Research Article

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One-phase Stefan problem with temperature-dependent thermal conductivity and a boundary condition of Robin type

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Abstract: We study a one-phase Stefan problem for a semi-infinite material with temperature-dependent thermal conductivity with a boundary condition of Robin type at the fixed face $x = 0$. We obtain sufficient conditions for data in order to have a parametric representation of the solution of similarity type for $t \geq t_0 > 0$ with t_0 an arbitrary positive time. This explicit solution is obtained through the unique solution of an integral equation with the time as a parameter.

Keywords: Stefan problem, free boundary problem, phase-change process, nonlinear thermal conductivity, similarity solution

MSC 2010: 35R35, 80A22, 35C05

1 Introduction

We will consider a phase-change problem (Stefan problem) for a nonlinear heat conduction equation for a semi-infinite region $x > 0$ with a nonlinear thermal conductivity $k(\theta)$ given by

$$k(\theta) = \frac{\rho c}{(a + b\theta)^2} \tag{1.1}$$

and phase change temperature $\theta_f = 0$, where a, b are positive parameters, c, ρ are the specific heat and the density of the medium respectively. This kind of thermal conductivity or diffusion coefficient was considered in [2, 3, 5, 6, 12, 14, 20, 22, 24]. The modeling of this kind of problems is a great mathematical and industrial significance problem. Phase-change problems appear frequently in industrial processes and other problems of technological interest [1, 7–11, 13, 15, 16]. ~~A large~~ bibliography on the subject was given ~~recently~~ [23].

Note 1:
Removed 'recent'.

The mathematical formulation of our free boundary (fusion process) problem consists in determining the evolution of the moving phase separation $x = s(t)$ and the temperature distribution $\theta = \theta(x, t) \geq 0$ satisfying the conditions

$$\rho c \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\Big(k(\theta)\frac{\partial \theta}{\partial x}\Big), \qquad 0 < x < s(t),\ t > 0, \tag{1.2}$$

$$k(\theta(0,t))\frac{\partial \theta}{\partial x}(0,t) = \frac{h_0}{\sqrt{t}}(\theta(0,t) - \theta_0), \quad h_0 > 0,\ t > 0, \tag{1.3}$$

$$k(\theta(s(t),t))\frac{\partial \theta}{\partial x}(s(t),t) = -\rho l \dot{s}(t), \qquad t > 0, \tag{1.4}$$

$$\theta(s(t),t) = 0, \qquad t > 0, \tag{1.5}$$

$$s(0) = 0, \tag{1.6}$$

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where $h_0 > 0$ is the thermal transfer coefficient, l is the latent heat of fusion of the medium and we assume

$$\theta_0(0,t) < \theta_0,$$

where $\theta_0 > 0$ is the ambient temperature.

An analogous problem to (1.2)–(1.6) was considered in [18] where the temperature and flux-type conditions on the fixed face $x = 0$ were studied. In order to have a parametric representation of the solution of similarity type, sufficient conditions for data were obtained.

In [4, 17, 19, 21] free boundary problems were considered for the equation

$$\rho c \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(k(\theta)\frac{\partial \theta}{\partial x}\right) - v(\theta)\frac{\partial \theta}{\partial x}, \quad 0 < x < s(t),\ t > 0, \tag{1.7}$$

where the thermal conductivity $k(\theta)$ and the velocity term $v(\theta)$ are given by (1.1) and

$$v(\theta) = \rho c \frac{d}{2(a + b\theta)^2} \tag{1.8}$$

respectively with $d > 0$. In those papers temperature and flux-type conditions on the fixed face $x = 0$ were studied. In [4] a Dirichlet boundary condition at the fixed face $x = 0$ is considered. Under the Bäcklund transformation the Stefan problem is reduced to an associated free boundary problem and the existence and uniqueness local in time, of the solution is proved by using the Friedman Rubinstein integral representation and the Banach contraction theorem. Furthermore, necessary and sufficient conditions for the existence of a parametric representation of the solution of similarity type was found in [17]. In [21] a Neumann boundary condition at the fixed face $x = 0$ is considered. A reciprocal transformation to the Stefan problem is applied and a parametric representation of similarity type of the solution is obtained. The results given in [21] are improved in [19] obtaining explicit solutions through the unique solution of a Cauchy problem.

Here we study the case without the velocity term, i.e. $d = 0$, in the differential equation (1.7) and we consider a convective boundary condition on the fixed face $x = 0$.

In Section 2 we prove the existence and uniqueness of an explicit solution of similarity type of the free boundary problem (1.2)–(1.6) for $t \geq t_0 > 0$ with t_0 an arbitrary positive time when data satisfy condition $\frac{a}{b} > \frac{l}{c}$. This type of exact solution to problems with parameters is useful to test by benchmarking with numerical methods for different data values.

In Section 3 we consider the case $\frac{a}{b} = \frac{l}{c}$. In both cases the explicit solutions are obtained through the unique solutions of the integral equations with the time as a parameter.

Besides, there does not exist any solution of similarity type to the free boundary problem (1.2)–(1.6) for the case $\frac{a}{b} < \frac{l}{c}$.

2 Existence and uniqueness of solution of the free boundary problem with boundary condition of Robin type on the fixed face for the case $\frac{a}{b} > \frac{l}{c}$

We consider the free boundary problem (1.2)–(1.6) with the parameters a, b and the coefficients l, c satisfy the condition

$$\frac{a}{b} > \frac{l}{c}. \tag{2.1}$$

Now, we give several transformations to obtain an equivalent problem to (1.2)–(1.6) which admits a similarity-type solution.

If we define

$$\Theta = \frac{1}{a + b\theta}, \tag{2.2}$$

~~then~~ problem (1.2)–(1.6) becomes

$$\frac{\partial \Theta}{\partial t} = \Theta^2 \frac{\partial^2 \Theta}{\partial x^2}, \quad 0 < x < s(t), \quad t > 0, \tag{2.3}$$

$$\frac{\partial \Theta}{\partial x}(0, t) = \frac{-h_0^*}{\sqrt{t}} \left[\frac{1}{\Theta(0, t)} - \Theta_0 \right], \quad t > 0, \tag{2.4}$$

$$\frac{\partial \Theta}{\partial x}(s(t), t) = \frac{bl}{c} \dot{s}(t), \quad t > 0, \tag{2.5}$$

$$\Theta(s(t), t) = \frac{1}{a}, \quad t > 0, \tag{2.6}$$

$$s(0)Q = 0, \tag{2.7}$$

where h_0^* is a constant defined by

$$h_0^* = \frac{h_0}{\rho c} \tag{2.8}$$

and

$$\Theta_0 = a + b\theta_0. \tag{2.9}$$

Let us perform the transformation

$$\chi(x, t) = \int_0^x \frac{d\eta}{\Theta(\eta, t)}, \quad \Psi(\chi, t) = \Theta(x, t) \tag{2.10}$$

and

$$S(t) = \chi(s(t), t). \tag{2.11}$$

Problem (2.3)–(2.7) becomes

$$\frac{\partial \Psi}{\partial t} = \frac{\partial^2 \Psi}{\partial \chi^2} + \frac{h_0^*}{\sqrt{t}} \left[\frac{1}{\Psi(0, t)} - \Theta_0 \right] \frac{\partial \Psi}{\partial \chi}, \quad 0 < \chi < S(t), \quad t > 0, \tag{2.12}$$

$$\frac{\partial \Psi}{\partial \chi}(0, t) = -\frac{h_0^*}{\sqrt{t}} \left[\frac{1}{\Psi(0, t)} - \Theta_0 \right] \Psi(0, t), \quad t > 0, \tag{2.13}$$

$$\frac{\partial \Psi}{\partial \chi}(S(t), t) = \frac{1}{a(\frac{ca}{bl} - 1)} \left[\dot{S}(t) + \frac{h_0^*}{\sqrt{t}} \left(\frac{1}{\Psi(0, t)} - \Theta_0 \right) \right], \quad t > 0, \tag{2.14}$$

$$\Psi(S(t), t) = \frac{1}{a}, \quad t > 0, \tag{2.15}$$

$$S(0) = 0, \tag{2.16}$$

where

$$\dot{S}(t) = \left(a - \frac{bl}{c} \right) \dot{s}(t) + \frac{w}{\sqrt{t}} \tag{2.17}$$

and

$$w = -h_0^* \left[\frac{1}{\varphi(0)} - \Theta_0 \right]. \tag{2.18}$$

We remark that by hypothesis $\theta_0(0, t) < \theta_0$ and $h_0 > 0$ results $w > 0$.

If we introduce the similarity variable

$$\xi = \frac{\chi}{2\sqrt{t}}, \tag{2.19}$$

and the solution is sought of type

$$\Psi(\chi, t) = \varphi(\xi) = \varphi\left(\frac{\chi}{2\sqrt{t}} \right), \tag{2.20}$$

then the free boundary $S(t)$ of problem (2.12)–(2.16) must be of the type

$$S(t) = 2\Lambda\sqrt{t}, \quad t > 0, \tag{2.21}$$

with $\Lambda > 0$ an unknown coefficient to be determined. Problem (2.12)–(2.16) yields

$$\varphi''(\xi) + 2\varphi'(\xi)(\xi - w) = 0, \quad 0 < \xi < \Lambda, \tag{2.22}$$

$$\varphi'(0) = 2w\varphi(0), \tag{2.23}$$

$$\varphi(\Lambda) = \frac{1}{a}, \tag{2.24}$$

$$\varphi'(\Lambda) = 2\alpha^*(\Lambda - w), \tag{2.25}$$

where

$$\alpha^* = \frac{bl}{a(ac - bl)} > 0. \tag{2.26}$$

Taking into account expressions (2.17) and (2.21), we have

$$s(t) = 2\lambda\sqrt{t} \tag{2.27}$$

with

$$\lambda = \frac{\Lambda - w}{a - \frac{bl}{c}}. \tag{2.28}$$

If we integrate (2.22), we obtain

$$\varphi(\xi) = D\big[\operatorname{erf}(\xi - w) + \operatorname{erf}(w)\big] + C, \quad 0 < \xi < \Lambda. \tag{2.29}$$

From conditions (2.23)–(2.25) we have that

$$C = \varphi(0) = \frac{\alpha^*(\Lambda - w)\exp((\Lambda - w)^2)\exp(-w^2)}{w}, \tag{2.30}$$

$$D = \sqrt{\pi}\alpha^*(\Lambda - w)\exp((\Lambda - w)^2), \tag{2.31}$$

and the unknowns w and Λ must satisfy the equations

$$\frac{w\exp(-(\Lambda - w)^2)\exp(w^2)}{\sqrt{\pi}w\exp(w^2)[\operatorname{erf}(\Lambda - w) + \operatorname{erf}(w)] + 1} = a\alpha^*(\Lambda - w), \tag{2.32}$$

$$(\Lambda - w)\exp((\Lambda - w)^2) = \frac{h_0^* w \exp(w^2)}{\alpha^*[h_0^*\Theta_0 - w]}. \tag{2.33}$$

Assumption (2.1) implies that the unknowns $0 < w < h_0^*\Theta_0$ and Λ must verify $\Lambda > w$. Then, for each $0 < w < h_0^*\Theta_0$ we solve equation (2.32) in the unknown $\Lambda > w$.

For this, we define the real functions

$$W_1(x) = \frac{w\exp(w^2)\exp[-(x - w)^2]}{1 + w\exp(w^2)\sqrt{\pi}(\operatorname{erf}(x - w) + \operatorname{erf}(w))} \tag{2.34}$$

and

$$W_2(x) = a\alpha^*(x - w). \tag{2.35}$$

Now, we have that $\Lambda > w$ must be the solution of the equation

$$W_1(x) = W_2(x), \quad x > w, \tag{2.36}$$

which is equivalent to equation (2.32).

Lemma 2.1. *For each* $0 < w < h_0^*\Theta_0$ *there exists a unique solution* $\Lambda = \Lambda(w)$ *of equation* (2.36).

Proof. It is easy to prove that

$$W_1(w) = \frac{w\exp(w^2)}{1 + \sqrt{\pi}w\exp(w^2)\operatorname{erf}(w)} > 0, \quad W_1(+\infty) = 0,$$

W_1 is a decreasing function and from (2.1) we have that W_2 is an increasing function. So, there exists a unique solution $\Lambda(w)$ of equation (2.36). □

Taking into account (2.32) and Lemma 2.1 we can rewrite equation (2.33) as

$$V_1(w) = V_2(w), \quad 0 < w < h_0^*\Theta_0, \tag{2.37}$$

where

$$V_1(x) = -x + h_0^* b\theta_0$$

and

$$V_2(x) = \sqrt{\pi} h_0^* a x \exp(x^2)[\operatorname{erf}(\Lambda(x) - x) + \operatorname{erf}(x)]$$

with $\Lambda(x)$ the unique solution of (2.36).

Lemma 2.2. *There exists a unique $w \in [0, h_0^* b\theta_0)$ solution of* (2.37).

Proof. The function V_2 satisfies $V_2(0) = 0$, $V_2(x) > 0$ for $x > 0$. Moreover, V_1 is a decreasing function and $V_1(x) > 0$ for $x \in [0, h_0^* b\theta_0)$, $V_1(h_0^* b\theta_0) = 0$ and $V_1(x) < 0$ for $x \in (h_0^* b\theta_0, h_0^*\Theta_0)$. Then there exists at least one $w \in [0, h_0^* b\theta_0)$ which is ~~a~~ solution of (2.37).

To prove uniqueness we suppose that there exist two solutions w_1 and w_2 to (2.33) such that $w_1 < w_2$. Then

$$\frac{h_0^* w_1 \exp(w_1^2)}{\alpha^*[h_0^*\Theta_0 - w_1]} < \frac{h_0^* w_2 \exp(w_2^2)}{\alpha^*[h_0^*\Theta_0 - w_2]}$$

and from (2.33) we obtain

$$(\Lambda(w_1) - w_1)\exp((\Lambda(w_1) - w_1)^2) < (\Lambda(w_2) - w_2)\exp((\Lambda(w_2) - w_2)^2).$$

Taking into account that the function $f(x) = x\exp(x^2)$ is an increasing function, we have that

$$\Lambda(w_1) - w_1 < \Lambda(w_2) - w_2.$$

Moreover, also since $V_1(w_1) = V_2(w_1)$ and $V_1(w_2) = V_2(w_2)$, after some calculations we obtain

$$\frac{w_2 - w_1}{\sqrt{\pi} h_0^* a} = w_1 \exp(w_1^2)[\operatorname{erf}(\Lambda(w_1) - w_1) + \operatorname{erf}(w_1)] - w_2 \exp(w_2^2)[\operatorname{erf}(\Lambda(w_2) - w_2) + \operatorname{erf}(w_2)]$$

which contradicts our hypothesis since the right-hand side is negative. ~~Thus $w_1 = w_2$~~. □

Theorem 2.3. *Let us assume the hypothesis* (2.1).

(i) *If (Θ, s) is a solution of the free boundary problem* (2.3)–(2.7), *then $\Theta = \Theta(x, t)$ is a solution, in variable x, of the integral equation*

$$\Theta(x, t) = C + D\left[\operatorname{erf}\left(\frac{\int_0^x \frac{d\eta}{\Theta(\eta, t)}}{2\sqrt{t}} - w\right) + \operatorname{erf}(w)\right], \quad 0 \le x \le s(t), \tag{2.38}$$

where $t > 0$ is a parameter, ~~and~~ C and D are defined by (2.30) *and* (2.31) *respectively, where w and Λ are the unique solutions of* (2.32) *and* (2.33). *The free boundary $s(t)$ is given by* (2.27) *and* (2.28). *Moreover, the function $Y(x, t)$ defined by*

$$Y(x, t) = \frac{1}{2\sqrt{t}} \int_0^x \frac{d\eta}{\Theta(\eta, t)} - w, \quad 0 \le x \le s(t), \quad t > 0, \tag{2.39}$$

satisfies the conditions

$$\frac{\partial Y}{\partial x}(x, t) = \frac{1}{2\sqrt{t}} \frac{1}{\Theta(x, t)}, \qquad 0 < x < s(t), \quad t > 0, \tag{2.40}$$

$$Y(0, t) = -w, \qquad t > 0, \tag{2.41}$$

$$\frac{\partial Y}{\partial t}(x, t) = -\frac{1}{2t}\left(Y(x, t) + \frac{D}{\sqrt{\pi}} \frac{\exp(-Y^2(x, t))}{\Theta(x, t)}\right), \qquad 0 < x < s(t), \quad t > 0, \tag{2.42}$$

$$Y(s(t), t) = \Lambda - w, \qquad t > 0. \tag{2.43}$$

(ii) *Conversely, if* Θ *is a solution of the integral equation* (2.38) *with* s *given by* (2.27) *and function* Y *defined by* (2.39) *satisfies conditions* (2.40)–(2.43), C *and* D *are defined by* (2.30) *and* (2.31) *respectively, where* w *and* Λ *are the unique solutions of equations* (2.32) *and* (2.33), *then* (Θ, s) *is a solution of the free boundary problem* (2.3)–(2.7).

Proof. (i) From the previous computation we have

$$\Theta(x,t)=\varphi(\xi)=C+D[\operatorname{erf}(\xi-w)+\operatorname{erf}(w)]=C+D\left[\operatorname{erf}\left(\frac{\int_0^x\frac{d\eta}{\Theta(\eta,t)}}{2\sqrt{t}}-w\right)+\operatorname{erf}(w)\right],$$

that is Θ is a solution of the integral equation (2.38). By elementary computations we have that ~~the~~ function Y, defined by (2.39), satisfies conditions (2.40), (2.41) and

$$\begin{aligned}\frac{\partial Y}{\partial t}(x,t)&=-\frac{1}{4t\sqrt{t}}\int_0^x\frac{d\eta}{\Theta(\eta,t)}-\frac{1}{2\sqrt{t}}\int_0^x\Theta_{xx}(\eta,t)\,d\eta\\&=-\frac{1}{2t}(Y(x,t)+w)-\frac{1}{2\sqrt{t}}(\Theta_x(x,t)-\Theta_x(0,t))\\&=-\frac{1}{2\sqrt{t}}\left(\frac{Y(x,t)}{\sqrt{t}}+\Theta_x(x,t)\right)\\&=-\frac{1}{2t}\left(Y(x,t)+\frac{D}{\sqrt{\pi}}\frac{\exp(-Y^2(x,t))}{\Theta(x,t)}\right),\end{aligned}$$

that is (2.42). Finally, we get

$$Y(s(t),t)=\frac{1}{2\sqrt{t}}\int_0^{s(t)}\frac{d\eta}{\Theta(\eta,t)}-w=\frac{\chi(s(t),t)}{2\sqrt{t}}-w=\frac{S(t)}{2\sqrt{t}}-w=\Lambda-w,$$

that is (2.43).

(ii) In order to proof that (Θ, s) is a solution of the free boundary problem (2.3)–(2.7), we get

$$\Theta_{xx}(x,t)=\left(\frac{D}{\sqrt{\pi t}}\frac{\exp(-Y^2(x,t))}{\Theta(x,t)}\right)_x=-\frac{D}{\sqrt{\pi t}}\frac{\exp(-Y^2(x,t))}{\Theta^2(x,t)}\left(Y(x,t)+\frac{D}{\sqrt{\pi}}\frac{\exp(-Y^2(x,t))}{\Theta(x,t)}\right).$$

~~We have~~

$$\Theta_t(x,t)=\frac{2D}{\sqrt{\pi}}\exp(-Y^2(x,t))Y_t(x,t)=-\frac{D}{\sqrt{\pi t}}\exp(-Y^2(x,t))\left(Y(x,t)+\frac{D}{\sqrt{\pi}}\frac{\exp(-Y^2(x,t))}{\Theta(x,t)}\right);$$

then (2.3) holds. Furthermore,

$$\Theta(0,t)=C. \tag{2.44}$$

Taking into account (2.30), (2.31), (2.40), (2.41) and (2.44), we obtain

$$\Theta_x(0,t)=\frac{D}{\sqrt{\pi t}}\frac{\exp(-Y^2(0,t))}{\Theta(0,t)}=\frac{w}{\sqrt{t}},$$

that is (2.4). Moreover

$$\Theta(s(t),t)=C+D[\operatorname{erf}(Y(s(t),t))+\operatorname{erf}(w)]=C+D[\operatorname{erf}(\Lambda-w)+\operatorname{erf}(w)]=\frac{1}{a},$$

that is (2.6). ~~Finally, by~~ using (2.26), (2.31), (2.40) and (2.43), we have

$$\begin{aligned}\Theta_x(s(t),t)&=\frac{D}{\sqrt{\pi t}}\frac{\exp(-Y^2(s(t),t))}{\Theta(s(t),t)}=\frac{aD}{\sqrt{\pi t}}\exp(-(\Lambda-w)^2)=\frac{a\alpha^*(\Lambda-w)}{\sqrt{t}}\\&=\frac{1}{\sqrt{t}}\frac{bl}{ca-bl}(\Lambda-w)=\frac{bl\lambda}{c\sqrt{t}}=\frac{bl}{c}\dot{s}(t),\end{aligned}$$

that is (2.5). ∎

Theorem 2.4. *Let us assume hypothesis* (2.1).

(i) *The integral equation* (2.38) *has a unique solution for* $t \geq t_0 > 0$ *with* t_0 *an arbitrary positive time.*

(ii) *The free boundary problem* (1.2)–(1.5) *satisfying hypothesis* (2.1) *has a unique similarity-type solution* (θ, s) *for* $t \geq t_0 > 0$ *(with* t_0 *an arbitrary positive time) which is given by*

$$\theta(x,t) = \frac{1}{b}\left[\frac{1}{\Theta(x,t)} - a\right], \quad 0 < x < s(t), \quad t \geq t_0 > 0, \tag{2.45}$$

$$s(t) = \frac{2(\Lambda - w)}{a - \frac{bl}{c}}\sqrt{t}, \qquad t \geq t_0 > 0, \tag{2.46}$$

where Θ *is the unique solution of the integral equation* (2.38), *where* C *and* D *are defined by* (2.30) *and* (2.31) *respectively,* w *and* Λ *are the unique solutions of equations* (2.32) *and* (2.33).

Proof. (i) If we define $Y(x,t)$ by (2.39), then (2.38) is equivalent to the Cauchy differential problem

$$\begin{cases} \dfrac{\partial Y}{\partial x}(x,t) = \dfrac{1}{2\sqrt{t}}\dfrac{1}{(C + D\operatorname{erf}(Y(x,t)))} \equiv G(x,t,Y(x,t)), \quad 0 < x < s(t), \quad t > 0, \\ Y(0,t) = -w, \end{cases} \tag{2.47}$$

with a parameter $t > 0$. We have

$$\left|\frac{\partial G}{\partial Y}\right| \leq \frac{D}{C^2\sqrt{\pi t}}$$

which is bounded for all $t \geqslant t_0 > 0$, $0 \leq x \leq s(t)$, for an arbitrary positive time t_0. Then, problem (2.47) (i.e. the integral equation (2.38)) has a unique solution for $t \geq t_0 > 0$, for an arbitrary positive time t_0.

(ii) ~~This~~ follows taking into account Theorem 2.3 and from elementary but tedious computations. □

Remark 2.5. The free boundary given by (2.46) satisfies $s(0) = 0$.

Remark 2.6. Observe that $Y(x,t)$ ~~does~~ not possess a limit at $(0,0)$ because $Y(0,t) = -w < 0$ for $t > 0$ and $\lim Y(s(t),t) = \Lambda - w > 0$ when t goes to 0.

3 Existence and uniqueness of solution of the free boundary problem with boundary condition of Robin type on the fixed face for the case $\frac{a}{b} = \frac{l}{c}$

We consider

$$\frac{a}{b} = \frac{l}{c}. \tag{3.1}$$

~~Similarly to what made in the~~ previous section, problem (1.2)–(1.6) becomes (2.3)–(2.9).

By using (2.10) and (2.11) we have (2.12), (2.13), (2.15), (2.16) and

$$\dot{S}(t) = -\frac{h_0^*}{\sqrt{t}}\left[\frac{1}{\Psi(0,t)} - \Theta_0\right]. \tag{3.2}$$

We introduce the variable (2.19) and the similarity solution given by (2.20), that is

$$\Psi(\chi,t) = \varphi(\xi) = \varphi\left(\frac{\chi}{2\sqrt{t}}\right).$$

The free boundary $S(t)$ of problem (2.12), (2.13), (2.15), (2.16) and (3.2) must be of the type (2.21) with $\Lambda > 0$ an unknown coefficient to be determined and from (3.2) we have that

$$\varphi(0) = \frac{h_0^*}{h_0^*\Theta_0 - \Lambda}. \tag{3.3}$$

Then problem (2.12), (2.13), (2.15), (2.16) and (3.2) yields

$$\varphi''(\xi) + 2\varphi'(\xi)(\xi - \Lambda) = 0, \quad 0 < \xi < \Lambda, \tag{3.4}$$

$$\varphi'(0) = \frac{2h_0^*\Lambda}{h_0^*\Theta_0 - \Lambda}, \tag{3.5}$$

$$\varphi(\Lambda) = \frac{1}{a} \tag{3.6}$$

and (3.3).

If we integrate (3.4), we obtain

$$\varphi(\xi) = D[\operatorname{erf}(\xi - \Lambda) + \operatorname{erf}(\Lambda)] + C, \quad 0 < \xi < \Lambda. \tag{3.7}$$

From the boundary conditions we have that

$$C = \varphi(0), \quad D = \frac{\sqrt{\pi}h_0^*\Lambda\exp(\Lambda^2)}{h_0^*\Theta_0 - \Lambda}, \tag{3.8}$$

and Λ is the unique solution of the equation

$$\sqrt{\pi}x\exp(x^2)\operatorname{erf}(x) = \frac{h_0^* b\theta_0 - x}{ah_0^*}, \quad 0 \le x \le h_0^* b\theta_0. \tag{3.9}$$

Theorem 3.1. *Let us consider hypothesis* (3.1).

(i) *If* (Θ, s) *is a solution of the free boundary problem* (2.3)–(2.7)*, then* $\Theta = \Theta(x,t)$ *is a solution, in variable* x*, of the integral equation:*

$$\Theta(x,t) = C + D\left[\operatorname{erf}\left(\frac{\int_0^x \frac{d\eta}{\Theta(\eta,t)}}{2\sqrt{t}} - \Lambda\right) + \operatorname{erf}(\Lambda)\right], \quad 0 \le x \le s(t), \tag{3.10}$$

where $t > 0$ *is a parameter* C *and* D *are defined by* (3.8) *and* Λ *is the unique solution of equation* (3.9)*. The free boundary* $s(t)$ *is given by*

$$s(t) = \frac{2h_0^*\Lambda\exp(\Lambda^2)}{h_0^*\Theta_0 - \Lambda}\sqrt{t}. \tag{3.11}$$

Moreover, the function $Y(x,t)$ *defined by*

$$Y(x,t) = \frac{1}{2\sqrt{t}}\int_0^x \frac{d\eta}{\Theta(\eta,t)} - \Lambda, \quad 0 \le x \le s(t), \quad t > 0, \tag{3.12}$$

satisfies the conditions

$$\frac{\partial Y}{\partial x}(x,t) = \frac{1}{2\sqrt{t}}\frac{1}{\Theta(x,t)}, \quad 0 < x < s(t), \quad t > 0, \tag{3.13}$$

$$Y(0,t) = -\Lambda, \quad t > 0, \tag{3.14}$$

$$\frac{\partial Y}{\partial t}(x,t) = -\frac{1}{2t}\left(Y(x,t) + \frac{D}{\sqrt{\pi}}\frac{\exp(-Y^2(x,t))}{\Theta(x,t)}\right), \quad 0 < x < s(t), \quad t > 0, \tag{3.15}$$

$$Y(s(t),t) = 0, \quad t > 0. \tag{3.16}$$

(ii) *Conversely, if* Θ *is a solution of the integral equation* (3.10) *with* s *given by* (3.11) *and function* Y *defined by* (3.12) *satisfies conditions* (3.13)–(3.16)*,* D *and* C *are defined by* (3.8) *and* Λ *is the unique solution of equation* (3.9)*, then* (Θ, s) *is a solution of the free boundary problem* (2.3)–(2.7).

Theorem 3.2. *Let us consider hypothesis* (3.1).

(i) *The integral equation* (3.10) *has a unique solution for* $t \ge t_0 > 0$ *with* t_0 *an arbitrary positive time.*

(ii) *The free boundary problem* (1.2)–(1.5) *has a unique similarity-type solution* (θ, s) *for* $t \ge t_0 > 0$ *(with* t_0 *an arbitrary positive time) which is given by*

$$\theta(x,t) = \frac{1}{b}\left[\frac{1}{\Theta(x,t)} - a\right], \quad 0 < x < s(t), \quad t \ge t_0 > 0, \tag{3.17}$$

and (3.11)*, where* Θ *is the unique solution of the integral equation* (3.10)*, where* C *and* D *are defined by* (3.8) *and* Λ *is the unique solution of equation* (3.9).

Remark 3.3. The free boundary given by (3.11) satisfies $s(0) = 0$.

Remark 3.4. The free boundary problem (1.2)–(1.6) has not a similarity-type solution for the case $\frac{a}{b} < \frac{l}{c}$. We note that in this case from (2.26) we have $\alpha^* < 0$, then from (2.30) we obtain $\varphi(0) < 0$ and taking into account (2.18) and (2.33) we have $\Lambda > w > (a + b\theta_0)h_0^*$. Then (2.32) has not solution.

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