THE COMPLETE VERSION OF THE GORRITI HAT GAME

by

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Abstract: In this paper we introduce two simple versions of the hat game for three players and a generalization for n players. We compute the extensive version of the game and all the equilibrium and friendly equilibrium points as a simple example of the structure function.

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Acknowledgement: I am indebted with some friends of the Gorriti Squash Club in Buenos Aires, especially to Victor and Alejandra Car Merkley. In an "asado", after some discussion, nobody could thoroughly remember the classical hat problem. As a consequence of that, we developed the version of the hat game presented here and subsequently a further version.

1. **The Gorriti version of the hat game**
   We have n players and each player has a white or black hat on his or her head. They do not know the color of their own hat. The game is played as follows: the players take turns trying to guess the color of their hat. They might guess correctly (their hat's color is the color they have guessed) or incorrectly.

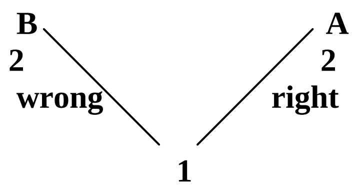

fig. 1

If the game reaches the point **A**, the second player tries to guess the color of his hat. If the second player guesses correctly, the procedure is repeated as in the first step. An analogous mechanism is applied if the second player does not guess correctly. It is clear how the truncated tree is defined in position **B**.

In order to avoid confusion, we show the following partial tree.

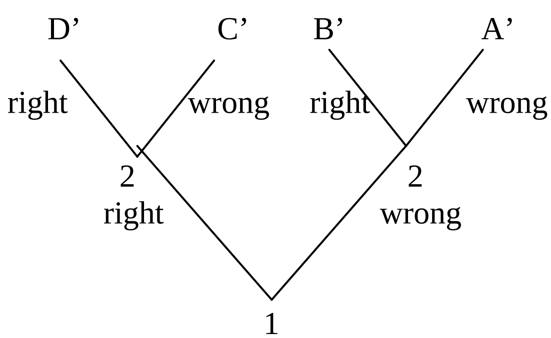

fig. 2

The game continues repeating the same mechanism with the remaining players. Thus we obtain an extensive tree where all branches have the same length. The payoff of the players is given as follows:

$$\begin{matrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{matrix}$$

where a_i is either 0 or 1 depending on whether player i has guessed correctly the color of his hat (0 if the guess is incorrect, 1 if the guess is correct).

As a comment we would like to mention that our Gorriti version could be related to other versions of the hat game presented in some conferences at the Kellog Center of Northwestern University by professor Selten several years ago. This reference was given to us by professor Kalai (1989) during a visit to the Instituto de Matemática Aplicada San Luis.

Now it is clear that the game is well defined. We wish to compute an equilibrium point and afterwards all the equilibrium points.

We will concentrate our attention to the case of three players. The game is depicted in the next figure

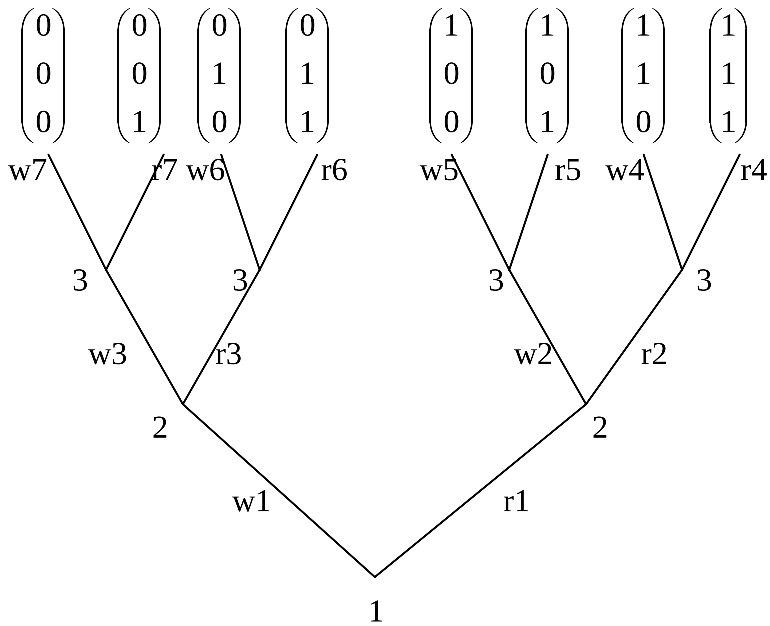

fig. 3

It is interesting to observe that the payoff functions given in this game with three players are evidently related to all the possibilities of taking two objects (0 and 1) and arranging them in groups of three. Therefore, this simple observation allows us to generalize the game tree for the case of n players.

2. Equilibrium Points

In order to make this paper easy to understand we remind the reader that an equilibrium point in an extensive game is a point $\sigma_i \in \Sigma_i$ where Σ_i is the set of strategies for a player $i \in P$ such that for each $\sigma_\nu \in P$ and $\sigma_i \in \Sigma_i$

$$A_i(\sigma_i, \sigma_{-i}) \geq A_i(\sigma_\nu, \sigma_{-i})$$

where $\sigma_{-i} = (\sigma_1, ..., \sigma_{i-1,} \sigma_{i+1}, ..., \sigma_n)$ for each i.

Now we immediately see that the strategy

$\sigma_1 = \{r_1\}$, $\sigma_2 = \{r_2, r_3\}$, $\sigma_3 = \{r_4, r_5, r_6, r_7\}$ is an equilibrium point since

$A_1(\sigma_1, \sigma_2, \sigma_3) = A_1(r_1, r_2, r_3, r_4, r_5, r_6, r_7) = 1$

$$\begin{aligned}
&\geq A_1(w_1, r_2, r_3, r_4, r_5, r_6, r_7) = 0 \\
A_2(\sigma_1, \sigma_2, \sigma_3) = &\ A_2(r_1, r_2, r_3, r_4, r_5, r_6, r_7) = 1 \\
&\geq A_2(r_1, r_2, w_3, r_4, r_5, r_6, r_7) = 1 \\
&\geq A_2(r_1, w_2, r_3, r_4, r_5, r_6, r_7) = 0 \\
&\geq A_2(r_1, w_2, w_3, r_4, r_5, r_6, r_7) = 0 \\
A_3(\sigma_1, \sigma_2, \sigma_3) = &\ A_3(r_1, r_2, r_3, r_4, r_5, r_6, r_7) = 1 \\
&\geq A_3(r_1, r_2, r_3, r_4, r_5, r_6, w_7) = 1 \\
&\geq A_3(r_1, r_2, r_3, r_4, r_5, w_6, r_7) = 1 \\
&\geq A_3(r_1, r_2, r_3, r_4, w_5, r_6, r_7) = 1 \\
&\geq A_3(r_1, r_2, r_3, w_4, r_5, r_6, r_7) = 0 \\
&\geq A_3(r_1, r_2, r_3, r_4, r_5, w_6, w_7) = 1 \\
&\geq A_3(r_1, r_2, r_3, r_4, w_5, r_6, w_7) = 1 \\
&\geq A_3(r_1, r_2, r_3, r_4, w_5, w_6, r_7) = 1 \\
&\geq A_3(r_1, r_2, r_3, w_4, z_5, z_6, z_7) = 0 \qquad \forall z_5, z_6, z_7
\end{aligned}$$

Now we are going to find all the equilibrium points of this game.

First, consider a strategy $\sigma_2 = (r_1, r_2, r_3, r_4, z_5, z_6, z_7)$; then performing similar operations to those done to σ_1, σ_2, σ_3 it is easy to see that the point σ_2 for any $z = (z_4, z_5, z_6, z_7)$ is an equilibrium point.

Now we will prove that such a set of points is the set of equilibrium points, in other words that any equilibrium point has to be of the form $\{r_1, r_2, z_3, r_4, z_5, z_6, z_7\}$ for any z_3, z_5, z_6, z_7.

Let us assume the contrary, this is, that σ_1, σ_2, σ_3 is of the form $\{r_1, w_2, z_3, r_4, z_5, z_6, z_7\}$ then if $z_5 = r_5$ we have

$A_1(\sigma_1, \sigma_2, \sigma_3) = A_1(r_1, w_2, z_3, r_4, r_5, z_6, z_7) = 1$

$A_2(\sigma_1, \sigma_2, \sigma_3) = A_2(r_1, w_2, z_3, r_4, r_5, z_6, z_7) = 0$

and

$A_3(\sigma_1, \sigma_2, \sigma_3) = A_3(r_1, w_2, z_3, r_4, r_5, z_6, z_7) = 1$

Now take $(\sigma_1, \sigma_2, \sigma_3) = (r_1, w_2, z_3, r_4, r_5, z_6, z_7)$ and compute

$A_2(\sigma_1, \sigma_2, \sigma_3) = A_2(r_1, r_2, z_3, r_4, r_5, z_6, z_7) = 1$

which implies that the point $(\sigma_1, \sigma_2, \sigma_3)$ is not an equilibrium point. On the other hand if the point is of the form $\{w_1, r_2, z_3, r_4, z_5, z_6, z_7\}$ it is clear that it is not an equilibrium point since all the branches stemming from the roots using w_1 give rise the value 0 for the payoff function of the first player.

Finally if the point is $\{r_1, r_2, z_3, w_4, z_5, z_6, z_7\}$ then

$A_3(r_1, r_2, z_3, w_4, z_5, z_6, z_7) = 0 \leq A_3(r_1, r_2, z_3, r_4, z_5, z_6, z_7) = 1$

and therefore cannot be an equilibrium point. The conjecture made above is true.

The form for all equilibrium points for the three person game given in the previous presentation suggests that all equilibrium points in a n-players Gorriti version of the hat game are of the form

$$\left(r_1, r_2, z_3, r_4, z_5, z_6, z_7, r_8, z_9, z_{11}, \dots, r_{2^j}, z_{2^{j+1}}, \dots, r_{2^{j+1}}, \dots, r_{2^n}, \dots\right)$$

For notational easiness this has to be true!

3. Friendly equilibrium points

Consider a friendly structure similar to the one of the previous game. For instance $e(1) = \{2\}$, $e(2) = \{1\}$, $e(3) = \Phi$, where Φ indicates the empty set. We remind the reader that the friendly equilibrium point was introduced by Marchi(2005).

In our case a friendly equilibrium point with the friendly structure function is a point $(\sigma_1, \sigma_2, \sigma_3)$ such that

$$A_1(\sigma_1, \sigma_2, \sigma_3) \geq A_1(\overline{\sigma}_1, \sigma_2, \sigma_3) \quad \forall \overline{\sigma}_1$$

$$A_2(\sigma_1, \sigma_2, \sigma_3) \geq A_2(\hat{\sigma}_1, \sigma_2, \sigma_3)$$

for all $\hat{\sigma}_1 \in \Psi_1 = \{\overline{\sigma}_1 : \mathrm{A}_1(\sigma_1, \sigma_2, \sigma_3) = A_1(\overline{\sigma}_1, \sigma_2, \sigma_3)\}$

$$A_2(\sigma_1, \sigma_2, \sigma_3) \geq A_2(\sigma_1, \overline{\sigma}_2, \sigma_3) \quad \forall \overline{\sigma}_2$$

and

$$A_1(\sigma_1, \sigma_2, \sigma_3) \geq A_1(\hat{\sigma}_1, \sigma_2, \sigma_3)$$

$$\forall \hat{\sigma}_1 \in \Psi_2 = \{\overline{\sigma}_2 : \mathrm{A}_2(\sigma_1, \sigma_2, \sigma_3) = A_2(\sigma_1, \overline{\sigma}_2, \sigma_3)\}$$

and finally

$$A_3(\sigma_1, \sigma_2, \sigma_3) \geq A_3(\sigma_1, \sigma_2, \overline{\sigma}_3) \quad \forall \overline{\sigma}_3$$

The existence theorem is given clearly by Marchi in (2005)for an extensive game with perfect information.

The point $(\sigma_1, \sigma_2, \sigma_3) = (r_1, r_2, r_3, r_4, r_5, r_6, r_7)$ is a friendly equilibrium point. Indeed take the first player, since it is in an equilibrium point, it fulfills the first condition written above for the first player.

The second inequality is also satisfied. Take any r_1. If $\sigma_1 = \overline{\sigma}_1 = r_1$ the second inequality is also satisfied. If $\sigma_1 \neq r_1$ then

$$A_2(\sigma_1, \sigma_2, \sigma_3) = 1 \geq A_2(\overline{\sigma}_1, \sigma_2, \sigma_3) = A_2(w_1, r_2, r_3, r_4, r_5, r_6, r_7) = 1$$

and the second inequality for the first player is also satisfied.

For the second player the inequality of the equilibrium point is automatically satisfied. The second one is verified by the following reasoning: take $\overline{\sigma}_2 = (r_2, w_3)$ then

$$A_1(\sigma_1, \sigma_2, \sigma_3) = 1 \geq A_1(\sigma_1, \overline{\sigma}_2, \sigma_3) = A_1(r_1, r_2, w_3, r_4, r_5, r_6, r_7) = 1$$

Similarly for $\sigma_2 = (r_2, w_3)$ and $\sigma_2 = (w_2, w_3)$

4. The Natural Hat Game

The very well known hat game can give rise to different interpretations of extensive games with complete information. In the first paragraphs we have written a version. However it is possible to propose a different natural hat game.

We have n players $i \in N = \{1, \dots, n\}$. Each player has a hat on its head. The hats can be either black or white. The players do not know the color of their hats. The rules of the game are as follows: the first player has the chance to announce the color of his hat or to pledge ignorance. In a similar manner the game is constructed accordingly to the previous version . The payoff functions are given as follows

$$\begin{matrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{matrix}$$

where a_i is either 0, ½ or 1, according to the rule that if player i in his or her choice in the determination of the branch α has chosen w, p or r where:
w = a different color than given in his head
p = plead ignorance
r = the color of the hat on his head

This version of the game is very different to the Gorriti version.
Now consider a case for three players. In this case the game is expressed by the diagram fig. 4 which describes the extensive game.

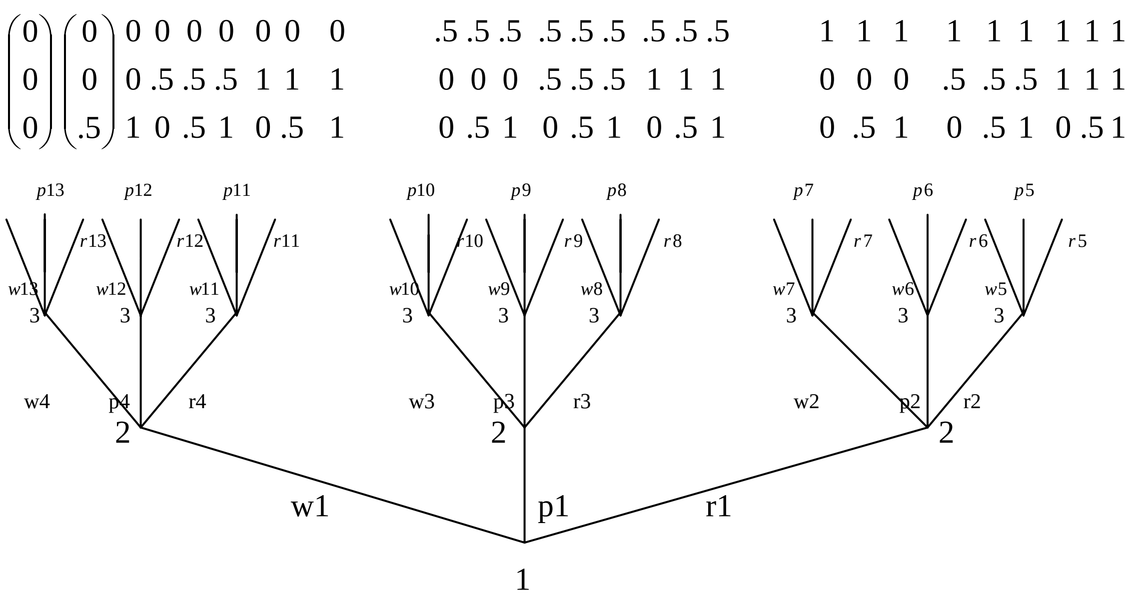

fig. 4

It is possible to prove that the points

$(r_1, r_2, r_3) = \{r_1, r_2, z_3, z_4, r_5, z_6, z_7, z_8, z_9, z_{10}, z_{11}, z_{12}, z_{13}\}$ are all the equilibrium points.
The generalization to n players is interesting and rather straightforward. It is a notation question!
One can observe the relationship of this game with the real hat problem.

As a final comment we would like to say that it is possible to study the material given in the refinement topic of the book by Van Damme (1992).

5. Bibliography

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