\phi\left(\frac{1}{\varepsilon_l^{i_l-1}}\right)\frac{\phi(\varepsilon_j^{i_j})\varepsilon_j}{\varepsilon_l}},$$

and the last expression tends to infinite, so $< i'_k >$ is not $\phi-$balanced.

In [2] it was given an algorithm which generates all balanced integers in L^p spaces. Now, we will present an algorithm that inductively generates all the integers m which can be $\phi-$balanced in L^{ϕ}, say, it generates $n-$tuple $< i_k^{(m)} >$ such that $\sum_{k=1}^{n} i_k^{(m)} = m$.

Beginning with the $\phi-$balanced $n-$tuple $< i_k^{(0)} > = < 0 >$ correspon-ding to the $\phi-$balanced integer 0 and given $< i_k^{(m)} >$, determine a maximal element of $< \phi(\varepsilon_k^{i_k^{(m)}})\varepsilon_k >$, say $\phi(\varepsilon_{k*}^{i_{k*}^{(m)}})\varepsilon_{k*} = \max\{\phi(\varepsilon_k^{i_k^{(m)}})\varepsilon_k\}$ and define $i_k^{(m+1)} = i_k^{(m)}$ for $k \neq k*$ and $i_k^{(m+1)} = i_k^{(m)} + 1$ for $k = k*$.

Remark 4. The algorithm reduces, at each step, the largest value of $\phi(\varepsilon_k^{i_k})$ by incrementing the exponent by 1.

Lemma 3.3.

a) *The above algorithm generates all* $\phi-$*balanced* $< i_k >$.

b) *If a* $n-$*tuple* $< i_k^{(m)} >$ *generated by the algorithm* ($m \geq 1$) *is* $\phi-$*balanced, then there is a unique maximal element of* $< \phi(\varepsilon_k^{i_k^{(m-1)}})\varepsilon_k >$.

Proof. First we prove a). Suppose that $< i_k >$ is $\phi-$balanced with $\sum_{k=1}^{n} i_k = m$ and $< i_k^{(m)} >$ as obtained by the algorithm is not $\phi-$balanced. Then, there exist indices r, s with $i_r^{(m)} > i_r$ and $i_s > i_s^{(m)}$ such that, using the $\phi-$balanced integer definition,

$$\phi(\varepsilon_r^{i_r^{(m)}-1})\varepsilon_r = O(\phi(\varepsilon_r^{i_r})\varepsilon_r) = O\left(\max\{\phi(\varepsilon_k^{i_k})\varepsilon_k\}\right) = o\left(\frac{\varepsilon_s}{\phi(\frac{1}{\varepsilon_s^{i_s-1}})}\right),$$

and from the Δ' condition on ϕ, the last expression is an $o(\phi(\varepsilon_s^{i_s-1})\varepsilon_s)$, so

$$\phi(\varepsilon_r^{i_r^{(m)}-1})\varepsilon_r = o(\phi(\varepsilon_s^{i_s^{(m)}})\varepsilon_s).$$

Since $i_r^{(m)} > 0$ at some previous step $\phi(\varepsilon_r^{i_r^{(m)}-1})\varepsilon_r$ was maximal of $< \phi(\varepsilon_k^{i_k^{(m')}})\varepsilon_k >$ with $m' < m$, so $\phi(\varepsilon_r^{i_r^{(m)}-1})\varepsilon_r = \max\{\phi(\varepsilon_k^{i_k^{(m)}})\varepsilon_k\}$ because the exponents are non-decreasing at each step of the algorithm. Thus for any k

$$\phi(\varepsilon_k^{i_k^{(m)}})\varepsilon_k = O(\phi(\varepsilon_r^{i_r^{(m)}-1})\varepsilon_r),$$

which is a contradiction.

Now we will prove b). If $\max\{\phi(\varepsilon_k^{i_k^{(m-1)}})\varepsilon_k\}$ is not unique then $< i_k^{(m)} >$ cannot be $\phi-$balanced because if the indeces j and s gives a maximal element of $< \phi(\varepsilon_k^{i_k^{(m-1)}})\varepsilon_k >$ then there exist two constants M, N such that

$$M \leq \frac{\phi(\varepsilon_j^{i_j^{(m-1)}})\varepsilon_j}{\phi(\varepsilon_s^{i_s^{(m-1)}})\varepsilon_s} \leq N.$$

Suppose that $i_k^{(m)} = i_k^{(m-1)}$ for $k \neq s$ and $i_k^{(m)} = i_k^{(m-1)} + 1$ for $k = s$, so $i_s^{(m)} > 0$ and then

$$\begin{aligned}
\phi\left(\frac{1}{\varepsilon_s^{i_s^{(m)}-1}}\right)\max\left\{\frac{\phi(\varepsilon_k^{i_k^{(m)}})\varepsilon_k}{\varepsilon_s}\right\} &= \phi\left(\frac{1}{\varepsilon_s^{i_s^{(m-1)}}}\right)\frac{\phi(\varepsilon_j^{i_j^{(m-1)}})\varepsilon_j}{\varepsilon_s} \\
&\geq A\frac{\phi(\varepsilon_j^{i_j^{(m-1)}})\varepsilon_j}{\phi(\varepsilon_s^{i_s^{(m-1)}})\varepsilon_s} \geq AM,
\end{aligned}$$

by the Δ' condition. This show that $< i_k^{(m)} >$ cannot be $\phi-$balanced.

Remark 5. The inverse inequality hand of the Lemma 3.3 does not hold.

For example, if we take $\phi(x) = x^3(|\ln x| + 1)$ with $\phi(0) = 0$ and $< \varepsilon_k >=< \delta, \delta^4 >$, then in the first step, the algorithm generate $< 1, 0 >$ with a unique corresponding maximal element $\max\{\phi(\varepsilon_k^{i_k})\varepsilon_k\} = \phi(\varepsilon_1^1)\varepsilon_1$, however the bi-tuple $< 2, 0 >$ is not $\phi-$balanced. In [2] it is proved the inverse inequality of this Lemma for the L^p case assuming a stronger condition on the $n-$tuple $< \varepsilon_k >$.

4 Best Local $\phi-$approximation in Orlicz Spaces with $\Phi-$balanced Neighborhood

We now turn to the main result concerning the behavior of the net $\{g_\delta\}$ of best $\phi-$approximations and for its proof we need the following three Lemmas.

Lemma 4.1. *Given $< \varepsilon_k >$ and $< i_k >$, define $m = \max\{i_k\}$. Let $f \in PC^m(X)$ and $S_N \subseteq PC^m(X)$. If we consider a net $\{g_\delta\}$ of best $\phi-$approximation such that*

$$\int_X \phi(|g_\delta|)dx \longrightarrow \infty, \qquad as \qquad \delta \longrightarrow 0, \tag{1}$$

then the net of functions

$$h_\delta = \frac{g - g_\delta}{\int_X |g - g_\delta| dx},$$

where $g \in S_N$ interpolate $f^{(j)}(x_k)$, $\quad 0 \le j \le i_k - 1,\ 1 \le k \le n$, satisfies the following properties

i) $\int_X \phi(|h_\delta|)\mathrm{d}x \ge A > 0$;

ii) $\int_V \phi(|h_\delta|)\mathrm{d}x = o\left(\max\{\phi(\varepsilon_k^{i_k})\varepsilon_k\}\right)$.

Proof. Property $i)$ follows by using Jensen's inequality, and the pro-perty $ii)$ is a consequence of the hypothesis. In fact, since ϕ satisfies the Δ_2 condition

$$\int_X \phi(|g_\delta|)\mathrm{d}x \le M\left(\int_X \phi(|g_\delta - g|)\mathrm{d}x + \int_X \phi(|g|)\mathrm{d}x\right),$$

so, using (1) we obtain

$$\int_X \phi(|g - g_\delta|)\mathrm{d}x \longrightarrow \infty, \tag{2}$$

as $\delta \longrightarrow 0$, or

$$\int_X |g - g_\delta|\mathrm{d}x \longrightarrow \infty,$$

as $\delta \longrightarrow 0$, because if there were a sequence $\{\delta_s\}$ such that $\int_X |g-g_{\delta_s}|\mathrm{d}x \leq M$ for all δ_s, then $\|g-g_{\delta_s}\|_\infty \leq M$ for all δ_s since the norms are equivalent, and so $\int_X \phi(|g-g_{\delta_s}|)\mathrm{d}x \leq M'$, which is a contradiction.

On the other hand, by the Δ_2 condition on ϕ and Lemma **??**

$$\begin{aligned} \int_V \phi(|g-g_\delta|)\mathrm{d}x &\leq M\int_V \phi(|g-f|)\mathrm{d}x + M\int_V \phi(|f-g_\delta|)\mathrm{d}x \\ &= O\left(\max\{\phi(\varepsilon_k^{i_k})\varepsilon_k\}\right), \end{aligned}$$

then, since ϕ is a convex function

$$\int_V \phi(|h_\delta|)\mathrm{d}x \leq \frac{1}{\int_X |g-g_\delta|\mathrm{d}x}\int_V \phi(|g-g_\delta|)\mathrm{d}x = o\left(\max\{\phi(\varepsilon_k^{i_k})\varepsilon_k\}\right),$$

as required.

Lemma 4.2. *Given $<\varepsilon_k>$, set a $\phi-$balanced $n-$tuple $<i_k>$ such that $N=\sum_{k=1}^{n} i_k$ and define $m=\max\{i_k\}$. If $f\in PC^m(X)$, $S_N\subseteq PC^m(X)$ and $\{g_\delta\}$ is a net of best $\phi-$approximation, then there exists $M>0$ such that for all $\delta>0$,*

$$\int_X \phi(|g_\delta|)\ \mathrm{d}x \leq M.$$

Proof. Suppose $\{\delta_r\}$ is a sequence such that

$$\int_X \phi(|g_{\delta_r}|)\mathrm{d}x \longrightarrow \infty,$$

as $\delta_r \longrightarrow 0$. Let g be a fixed function in S_N interpolating the derivatives $f^{(j)}(x_k)$, $0\leq j\leq i_k-1,\ 1\leq k\leq n$ and define

$$h_{\delta_r} = \frac{g-g_{\delta_r}}{\int_X |g-g_{\delta_r}|\mathrm{d}x},$$

then, using Lemma 3.1, we have

i) $\int_X \phi(|h_{\delta_r}|)\mathrm{d}x \geq A>0$, for all δ_r.

ii) $\int_V \phi(|h_{\delta_r}|)\mathrm{d}x = o\left(\max\{\phi(\varepsilon_k^{i_k})\varepsilon_k\}\right)$.

Expanding h_{δ_r} by Taylor polynomials we obtain for each k

$$h_{\delta_r}(x) = \sum_{j=1}^{i_k-1}\frac{h_{\delta_r}^{(j)}(x_k)}{j!}(x-x_k)^j + R_{\delta_r}(x),$$

where $R_{\delta_r}(x) = O((x-x_k)^{i_k})$ uniformly in δ_r. In fact, since $\|h_{\delta_r}\|_{L^1(X)} \leq 1$ for all δ_r and using the fact that the norms are equivalent in S_N, we can choose the norm

$$\|h_{\delta_r}\| := \sup_{x\in X}\{|h_{\delta_r}(x)| + \cdots + |h_{\delta_r}^{(i_k)}(x)|\}$$

to show the statement.

This uniform bound of $R_{\delta_r}(x)$ leads to

$$\int_{V_k} \phi(|R_{\delta_r}(x)|)\mathrm{d}x = O(\varepsilon_k \phi(\varepsilon_k^{i_k})),$$

for all r, taking $x - x_k = \varepsilon_k y$ and using that the sets A_k are uniformly bounded. Thus, using property ii) we obtain

$$\begin{aligned}
&\int_{V_k} \phi\left(|\sum_{j=0}^{i_k-1} \frac{h_{\delta_r}^{(j)}(x_k)}{j!}(x-x_k)^j|\right)\mathrm{d}x \\
&\leq M\int_{V_k} \phi\left(|\sum_{j=0}^{i_k-1} \frac{h_{\delta_r}^{(j)}(x_k)}{j!}(x-x_k)^j + R_{\delta_r}(x)|\right)\mathrm{d}x + M\int_{V_k}\phi(|R_{\delta_r}(x)|)\mathrm{d}x \\
&= M\int_{V_k}\phi(|h_{\delta_r}|)\mathrm{d}x + O(\varepsilon_k\phi(\varepsilon_k^{i_k})) \leq M\int_V \phi(|h_{\delta_r}|)dx + O(\varepsilon_k\phi(\varepsilon_k^{i_k})) \\
&= o\left(\max\{\phi(\varepsilon_l^{i_l})\varepsilon_l\}\right) + O(\varepsilon_k\phi(\varepsilon_k^{i_k})),
\end{aligned}$$

if we substitute $x - x_k = \varepsilon_k y$ again we obtain

$$\int_{A_k} \phi\left(|\sum_{j=0}^{i_k-1} \frac{h_{\delta_r}^{(j)}(x_k)}{j!}(\varepsilon_k)^j y^j|\right)\mathrm{d}y = O\left(\max\{\frac{\phi(\varepsilon_l^{i_l})\varepsilon_l}{\varepsilon_k}\}\right),$$

$1 \leq k \leq n$. From Corollary 2.3

$$\phi\left(|\frac{h_{\delta_r}^{(j)}(x_k)}{j!}(\varepsilon_k)^j|\right) = O\left(\max\left\{\frac{\phi(\varepsilon_l^{i_l})\varepsilon_l}{\varepsilon_k}\right\}\right),$$

$j = 0,\cdots,i_k-1,\ 1 \leq k \leq n$, using that ϕ satisfies the Δ' condition, for each k we get

$$\phi(|h_{\delta_r}^{(j)}(x_k)|) \leq M\phi\left(\frac{1}{\varepsilon_k^{i_k-1}}\right) O\left(\max\left\{\frac{\phi(\varepsilon_l^{i_l})\varepsilon_l}{\varepsilon_k}\right\}\right),$$

$j = 0,\cdots,i_k-1$, finally. Since N is a $\phi-$balanced integer and ϕ an increasing function, we obtain

$$h_{\delta_r}^{(j)}(x_k) = o(1),$$

$0 \leq j \leq i_k - 1$, $1 \leq k \leq n$, and when $\delta_r \longrightarrow 0$. Now, since $h_{\delta_r} \in S_N$ we get for $h_{\delta_r} = \sum_{i=1}^{N} a_{\delta_r,i} \widetilde{h_i}$ where $\{\widetilde{h_1}, \cdots, \widetilde{h_N}\}$ is a basis of S_N. So h_{δ_r} is uniquely determined by the N values $h_{\delta_r}^{(j)}(x_k)$, $0 \leq j \leq i_k - 1$, $1 \leq k \leq n$, using the fixed linear transformation

$$\begin{pmatrix} \widetilde{h_1}(x_1) & . & . & \widetilde{h_N}(x_1) \\ . & . & . & . \\ . & . & . & . \\ \widetilde{h_1}^{(i_n-1)}(x_n) & . & . & \widetilde{h_N}^{(i_n-1)}(x_n) \end{pmatrix},$$

then $h_{\delta_r} \longrightarrow 0$ as $\delta_r \longrightarrow 0$ and then

$$\lim_{\delta_r \longrightarrow 0} \int_X \phi(|h_{\delta_r}|) \mathrm{d}x = 0$$

which contradicts property i). Thus g_δ must be bounded for all δ and the proof of the Lemma is complete.

Lemma 4.3. *Given $< \varepsilon_k >$ and a $\phi-$balanced $n-$tuple $< i_k >$ such that $N = \sum_{k=1}^{n} i_k$, define $m = \max\{i_k\}$. If $f \in PC^m(X)$, $S_N \subseteq PC^m(X)$ and $\{g_\delta\}$ is a net of best $\phi-$approximation, then for each k*

$$\phi(|(f-g_\delta)^{(j)}(x_k)|\ \varepsilon_k^j) \ = \ O\left(\max\left\{\frac{\phi(\varepsilon_l^{i_l})\varepsilon_l}{\varepsilon_k}\right\}\right),$$

$0 \leq j \leq i_k - 1$.

Proof. For each k, expanding $f - g_\delta$ using Taylor polynomials, we obtain

$$(f-g_\delta)(x) \ = \ \sum_{j=1}^{i_k-1} \frac{(f-g_\delta)^{(j)}(x_k)}{j!}(x-x_k)^j + R_\delta(x).$$

We may use the above argument to show that $R_{\delta_r}(x) = O((x-x_k)^{i_k})$ uniformly in δ. In fact, it follows from Lemma 3.2 that $\int_X \phi(|g_\delta|)\mathrm{d}x \leq M$ for all δ, thus $||g_\delta||_\phi \leq M$ for all δ and using that the norms are equi-valent in S_N, we can choose again the norm

$$\|h_{\delta_r}\| = \sup_{x \in X}\{|h_{\delta_r}(x)| + \cdots + |h_{\delta_r}^{(i_k)}(x)|\}$$

to show the statement.

These uniform bound of $R_{\delta_r}(x)$ and of the sets A_k, substituting $x - x_k = \varepsilon_k y$, leads to

$$\int_{V_k} \phi(|R_\delta(x)|)\mathrm{d}x = O(\phi(\varepsilon_k^{i_k})\varepsilon_k). \tag{3}$$

On the other hand, using a fixed $g \in S_N$ which interpolates the derivatives $f^{(j)}(x_k)$, $0 \leq j \leq i_k - 1$, $1 \leq k \leq n$, from Lemma 1.1 we obtain

$$\int_V \phi(|f - g_\delta|)\mathrm{d}x \;=\; O\left(\max\{\phi(\varepsilon_k^{i_k})\varepsilon_k\}\right),$$

thus, using (3) and the Δ_2 condition, we obtain for each k

$$\begin{aligned}\int_{V_k} \phi\left(|\sum_{j=0}^{i_k-1} \frac{(f-g_\delta)^{(j)}(x_k)}{j!}(x-x_k)^j|\right)\mathrm{d}x &\leq M\int_{V_k}\phi(|f-g_\delta|)\mathrm{d}x \\ +M\int_{V_k}\phi(|R_\delta(x)|)\mathrm{d}x &= O\left(\max\{\phi(\varepsilon_l^{i_l})\varepsilon_l\}\right) + O(\varepsilon_k\phi(\varepsilon_k^{i_k})) \\ &= O\left(\max\{\phi(\varepsilon_l^{i_l})\varepsilon_l\}\right),\end{aligned}$$

and finally if we substitute $x - x_k = \varepsilon_k y$ then for each k

$$\int_{A_k} \phi\left(|\sum_{j=0}^{i_k-1} \frac{(f-g_\delta)^{(j)}(x_k)}{j!}\varepsilon_k^j y^j|\right)\mathrm{d}y = O\left(\max\left\{\frac{\phi(\varepsilon_l^{i_l})\varepsilon_l}{\varepsilon_k}\right\}\right).$$

Now we conclude from Corollary 1.3 and the Δ_2 condition that

$$\phi(|(f-g_\delta)^{(j)}(x_k)\varepsilon_k^j|) = O\left(\max\left\{\frac{\phi(\varepsilon_l^{i_l})\varepsilon_l}{\varepsilon_k}\right\}\right),$$

$0 \leq j \leq i_k - 1$, for each k, as we required.

Now we present the following result.

Theorem 4.4. *If N is a $\phi-$balanced integer with $\phi-$balanced $< i_k >$ and $f \in PC^m(X)$, $S_N \in PC^m(X)$, ($m = \max\{i_k\}$), then the best local $\phi-$approximation of f from S_N is the unique $g \in S_N$ defined by the N interpolation conditions*

$$f^{(j)}(x_k) = g^{(j)}(x_k),$$

$0 \leq j \leq i_k - 1, \quad 1 \leq k \leq n.$

Proof. From Lemma 3.3

$$\phi(|(f-g_\delta)^{(j)}(x_k)\varepsilon_k^j|) = O\left(\max\left\{\frac{\phi(\varepsilon_l^{i_l})\varepsilon_l}{\varepsilon_k}\right\}\right)$$

for $0 \leq j \leq i_k - 1$, $1 \leq k \leq n$. Using the Δ' condition and the $\phi-$balanced integer definition we have for each k

$$\phi(|(f-g_\delta)^{(j)}(x_k)|) = \phi\left(\frac{1}{\varepsilon_k^{i_k-1}}\right) O\left(\max\left\{\frac{\phi(\varepsilon_l^{i_l})\varepsilon_l}{\varepsilon_k}\right\}\right) = o(1)$$

for $0 \leq j \leq i_k - 1$, thus, since ϕ is a increasing function with $\phi(x) = 0$ if and only if $x = 0$

$$\lim_{\delta \longrightarrow 0} g_\delta^{(j)}(x_k) = f^{(j)}(x_k)$$

for $0 \leq j \leq i_k - 1, \quad 1 \leq k \leq n$. Now we will do a similar above analysis. As g_δ is uniquely determined via a fixed linear transformation with rank N from the N values $g_\delta^{(j)}(x_k), \quad 0 \leq j \leq i_k - 1, \ 1 \leq k \leq n$, then g_δ must converge to the unique g satisfying

$$g^{(j)}(x_k) = f^{(j)}(x_k)$$

for $0 \leq j \leq i_k - 1, \quad 1 \leq k \leq n$. This g is by definition the best local $\phi-$approximation of f from S_N.

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