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Histories in quantum mechanics: distinguishing between formalism and interpretation

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Abstract In spite of being a well articulated proposal, the theory of quantum histories (TQH), in its different versions, suffers from certain difficulties that have been pointed out in the literature. Nevertheless, two facets of the proposal have not been sufficiently stressed. On the one hand, it is a non-collapse formalism that should be technically appropriate to supply descriptions based on quantum properties at different times. On the other hand, it intends to provide an interpretation of quantum mechanics that solves the traditional puzzles of the theory. In this article we spell out the main criticisms to TQH and classify them into two groups: theoretical and interpretive. Whereas the latter might be ignored if the TQH were considered as a quantum formalism with its minimum interpretation, the former seems to point toward technical difficulties that must be faced in a theoretically adequate proposal. Precisely with the purpose of solving these difficulties, we introduce a different perspective, called Formalism of Generalized Contexts or Formalism of Contextual Histories (FCH), which supplies a precise condition for consistently talking of quantum properties at different times without the theoretical shortcomings of the TQH.

Keywords Quantum histories · Consistency condition · Formalism of contextual histories · Time translation of properties · Generalized context · Compatibility condition

1 Introduction

According to the standard formulation of quantum mechanics, a quantum system is represented by a Hilbert space and every pure state of the system is represented by a normalized vector in that space (von Neumann 1932). States follow two types of time evolutions. When there is no measurement, evolutions are governed by the Schrödinger

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equation, leading to a continuous and deterministic dynamics. When a measurement is performed on the system, the state collapses onto one of the eigenstates of the measured observable in a discontinuous and non-deterministic process.

From this standard perspective, the properties of the system are represented by subspaces of the Hilbert space or by their corresponding orthogonal projectors. The set of all possible properties has an orthocomplemented lattice structure, on which logical operations between properties can be defined (Birkhoff and von Neumann 1936). The probability of measuring a property of the system is given by the Born rule; on this basis, the probabilities corresponding to logical operations between properties can also be computed.

The standard formulation of quantum mechanics has a remarkable predictive success. However, two aspects were considered conceptually unsatisfactory by several authors:

- The first one is the distinction between ordinary physical processes and measurement processes. On the one hand, since probability is defined only for measurements, it cannot be applied to systems which do not admit of being measured by an external apparatus, as those studied in quantum cosmology. On the other hand, since measuring devices are made of the same components as the remaining physical systems, it is expected that measurement processes are not essentially different from ordinary physical processes.
- The second unsatisfactory aspect is the impossibility of defining logical operations between properties at different times. This is a shortcoming because in many cases it is interesting to consider conjunctions or disjunctions of properties at different times; for instance, in the measurement process, a property of the measured system before a measurement needs to be logically related to a property of the measuring apparatus after measurement.

The idea of *quantum history* was mainly motivated by these two unsatisfactory aspects of standard quantum mechanics. In 1984, Robert Griffiths presented the first version of his Theory of Consistent Histories (Griffiths 1984); some years later, he introduced some modifications to that original version (Griffiths 2002, 2013). Omnès (1987, 1988a, b, 1994, 1999) also published a series of articles that contributed to the development of this theory. Simultaneously, Murray Gell-Mann and James Hartle developed a similar formalism, called Decoherent Histories Interpretation (Gell-Mann and Hartle 1990, 1993; Hartle 1991). Although these proposals do not agree in every detail, their strong similarities justify to subsume all of them under the label "*Theory of Quantum Histories*" (TQH).

All the versions of the TQH claim to have solved the two difficulties mentioned above:

- On the one hand, they provide a formulation of quantum mechanics in which measurements are treated in the same way as other physical processes. In particular, measurements are considered ordinary physical interactions between a system to be measured and a measuring apparatus. Therefore, the quantum dynamics is always described by the Schrödinger equation: an additional projection postulate is not required to account for collapse. As a result, the formalism is applicable to closed systems, such as those studied by cosmology.

– On the other hand, they extend the standard formalism of quantum mechanics in order to be able to define logical operations between properties at different times. For this purpose, they introduce the notion of history, which generalizes the notion of event: an elemental history is defined as a sequence of events at different times, where an event is the occurrence of a property. But since it is not possible to assign probabilities to the set of all histories, it is necessary to select a subset of histories that satisfies additional conditions.

Since there is no collapse of the state vector, measurements are treated in the same way as other physical processes. Due to this reason, advocates of the TQH claim that their theory provides a *realist interpretation* of quantum mechanics.

In spite of being a well articulated proposal, the TQH in its different versions suffers from certain difficulties that have been pointed out in the literature. Nevertheless, two facets of the proposal should be distinguished. On the one hand, it is a non-collapse formalism that should be technically appropriate to supply descriptions based on quantum properties at different times. On the other hand, it intends to provide an interpretation of quantum mechanics that solves the traditional puzzles of the theory. One might reject the TQH as an acceptable interpretation of quantum mechanics, but still consider it an adequate non-collapse formalism to describe possible sequences of quantum properties in time. In general, this distinction is not sufficiently taken into account in the literature, and the TQH is criticized as a whole. However, the objections should discriminate those two aspects if the limitations of the theory have to be overcome.

On this basis, in the first part of this article we will spell out some traditional and new criticisms to the TQH, and we will classify them into two groups: theoretical and interpretive. Whereas the latter might be ignored if the TQH were considered a quantum formalism endowed with a minimum interpretation, the former seems to point toward technical difficulties that must be faced in a theoretically adequate proposal. Precisely with the purpose of solving these difficulties, in the second part of this article we will introduce a different perspective, called *Formalism of Generalized Contexts* or *Formalism of Contextual Histories* (FCH), which supplies a precise condition for consistently talking of quantum properties at different times without the theoretical shortcomings of the TQH. Of course, since the FCH is a formalism and does not intend to supply an interpretation of quantum mechanics, its scope is more restricted than that of the TQH. Nevertheless, the FCH may turn out to be interpretively relevant if supplemented with an adequate interpretation.

2 Theory of quantum histories

In this section we will briefly introduce the main tenets of the TQH. We will follow Griffiths' version (1984, 2002, 2013), but many of the discussions developed in this article can be extended to the other versions.

As mentioned above, in standard quantum mechanics the properties of a system are represented by subspaces of the Hilbert space of the system or by their corresponding orthogonal projectors. In the TQH, an *elementary history* is defined as a sequence of events −occurrences of properties− at different times, and a *composite history* is defined as the result of logical operations between elemental histories. For example, if we consider n times, $t_1 < \ldots < t_n$, and properties p_i and q_i at each time t_i, then $\breve{p} = (p_1, \ldots, p_n)$ is the elementary history in which, at each time t_i, the property p_i is the case, and $\breve{q} = (q_1, \ldots, q_n)$ is the elementary history in which, at each time t_i, the property q_i is the case. Moreover, $\breve{p} \vee \breve{q}$, $\breve{p} \wedge \breve{q}$ and $\neg \breve{p}$ are composite histories that result from logical operations (disjunction, conjunction and negation, respectively) between histories $\breve{p}$ and $\breve{q}$.

An n-time elementary history, such as $\breve{p} = (p_1, \ldots, p_n)$, is represented by the projector $\breve{\Pi} = \Pi_1 \otimes \ldots \otimes \Pi_n$, where each Π_i corresponds to the property p_i. Projector $\breve{\Pi}$ is defined on the Hilbert space $\breve{\mathcal{H}} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n$, i.e., on the tensor product of n copies of the Hilbert space of the system. As $\breve{\mathcal{H}}$ is a Hilbert space, the set of all the projectors on it has an orthocomplemented and non-distributive lattice structure. In order to define probabilities on quantum histories, it is necessary to select a Boolean sub-lattice of the full lattice, that is, a distributive and orthocomplemented lattice. In this Boolean sub-lattice of quantum histories, the logical operations in terms of the projectors are expressed as follows:

$$\breve{p} \vee \breve{q} \longleftrightarrow \breve{\Pi}_{\breve{p} \vee \breve{q}} = \breve{\Pi}_{\breve{p}} + \breve{\Pi}_{\breve{q}} - \breve{\Pi}_{\breve{p}} \breve{\Pi}_{\breve{q}} \tag{1}$$

$$\breve{p} \wedge \breve{q} \longleftrightarrow \breve{\Pi}_{\breve{p} \wedge \breve{q}} = \breve{\Pi}_{\breve{p}} \breve{\Pi}_{\breve{q}} \tag{2}$$

$$\neg \breve{p} \longleftrightarrow \breve{\Pi}_{\neg \breve{p}} = \breve{I} - \breve{\Pi}_{\breve{p}} \tag{3}$$

where $\breve{I}$ is the identity of $\breve{\mathcal{H}}$.

In order to obtain a distributive sub-lattice of histories, a projective decomposition of the identity of $\breve{\mathcal{H}}$, at each time t_j ($j = 1, \ldots, n$), must be first selected, i.e., a set of projectors $\left\{\Pi_j^{k_j}\right\}_{k_j \in \sigma_j}$ which are mutually orthogonal and sum the identity of $\breve{\mathcal{H}}$:

$$\Pi_j^{k_j} \Pi_j^{k_j'} = \delta_{k_j k_j'} \Pi_j^{k_j}, \quad \sum_{k_j} \Pi_j^{k_j} = I, \quad k_j, k_j' \in \sigma_j, \quad j = 1, \ldots, n \tag{4}$$

Then, all possible combinations of quantum histories are built by picking one projector $\Pi_j^{k_j}$ at each time t_j:

$$\breve{\Pi}^K = \Pi_1^{k_1} \otimes \ldots \otimes \Pi_n^{k_n}, \qquad K = (k_1, \ldots k_n) \in \sigma_1 \times \ldots \times \sigma_n \tag{5}$$

These histories $\breve{\Pi}^K$ are called *atomic histories*, and they form a projective decomposition of the identity of $\breve{\mathcal{H}}$:

$$\breve{\Pi}^K \breve{\Pi}^{K'} = \delta_{KK'} \breve{\Pi}^K, \quad \sum_K \breve{\Pi}^K = \breve{I}, \quad K \in \sigma_1 \times \ldots \times \sigma_n \tag{6}$$

Finally, the set obtained by making arbitrary disjunctions between atomic histories is called *family of histories*:

$$\left\{\breve{\Pi} \mid \breve{\Pi}= \sum_K \alpha_K \breve{\Pi}^K, \alpha_K = 0,1 \text{ and } K \in \sigma_1 \times \ldots \times \sigma_n\right\} \tag{7}$$

Families of histories have a Boolean lattice structure; therefore, it is possible to define a probability on them. For this purpose, an operator called *chain operator* is introduced: for each atomic history represented by $\breve{\Pi}^K$, the chain operator is given by

$$C\left(\breve{\Pi}^K\right) = \Pi^{k_1}_{1,0}\Pi^{k_2}_{2,0}\ldots\Pi^{k_n}_{n,0} \quad \text{with} \quad \Pi^{k_j}_{j,0} = U\left(t_0, t_j\right)\Pi^{k_j}_j U^{-1}\left(t_0, t_j\right) \tag{8}$$

where $U(t_0, t_j)$ is the time-evolution operator $U(t_0, t_j) = \exp[-iH(t_0 - t_j)/\hbar]$ (where H is the system's Hamiltonian) and t_0 is the initial time. The chain operator for non-atomic histories is obtained by a linear extension of the atomic case: for a generic history, the chain operator is given by

$$C\left(\breve{\Pi}\right) = \sum_K \alpha_K C\left(\breve{\Pi}^K\right) \quad K \in \sigma_1 \times \cdots \times \sigma_n \quad \alpha_K = 0,1 \tag{9}$$

Then, the probability of a history is defined as follows:

$$\Pr\left(\breve{\Pi}\right) = \mathrm{Tr}\left(C^\dagger\left(\breve{\Pi}\right)\rho_0 C\left(\breve{\Pi}\right)\right) \tag{10}$$

where ρ_0 is the state operator at the initial time t_0.

However, in general the probability so defined does not satisfy the additivity axiom of Kolmogorov. Therefore, in order to obtain a well-defined probability, it is necessary that the family of histories satisfy an additional condition. In his Theory of Consistent Histories, Griffiths calls it *consistency condition:*

$$\mathrm{Re}\left[\mathrm{Tr}\left(C^\dagger\left(\breve{\Pi}^K\right)\rho_0 C\left(\breve{\Pi}^{K'}\right)\right)\right] = 0 \quad \text{for all} \quad K \neq K' \tag{11}$$

In the Decoherent Histories Interpretation version of Gell-Mann and Hartle (1990), the consistency condition does not need to be satisfied exactly, but only approximately, that is,

$$\mathrm{Re}\left[\mathrm{Tr}\left(C^\dagger\left(\breve{\Pi}^K\right)\rho_0 C\left(\breve{\Pi}^{K'}\right)\right)\right] \approx 0 \quad \text{for all} \quad K \neq K' \tag{12}$$

A family of histories that satisfies the consistency condition is called *consistent family of histories* (also *family of consistent histories*) or *realm*. Once a realm is defined, probabilities can be assigned to its quantum histories, and physical situations involving properties at different times can be described.

It is worth noticing that different consistent families of histories can be obtained for a single physical system. When two consistent families can be combined into another consistent family that includes both of them, it is said that those families are *compatible*. Otherwise, the consistent families of histories are *incompatible*.

A central element of the Theory of Consistent Histories is the *Single-Framework Rule* (Griffiths 2002), which states that incompatible realms cannot be combined into a single description of a quantum system: probabilistic reasoning is valid only if it is carried out in the framework of a single realm. However, all realms are equally valid to describe a physical system −*Principle of Equality*− and, therefore, physicists are free to select any realm considered appropriate in a specific situation −*Principle of Liberty*− (Griffiths 2013). Nevertheless, not all realms are equally useful to describe a particular system −*Principle of Utility*− (Griffiths 2013).

3 Traditional objections

The TQH introduces interesting ideas that might work in the attempt to find solutions to the two conceptual problems of standard quantum mechanics pointed out in the Introduction. On the one hand, the system always evolves unitarily: a special postulate is not introduced to account for measurements. On the other hand, the fact that it relies on a lattice of histories instead of on a lattice of properties-at-a-time makes it apt to describe the quantum system at different times in a consistent way.

However, in spite of its advantages, some authors consider that the TQH suffers from certain shortcomings that cannot be ignored. In this section we will briefly review some of the traditional objections that have been advanced against the TQH, along with the usual responses. We will come back to them later, in the next sections.

3.1 Contrary retrodictions

Two projectors Π_a and Π_b are said to be *complementary* when they do not commute: $\Pi_a\Pi_b \neq \Pi_b\Pi_a$; they are said to be *contradictory* when they sum to the identity, and *contrary* when they are orthogonal and non-contradictory. In standard quantum mechanics, given two contrary properties p and q, that is, represented by contrary projectors, there are no property r and state ρ such that, in the state ρ, the probability of p conditional to r and the probability of q conditional to r are both equal to one. That is, there is no possibility of contrary inferences in standard quantum mechanics.

This is not the case in the TQH. Kent (1997) pointed out that, in this case, probabilistic retrodictions depend on the choice of the realm. This freedom allows the theory, from a given initial state, to retrodict two contrary properties from different realms. In other words, given the initial state ρ_0 of the system, it is possible to infer, with certainty, two inconsistent facts about the past from two different consistent families of histories.

Hartle (2007) developed an example of retrodiction of contrary properties that is a special case of the example offered by Kent. He focused on a quantum system in a state represented by the vector $|\psi\rangle = (|A\rangle + |B\rangle + |C\rangle)\ /\sqrt{3}$ at time t_2, where $|A\rangle$, $|B\rangle$ and $|C\rangle$ are three orthogonal and normalized vectors. For simplicity, the Hamiltonian is chosen to be zero. At time t_2, the property Φ, represented by the projector $P_\Phi = |\Phi\rangle\langle\Phi|$, is considered, with $|\Phi\rangle = (|A\rangle + |B\rangle - |C\rangle)\ /\sqrt{3}$. A suitable consistent family of two-time histories can be defined on the basis of the properties generated by the projectors $P_A = |A\rangle\langle A|$ and $P_{\overline{A}} = I - |A\rangle\langle A|$ at time $t_1 < t_2$ and the properties generated by the projectors $P_\Phi = |\Phi\rangle\langle\Phi|$ and $P_{\overline{\Phi}} = I - |\Phi\rangle\langle\Phi|$ at time t_2. In the framework of this realm, the following results are obtained:

$$\Pr_\psi(A, t_1|\Phi, t_2) = 1 \qquad \Pr_\psi\left(\overline{A}, t_1|\Phi, t_2\right) = 0 \tag{13}$$

Therefore, the property Φ at time t_2 implies the property A at the previous time t_1. An analogous inference can be performed in a different consistent family of histories, including the properties generated by projectors $P_B = |B\rangle\langle B|$ and $P_{\overline{B}} = I - |B\rangle\langle B|$ at time $t_1 < t_2$ and the properties generated by projectors $P_\Phi = |\Phi\rangle\langle\Phi|$ and $P_{\overline{\Phi}} = I - |\Phi\rangle\langle\Phi|$ at time t_2. In this case, the following results are obtained:

$$\Pr_\psi(B, t_1|\Phi, t_2) = 1 \qquad \Pr_\psi\left(\overline{B}, t_1|\Phi, t_2\right) = 0 \tag{14}$$

This means that the property Φ at time at time t_2 also implies the property B at the previous time t_1. But properties A and B are contrary since represented by orthogonal vectors. Therefore, from a given property Φ at time t_2, contrary properties can be inferred for a previous time t_1.

In summary, according to the TQH, in general there is not a unique past for given present data (although not the same criticism, an objection of the same kind can be found in Goldstein 1998 and Bassi and Ghirardi 1999). The advocates of the theory claim that this is not a real problem (Griffiths and Hartle 1998), because each retrodiction is obtained in a different consistent family of histories; but different incompatible realms cannot be considered simultaneously since they correspond to different event spaces for probabilities (Single-Framework Rule).

3.2 Lack of predictive power

A second criticism to the TQH is that the theory seems to lack predictive power. The previous objection showed that, from the present situation of the system, contrary properties can be retrodicted with certainty in different realms. An analogous objection can be raised about the future: it is possible to predict two contrary properties with certainty in two different consistent families of histories (Dowker and Kent 1996). This means that predictions about the future properties of the system depend on the choice of the realm. Since, according to the TQH, all realms are on a par with each other regarding the description of a physical system, the theory seems to be useless to predict the future properties of a system.

For the advocates of the TQH this is not a problem because, as in the first objection, each prediction is obtained in a different realm. Then, once the specific experiment to be performed is decided, only one realm turns out to be appropriate. This means that, for each experiment, the theory makes the correct predictions. However, this response needs to count first with a criterion to decide when a measuring experiment is performed, that is, to distinguish between quantum measurement processes and ordinary quantum non-measurement processes. But, the need of such a distinction was one of the interpretive problems due to which the TQH was originally proposed and that the theory claims to have dissolved by dispensing with collapse (see Okon and Sudarsky 2014b).

3.3 Discontinuity in property ascription

Let us consider a spin system prepared at time t_0 in a pure state represented by $|+x\rangle$ or $\rho_0 = |+x\rangle\langle +x|$, and a family of histories that includes the histories constituted by the two possible spin values along the x axis at time t_1, represented by the projectors $\Pi_x = |+x\rangle\langle +x|$ and $\overline{\Pi}_x = |-x\rangle\langle -x|$, together with the two possible spin values along the z axis at time t_2, represented by the projectors $\Pi_z = |+z\rangle\langle +z|$ and $\overline{\Pi}_z = |-z\rangle\langle -z|$. By means of Eq. (10), well defined probabilities can be assigned to all the members of the family.

The probability of the spin to have the value $+(1/2)\,\hbar$ along the x axis at time t_1*and* to have the value $-(1/2)\,\hbar$ along the z axis at time t_2 can be computed as (see Eq. (10)):

$$\Pr\left(\Pi_1^{+x}, \Pi_2^{+z}\right) = \mathrm{Tr}\left(\Pi_2^{+z}\Pi_1^{+x}\rho_0\Pi_1^{+x}\Pi_2^{+z}\right) = |\langle +x|+z\rangle|^2 = {}^1/_2 \tag{15}$$

But since there is no restriction on the selection of time t_1, it can be chosen very close to time t_2. Therefore, the following limit can be computed:

$$\lim_{t_1 \to t_2} \Pr\left(\Pi_1^{+x}, \Pi_2^{+z}\right) = {}^1/_2 \tag{16}$$

In standard quantum mechanics, assigning well-defined values to different components of the spin is forbidden by the Heisenberg Principle, since operators representing different components of spin do not commute. As a consequence, the limit of Eq. (16) cannot be understood as the probability of the conjunction of the spin values $+(1/2)\,\hbar$ along the x axis and the value $-(1/2)\,\hbar$ along the z axis at the same time t_2. As Laura and Vanni (2009: 164) correctly notice, this means that the TQH leads to discontinuity in the assignment of properties of the system at different times.

3.4 Non-persistence of quasi-classicality

Another objection against the TQH has to do with how the theory explains our perception of a persisting nearly classical world. The TQH's explanation is based on selecting suitable quasi-classical realms in which projective decompositions at each time correspond to macroscopic properties. The appropriate projective decompositions will be coarse-grained decompositions such that: (i) the resulting projectors correspond to subspaces of large dimension and (ii) those projectors are maximally refined, i.e., if they are further fine-grained, they cease to satisfy the consistency condition. Defenders of the TQH consider that it is possible to find quasi-classical realms containing one history that approximates the results of classical mechanics and that is endowed with a probability very close to one.

However, Dowker and Kent (1996) argued that, in general, histories belonging to a quasi-classical realm that are quasi-classical up to a certain time, stop being quasi-classical in the future of that time. In other words, quasi-classicality in a consistent family of histories does not persist in consistent future extensions of the histories. This is particularly serious in a theory that intends to explain measurements as quasi-classical results with no need of collapse: as the authors claim, "*if we have a theory describing our own quasi-classical domain up to the present time, and if the theory tells us that the results of some quantum experiment we are about to perform are genuinely unpredictable, then we can find another theory which reproduces the description up to the present time, but in which the standard quasi-classical description of the experimental results cannot be made.*" (Dowker and Kent 1996: 1633).

A response to this objection is given by Griffiths (2013). He recognizes that there is no guarantee that quasi-classicality in a realm will persist in the future, but argues that the TQH does not need to supply such a guarantee: quasi-classicality is not a property of the world but a property of a description of the world. Therefore, the use of a quasi-classical realm up to one time and another quasi-classical realm after that time poses no conceptual difficulty.

3.5 Violation of probability axioms

The last objection considered in this section points to the fact that, in some versions of the TQH, such as the Decoherent Histories Interpretation, the consistency condition holds approximately and, as a consequence, the probability sum rules do not hold (Barrett 1999).

According to Gell-Mann and Hartle (1990; see also Hartle 1991), it is natural to consider those sets for which the probability sum rules are slightly violated on an equal footing with the sets of histories for which the consistency condition holds. They point out that, if the violation is sufficiently small, no experiment could detect the discrepancy, and that in any case one can remove the sum rule violation by an ad hoc but equally undetectable renormalization of the probabilities.

However, the authors' response misses the point. The difficulty is not that the theory provides approximated probabilities for quantum histories, which would not be a real problem if the formalism guarantees that discrepancies between approximated probabilities and exact probabilities are maintained small (see Okon and Sudarsky 2014a). The problem here is that the precise consistency condition is necessary to have *well-defined probabilities*: approximation in the consistency condition implies that, in general, the rule to compute probabilities does not satisfy the axioms of Kolmogorov. Therefore, if the consistency condition holds approximately, it is not clear whether the quantities defined by the theory by means of the Born Rule can genuinely be called probabilities. As Dowker and Kent (1996: 1577) stress, "*this seems a rather casual disruption of the mathematical structure of a fundamental theory.*"

A possible reply is to consider only theories of quantum histories that exactly satisfy the consistency condition, such Griffiths' Theory of Consistent Histories. Moreover, Dowker and Kent (1996) suggested that there is no need to consider approximately consistent sets in any fundamental formulation of the TQH, because an exactly consistent set can always be found in the neighborhood of any approximately consistent set of histories.

4 Recent objections

In two recent articles, Okon and Sudarsky (2014a, b) analyzed the conceptual implications of the TQH and, as the result, raised further objections. In this section we will briefly review them in the order in which they were introduced by the authors, and will add a further difficulty usually not considered in the literature.

4.1 Realm-dependent reality

The first objection introduced by Okon and Sudarsky (2014a) has to do with the fact that the TQH proposes a realm-dependent reality: what is real depends on the choice of the consistent family of histories. In fact, according to the TQH, there is no privileged realm. The Principle of Equality states that all realms are equally valid to describe a physical system, and the Principle of Liberty allows the physicist to choose any realm considered appropriate for practical reasons. This freedom implies that the actual history and the actual properties depend on the selected realm: reality becomes relative to the consistent family of histories picked out in each case.

Regarding this point, Griffiths (2013) argues that selecting a realm is, in a certain sense, like selecting a reference frame in special relativity: there is no preferred reference frame, all frames are equally valid to describe a system, but not all frames are equally useful when a particular problem is considered. However, as Okon and Sudarsky (2014a) stress, the analogy is not accurate enough. In the case of special relativity, different relative descriptions can be combined into a unified non-relative description that might be conceived as the description of an underlying, non-relative reality. This is not the case in the TQH, since two different realms, in general, cannot be combined into a single unified description. This means that the realm-relativity of the TQH is very different from other kinds of relativity in physics. Griffiths (2013) acknowledges the disanalogy, but he does not offer a way out of the problem.

4.2 Unclear definition of initial conditions

The second objection raised by Okon and Sudarsky (2014a) is related with how the Hamiltonian and the initial state, subsumed under the label 'initial conditions' by the authors, are determined in the framework of the TQH. In fact, according to the TQH, the initial state and the Hamiltonian of the system are the same for all realms, i.e., they do not depend on each consistent family of histories. However, the realm-dependence of reality makes difficult to understand how one could have access to these two realm-independent elements of the theory. In a sense, this objection is a refinement of the previous criticism, directed to the realm-dependence of reality.

Proponents of the Decoherent Histories Interpretation argue that the initial conditions must be determined by an external theory (Gell-Mann and Hartle 1990). However, as Okon and Sudarsky (2014a) emphasize, if what is real is relative to a realm, no observation can count as an evidence for this external theory. Moreover, since past is relative to the realm, whatever it takes to be the initial state in a given realm, it might not be so in a different one.

A possible answer to the objection would adduce that the TQH generalizes standard quantum mechanics and, therefore, if the initial state and the Hamiltonian can legitimately be selected in standard quantum theory, then the same should be possible in the TQH. But Okon and Sudarsky (2014a) correctly point out that the realm-dependence of reality is not an aspect of standard quantum mechanics; so, that feature cannot legitimately be extrapolated to the TQH.

4.3 Persistence of the measurement problem

As we said in the Introduction, standard quantum mechanics introduces two types of temporal evolution for the states of a quantum system: Schrödinger evolution and collapse. The former is a continuous and deterministic evolution that applies to ordinary physical processes. The latter applies to measurements: when a measurement on a system is performed, the state collapses to the eigenstate corresponding to the observed value of the measured observable. Unlike the Schrödinger evolution, collapse evolution is discontinuous and nondeterministic.

In order to apply this two-evolution theory, it is necessary to previously know which processes are ordinary physical processes and which ones are measurements. However, the concept of measurement is not well defined within the theory. The only way to identify measurement processes as different from ordinary processes is by appealing to elements external to the theory. This is one of the aspects of the measurement problem of standard quantum mechanics.

Proponents of the TQH claim that the theory has solved the measurement problem, since it is capable of making predictions without the addition of the collapse postulate. However, according to Okon and Sudarsky (2014a, see 2014b for a detailed argumentation), this is not the case: the TQH does not solve the measurement problem due to, at least, two different reasons. First, the theory does not specify how the relevant realm is to be chosen beyond stating that it has to be selected according to the questions one is trying to answer. But, as in the case of the identification of measurement processes, this criterion is also clearly external to the theory. Second, even after (somehow) fixing a realm, the theory does not offer a way to decide about the status of the different histories within it. If only one of the histories within the selected realm is actual, then the theory is descriptively incomplete, since it offers no conceptual reason to ascribe a privileged status to the actual history. If, on the contrary, all the histories within the selected realm are actual, it is not legitimate to interpret the numbers generated by the Born rule in the context of the realm as probabilities, since all the alternatives are realized.

4.4 Weak explanation of the experience of quasi-classicality

The last objection raised by Okon and Sudarsky (2014a) has to do with a particular element of Gell-Mann and Hartle's version of the TQH (Gell-Mann and Hartle 1990, 1994; Hartle 2007): *information gathering and utilizing system* (IGUS). The term 'IGUS' refers to a complex adaptive system that has evolved to exploit the predictability of a quasi-classical domain. Human beings constitute a particular type of IGUSes. The most relevant feature of these adaptive systems is that they are able to use, at least rudimentarily, a physical theory in order to make predictions about their environment.

In the Decoherent Histories Interpretation, the notion of IGUS is introduced to explain why, among all the possible realms allowed by the theory, we only experience quasi-classical realms. Gell-Mann and Hartle argue that IGUSes have evolved to make predictions in quasi-classical domains because this behavior is adaptive and because quasi-classical domains have enough regularity to make predictions with very simple means.

Okon and Sudarsky (2014a) correctly remark that the appeal to an evolutionary explanation does not fit in this case, since the Decoherent Histories Interpretation does not include any of the essential elements of an evolutionary theory: a varied initial population, an external environment, heredity and selection. In particular, there is no external environment in the context of which IGUSes evolve since, according to Gell-Mann and Hartle's proposal, the quasi-classical realm is a part of what should be adaptively selected.

4.5 Dependence of the consistency condition on the initial state

In the TQH, the consistency condition selects the consistent families of histories, that is, the sets of histories where probabilities can be correctly defined. As shown in Eq. (11), this condition depends on the initial state ρ_0 of the system; therefore, the sets of histories where probabilities can be correctly defined changes from initial state to initial state. This is completely different from what happens in standard quantum mechanics, in which the contexts where probabilities are well defined are all the possible distributive sublattices of the Hilbert space, which do not depend on the initial state.

Moreover, in the axiomatic approaches to standard quantum mechanics, once the observables of the system are identified, states are defined as functionals acting on the space of observables: the basic element of the theory is the space of possible properties, and states are secondary from a logical viewpoint. Since quantum histories play the role of properties at different times, it seems reasonable to require that the selected families of histories satisfy a condition independent from a logically secondary element of the theory such as the state of the system.

The fact that the consistency condition depends on the initial state of the system adds a new level of dependence of reality to that pointed out by Okon and Sudarsky (2014a). In fact, as discussed above, what is real depends on the particular consistent family of histories selected in the set of all consistent families defined by the consistency condition. But this set of consistent families depends, in turn, on the initial state of the system. Therefore, the initial state constrains reality, not merely by selecting the future evolution of the system among all the possible evolutions, as in classical mechanics, but in the sense that it takes part in the definition of the domain of possibility itself.

5 Rearranging the criticisms

Okon and Sudarsky (2014a, b) made a significant contribution to the discussion about the TQH, not only by clearly summarizing the main traditional objections directed to the theory, but also by formulating new ones. However, in this paper we are not interested in the difference between traditional and new objections. What is relevant to our purpose is to distinguish between those criticisms that point to theoretical shortcomings of the TQH and the criticisms directed to interpretive limitations of the theory.

Let us recall that a physical theory is constituted by a formal structure and a semantic interpretation that endows the formalism with physical content. In the semantic interpretation, in turn, two levels can be distinguished: the so-called 'minimal interpretation', that is, the physical content necessary to apply the formalism to the world, and what is commonly conceived as the "interpretation" of the theory, which supplies the content of the elements of the theory not accessible by experimentation in a direct way. As usual in the literature on quantum mechanics, we will consider the formalism with its minimal interpretation as the *theoretical* core of the physical theory, and will reserve the name '*interpretation*' for the remaining content.

It is quite clear that the TQH is presented as a complete theory, with its theoretical core and endowed with an interpretation that attempts to solve the traditional interpretive difficulties of quantum mechanics. Therefore, the criticisms raised against it can be classified into *theoretical* and *interpretive*, according to which of the two aspects is questioned. On this basis, let us reorganize the objections presented in the previous sections as follows:

Theoretical objections.

– *Contrary retrodictions*: it is possible to retrodict, with certainty, the occurrence of two contrary properties in the past from two different consistent families of histories.
– *Lack of predictive power*: predictions about the future properties of the system depend on the choice of one realm among all the realms, which are all equally legitimate for describing the physical system.
– *Discontinuity in property ascription*: in the limit of infinitely close times, contrary properties can be assigned to the system.
– *Violation of probability axioms*: if the consistency condition holds approximately, then the rule to compute probabilities does not satisfy the axioms of probability; therefore, it is not clear why the quantities defined by the theory should genuinely be called probabilities.
– *Dependence of the consistency condition on the initial state*: the sets of histories where probabilities can be correctly defined −the consistent families of histories− depend on the initial state of the system;

Interpretive objections.

– *Realm-dependent reality*: reality becomes relative to the consistent family of histories selected in each case.
– *Unclear definition of initial conditions*: since the initial conditions are realm-independent, it is not clear how one can have access to them.
– *Persistence of the measurement problem*: the measurement problem is not solved because (a) the theory does not specify how the relevant realm is to be chosen, and (b) the theory does not supply any reason to ascribe a privileged status to the actual history.

- *Non-persistence of quasi-classicality*: histories belonging to a consistent family that are quasi-classical up to a certain time may stop being quasi-classical in the future.
- *Weak explanation of the experience of quasi-classicality*: quasi-classicality cannot be explained in evolutionary terms because the theory does not include any of the essential elements of an acceptable evolutionary explanation.

Assuming that these objections are well founded, a comprehensive view based on quantum histories should successfully face all of them. The first step toward this goal is to devise a formalism of quantum histories that solves the theoretical difficulties that threaten the TQH. In the following section we will introduce a new formalism, the Formalism of Contextual Histories (FCH), and in Section 7 we will argue that it provides those required solutions.

6 The formalism of contextual histories

In the TQH, the quantum histories belonging to a realm must satisfy the consistency condition. Since this condition includes the initial state, the consistent families of histories depend on the initial conditions of the system. Therefore, a family of histories may be consistent for one initial state, but not for another. As stressed above, this situation is different from what happens in standard quantum mechanics, in which the contexts of "consistent" properties −sets of properties where the axioms of probability can be applied− are given by all the possible distributive sublattices generated by the Hilbert space, and they do not depend on the initial state of the system.

In order to avoid that dependence on the initial state, Roberto Laura and Leonardo Vanni developed an alternative approach to quantum histories, called *Formalism of Contextual Histories*. This formalism extends standard quantum mechanics with the purpose of assigning legitimate probabilities to quantum histories (Laura and Vanni 2008, 2009; Losada and Laura 2014a). It has proved to be useful to supply the time-dependent description of quantum decay processes (Losada and Laura 2013), to describe the logics of quantum measurements (Vanni and Laura 2013; Losada et al. 2016), and to explain the double slit experiment with and without measurement instruments (Losada et al. 2013).

As in the case of the TQH, in the framework of the FCH measurements do not have a privileged status. They are described as ordinary physical interactions between a system and a measuring apparatus. The dynamics of measurements is governed by the Schrödinger equation; therefore, no additional postulate to account for the collapse of the state vector is required.

The FCH is based on the notion of *time translation of properties*. As in the TQH, an additional condition on the sets of histories is necessary to have well-defined probabilities. In this case, the additional condition imposed on properties at different times is the commutation of the corresponding projectors *translated to a common time*. In this way, the traditional concept of *context* of standard quantum mechanics, defined for simultaneous properties, is extended to properties at different times. In the rest of this section we will briefly describe the main ideas supporting this formalism.

6.1 The logical structure of temporal equivalent classes

From a formal viewpoint, in this approach it is not necessary to define the tensor product of n copies of the Hilbert space: it is sufficient to consider the Hilbert space of the physical system under consideration. The states of the system are defined in the same way as in standard quantum mechanics, and their dynamics is governed by the time-evolution operator $U(t', t) = \exp[-iH(t' - t)/\hbar]$, where H is the system's Hamiltonian. Also as usual, the properties of the system are represented by subspaces of Hilbert space or by their respective projectors: a value-property p_a is represented by the projector Π_a.

The starting point of the FCH consists in considering properties at a certain time and their time evolution. In order to develop this idea, it is convenient to work in the Heisenberg framework, in which the projectors representing quantum properties evolve in time. The suggestion of using the Heisenberg representation for the conjunction of properties at different times was originally proposed by Leslie Ballentine (1998: Section 9.6), but he did not discuss the conditions for well-defined probabilities.

In the FCH, a value-property p_a at time t_a is represented by the pair (Π_a, t_a). In turn, the value-property p_b at time t_b, represented by the pair (Π_b, t_b), is defined as a *time translation* of the value-property p_a at time t_a, represented by the pair (Π_a, t_a), when

$$\Pi_b = U(t_b, t_a)\,\Pi_a\,U^{-1}(t_b, t_a) \tag{17}$$

It can be proved (Laura and Vanni 2009) that the relationship between (Π_a, t_a) and its time translations is transitive, reflexive and symmetric and, therefore, is an equivalence relation that defines a *temporal equivalence class*:

$$[\Pi_a, t_a] = \left\{(\Pi_b, t_b) | \Pi_b = U(t_b, t_a)\,\Pi_a\,U^{-1}(t_b, t_a)\right\} \tag{18}$$

It can be said that the projector Π_a represents the class at t_a as well as that each projector Π_b represents the same class at the corresponding time t_b. In physical terms, a temporal equivalence class includes all the properties-at-a-time that can be transformed into each other by the time evolution of the system. Since the Born rule assigns the same probability value to all the properties-at-a-time connected by time translation (see Vanni and Laura 2013: 2387), the equivalence class might also be conceived as a single property extended in time.

If the set of all temporal equivalence classes is denoted by $[E]$, an *order relation* between classes can be defined on it:

$$[\Pi_1, t_1] \leq [\Pi_2, t_2] \quad \text{if and only if} \quad V_{1,0} \subseteq V_{2,0} \tag{19}$$

where $V_{1,0}$ and $V_{2,0}$ are the subspaces associated to $\Pi_{1,0}$ and $\Pi_{2,0}$, respectively, and

$$\Pi_{1,0} = U(t_0, t_1)\,\Pi_1\,U^{-1}(t_0, t_1) \qquad \Pi_{2,0} = U(t_0, t_2)\,\Pi_2\,U^{-1}(t_0, t_2) \tag{20}$$

In turn, *complementation* can be defined on $[E]$ as follows:

$$\overline{[\Pi, t]} = [\Pi^{\perp}, t] \tag{21}$$

where $\perp$ is the orthogonal complement of Π, i.e. $\Pi^{\perp}=I-\Pi$, where I is the identity operator of the Hilbert space.

On the basis of the above definitions of the order relation and of the complement operation, the structure $\langle [E], \leq, ^{-} \rangle$ turns out to be an *orthocomplemented lattice*. Therefore, the basic logical operations between temporal equivalence classes can be defined. In particular, disjunction and conjunction between temporal equivalence classes are given by the disjunction and the conjunction of the projectors representative of those classes but translated to a common time t_0:

- Negation $\neg[\Pi_1, t] = \overline{[\Pi, t]} = \left[\Pi_1^{\perp}, t\right] = [I - \Pi_1, t]$ (22)

- Disjunction $[\Pi_1, t_1] \vee [\Pi_2, t_2] = \left[\Pi_{1,0} \vee \Pi_{2,0}, t_0\right] = \left[I - \lim_{n\to\infty} \left\{ \left(I - \Pi_{1,0}\right)\left(I - \Pi_{2,0}\right) \right\}^n, t_0\right]$ (23)

- Conjuction $[\Pi_1, t_1] \wedge [\Pi_2, t_2] = \left[\Pi_{1,0} \wedge \Pi_{2,0}, t_0\right] = \left[\lim_{n\to\infty} \left(\Pi_{1,0}\Pi_{2,0}\right)^n, t_0\right]$ (24)

where t_0 is an arbitrary time, and $\Pi_{1,\ 0}$, $\Pi_{2,\ 0}$ are defined as above, in Eq. (20).

6.2 Generalized contexts

In order to correctly define probabilities, an orthocomplemented lattice is not sufficient: a *Boolean* lattice is necessary, i.e., a *distributive* orthocomplemented lattice.

In standard quantum mechanics, the word 'context' refers to a set of projectors representing properties that can be simultaneously applied for the description of a quantum system at a given time, and that can be endowed with a Boolean structure with well-defined probabilities. The FCH supplies a prescription to obtain, from the orthocomplemented lattice of temporal equivalent classes, a Boolean sublattice which will be called *generalized context*.

For this purpose, let us select an arbitrary time t_i and a projective decomposition of the identity of the Hilbert space, i.e., a set of projectors $\Pi_i^{k_i}$, which satisfy

$$\Pi_i^{k_i}\Pi_i^{k'_i} = \delta_{k_i k'_i}\Pi_i^{k_i} \qquad \sum_{k_i}\Pi_i^{k_i} = I \quad \text{with } k_i, k_i^{'} \in \sigma_i \tag{25}$$

Then, we define the context of properties C_i generated by the projectors $\Pi_i^{k_i}$, i.e., the set of arbitrary sums of $\Pi_i^{k_i}$:

$$C_i = \left\{ \Pi_i : \Pi_i = \sum_{k_i} \alpha_{k_i}\Pi_i^{k_i}, \alpha_{k_i} = 0, 1 \right\}. \tag{26}$$

On this basis, the *generalized context* $[E]_{C_i}$ can be defined as:

$$[E]_{C_i} = \{[\Pi_i, t_i] \in [E] : \Pi_i \in C_i\} \tag{27}$$

As a result, the structure $\langle [E]_{C_i}, \leq, ^{-} \rangle$ is a generalized context, that is, a distributive sublattice of the orthocomplemented lattice $\langle [E], \leq, ^{-} \rangle$.

Once a generalized context is so defined, the probabilities of its temporal equivalence classes can be computed by means of a natural generalization of the Born rule. Given the classes $[\Pi^k, t]$ belonging to the generalized context $[E]_C$, their probability is given by

$$\Pr(\Pi_i, t_i) = \mathrm{Tr}\left(\rho_{t_i}\Pi_i\right) \tag{28}$$

where ρ_{t_i} is the state of the system at time t_i. These probabilities are well defined and satisfy the axioms of Kolmogorov.

6.3 The logical structure of contextual histories

The FCH, based on the idea of properties-at-a-time and their time translation, has the capability of representing histories that involve properties at different times.

Let us consider the contexts of properties C_1 at time t_1 and C_2 at time t_2, generated by the projectors $\Pi_1^{k_1}$ at t_1 and $\Pi_2^{k_2}$ at t_2 respectively, such that:

$$\Pi_1^{k_1}\Pi_1^{k_1'} = \delta_{k_1k_1'}\Pi_1^{k_1} \text{ and } \textstyle\sum_{k_1}\Pi_1^{k_1} = I \quad \text{with } k_1, k_1' \in \sigma_1 \tag{29}$$

$$\Pi_2^{k_2}\Pi_2^{k_2'} = \delta_{k_2k_2'}\Pi_2^{k_2} \text{ and } \textstyle\sum_{k_2}\Pi_2^{k_2} = I \quad \text{with } k_2, k_2' \in \sigma_2 \tag{30}$$

Let us now consider the *two-time history* $h^{k_1k_2}$, denoted by the expression 'the property represented by $\Pi_1^{k_1}$ at t_1*and* the property represented by $\Pi_2^{k_2}$ at t_2'. According to the definition introduced above (see Eq. (24)), the conjunction between time equivalence classes with representative elements $\Pi_1^{k_1}$ at t_1 and $\Pi_2^{k_2}$ at t_2 is the conjunction of the classes represented by the projectors $\Pi_{1,0}^{k_1}$ and $\Pi_{2,0}^{k_2}$, time translations of $\Pi_1^{k_1}$ and $\Pi_2^{k_2}$ to a common time t_0.

In standard quantum mechanics, the conjunction of simultaneous properties represented by non-commuting operators is not well defined: conjunctions make sense only in the commuting case. When the task is to deal with properties at different times, the commuting requirement must be generalized. A natural way to do it is to accept the descriptions involving the projectors $\Pi_1^{k_1}$ at t_1 *and* $\Pi_2^{k_2}$ at t_2 only in the case in which the projectors commute when translated to an arbitrary common time t_0:

$$\left[\Pi_{1,0}^{k_1}, \Pi_{2,0}^{k_2}\right] = 0 \tag{31}$$

When this *compatibility condition* holds, the two-time history $h^{k_1k_2}$ turns out to be (see the definition of conjunction in Eq. (24)):

$$h^{k_1k_2} = \left[\Pi_1^{k_1}, t_1\right] \wedge \left[\Pi_2^{k_2}, t_2\right] = \left[\Pi_{1,0}^{k_1}\Pi_{2,0}^{k_2}, t_0\right] \tag{32}$$

On the basis of Eq. (31), it is easy to see that the conjunction of properties at different times, represented by $\Pi_1^{k_1}$ at t_1 and $\Pi_2^{k_2}$ at t_2, is equivalent to a single property at the common time t_0, represented by the projector $\Pi_0^{k_1k_2}$ such that:

$$\Pi_0^{k_1k_2} = \Pi_{1,0}^{k_1}\Pi_{2,0}^{k_2} \quad \Rightarrow \quad h^{k_1k_2} = \left[\Pi_0^{k_1k_2}, t_0\right] \tag{33}$$

For all the two-time histories $h^{k_1k_2}$ of the form given by Eq. (33), it can be proved that

$$\Pi_0^{k_1k_2}\Pi_0^{k_1'k_2'} = \delta_{k_1k_1'}\delta_{k_2k_2'}\Pi_0^{k_1k_2} \quad and \quad \Sigma_{k_1k_2}\Pi_0^{k_1k_2} = I \quad with \ k_1, k_1' \in \sigma_1 \ and \ k_2, k_2' \in \sigma_2 \tag{34}$$

This means that the histories $h^{k_1k_2}$, represented by the complete and exclusive set of projectors $\Pi_0^{k_1k_2}$, can be viewed as properties that generate a generalized context at t_0. It is in this sense that the histories that satisfy the compatibility condition can be called *contextual histories* belonging to a *context of histories*.

In turn, since a generalized context is a distributive sublattice of an orthocomplemented lattice, if the state of the system at t_0 is ρ_0, the probabilities of the histories $h^{k_1k_2}$ belonging to the same context of histories can be computed by the generalization of the Born rule introduced above (see Eq. (28)):

$$\Pr\left(h^{k_1k_2}\right) = \Pr\left(\left[\Pi_0^{k_1k_2}, t_0\right]\right) = \operatorname{Tr}\left(\rho_0\Pi_0^{k_1k_2}\right) \tag{35}$$

In the standard formalism of quantum mechanics, the order relation $p_1 \leq p_2$ between two properties is represented by the inclusion of the Hilbert subspaces corresponding to the respective projectors Π^{p_1} and Π^{p_2}. If the two properties, considered at the same time t, belong to the same context, the implication corresponds to the conditional probability

$$\Pr\left({}^{\Pi^{p_2}}\big/_{\Pi^{p_1}}\right) = \frac{\Pr(\Pi^{p_2} \wedge \Pi^{p_1})}{\Pr(\Pi^{p_1})} = \frac{\operatorname{Tr}(\rho_t\Pi^{p_2}\Pi^{p_1})}{\operatorname{Tr}(\rho_t\Pi^{p_1})} = \frac{\operatorname{Tr}(\rho_t\Pi^{p_1})}{\operatorname{Tr}(\rho_t\Pi^{p_1})} = 1 \tag{36}$$

where ρ_t is the state of the system at time t. Equation (36) can be interpreted as saying that if property p_1 is the case at time t, then property p_1 is also the case at the same time. The FCH endows the implication between properties at different times with a clearly analogous meaning. The properties p_1 at t_1 and p_2 at t_2 are represented by the temporal equivalence classes $\left[\Pi_1^{p_1}, t_1\right]$ and $\left[\Pi_2^{p_2}, t_2\right]$, respectively, and the order relation between them is represented by the relation $\left[\Pi_1^{p_1}, t_1\right] \leq \left[\Pi_2^{p_2}, t_2\right]$ between the corresponding classes (see Eqs. (19) and (20)). If the two classes belong to the same context of histories, then:

$$\begin{aligned}\Pr\left({}^{\left[\Pi_2^{p_2}, t_2\right]}\Big/_{\left[\Pi_1^{p_1}, t_1\right]}\right) &= \frac{\Pr\left(\left[\Pi_2^{p_2}, t_2\right] \wedge \left[\Pi_1^{p_1}, t_1\right]\right)}{\Pr\left(\left[\Pi_1^{p_1}, t_1\right]\right)} = \frac{\Pr\left(\left[\Pi_{2,0}^{p_2}, t_0\right] \wedge \left[\Pi_{1,0}^{p_1}, t_0\right]\right)}{\Pr\left(\left[\Pi_{1,0}^{p_1}, t_0\right]\right)} = \\ &= \frac{\Pr\left(\left[\Pi_{2,0}^{p_2}\,\Pi_{1,0}^{p_1}, t_0\right]\right)}{\Pr\left(\left[\Pi_{1,0}^{p_1}, t_0\right]\right)} = \frac{\Pr\left(\left[\Pi_{1,0}^{p_1}, t_0\right]\right)}{\Pr\left(\left[\Pi_{1,0}^{p_1}, t_0\right]\right)} = 1\end{aligned} \tag{37}$$

where $\Pi^{p_1}_{1,0}$ and $\Pi^{p_2}_{2,0}$ are defined analogously to Eq. (20). Equation (37) can be interpreted as saying that if property p_1 is the case at time t_1, then property p_1 is also the case at time t_2.

What was explained in the previous paragraphs for two-time histories can be generalized to many-time histories. Given n times $t_1 < \ldots < t_n$, let us consider a context of properties C_i at each time t_i, generated by the properties represented by the projectors $\Pi^{k_i}_i$, which satisfy

$$\Pi^{k_i}_i \Pi^{k'_i}_i = \delta_{k_i k'_i} \Pi^{k_i}_i \quad \text{and} \quad \sum_{k_i} \Pi^{k_i}_i = I \quad \text{with } k_i, k'_i \in \sigma_i \tag{38}$$

In order to define contextual histories involving properties coming from any of these contexts, it is necessary that those histories belong to the same context of histories. In other words, it is necessary that the compatibility condition holds, that is, the projectors representing the properties at issue must commute when translated to an arbitrary common time t_0:

$$\left[\Pi^{k_i}_{i,0}, \Pi^{k_j}_{j,0}\right] = 0 \quad k_i \in \sigma_i, k_j \in \sigma_j, 1 \leq i, j \leq n \tag{39}$$

with

$$\Pi^{k_i}_{i,0} = U(t_0, t_i)\, \Pi^{k_i}_i \, U^{-1}(t_0, t_i) \qquad \Pi^{k_j}_{j,0} = U\left(t_0, t_j\right) \Pi^{k_j}_j \, U^{-1}\left(t_0, t_j\right) \tag{40}$$

The conjunction of Eqs. (39) and (40) can be considered the *general compatibility condition* of the FCH.

7 Comparison and relationships between the two approaches

Although both the TQH and the FCH can represent quantum histories and can assign probabilities to them, they are different proposals and, as a consequence, the analysis of their differences and their relationships is a task that deserves to be carried out.

7.1 Comparing the TQH and the FCH

The first difference between the two approaches lies in the condition used to restrict the set of histories. The *compatibility condition* of the FCH is a natural generalization of the commutation condition for compatible properties belonging to a same context at a fixed time: according to that condition, compatible histories belong to a common generalized context, and probabilities are defined on them by a generalization of the Born rule. The *consistency condition* of the TQH, on the contrary, has no immediate conceptual counterpart in standard quantum mechanics.

The above feature is related with the fact that the compatibility condition is necessary in the framework of the FCH from a logical point of view, that is, to define a Boolean lattice of histories. The consistency condition of the TQH, by contrast, has no direct connection with the logical structure of quantum histories. All families of histories, consistent or not, are Boolean lattices; therefore, an additional condition is not required to obtain Booleanity. Nevertheless, the consistency condition is necessary to have a well-defined probability.

Another relevant difference concerns the role played by the *initial state* in both approaches. In the TQH, the consistency condition depends on the initial state of the system; therefore, the consistent families of histories also depend on the initial state. On the contrary, in the FCH the compatibility condition is independent of the initial state of the system. Therefore, the logical structure of the sets of contextual histories is a distributive orthocomplemented lattice independent of the initial state. This stands in clear analogy to standard quantum mechanics, in which the contexts of properties are all the possible distributive sublattices of the orthocomplemented lattice corresponding to the Hilbert space of the system and do not depend on the initial state of the system.

Moreover, as explained above, the FCH endows the *order relation* between properties at different times with a probabilistic meaning, which is manifestly analogous to that of the order relation between properties-at-a-same-time in standard quantum mechanics. This allows the new formalism to conceive the *implication* between properties at different times as the relation linking them when the corresponding *conditional probability* is one, analogously to the implication between properties-at-a-same-time in standard quantum mechanics. In the TQH, by contrast, there is not such a strong connection between order relation, conditional probabilities and implication, unless the consistency condition be imposed for all states (we will come back to this point below).

Although the above differences are sufficient to clearly show the divergence between the two proposals, perhaps the main distinction between the TQH and the FCH lies in their general aim. As explicitly pointed out in the Introduction, the TQH in its different versions intends to be a generalization of quantum mechanics with a precise interpretive goal: solving the measurement problem of standard quantum mechanics, derived from the collapse hypothesis. The purpose of the FCH is more modest: it intends to be not a theory, but a formalism devoted to "*deal in a consistent way with expressions involving different properties of the system at different times. For example, it is necessary to relate a property of a microscopic system at a given time, previous to a measurement, with a property of an instrument when the measurement is finished*." (Laura and Vanni 2009: 160). In other words, the FCH has no interpretive aspirations and, as a consequence, tries to remain as close to standard quantum mechanics as possible. For this reason, it is usually introduced as a *formalism*, and not as a theory.

7.2 Relating the TQH and the FCH

Despite the formal and conceptual differences between the TQH and the FCH, at least two important relationships can be established between them.

First, there is a significant relation between the conditions proposed by the two approaches: it can be proved that *the compatibility condition of the FCH implies the consistency condition of the TQH*, but in general the converse conditional is not true (Laura and Vanni 2009; Losada and Laura 2014a, b). This means that the compatibility condition is stronger than the consistency condition: a context of histories obtained in the FCH is also a consistent family of histories in the TQH, with the same probabilities defined on them. In the light of the criticisms directed to the TQH, the strength of the

FCH may be considered an advantage. For instance, after reporting the possible responses to some objections, Kent concludes: "*If we reject these defences, we seem to be left with the conclusion that the contrary inferences implied by the consistent histories formalism make it hard to take it seriously as a fundamental theory in its present form. This means that further constraints beyond consistency are needed in order to construct a natural generalisation of the Copenhagen interpretation to closed systems*." (Kent 1997: 2876).

The second relation, which in a certain sense explains conceptually the previous one, is that the consistency condition of the TQH, when imposed to all possible initial states, is equivalent to the compatibility condition of the FCH (Losada and Laura 2014a, b). This means that the two following situations are satisfied. First, if a quantum history satisfies the consistency condition for all initial states, then it satisfies the compatibility condition, and vice versa: the consistent families of histories in the TQH are the same sets as the contexts of histories in the FCH. Second, both the FCH and the TQH with the consistency condition valid for all states assign the same probability values to the selected quantum histories. This result shows that, although the TQH and the FCH are significantly different approaches, from a formal viewpoint the FCH is a restriction of the TQH. Such a formal relation will be relevant in the explanation of how the theoretical objections raised against the TQH can be managed by the FCH.

8 Reconsidering the criticisms from the new perspective

The FCH was developed in several of papers and has proved to be useful to describe different physical processes, as the quantum measurement, the decay process and the double-slit experiment. Moreover, it was compared with the TQH and the relations between the two approaches were formally proved. Nevertheless, the comparison was never considered with the purpose of facing the objections raised against the TQH. This is the task to be undertaken in this section.

8.1 Contrary retrodictions

As explained above, there is no possibility of contrary inferences in standard quantum mechanics: no quantum state leads to assign probability 1 to the occurrence of contrary properties conditioned to the occurrence of a single property. On the contrary, according to the TQH, from a given initial state two contrary properties, from different realms, can be retrodicted.

Being a natural generalization of standard quantum mechanics, it can be expected that the FCH faces this criticism successfully. In fact, the compatibility condition imposes the commutation of the projectors corresponding to the time translation of the properties to a single common time. As a consequence, a generalized context of quantum histories has the logical structure of a distributive orthocomplemented lattice of subspaces of a Hilbert space (the same logical structure of the properties-at-a-time in standard quantum mechanics). Because of this logical structure, in the FCH it is not possible to infer contrary properties from different generalized contexts (see Losada and Laura 2014b).

8.2 Lack of predictive power

This criticism points to the fact that, according to the TQH, it is possible to predict contrary properties with certainty in different consistent families of histories: predictions depend on the choice of the realm, in a set where all the realms are equally legitimate.

The FCH can face this problem on the basis of the same feature that solves the previous objection: there is no possibility of contrary inferences from different generalized contexts, neither toward the past nor toward the future (Losada and Laura 2014b). Therefore, in the framework of the FCH the objection is answered with no need to remain confined to a single realm, as in the case of the TQH.

8.3 Discontinuity in property ascription

This objection points to the fact that, according to the TQH, incompatible properties can be assigned to a system in the limit of infinitely close times. This result does not fit comfortably in a theory that admits the Heisenberg Principle, since implies a sort of time discontinuity in the adscription of properties to the system.

Laura and Vanni (2009) prove that the FCH solves the difficulty since rules out that kind of discontinuity. In the particular case of the assignment of spin values in orthogonal space directions, as introduced in Subsection 3.3, the authors show that, according to the compatibility condition, the z-components of the spin at time t_1 are the only choice compatible with the z-components of the spin at time t_2, and this holds for any initial state ρ_0.

8.4 Violation of probability axioms

In the Decoherent Histories version of the TQH, the consistency condition holds approximately since, according Gell-Mann and Hartle, if the violation is sufficiently small, no experiment could detect the discrepancy. But, as it was correctly stressed, the difficulty here is not that the probabilities are approximate, but that they are not well defined if the consistency condition is not satisfied with complete precision.

In the FCH, the compatibility condition, which selects the set where probabilities can be correctly defined, is not a conceptual addition to standard quantum mechanics. Since the compatibility condition requires commutation, saying that the condition is satisfied only approximately makes no sense. It would be like admitting, in standard quantum mechanics, that two observables A and B commute when AB is approximately equal to BA: this would break down the mathematical structure of the theory, as noticed by Dowker and Kent (1996) when discussing the Decoherent Histories Interpretation.

In other words, the FCH faces this criticism from the same perspective as that of Griffits' Theory of Consistent Histories, but in the FCH case on the basis of a strong analogy to standard quantum mechanics.

8.5 Dependence of the consistency condition on the initial state

Let us recall that, in the TQH, the consistency condition depends on the initial state of the system. In this sense, the TQH is completely different from standard quantum mechanics, in which the contexts where probabilities are well defined do not depend on the initial state. This sounds particularly odd in the axiomatic approaches to standard quantum mechanics, where the states are defined as functionals acting on the space of the properties.

The FCH faces this objection successfully for the simple reason that the compatibility condition is not a function of the initial state of the system. As a consequence, the contexts of histories on which probabilities can be defined are independent of the initial state. Moreover, as explained above, when the requirement that the consistency condition is independent of the initial state is added to the condition itself, the formalism of the TQH collapses onto the FCH.

8.6 Interpretive objections

The interpretive criticisms reported above, mainly proposed by Okon and Sudarsky (2014a, b), make sense independently of considering them as acceptable or not: the TQH is proposed as an interpretation of quantum mechanics that intends to offer a solution to the main interpretive problems of the theory. In fact, an interpretation should take a position about how reality is as described by the theory, should propose a solution to the quantum measurement problem, should give an adequate account of the classical limit of quantum mechanics, etc.

On the contrary, the FCH does not intend to be a theory endowed with an interpretive content. It is introduced with the only purpose of supplying a formal complement to standard quantum mechanics that allows the theory to relate propositions about properties at different times. According to the authors, the aim is to propose "*a formalism suitable to deal with descriptions and reasonings about physical systems involving quantum properties at different times*." (Laura and Vanni 2009: 172). In other words, the FCH intends to define the *universes of discourse* in whose framework a meaningful talk is possible: "*Each family of consistent histories generates a possible universe of discourse about a quantum system*." (Laura and Vanni 2009: 162; see also Losada et al. 2013: 1).

As a consequence, the interpretive criticisms raised against the TQH on the basis of its interpretive aspirations are beyond the scope of the FCH: it cannot be said that the FCH is superior to the TQH as an interpretation of quantum mechanics. Who wants to discuss interpretive issues from the perspective of the FCH needs to associate it with some specific interpretation of quantum mechanics. Only in that case the evaluation of the behavior of formalism plus interpretation will be possible.

9 Some applications of the formalism of contextual histories

As pointed out in Subsection 7.2, although the TQH and the FCH are significantly different approaches, from a formal viewpoint the FCH is a restriction of the TQH, and this fact is essential in the way in which the FCH can manage the objections raised against the TQH. This fact opens up the question about whether the FCH may account for the physical situations that are correctly explained in the context of the TQH. In the following subsection we will introduce the application of the FCH to some examples that are traditional in the literature on quantum mechanics.

9.1 Non ideal quantum measurement

Let us consider the generic case of a non-ideal measurement, because the ideal measurement is a particular case of that one.

A measurement is an interaction between a system S, represented by a Hilbert space $\mathcal{H}_A$, and a measuring apparatus M, represented by a Hilbert Space $\mathcal{H}_M$. The interaction, during the time interval (t_0, t_1), intends to establish a correlation between the eigenstates $|a_i\rangle$ of the observable A of S and the eigenstates $|p_i\rangle$ of the pointer observable P of M. If the state of the system S at time t_0 is $|\psi_0\rangle = \sum_i c_i |a_i\rangle$, the interaction is represented in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_M$ of the composite system $S+M$ by a unitary transformation $U_{SM}(t_1, t_0)$ such that:

$$|\Psi_0\rangle = \sum_i c_i |a_i\rangle \otimes |p_0\rangle \xrightarrow{U_{SM}(t_1,t_0)} |\Psi_1\rangle = \sum_i c_i \left|a_i^{'}\right\rangle \otimes |p_i\rangle \tag{41}$$

where $|\Psi_0\rangle$ and $|\Psi_1\rangle$ are the states of the composite system at t_0, and t_1 respectively, $|a_i\rangle$ is the eigenstate with eigenvalue a_i of the observable A, $|p_0\rangle$ is the initial reference eigenstate of the pointer P, $|p_i\rangle$ is the eigenstate with eigenvalue p_i of P, and $\left|a_i^{'}\right\rangle$ is the result of the non-ideality of the measurement.

The FCH provides the description of the process involving the possible values a_i of the observable A of the system S at time t_0 and the possible values p_i of the pointer P of the apparatus M at time t_1. These properties are represented by the projectors:

$$\Pi_{a_i} = |a_i\rangle\langle a_i| \otimes |p_0\rangle\langle p_0| \qquad \Pi_{p_i} = I_S \otimes |p_i\rangle\langle p_i| \tag{42}$$

These projectors satisfy the compatibility condition when translated to the common time t_0 (see Eq. (31)):

$$\left[\Pi_{a_i}, U_{SM}(t_0, t_1)\Pi_{p_j} U_{SM}^{-1}(t_0, t_1)\right] = 0 \tag{43}$$

Therefore, according to the FCH, the following conditional probability can be computed:

$$\Pr\left({}^{(a_i,t_0)}\Big/_{(p_j,t_1)} \right) = \frac{\Pr\left((a_i, t_0) \wedge \left(p_j, t_1\right)\right)}{\Pr\left(\left(p_j, t_1\right)\right)} = \frac{\langle \Psi_0 | \Pi_{a_i} U_{SM}(t_0, t_1) \Pi_{p_j} U_{SM}^{-1}(t_0, t_1) | \Psi_0 \rangle}{\langle \Psi_0 | U_{SM}(t_0, t_1) \Pi_{p_j} U_{SM}^{-1}(t_0, t_1) | \Psi_0 \rangle} = \delta_{ij} \tag{44}$$

This result can be interpreted by saying that if the pointer has the value p_j after the measurement, then the system S had the property $A = a_i$ before the measurement (for a detailed presentation, see Losada and Laura 2013).

9.2 Consecutive measurements

Let us now suppose that, after the first measurement of the observable A of the system S by means of the apparatus M at time t_1, a second measurement is performed: at $t_2 > t_1$, the observable B of the system S is measured by means of the apparatus N, which has a pointer Q with eigenstates $|q_j\rangle$. In the general case, the observables A and B do not commute; therefore, their respective eigenstates are related to each other by a change of basis:

$$|a_i\rangle = \sum_j d_{ij} |b_j\rangle \tag{45}$$

For simplicity we will consider only ideal measurements (non-ideality only introduces complexity in computation). In the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_M \otimes \mathcal{H}_N$ of the composite system $S+M+N$, the two interactions are represented by unitary transformations $U_{SM}(t_1, t_0)$ and $U_{SN}(t_2, t_1)$ such that:

$$|\Psi_0\rangle = \sum_i c_i |a_i\rangle \otimes |p_0\rangle \otimes |q_0\rangle \xrightarrow{U_{SM}(t_1,t_0)} |\Psi_1\rangle = \sum_i c_i |a_i\rangle \otimes |p_i\rangle \otimes |q_0\rangle \tag{46}$$

$$\begin{aligned} |\Psi_1\rangle = \sum_i c_i |a_i\rangle \otimes |p_i\rangle \otimes |q_0\rangle = \sum_i \sum_j c_i d_{ij} |b_j\rangle \otimes |p_i\rangle \otimes |q_0\rangle \xrightarrow{U_{SN}(t_2,t_1)} \\ \xrightarrow{U_{SN}(t_2,t_1)} |\Psi_2\rangle = \sum_i \sum_j c_i d_{ij} |b_j\rangle \otimes |p_i\rangle \otimes |q_j\rangle \end{aligned} \tag{47}$$

where $|q_0\rangle$ is the initial reference eigenstate of the pointer Q of the second measuring apparatus N.

In the composite system $S+M+N$, the possible values p_i of the pointer P of the apparatus M at time t_1 and the possible values q_i of the pointer Q of the apparatus N at time t_2 are represented by the projectors:

$$\Pi_{p_i} = I_S \otimes |p_i\rangle\langle p_i| \otimes I_N \qquad \Pi_{q_j} = I_S \otimes I_M \otimes |q_j\rangle\langle q_j| \tag{48}$$

Also in this case, these projectors satisfy the compatibility condition when translated to the common time t_1:

$$\left[\Pi_{p_i}, U_{SN}(t_1, t_2)\Pi_{q_j} U_{SN}^{-1}(t_1, t_2)\right] = 0 \tag{49}$$

Therefore, according to the FCH, the conditional probability that the second apparatus N measures q_i at t_2 given that the first apparatus measured p_j at t_1 is given by:

$$\begin{aligned} \Pr\left({}^{(q_j,t_2)}\big/_{(p_i,t_1)}\right) &= \frac{\Pr\left((q_j,t_2)\wedge(p_i,t_1)\right)}{\Pr((p_i,t_1))} = \\ &= \frac{\langle\Psi_0|U_{SM}(t_0,t_1)U_{SN}(t_1,t_2)\Pi_{q_j}U_{SN}^{-1}(t_1,t_2)U_{SM}^{-1}(t_0,t_1)U_{SM}(t_0,t_1)\Pi_{p_i}U_{SM}^{-1}(t_0,t_1)|\Psi_0\rangle}{\langle\Psi_0|U_{SM}(t_0,t_1)\Pi_{p_i}U_{SM}^{-1}(t_0,t_1)|\Psi_0\rangle} = \\ &= \left|\langle b_j|a_i\rangle\right|^2 \end{aligned} \tag{50}$$

It can be proved that this probability has the same value as that obtained under the assumption that the first measurement collapses the state $|\Psi_1\rangle = \sum_i c_i |a_i\rangle \otimes |p_i\rangle \otimes |q_0\rangle$ to one of its components (see Losada et al. 2016).

9.3 Further physical situations

As it is well known, in the double-slit experiment it is impossible to know which slit the particle crossed when the interference pattern is observed. In the context of the TQH, Omnès (1988b) proved that, with no measuring apparatus to detect the particle crossing one of the slits, there is no family of consistent histories that includes a history of the particle with the property of crossing a definite slit and arriving to a definite position of the screen. When the FCH is applied to this physical situation, the correct results are also obtained (see details in Losada et al. 2013). On the one hand, without measuring devices before the detection of the particle in the screen, the formalism proves the impossibility of describing the trajectory of the particle. On the other hand, with an apparatus recording which slit the particle crossed and another instrument recording the particle in different positions of the screen, there is a generalized context in which the records of both instruments can be described. In particular, the conditional probability of detecting the particle in a definite position of the screen given that it crossed one particular slit at an earlier time can be computed, and the non-interference pattern is obtained. The correct result so obtained is not surprising when this experiment is conceived as a particular case of consecutive measurements.

In the usual presentations of quantum decay processes (see, e.g. Davydov 1965), a quasi-stationary state of a quantum system with continuous spectrum is defined when the mean value of the energy has small dispersion. According to the orthodox collapse interpretation, the *survival probability* of the initial quasi-stationary state is defined as the probability for the system to be still found in the same quasi-stationary state in a future time. The time for which the survival probability is e^{-1} is called *lifetime*. For the description of a decay process that emphasizes the relevant properties of the system and its probabilities, it is possible to define the properties of *decay* and *non-decay,* mathematically represented by appropriate projectors. On this basis, the survival probability can be identified with the probability of the non-decay property at time t_2, conditional to the same property at a previous time t_1, for an arbitrary state prepared at time $t_0 < t_1 < t_2$. Since the conjunction of properties at different times is involved, it seems natural to appeal to a formalism of quantum histories. In fact, Omnès proposed to describe the decay process with the theory of consistent histories; however, he found the problem that the consistency condition is not satisfied (Omnès 1994). The application of the FCH to the decay process supplies a framework that not only explains Omnès' negative result, but also produces a positive account of the decaying situation. In fact, if the decaying system is considered as isolated, there is no generalized context that includes the decay and the non-decay properties at two different times. However, if the decaying is measured at two different times, it is possible to construct a generalized context that includes the pointer of the first measuring apparatus at a given time and the pointer of the second apparatus at a later time. Therefore, the survival probability can be interpreted as the conditional probability of measuring the non-decay property at a given time, given that the same non-decay property was measured at an earlier time (Losada et al. 2013). It is relevant to stress that the possibility of computing the conditional probability of measuring non-decay at a time given the measurement of non-decay at a previous time does not imply that the microscopic properties of decay and non-decay are "preexisting properties", that is, properties that are definite-valued independently of measurement.

10 Conclusions

In the present paper we considered the criticisms raised against the TQH, and we have distinguished between theoretical objections and interpretive objections. This task led us to analyze the FCH as an approach that is not a target of those criticisms.

The FCH intends to supply a formal complement to standard quantum mechanics that clearly establishes the conditions for a meaningful talk about properties at different times and about their probabilities of occurrence. For this purpose, the FCH replaces the consistency condition of the TQH with a compatibility condition that generalizes the commutation condition for compatible properties at a fixed time. In standard quantum theory, properties are compatible when represented by commuting projectors. In the FCH, histories are compatible if they are constituted by properties that are compatible when translated to a common time. The histories that satisfy the compatibility condition are called 'contextual histories' and belong to a context of histories.

We argued that the FCH supplies adequate solutions to the theoretical objections raised against the TQH. However, it is not a theory with interpretive import: the FCH only intends to be a formalism that establishes the boundaries of the meaningful discourse about a quantum system. In order to endow this formalism with interpretive content, it is necessary to associate it with a specific interpretation able to be consistently combined with the compatibility condition. For instance, in the field of the realist perspectives, if the interpretation selects a given context as the set of definite-valued properties at a certain time, and the FCH temporally translates the context to another time, then such an interpretation should select precisely that translated context as the set of definite-valued properties at the second time. But the task of finding an interpretation that satisfies this requirement is still a work in progress.

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