WEIGHTED BEST LOCAL APPROXIMATION IN ORLICZ SPACES

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Abstract. In this paper we prove the existence of best multipoint local $\phi-$approximation to a function f in an Orlicz space L^ϕ from an $N-$dimensional space S_N for a suitable integer N. The concept of $\phi-$balanced integer is considered since we deal with different speed of approaching the data on each of the real points $x_1, ..., x_n$.

Introduction

The best multipoint local approximations of a given data has been studied by several authors. In [1] Beatson and Chui considered the uniform norm when $n = 2$. In [6] Marano studied this pro-blem in L^p, where the same speed at each point was assumed. This problem was also considered by Chui et al. in [2], and in this case, they introduced the concept of balanced point in L^p, where a different speed of convergence is considered at each point and where this fact also depends on p. In [3] Favier studied the best local approximation by polynomials using the Luxemburg norm in an Orlicz space. In this article we study the best local approximations in Orlicz spaces with a generalized concept of balanced points which depends, also in this case, on the function ϕ.

We now introduce some notation. Let X be a bounded open set in $\mathbb{R}$ and $f : X \longrightarrow \mathbb{R}$ be a sufficiently smooth function. We consider a finite measure space $(X, \mathcal{A}, m)$, where m is the Lebesgue measure and denote $\mathcal{M} = \mathcal{M}(X, \mathcal{A}, m)$ the system of all equivalence classes of Lebesgue measurable real valued functions.

For each convex function $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, with $\phi(x) = 0$ if and only if $x = 0$, define

$$L^\phi(X) = \{f \in \mathcal{M} : \int_X \phi(\alpha|f(x)|)dx < \infty, \text{ for some } \alpha > 0\}.$$

This function spaces L^ϕ is called the Orlicz space determined by ϕ. The spaces L^ϕ can be endowed with the following norm

$$\|f\|_\phi := inf\{\lambda > 0 : \int_X \phi(\frac{|f(x)|}{\lambda})dx \leq 1\},$$

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called the Luxemburg norm. The spaces $(\mathrm{L}^{\phi}, \|.\|_{\phi})$ are Banach spaces (see e.g. [4]).

In this article we will use the following conditions. We say that the function ϕ satisfies the Δ_2 condition if there exists a constant $M > 0$ such that $\phi(2x) \le M\phi(x)$ for $x \ge 0$ and we say that ϕ satisfies the Δ' condition if there exists a constant $C > 0$ such that $\phi(xy) \le C\phi(x)\phi(y)$ for $x, y \ge 0$. There is some property about these conditions, for example, Δ' condition imply Δ_2 condition, furthermore it is well known that if ϕ satisfies the Δ_2 condition then the L^{ϕ} space can be defined as

$$L^{\phi}(X) = \{f \ : \ \int_X \phi(\alpha|f(x)|)dx < \infty, \ for\ any\ \alpha > 0\}.$$

We assume in this article that the convex function ϕ satisfies the Δ' condition. For a detailed study of Orlicz spaces the reader is referred to [4].

Given n real points $\{x_1, ..., x_n\}$ we define for $\delta > 0$ a net of measurable sets $V_k = V_k(\delta)$, $k = 1, ..., n$, where $V_k(\delta) = x_k + \varepsilon_k(\delta)\ A_k(\delta)$ and $A_k = A_k(\delta)$ is a measurable set with measure 1 and $\varepsilon_k = \varepsilon_k(\delta) \searrow 0$ as $\delta \longrightarrow 0$. We point out that the sets A_k are uniformly bounded for all $\delta > 0$.

For each $\delta > 0$ the function f will be approximated on the set $V = V(\delta) = \bigcup V_k$ by a function from a subspace S_N of L^{ϕ}. Denote $g_\delta \in S_N$ so that

$$\int_V \phi(|f(x) - g_\delta(x)|)dx \le \int_V \phi(|f(x) - h(x)|)dx,$$

for all $h \in S_N$. Such a function g_δ is called the best $\phi-$approximation of f from S_N.

Given f, S_N, ϕ and the $n-$tuples $< x_k > := < x_1, ..., x_n >$ and $< V_k > := < V_1, ..., V_n >$, if we consider a net of best $\phi-$approximation functions $\{g_\delta\}$ and it has a limit in S_N as $\delta \longrightarrow 0$ then this limit is called *the best local* $\phi-$*approximation of* f *from* S_N. Under certain conditions the best local $\phi-$approximation can be obtained by Hermite interpolation and it can be calculated explicitly without having to find the elements of the net $\{g_\delta\}$. This result will be presented in sections 3.

Now we make an assumption on the $n-$tuple $< \varepsilon_k >$ which will guarantee us that the terms of the form $\phi(\varepsilon_k^{\alpha})\varepsilon_k$ can be compared with each other as functions of δ which is a weaker condition to those in [2] for the L^P case. However in our case it depends also on ϕ. Namely, for any $\alpha, \beta \ge 0$ and any j, k such that $1 \le j, k \le n$, we assume that either $\phi(\varepsilon_j^{\beta})\ \varepsilon_j = O(\phi(\varepsilon_k^{\alpha})\ \varepsilon_k)$ or $\phi(\varepsilon_k^{\alpha})\ \varepsilon_k = O(\phi(\varepsilon_j^{\beta})\ \varepsilon_j)$ or both. Given a $n-$tuple of functions $< \phi(\varepsilon_k^{\alpha_k})\ \varepsilon_k >$ where $\alpha_k \ge 0$ is a positive real number, $\phi(\varepsilon_j^{\alpha_j})\ \varepsilon_j$ is said to be maximal if for all k, $1 \le k \le n$, $\phi(\varepsilon_k^{\alpha_k})\ \varepsilon_k = O(\phi(\varepsilon_j^{\alpha_j})\ \varepsilon_j)$. We denote it by $max\{\phi(\varepsilon_k^{\alpha_k})\varepsilon_k\}$.

Remark 1. The assumption over the $n-$tuple $< \varepsilon_k >$ imply the existence of $max\{\phi(\varepsilon_k^{\alpha_k})\varepsilon_k\}$ for any $n-$tuple $< \alpha_k >$ of positive real numbers.

We assume that f and S_N are in $PC^m(X)$, where $PC^m(X)$ is the class of functions with $m-1$ continuous derivatives and with piecewise continuous m^{th} derivative. The space S_N is assumed to be fully interpolating at the set $< x_k >$, that is if $S_N \in PC^m(X)$ and $i_1, .., i_n$ are nonnegative integers with $i_k \leq m$ and $\sum_{k=1}^n i_k = N$ then there exists a unique $g \in \mathrm{S_N}$ such that $g^{(j)}(x_k) = a_{j,k}$, $0 \leq j \leq i_k - 1$, $1 \leq k \leq n$, where $a_{j,k}$ are an arbitrary set of real numbers.

1. Preliminary Results

In this section we prove two Lemmas, which will be used to obtain the main results. This pattern follows the way used in [2] for the L^p case.

The following Lemma provides an order of the error $\int_V \phi(|f-g|)dx$ when $g \in \mathrm{S_N}$ and satisfies $f^{(j)}(x_k) = g^{(j)}(x_k)$, $0 \leq j \leq i_k - 1$, $1 \leq k \leq n$.

Lemma 1.1. *Let* $i_1, .., i_n$ *be nonnegative integers. Suppose* $h \in PC^m(X)$, *where* $m = max\{i_k\}$, *and* $\mathrm{h^{(j)}(x_k)} = 0$, $0 \leq \mathrm{j} \leq \mathrm{i_k}$-1, $1 \leq \mathrm{k} \leq \mathrm{n}$. *Then*

$$\int_V \phi(|h|)dx \; = \; O(max\{\phi(\varepsilon_k^{i_k})\varepsilon_k\}).$$

Proof. Approximating h by the Taylor polynomial about x_k, we have

$$h(x) = \sum_{j=0}^{i_k-1} \frac{h^{(j)}(x_k)}{j!}\,(x-x_k)^j \; + R_k(x),$$

where $R_k(x) = \frac{h^{(i_k)}(\xi)}{i_k!}(x-x_k)^{i_k}$, with ξ between x and x_k. It means $R_k(x) = O((x-x_k)^{i_k})$. Since $\mathrm{h^{(j)}(x_k)} = 0$, $0 \leq j \leq i_k - 1$, we have $h(x) = O((x-x_k)^{i_k})$, $x \in V_k$. Thus, by the Δ_2 condition on ϕ and setting
$x - x_k = \varepsilon_k y$, we obtain $\int_{V_k} \phi(|h(x)|)\; dx \; \leq M \int_{V_k} \phi(|x-x_k|^{i_k})\; dx \leq M \int_{A_k} \varepsilon_k \; \phi(\varepsilon_k^{i_k}\; |y|^{i_k})\; dy \; \leq M'\varepsilon_k\phi(\varepsilon_k^{i_k})$, since A_k are uniformly bounded for all $\delta > 0$. Finally there exist constant M and M' such that

$$\int_V \phi(|h|)dx \leq \; M \sum_{k=1}^n \phi(\varepsilon_k^{i_k})\varepsilon_k \leq M' \; max\{\phi(\varepsilon_k^{i_k})\varepsilon_k\},$$

or

$$\int_V \phi(|h|)dx \; = \; O(max\{\phi(\varepsilon_k^{i_k})\varepsilon_k\}),$$

as required. □

We now cite the Lemma 3 stated in [2] which will be used in the sequel.

Lemma 1.2. *Let Λ be a family of uniformly bounded measurable subsets of the real line with measure* 1*. Let $P(x) = c_0 + c_1 x + ... + c_m x^m$ be an arbitrary polynomial of degree m. Then there exists a constant M (depending on m) such that for all $P(x)$ and all $A \in \Lambda$,*

$$|c_k| \leq M \ \ \|P(x)\|_{L_p(A)},$$

$1 \leq p \leq \infty,\ 0 \leq k \leq m.$

As a consequence of this Lemma we obtain a similar result in L^ϕ.

Corollary 1.3. *Let Λ be a family of uniformly bounded measurable subsets of the real line with measure* 1*. Let $P(x) = c_0+c_1x+....+c_mx^m$ be an arbitrary polynomial of degree m. Then there exists a constant M (depending on m and ϕ) such that for all $P(x)$ and all $A \in \Lambda$,*

$$\phi(|c_k|) \leq M \int_A \phi(|P(x)|)dx,$$

$0 \leq k \leq m.$

Proof. In the following proof the constant M can be different in each occurrence. We know there is a constant M such that for all $P(x)$ and all $A \in \Lambda$, $|c_k| \leq M \ \int_A |P(x)|dx$, $0 \leq k \leq m$, then

$$\phi(|c_k|) \leq M'\phi(M)\phi\left(\int_A |P(x)|dx\right),$$

since ϕ is an increasing function which satisfies the Δ' condition. Now, using the Jensen's Inequality we obtain

$$\phi(|c_k|) \leq M \int_A \phi(|P(x)|)dx,\ 0 \leq k \leq m,$$

for all $A \in \Lambda$. This completes the proof. □

2. $\Phi-$Balanced Neighborhood in L^ϕ

The following definition generalizes a concept given in [2] for the L^p space.

Definition 2.1. *Given the $n-$tuple $< \varepsilon_k >$; an $n-$tuple $< i_k >$ of nonnegative integers is said to be $\phi-$balanced if for each j such that $i_j > 0$,*

$$\phi\left(\frac{1}{\varepsilon_j^{i_j-1}}\right) max\left\{\frac{\phi(\varepsilon_k^{i_k})\varepsilon_k}{\varepsilon_j}\right\} \ = \ o(1).$$

If $< i_k >$ is $\phi-$balanced, then $\sum_{k=1}^n i_k$ is said to be a $\phi-$balanced integer.
The $n-$tuple $< V_k >$ is said to be $\phi-$balanced neighborhoods if the dimension N of the space S_N is a $\phi-$balanced integer.

Remark 2. If $\phi(x) = x^p$, $1 \leq p < \infty$ the last definition of $\phi-$balanced is equivalent to those considered by Chui et al. in [2].

Example 2.2. *Let* $\phi(x) = x^3(1 + |\ln x|)$ *with* $\phi(0) = 0$*. This convex function satisfies the* Δ' *condition and* $<\varepsilon_1, \varepsilon_2> = <\delta, e^{-1/\delta}>$ *. In this case any integer* N *is a* $\phi-$*balanced integer.*

Remark 3. To each $\phi-$balanced integer there corresponds exactly one $\phi-$balanced $< i_k >$. In fact, since ϕ satisfies the Δ' condition and we suppose $< i_k > \neq < i'_k >$ with $\sum_{k=1}^n i_k = \sum_{k=1}^n i'_k$ and $< i_k >$ is $\phi-$balanced, then there exist j, l with $i_j < i'_j$ and $i'_l < i_l$, such that

$$\phi\left(\frac{1}{\varepsilon_j^{i'_j-1}}\right)\frac{\phi(\varepsilon_l^{i'_l})\varepsilon_l}{\varepsilon_j} \geq \phi\left(\frac{1}{\varepsilon_j^{i_j}}\right)\frac{\phi(\varepsilon_l^{i_l-1})\varepsilon_l}{\varepsilon_j} \geq M\,\frac{1}{\phi\left(\frac{1}{\varepsilon_l^{i_l-1}}\right)\frac{\phi(\varepsilon_j^{i_j})\varepsilon_j}{\varepsilon_l}},$$

and the last expression tends to infinite, so $< i'_k >$ is not $\phi-$balanced.

In [2] it was given an algorithm which generates all balanced integers in L^p spaces. Now, we will present an algorithm that inductively generates all the integers m which can be $\phi-$balanced in L^ϕ, that is, it generates a $n-$tuple $< i_k^{(m)} >$ such that $\sum_{k=1}^n i_k^{(m)} = m$.
Beginning with the $\phi-$balanced $n-$tuple $< i_k^{(0)} > = < 0 >$ corresponding to the $\phi-$balanced integer 0 and given $< i_k^{(m)} >$, determine a maximal element of $< \phi(\varepsilon_k^{i_k^{(m)}})\varepsilon_k >$, say $\phi(\varepsilon_{k*}^{i_{k*}^{(m)}})\varepsilon_{k*} = max\{\phi(\varepsilon_k^{i_k^{(m)}})\varepsilon_k\}$ and define $i_k^{(m+1)} = i_k^{(m)}$ for $k \neq k*$ and $i_k^{(m+1)} = i_k^{(m)} + 1$ for $k = k*$.

Remark 4. The algorithm reduces, at each step, the largest value of $\phi(\varepsilon_k^{i_k})\varepsilon_k$ changing i_k by $i_k + 1$.

Lemma 2.3. *a) The above algorithm generates all* $\phi-$*balanced* $< i_k >$.
b) If a $n-$*tuple* $< i_k^{(m)} >$ *generated by the algorithm* ($m \geq 1$) *is* $\phi-$*balanced, then there is a unique maximal element of* $< \phi(\varepsilon_k^{i_k^{(m-1)}})\varepsilon_k >$.

Proof. First we will prove a). Suppose that $< i_k >$ is $\phi-$balanced with $\sum_{k=1}^n i_k = m$ and $< i_k^{(m)} >$ as obtained by the algorithm is not $\phi-$balanced. Then, there exist indices r, s with $i_r^{(m)} > i_r$ and $i_s > i_s^{(m)}$ such that, using the $\phi-$balanced integer definition,

$$\phi(\varepsilon_r^{i_r^{(m)}-1})\varepsilon_r = O(\phi(\varepsilon_r^{i_r})\varepsilon_r) = O\left(max\{\phi(\varepsilon_k^{i_k})\varepsilon_k\}\right) = o\left(\frac{\varepsilon_s}{\phi(\frac{1}{\varepsilon_s^{i_s-1}})}\right),$$

and from the Δ' condition on ϕ, the last expression is an $o(\phi(\varepsilon_s^{i_s-1})\varepsilon_s)$, so

$$\phi(\varepsilon_r^{i_r^{(m)}-1})\varepsilon_r = o(\phi(\varepsilon_s^{i_s^{(m)}})\varepsilon_s).$$

Since $i_r^{(m)} > 0$ at some previous step $\phi(\varepsilon_r^{i_r^{(m)}-1})\varepsilon_r$ was maximal of $< \phi(\varepsilon_k^{i_k^{(m')}})\varepsilon_k >$ with $m' < m$, so $\phi(\varepsilon_r^{i_r^{(m)}-1})\varepsilon_r = max\{\phi(\varepsilon_k^{i_k^{(m)}})\varepsilon_k\}$ because the exponents are non-decreasing at each step of the algorithm.

Thus for any k

$$\phi(\varepsilon_k^{i_k^{(m)}})\varepsilon_k = O(\phi(\varepsilon_r^{i_r^{(m)}-1})\varepsilon_r),$$

which is a contradiction.
Now we will prove b). If $max\{\phi(\varepsilon_k^{i_k^{(m-1)}})\varepsilon_k\}$ is not unique then $< i_k^{(m)} >$ cannot be $\phi-$balanced because if the index j and s gives a maximal element of $< \phi(\varepsilon_k^{i_k^{(m-1)}})\varepsilon_k >$ then there exist two constant M, N such that

$$M \leq \frac{\phi(\varepsilon_j^{i_j^{(m-1)}})\varepsilon_j}{\phi(\varepsilon_s^{i_s^{(m-1)}})\varepsilon_s} \leq N.$$

Suppose that $i_k^{(m)} = i_k^{(m-1)}$ for $k \neq s$ and $i_k^{(m)} = i_k^{(m-1)} + 1$ for $k = s$, so $i_s^{(m)} > 0$ and then

$$\phi\left(\frac{1}{\varepsilon_s^{i_s^{(m)}-1}}\right) max\left\{\frac{\phi(\varepsilon_k^{i_k^{(m)}})\varepsilon_k}{\varepsilon_s}\right\} = \phi\left(\frac{1}{\varepsilon_s^{i_s^{(m-1)}}}\right)\frac{\phi(\varepsilon_j^{i_j^{(m-1)}})\varepsilon_j}{\varepsilon_s}$$

$$\geq A\frac{\phi(\varepsilon_j^{i_j^{(m-1)}})\varepsilon_j}{\phi(\varepsilon_s^{i_s^{(m-1)}})\varepsilon_s} \;\geq\; AM,$$

by the Δ' condition. This show that $< i_k^{(m)} >$ cannot be $\phi-$balanced. □

Remark 5. The reverse hand of the Lemma 2.3 does not hold.
We take $\phi(x) = x^3(|\ln x| + 1)$ with $\phi(0) = 0$ and $< \varepsilon_k > = < \delta, \delta^4 >$ then, in the first step, the algorithm generate
$< 1, 0 >$ with a unique corresponding maximal element $max\{\phi(\varepsilon_k^{i_k})\varepsilon_k\} = \phi(\varepsilon_1^1)\varepsilon_1$, however the bi-tuple $< 2, 0 >$ is not $\phi-$balanced.

We point out that the last Remark shows a difference with the L^p case.In [2] it is proved the reverse hand of this Lemma for the L^p case assuming a stronger condition on the $n-$tuple $< \varepsilon_k >$.

3. Best Local $\phi-$approximation in Orlicz Spaces with $\phi-$balanced Neighborhood.

We now come to the main result concerning to the behavior of a net $\{g_\delta\}$ of best $\phi-$approximations. For this purpose we set the following three Lemmas.

Lemma 3.1. *Given $< \varepsilon_k >$ and $< i_k >$, define $m = \max\{i_k\}$. Let $f \in PC^m(X)$ and $S_N \subseteq PC^m(X)$. If we consider a net $\{g_\delta\}$ of best $\phi-$approximation such that*

(1) $$\int_X \phi(|g_\delta|)dx \longrightarrow \infty, \; as \; \delta \longrightarrow 0,$$

then the net of functions

$$h_\delta = \frac{g - g_\delta}{\int_X |g - g_\delta| dx},$$

where $g \in S_N$ *interpolate* $f^{(j)}(x_k),\ 0 \le j \le i_k - 1,\ 1 \le k \le n$, *satisfies the following properties*

i) $\int_X \phi(|h_\delta|)dx \ge A > 0$,

ii) $\int_V \phi(|h_\delta|)dx = o\left(max\{\phi(\varepsilon_k^{i_k})\varepsilon_k\}\right)$.

Proof. Property $i)$ follows using the Jensen's Inequality and property $ii)$ is a consequence of the hypothesis. In fact, since ϕ satisfies the Δ_2 condition

$$\int_X \phi(|g_\delta|)dx \le M\left(\int_X \phi(|g_\delta - g|)dx + \int_X \phi(|g|)dx\right),$$

so, using (1) we obtain

$$\int_X \phi(|g - g_\delta|)dx \longrightarrow \infty, \tag{2}$$

as $\delta \longrightarrow 0$, or

$$\int_X |g - g_\delta| dx \longrightarrow \infty,$$

as $\delta \longrightarrow 0$, because if there were exist a sequence $\{\delta_s\}$ such that $\int_X |g - g_{\delta_s}|dx \le M$ for all δ_s, then $\|g - g_{\delta_s}\|_\infty \le M$ for all δ_s since the norms are equivalent, and so $\int_X \phi(|g - g_{\delta_s}|)dx \le M'$, which is a contradiction.

On the other hand, by the Δ_2 condition on ϕ and Lemma 1.1

$$\begin{aligned}\int_V \phi(|g - g_\delta|)dx &\le M\int_V \phi(|g - f|)dx + M\int_V \phi(|f - g_\delta|)dx \\ &= O\left(max\{\phi(\varepsilon_k^{i_k})\varepsilon_k\}\right),\end{aligned}$$

then, since ϕ is a convex function

$$\int_V \phi(|h_\delta|)dx \le \frac{1}{\int_X |g - g_\delta| dx}\int_V \phi(|g - g_\delta|)dx = o\left(max\{\phi(\varepsilon_k^{i_k})\varepsilon_k\}\right),$$

as required. □

Lemma 3.2. *Given* $< \varepsilon_k >$ *set a* $\phi-$*balanced* $n-$*tuple* $< i_k >$ *such that* $N = \sum_{k=1}^n i_k$ *and define* $m = \max\{i_k\}$. *If* $f \in PC^m(X)$, $S_N \subseteq PC^m(X)$ *and given* $\delta > 0$ *set* $\{g_\delta\}$ *for a net of best* $\phi-$*approximation of* f, *then there exists* $M > 0$ *such that*

$$\int_X \phi(|g_\delta|)\ dx \le M,$$

for all $\delta > 0$.

Proof. Suppose $\{\delta_r\}$ is a sequence such that

$$\int_X \phi(|g_{\delta_r}|)dx \longrightarrow \infty,$$

as $\delta_r \longrightarrow 0$. Let g be a fixed function in $\mathrm{S_N}$ interpolating the derivatives $\mathrm{f^{(j)}(x_k)}$, $0 \leq \mathrm{j} \leq \mathrm{i_k}$-1, $1 \leq \mathrm{k} \leq \mathrm{n}$ and define

$$h_{\delta_r} = \frac{g - g_{\delta_r}}{\int_X |g - g_{\delta_r}|dx},$$

then, using Lemma 3.1, we have

i) $\int_X \phi(|h_{\delta_r}|)dx \geq A > 0$, for all δ_r.

ii) $\int_V \phi(|h_{\delta_r}|)dx = o\left(max\{\phi(\varepsilon_k^{i_k})\varepsilon_k\}\right)$.

Expanding h_{δ_r} using Taylor polynomials we obtain for each k

$$h_{\delta_r}(x) \;=\; \sum_{j=1}^{i_k-1} \frac{h_{\delta_r}^{(j)}(x_k)}{j!}(x - x_k)^j + R_{\delta_r}(x),$$

where $R_{\delta_r}(x) = O((x-x_k)^{i_k})$ uniformly in δ_r. In fact, since $\|h_{\delta_r}\|_{L^1(X)} \leq 1$ for all δ_r and using the fact that the norms are equivalent in S_N, we can choose the norm $\|h_{\delta_r}\| := \sup_{x\in X}\{|h_{\delta_r}(x)| + ... + |h_{\delta_r}^{(i_k)}(x)|\}$ to show the statement.

This uniform bound of $R_{\delta_r}(x)$, leads to

$$\int_{V_k} \phi(|R_{\delta_r}(x)|)dx = O(\varepsilon_k \phi(\varepsilon_k^{i_k})),$$

for all r, taking $x - x_k = \varepsilon_k y$ and using that the sets A_k are uniformly bounded. Thus, using property *ii*) we obtain

$$\int_{V_k} \phi\left(|\sum_{j=0}^{i_k-1} \frac{h_{\delta_r}^{(j)}(x_k)}{j!}(x - x_k)^j|\right)dx$$

$$\leq M\int_{V_k} \phi\left(|\sum_{j=0}^{i_k-1} \frac{h_{\delta_r}^{(j)}(x_k)}{j!}(x - x_k)^j + R_{\delta_r}(x)|\right)dx + M\int_{V_k} \phi(|R_{\delta_r}(x)|)dx$$

$$= M\int_{V_k} \phi(|h_{\delta_r}|)dx + O(\varepsilon_k\phi(\varepsilon_k^{i_k})) \leq M\int_V \phi(|h_{\delta_r}|)dx + O(\varepsilon_k\phi(\varepsilon_k^{i_k}))$$

$$= o\left(max\{\phi(\varepsilon_l^{i_l})\varepsilon_l\}\right) + O(\varepsilon_k\phi(\varepsilon_k^{i_k})),$$

if we substitute $x - x_k = \varepsilon_k y$ again we obtain

$$\int_{A_k} \phi\left(|\sum_{j=0}^{i_k-1} \frac{h_{\delta_r}^{(j)}(x_k)}{j!}(\varepsilon_k)^j y^j|\right)dy = O\left(max\{\frac{\phi(\varepsilon_l^{i_l})\varepsilon_l}{\varepsilon_k}\}\right),$$

for $1 \leq k \leq n$. From Corollary 1.3

$$\phi\left(|\frac{h_{\delta_r}^{(j)}(x_k)}{j!}(\varepsilon_k)^j|\right) = O\left(max\left\{\frac{\phi(\varepsilon_l^{i_l})\varepsilon_l}{\varepsilon_k}\right\}\right),$$

$j = 0, ..., i_k - 1,\ 1 \le k \le n$, using that ϕ satisfies the Δ' condition, for each k we get

$$\phi(|h_{\delta_r}^{(j)}(x_k)|) \le M\phi\left(\frac{1}{\varepsilon_k^{i_k-1}}\right) O\left(max\left\{\frac{\phi(\varepsilon_l^{i_l})\varepsilon_l}{\varepsilon_k}\right\}\right),$$

for $j = 0, .., i_k - 1$. Finally, since N is a $\phi-$balanced integer and ϕ a strictly increasing function, we obtain

$$h_{\delta_r}^{(j)}(x_k) = o(1),$$

$0 \le j \le i_k - 1,\ 1 \le k \le n$, and when $\delta_r \longrightarrow 0$. Now, since $h_{\delta_r} \in \mathrm{S_N}$ we get $h_{\delta_r} = \sum_{i=1}^{N} a_{\delta_r,i}\widetilde{h_i}$ where $\{\widetilde{h_1}, ..., \widetilde{h_N}\}$ is a basis of $\mathrm{S_N}$. So h_{δ_r} is uniquely determined by the N values $h_{\delta_r}^{(j)}(x_k)$, $0 \le j \le i_k - 1$, $1 \le k \le n$, using the fixed linear transformation

$$\begin{pmatrix} \widetilde{h_1}(x_1) & . & . & \widetilde{h_N}(x_1) \\ . & . & . & . \\ . & . & . & . \\ \widetilde{h_1}^{(i_n-1)}(x_n) & . & . & \widetilde{h_N}^{(i_n-1)}(x_n) \end{pmatrix}.$$

Then $h_{\delta_r} \longrightarrow 0$ as $\delta_r \longrightarrow 0$, thus

$$\lim_{\delta_r \longrightarrow 0} \int_X \phi(|h_{\delta_r}|)dx = 0$$

which contradicts property i). Thus g_δ must be bounded for all δ and the proof of the Lemma is complete. □

Lemma 3.3. *Given* $< \varepsilon_k >$ *and a* $\phi-$*balanced* $n-$*tuple* $< i_k >$ *such that* $N = \sum_{k=1}^{n} i_k$, *define* $m = \max\{i_k\}$. *If* $f \in PC^m(X)$, $S_N \subseteq PC^m(X)$ *and* $\{g_\delta\}$ *is a net of best* $\phi-$*approximation, then for each* k

$$\phi(|(f - g_\delta)^{(j)}(x_k)|\ \varepsilon_k^j) \ = \ O\left(max\left\{\frac{\phi(\varepsilon_l^{i_l})\varepsilon_l}{\varepsilon_k}\right\}\right),$$

$0 \le j \le i_k - 1$.

Proof. For each k, expanding $f - g_\delta$ using Taylor polynomials, we obtain

$$(f - g_\delta)(x) \ = \ \sum_{j=1}^{i_k-1} \frac{(f - g_\delta)^{(j)}(x_k)}{j!}(x - x_k)^j + R_\delta(x).$$

We may use the above analysis to show that $R_{\delta_r}(x) = O((x - x_k)^{i_k})$ uniformly in δ. In fact, it follows from Lemma 3.2 that $\int_X \phi(|g_\delta|)dx \le M$ for all δ, thus $||g_\delta||_\phi \le M$ for all δ and using that the norms are equivalent in S_N, we can choose again the norm $\|h_{\delta_r}\| = \sup_{x \in X}\{|h_{\delta_r}(x)| + ... + |h_{\delta_r}^{(i_k)}(x)|\}$ to prove the statement.

These uniform bound of $R_{\delta_r}(x)$ and since the sets A_k are uniformly bounded, substituting $x - x_k = \varepsilon_k y$, leads to

$$\int_{V_k} \phi(|R_\delta(x)|)dx = O(\phi(\varepsilon_k^{i_k})\varepsilon_k). \tag{3}$$

On the other hand, using a fixed $g \in \mathrm{S_N}$ which interpolates the derivatives $\mathrm{f^{(j)}(x_k)}$, $0 \le \mathrm{j} \le \mathrm{i_k}\text{-}1$, $1 \le \mathrm{k} \le \mathrm{n}$, from Lemma 1.1 we obtain

$$\int_V \phi(|f - g_\delta|)dx \;=\; O\left(max\{\phi(\varepsilon_k^{i_k})\varepsilon_k\}\right),$$

thus, using (3) and the Δ_2 condition, we obtain for each k

$$\int_{V_k} \phi\left(|\sum_{j=0}^{i_k-1} \frac{(f-g_\delta)^{(j)}(x_k)}{j!}(x-x_k)^j|\right) dx \le M \int_{V_k} \phi(|f-g_\delta|)dx$$
$$+ M \int_{V_k} \phi(|R_\delta(x)|)dx = O\left(max\{\phi(\varepsilon_l^{i_l})\varepsilon_l\}\right) + O(\varepsilon_k \phi(\varepsilon_k^{i_k}))$$
$$= O\left(max\{\phi(\varepsilon_l^{i_l})\varepsilon_l\}\right),$$

and finally if we substitute $x - x_k = \varepsilon_k y$ then for each k

$$\int_{A_k} \phi\left(|\sum_{j=0}^{i_k-1} \frac{(f-g_\delta)^{(j)}(x_k)}{j!}\varepsilon_k^j y^j|\right) dy = O\left(max\left\{\frac{\phi(\varepsilon_l^{i_l})\varepsilon_l}{\varepsilon_k}\right\}\right).$$

Now we conclude from Corollary 1.3 that

$$\phi(|(f-g_\delta)^{(j)}(x_k)\varepsilon_k^j|) = O\left(max\left\{\frac{\phi(\varepsilon_l^{i_l})\varepsilon_l}{\varepsilon_k}\right\}\right),$$

$0 \le j \le i_k - 1$, for each k. □

Now we present the following result.

Theorem 3.4. *If N is a $\phi-$balanced integer with $\phi-$balanced $< i_k >$ and $f \in PC^m(X)$, $\mathrm{S_N} \subset PC^m(X)$, $(m = max\{i_k\})$, then the best local $\phi-$approximation of f from $\mathrm{S_N}$ is the unique $g \in \mathrm{S_N}$ defined by the N interpolation conditions*

$$f^{(j)}(x_k) = g^{(j)}(x_k),$$

$0 \le j \le i_k - 1,\ 1 \le k \le n.$

Proof. From Lemma 3.3

$$\phi(|(f-g_\delta)^{(j)}(x_k)\varepsilon_k^j|) = O\left(max\left\{\frac{\phi(\varepsilon_l^{i_l})\varepsilon_l}{\varepsilon_k}\right\}\right),$$

$0 \le j \le i_k - 1,\ 1 \le k \le n$. Using the Δ' condition and the $\phi-$balanced integer definition we have for each k

$$\phi(|(f-g_\delta)^{(j)}(x_k)|) = \phi\left(\frac{1}{\varepsilon_k^{i_k-1}}\right) O\left(max\left\{\frac{\phi(\varepsilon_l^{i_l})\varepsilon_l}{\varepsilon_k}\right\}\right) = o(1),$$

$0 \leq j \leq i_k - 1$, thus, since ϕ is a strictly increasing function with $\phi(x) = 0$ if and only if $x = 0$

$$\lim_{\delta \longrightarrow 0} g_\delta^{(j)}(x_k) = f^{(j)}(x_k),$$

$0 \leq j \leq i_k - 1,\ 1 \leq k \leq n$. Now we will do a similar analysis as above . As g_δ is uniquely determined via a fixed linear transformation with rank N from the N values $g_\delta^{(j)}(x_k),\ 0 \leq \mathrm{j} \leq \mathrm{i_k}\text{-}1,\ 1 \leq \mathrm{k} \leq \mathrm{n}$, then g_δ must converge to the unique g satisfying

$$g^{(j)}(x_k) = f^{(j)}(x_k),$$

$0 \leq j \leq i_k - 1,\ 1 \leq k \leq n$. This g is by definition the best local $\phi-$approximation of f from S_N.

□

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