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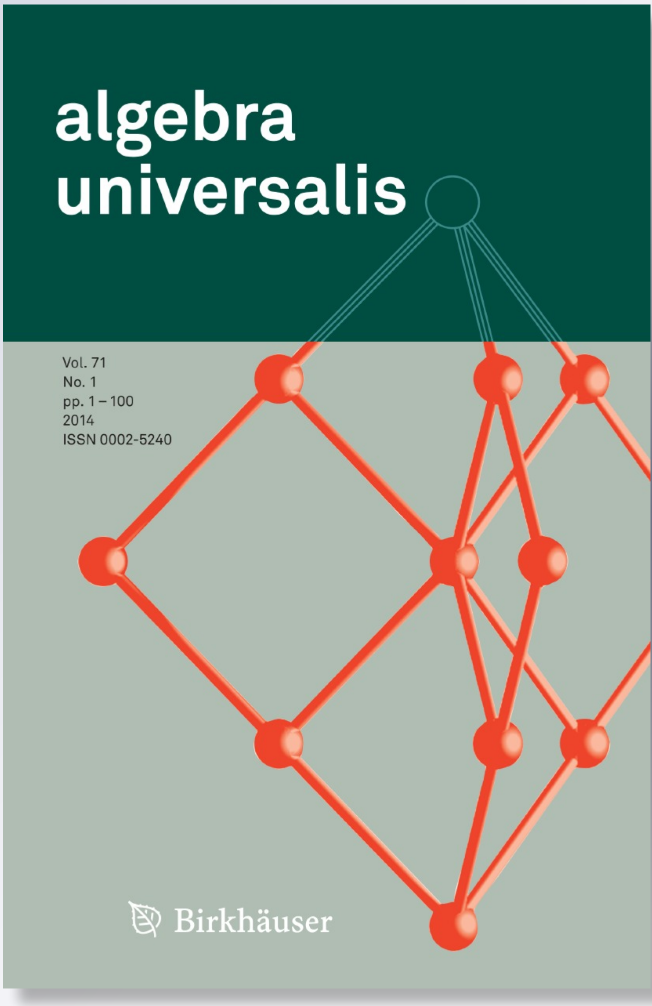

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Abstract. Given an integral commutative residuated lattice $\mathbf{L}$, the product $L \times L$ can be endowed with the structure of a commutative residuated lattice with involution that we call a *twist-product*. In the present paper, we study the subvariety $\mathbb{K}$ of commutative residuated lattices that can be represented by twist-products. We give an equational characterization of $\mathbb{K}$, a categorical interpretation of the relation among the algebraic categories of commutative integral residuated lattices and the elements in $\mathbb{K}$, and we analyze the subvariety of representable algebras in $\mathbb{K}$. Finally, we consider some specific class of bounded integral commutative residuated lattices $\mathbb{G}$, and for each fixed element $\mathbf{L} \in \mathbb{G}$, we characterize the subalgebras of the twist-product whose negative cone is $\mathbf{L}$ in terms of some lattice filters of $\mathbf{L}$, generalizing a result by Odintsov for generalized Heyting algebras.

1. Introduction

By a *commutative residuated lattice* we mean a *residuated lattice-ordered commutative monoid*, i.e., an algebra $\mathbf{A} = (A, \vee, \wedge, \cdot, \to, e)$ of type $(2, 2, 2, 2, 0)$ such that $(A, \vee, \wedge)$ is a lattice, $(A, \cdot, e)$ is a *commutative* monoid and the following residuation condition is satisfied:

$$x \cdot y \leq z \text{ if and only if } x \leq y \to z, \tag{1.1}$$

where x, y, z denote arbitrary elements of A and $\leq$ is the order given by the lattice structure.

It is well known that commutative residuated lattices form a variety that we shall denote by $\mathbb{CRL}$ (see, for instance, [3, 12, 14]).

An *involution* on $\mathbf{L} \in \mathbb{CRL}$ is a unary operation $\sim$ satisfying the equations $\sim\sim x = x$ and $x \to\ \sim y = y \to\ \sim x$. If $f := \sim e$, then $\sim x = x \to f$ and f satisfies the equation

$$(x \to f) \to f = x. \tag{1.2}$$

The element f in equation (1.2) is called a *dualizing element*.

Conversely, if $f \in L$ is a dualizing element and we define $\sim x = x \to f$ for all $x \in L$, then $\sim$ is an involution on $\mathbf{L}$ and $\sim e = f$. Hence, there is a bijective correspondence between involutions on $\mathbf{L}$ and dualizing elements in L (see [13, 22] for details).

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Taking $f = e$ in (1.2), we obtain an equation in the language of residuated lattices that determines a subvariety $\mathbb{I}_e\mathbb{CRL}$ of $\mathbb{CRL}$. We call the elements of this subvariety *e-involutive commutative residuated lattices* or *e-lattices* for short (they were called *residuated lattices with involution* in [4,6]).

By a *twist-product* of a lattice $\mathbf{L}$, we mean the cartesian product of $\mathbf{L}$ with its order-dual $\mathbf{L}^\partial$ equipped with the natural order involution $(x, y) \mapsto (y, x)$ for all $(x, y) \in L \times L^\partial$. As far as we know, the idea of considering this kind of construction to deal with order involutions on lattices goes back to Kalman's 1958 paper [15], but the name "twist" appeared thirty years later in Kracht's paper [16].

Kalman only referred to bare lattices, but later on several authors considered lattices with additional operations which allow the definition of new operations on the basic twist-product [4–8, 11, 16, 19–23].

Tsinakis and Wille [22], inspired by Chu's work on category theory [2, Appendix], endowed the twist-product of a residuated lattice $\mathbf{L}$ having greatest element $\top$ with a residuated monoid structure with unit $(e, \top)$ and such that the pair $(\top, e)$ is the dualizing element for the natural involution as a twist-product. Hence, when $\mathbf{L}$ is *integral*, i.e., when e is the greatest element of $\mathbf{L}$, the dualizing pair is (e, e) and if $\mathbf{L}$ is also commutative, we obtain an e-lattice that we denote $\mathbf{K}(\mathbf{L})$.

We define a *K-lattice* as an e-lattice isomorphic to a subalgebra of $\mathbf{K}(\mathbf{L})$ for some integral $\mathbf{L} \in \mathbb{CRL}$.

Our first aim is to show that the class of K-lattices is a subvariety $\mathbb{K}$ of $\mathbb{CRL}$ (more precisely, a proper subvariety of $\mathbb{I}_e\mathbb{CRL}$), characterized by a simple set of equations. Moreover, we show that the correspondence $\mathbf{L} \mapsto \mathbf{K}(\mathbf{L})$ can be lifted to a functor $\mathbf{K}$ from the algebraic category of integral commutative residuated lattices into the algebraic category of K-lattices, and that the functor $\mathbf{K}$ has a left adjoint (see Section 4).

An important property is that $\mathbf{L}$ is isomorphic to the negative cone of $\mathbf{K}(\mathbf{L})$. From this, it follows that the congruence lattices of $\mathbf{L}$ and $\mathbf{K}(\mathbf{L})$ are isomorphic. On the other hand, although $\mathbf{K}(\mathbf{L})$ is distributive if and only if $\mathbf{L}$ is distributive (Corollary 3.9), in general, equations are not transferred from $\mathbf{L}$ to $\mathbf{K}(\mathbf{L})$, as is shown by the equations characterizing representability (see Section 5).

By a *K-expansion* of an integral $\mathbf{L} \in \mathbb{CRL}$, we mean a K-lattice $\mathbf{A}$ such that the negative cone of $\mathbf{A}$ is isomorphic to $\mathbf{L}$. The K-expansions of $\mathbf{L}$ are in one to one correspondence with some subalgebras of $\mathbf{K}(\mathbf{L})$ that we call *admissible*. A natural question is whether it is possible to associate each admissible subalgebra with a substructure of $\mathbf{L}$. We could not solve this problem in full generality, but we found partial solutions inspired by previous works on Heyting algebras.

A *generalized Heyting algebra* (called *implicative lattice* by Odintsov [18–20]) is an integral residuated lattice that satisfies the equation $x \cdot y = x \wedge y$. Generalized Heyting algebras can be thought as the bottom-free reducts of Heyting algebras.

The subvariety of $\mathbb{K}$ formed by the isomorphic copies of subalgebras of $\mathbf{K}(\mathbf{L})$ for generalized Heyting algebras $\mathbf{L}$ was introduced in [4, 6] under the name of *NPc-lattices* to provide an algebraic semantics for paraconsistent Nelson logic [1] based on residuated lattices. NPc-lattices can be characterized as K-lattices that satisfy the equation $(x \wedge e)^2 = x \wedge e$. The e-less reducts of NPc-lattices are termwise equivalent to Odintsov's N4-lattices [18–20], the original algebraic semantics for paraconsistent Nelson logic. As a consequence, we obtained that *for a generalized Heyting algebra* $\mathbf{L}$*, the admissible subalgebras of* $\mathbf{K}(\mathbf{L})$ *are in bijective correspondence with the Peirce filters of* $\mathbf{L}$.

Inspired in this result, we consider bounded integral commutative residuated lattices satisfying the Glivenko condition (see Theorem 6.4). For such an $\mathbf{L}$, we find a correspondence between a class of lattice filters of $\mathbf{L}$ and some special admissible subalgebras of $\mathbf{K}(\mathbf{L})$. When $\mathbf{L}$ is a Heyting algebra, our results are analogous to those of Odinstov.

Although we assume that the reader is familiar with the theory of residuated lattices, in the first section we recall some basic notions to fix the notations and we give some details on congruences, stressing the role played by implicative filters. In Section 5, before considering the relations between filters and admissible subalgebras, we establish properties of negation in bounded residuated lattices of independent interest.

2. Preliminaries

We recall for later reference some basic properties of commutative residuated lattices. For details, see [3], [12] and [14].

The residuated condition (1.1) can be replaced by the following set of equations:

(R_1) $x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z)$,
(R_2) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$,
(R_3) $(x \cdot (x \to y)) \vee y = y$,
(R_4) $(x \to (x \cdot y)) \wedge y = y$.

If the underlying lattice of $\mathbf{A} \in \mathbb{CRL}$ is distributive, we say that $\mathbf{A}$ is a *commutative distributive residuated lattice.*

Lemma 2.1. *Let* $\mathbf{A}$ *be a commutative residuated lattice, and let* $x, y, z \in A$*. The following hold.*

(RL_1) $x \leq y$ *iff* $x \to y \geq e$,
(RL_2) $x \leq y$ *implies* $x \cdot z \leq y \cdot z$,
(RL_3) $x \to (y \to z) = (x \cdot y) \to z = y \to (x \to z)$,
(RL_4) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.

A residuated lattice $\mathbf{A}$ is called *integral* provided $x \leq e$ for all $x \in A$.

The *negative cone* of $\mathbf{A} \in \mathbb{CRL}$ is the set $A^- = \{x \in A : x \leq e\}$. From RL_2 in Lemma 2.1, A^- is closed under the operations $\vee, \wedge, \cdot$, and if the binary operation $\to_e$ is defined as $x \to_e y = (x \to y) \wedge e$, then it is easy to check that $\mathbf{A}^- = (A^-, \vee, \wedge, \cdot, \to_e, e)$ is an integral commutative residuated lattice.

A convex subalgebra of $\mathbf{A} \in \mathbb{CRL}$ is a subalgebra $\mathbf{S}$ of $\mathbf{A}$ such that if $x, y \in S$, then the whole segment $[x, y] = \{z \in A : x \leq z \leq y\}$ is in S. Given a congruence θ of $\mathbf{A}$, $S_\theta = \{x \in A : (x, e) \in \theta\}$ is a convex subalgebra of $\mathbf{A}$. The following result is proved in [14, Section 2].

Theorem 2.2. *The correspondence $\theta \mapsto S_\theta$ establishes an order isomorphism from the set* $\mathrm{Cong}(\mathbf{A})$ *of congruences of* $\mathbf{A}$ *onto the set* $\mathrm{Sub}_c(\mathbf{A})$ *of convex subalgebras of* $\mathbf{A}$ *when both sets are ordered by inclusion.*

An *implicative filter* (i-filter for short) of an integral commutative residuated lattice $\mathbf{A}$ is a subset $F \subseteq A$ such that $e \in F$ and is closed under modus ponens: $x \in F$ and $x \to y \in F$ imply $y \in F$. Implicative filters can also be characterized as subsets of A that are nonempty, upwards closed, and closed by product $\cdot$. It follows easily that *implicative filters are precisely the convex subalgebras of integral commutative residuated lattices.* Hence, by Theorem 2.2, there is an order isomorphism from $\mathrm{Cong}(\mathbf{A})$ onto the set $\mathrm{Filt}(\mathbf{A})$ of i-filters of $\mathbf{A}$, ordered by inclusion.

Observe that for an implicative filter F of an integral commutative residuated lattice $\mathbf{L}$, one has that

$$(x, y) \in \theta(F) \text{ if and only if } x \to y \in F \text{ and } y \to x \in F. \tag{2.1}$$

Let $\mathbf{A} \in \mathbb{CRL}$ and let $F \in \mathrm{Filt}(\mathbf{A}^-)$. It follows from [14, Lemma 2.7] that

$$C(F) = \{x \in A : z \leq x \leq z \to e \text{ for some } z \in F\}$$

is the universe of a convex subalgebra of $\mathbf{A}$. Moreover, from the results of [14], the following theorem can be deduced.

Theorem 2.3. *Let $\mathbf{A} \in \mathbb{CRL}$. The correspondence φ:* $\mathrm{Filt}(\mathbf{A}^-) \to \mathrm{Sub}_c(\mathbf{A})$ *given by $F \mapsto C(F)$ is an order isomorphism.*

Proof. Let F be an i-filter of the integral residuated lattice $\mathbf{A}^-$. We shall see that

$$F = C(F) \cap A^-. \tag{2.2}$$

Indeed, if $z \in F$, then $z \in A^-$ and $z \leq z \leq z \to e$; thus, $F \subseteq C(F) \cap A^-$. For the opposite inclusion, take $x \in C(F) \cap A^-$. By definition, there is $z \in F$ such that $z \leq x \leq z \to e$. Since F is upwards closed, we get $x \in F$. From (2.2), we conclude that φ is injective.

To check surjectivity, let $\mathbf{S} \in \mathrm{Sub}_c(\mathbf{A})$. First we see that $F = S \cap A^-$ is an implicative filter of the negative cone of $\mathbf{A}$. Clearly, $e \in S \cap A^-$ and if $x, y \in S \cap A^-$, then $x \cdot y \in S \cap A^-$. To see that $S \cap A^-$ is upwards closed, let $x \in S \cap A^-$ and $x \leq y \leq e$. Then $x \cdot y \leq e \cdot e = e$, and $y \leq x \to e$. Hence, we have $x \leq y \leq x \to e$, and since S is convex, we get $y \in S \cap A^-$.

Now we prove that $\mathbf{S} = C(F) = C(S \cap A^-)$. The inclusion $C(F) \subseteq S$ follows immediately from the convexity of S. For the opposite inclusion, take

$s \in S$. Since $\mathbf{S}$ is a subalgebra of $\mathbf{A}$, the element $h = s \wedge e \wedge (s \to e)$ belongs to $S \cap A^-$. We have

$$s \cdot h = s \cdot (s \wedge e \wedge (s \to e)) \leq s \cdot (s \to e) \leq e.$$

Then $h \leq s \leq h \to e$ and $S \subseteq C(F)$.

The reader can easily corroborate that φ is order preserving. □

Notice that the inverse of the isomorphism φ in the above theorem is the correspondence $\mathbf{S} \mapsto S \cap A^-$. As an immediate corollary we get the following (see [14]).

Corollary 2.4. *The lattices* $\mathrm{Cong}(\mathbf{A})$ *and* $\mathrm{Cong}(\mathbf{A}^-)$ *are isomorphic.*

3. e-Lattices and twist-products

As mentioned in the Introduction, by an e-lattice we mean a commutative residuated lattice $\mathbf{A}$ which satisfies the equation

$$(x \to e) \to e = x, \tag{3.1}$$

and it is easy to see that the involution $\sim$ given by the prescription $\sim x = x \to e$ for all $x \in A$, satisfies the following properties:

(M_1) $\sim\sim x = x$,
(M_2) $\sim(x \vee y) = \sim x \wedge \sim y$,
(M_3) $\sim(x \wedge y) = \sim x \vee \sim y$,
(M_4) $\sim(x \cdot y) = x \to \sim y$.

Moreover, we have $\sim e = e$.

Lattice-ordered abelian groups with $x \cdot y = x + y$, $x \to y = y - x$, and $e = 0$ are examples of e-lattices. Other examples of e-lattices are given by the following result, which is a particular case of [22, Corollary 3.6].

Theorem 3.1. *Let* $\mathbf{L} = (L, \vee, \wedge, \cdot, \to, e)$ *be an integral commutative residuated lattice. Then* $\mathbf{K}(\mathbf{L}) = (L \times L, \sqcup, \sqcap, \cdot_{K(L)}, \to_{K(L)}, (e, e))$ *with the operations* $\sqcup, \sqcap, \cdot, \to$ *given by*

$$(a, b) \sqcup (c, d) = (a \vee c, b \wedge d) \tag{3.2}$$
$$(a, b) \sqcap (c, d) = (a \wedge c, b \vee d) \tag{3.3}$$
$$(a, b) \cdot_{K(L)} (c, d) = (a \cdot c, (a \to d) \wedge (c \to b)) \tag{3.4}$$
$$(a, b) \to_{K(L)} (c, d) = ((a \to c) \wedge (d \to b), a \cdot d) \tag{3.5}$$

is an e-lattice. Moreover, the correspondence $(a, e) \mapsto a$ *defines an isomorphism from* $(\mathbf{K}(\mathbf{L}))^-$ *onto* $\mathbf{L}$.

Definition 3.2. We call $\mathbf{K}(\mathbf{L})$ the *full twist-product* obtained from $\mathbf{L}$, and every subalgebra $\mathbf{A}$ of $\mathbf{K}(\mathbf{L})$ containing the set $\{(a, e) : a \in L\}$ is called a *twist-product* obtained from $\mathbf{L}$.

With the notation of the previous theorem, notice that for every element $(a,b) \in K(L)$, we have

$$\sim(a,b) = (a,b) \to_{K(L)} (e,e) = (b,a). \tag{3.6}$$

From now on, without danger of confusion, we shall omit the subscript $K(L)$ from the operations in $\mathbf{K}(L)$.

Remark 3.3. Let τ be a lattice equation, i.e., an equation that only involves the operations $\wedge, \vee, e$. One can define the dual of τ to be the equation τ^d that arises by substituting every appearance of $\wedge$ in τ by $\vee$ and every appearance of $\vee$ by $\wedge$. The reader can check that if $\mathbf{L}$ satisfies τ and τ^d, then $\mathbf{K}(\mathbf{L})$ also satisfies both equations. Since $\mathbf{L}$ is isomorphic to a sublattice of $\mathbf{K}(\mathbf{L})$, $\mathbf{L}$ satisfies a lattice equation and its dual if and only if $\mathbf{K}(\mathbf{L})$ satisfies both equations. In particular, $\mathbf{L}$ is a distributive lattice if and only if $\mathbf{K}(\mathbf{L})$ is distributive.

Our next aim is to characterize the e-lattices that can be represented as twist-products obtained from their negative cones.

Definition 3.4. A commutative residuated lattice $\mathbf{L} = (L, \vee, \wedge, \cdot, \to, e)$ satisfies *distributivity at* e if the distributive laws

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \tag{3.7}$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{3.8}$$

hold whenever any of x, y, z is replaced by e.

Observe that any integral commutative residuated lattice $\mathbf{L}$ trivially satisfies the equation

$$(x \cdot y) \wedge e = (x \wedge e) \cdot (y \wedge e). \tag{3.9}$$

The reader can easily verify that (3.9) is also satisfied by the twist-product $\mathbf{K}(\mathbf{L})$. We also leave to the reader to check that $\mathbf{K}(\mathbf{L})$ is distributive at e and satisfies the following equation:

$$((x \wedge e) \to y) \wedge ((\sim y \wedge e) \to \sim x) = x \to y, \tag{3.10}$$

We shall prove that the two equations (3.9), (3.10) and distributivity at e are sufficient to guarantee representability by twist-products.

Definition 3.5. A K-lattice is an e-lattice satisfying (3.9), (3.10) and distributivity at e.

It follows from the definition that K-lattices form a variety, which we denote by $\mathbb{K}$. Since lattice-ordered abelian groups do not satisfy (3.9), they provide examples of e-lattices that are not K-lattices.

It is well known and easy to verify that distributivity at e implies the quasiequation

$$x \wedge e = y \wedge e \text{ and } x \vee e = y \vee e \text{ imply } x = y. \tag{3.11}$$

Since $\sim e = e$, by M_1 and M_3, (3.11) is equivalent to

$$\text{if } x \wedge e = y \wedge e \text{ and } \sim x \wedge e = \sim y \wedge e, \quad \text{then} \quad x = y. \tag{3.12}$$

Lemma 3.6. *The equation*

$$((x \wedge e) \rightarrow (y \wedge e)) \wedge e = ((x \wedge e) \rightarrow y) \wedge e \tag{3.13}$$

holds in every K-lattice.

For the sake of completeness we reproduce a proof of this lemma given in [4].

Proof. From the fact that $(x \wedge e) \rightarrow (y \wedge e) \leq (x \wedge e) \rightarrow y$, we get the inequality $((x \wedge e) \rightarrow (y \wedge e)) \wedge e \leq ((x \wedge e) \rightarrow y) \wedge e$. For the opposite direction, since $(x \wedge e) \cdot ((x \wedge e) \rightarrow y) \leq y$, we obtain $((x \wedge e) \cdot ((x \wedge e) \rightarrow y)) \wedge e \leq y \wedge e$, which by (3.9) can be written as $(x \wedge e) \cdot (((x \wedge e) \rightarrow y) \wedge e) \leq y \wedge e$. Hence, by (1.1), $((x \wedge e) \rightarrow y) \wedge e \leq (x \wedge e) \rightarrow (y \wedge e)$, which implies

$$((x \wedge e) \rightarrow y) \wedge e \leq ((x \wedge e) \rightarrow (y \wedge e)) \wedge e. \qquad \square$$

Theorem 3.7. *Let* $\mathbf{A}$ *be a K-lattice. The map* $\phi_{\mathbf{A}} \colon \mathbf{A} \rightarrow \mathbf{K}(\mathbf{A}^-)$ *given by* $x \mapsto (x \wedge e, \sim x \wedge e)$ *is an injective homomorphism.*

Proof. We first check that $\phi_{\mathbf{A}}$ is a homomorphism. The preservation of the lattice operations relies on M_2 and distributivity at e. In detail, for $x, y \in A$,

$$\begin{aligned}
\phi_{\mathbf{A}}(x \wedge y) &= ((x \wedge y) \wedge e, \sim(x \wedge y) \wedge e) = ((x \wedge e) \wedge (y \wedge e), (\sim x \vee \sim y) \wedge e) \\
&= ((x \wedge e) \wedge (y \wedge e), (\sim x \wedge e) \vee (\sim y \wedge e)) \\
&= (x \wedge e, \sim x \wedge e) \sqcap (y \wedge e, \sim y \wedge e) = \phi_{\mathbf{A}}(x) \sqcap \phi_{\mathbf{A}}(y).
\end{aligned}$$

Similarly, one can prove that $\phi_{\mathbf{A}}$ preserves the supremum. Observe that

$$\phi_{\mathbf{A}}(\sim x) = (\sim x \wedge e, \sim \sim x \wedge e) = (\sim x \wedge e, x \wedge e) = \sim(x \wedge e, \sim x \wedge e).$$

Due to M_4, it is only left to check that $\phi_{\mathbf{A}}$ preserves $\cdot$. To that end, notice that $\phi_{\mathbf{A}}(x \cdot y) = ((x \cdot y) \wedge e, \sim(x \cdot y) \wedge e)$, that thanks to equation (3.9) and M_4, can be rewritten as

$$((x \wedge e) \cdot (y \wedge e), (x \rightarrow \sim y) \wedge e). \tag{3.14}$$

On the other hand,

$$\begin{aligned}
&\phi_{\mathbf{A}}(x) \cdot \phi_{\mathbf{A}}(y) = \\
&\quad ((x \wedge e) \cdot (y \wedge e), ((x \wedge e) \rightarrow_e (\sim y \wedge e)) \wedge ((y \wedge e) \rightarrow_e (\sim x \wedge e))).
\end{aligned} \tag{3.15}$$

Since the first components of (3.14) and (3.15) coincide, to see that $\phi_{\mathbf{A}}(x \cdot y) = \phi_{\mathbf{A}}(x) \cdot \phi_{\mathbf{A}}(y)$ it remains to prove that the second components also coincide. Observe that from the definition of $\rightarrow_e$, Lemma 3.6, and (3.10), we have

$$\begin{aligned}
&((x \wedge e) \rightarrow_e (\sim y \wedge e)) \wedge ((y \wedge e) \rightarrow_e (\sim x \wedge e)) \\
&\quad = ((x \wedge e) \rightarrow (\sim y \wedge e)) \wedge ((y \wedge e) \rightarrow (\sim x \wedge e)) \wedge e \\
&\quad = ((x \wedge e) \rightarrow (\sim y)) \wedge e \wedge ((y \wedge e) \rightarrow (\sim x)) = (x \rightarrow \sim y) \wedge e.
\end{aligned}$$

Finally, the injectivity of $\phi_{\mathbf{A}}$ follows at once from (3.11). $\square$

Remark 3.8. Since for each $a \in A^-$ we have $\phi_{\mathbf{A}}(a) = (a, e)$, it follows by restriction that $\phi_{\mathbf{A}}$ defines an isomorphism from $\mathbf{A}^-$ onto $\phi_{\mathbf{A}}(\mathbf{A})^-$.

As an immediate consequence of the above theorem and Remark 3.3, we have the following.

Corollary 3.9. *A K-lattice satisfies a lattice identity τ if and only if its negative cone satisfies τ and τ^d. In particular, a K-lattice is distributive if and only if its negative cone is distributive.*

We say that a K-lattice $\mathbf{A}$ is *total* provided that the homomorphism $\phi_{\mathbf{A}}$ defined in Theorem 3.7 is an isomorphism from $\mathbf{A}$ onto $\mathbf{K}(\mathbf{A}^-)$. Since $\phi_{\mathbf{A}}$ is always injective, it is an isomorphism if and only if it is surjective, i.e., if and only if the following condition is satisfied:

$$\text{for all } x, y \in A^-, \text{ there is } z \in A \text{ such that } z \wedge e = x \text{ and } \sim z \wedge e = y. \tag{3.16}$$

4. Categorical interpretation

Let $\mathcal{I}$ and $\mathcal{K}$ be the algebraic categories of integral commutative residuated lattices and K-lattices, respectively. Notice that the operator $\mathbf{K}$ defined on each integral commutative residuated lattice $\mathbf{L}$ by $\mathbf{K}(\mathbf{L})$ can be extended to a functor from $\mathcal{I}$ into $\mathcal{K}$ by defining $\mathbf{K}(h)(x, y) = (h(x), h(y))$ for $h\colon \mathbf{L}_1 \to \mathbf{L}_2$.

On the other hand, we can define a functor $\mathbf{F}\colon \mathcal{K} \to \mathcal{I}$ by $\mathbf{F}(\mathbf{A}) = \mathbf{A}^-$, and for each $g\colon \mathbf{A}_1 \to \mathbf{A}_2$, $\mathbf{F}(g)$ is the restriction of g to $\mathbf{A}_1^-$.

For each K-lattice $\mathbf{A}$, the operator $\phi_{\mathbf{A}}$ defined in Theorem 3.7 gives an embedding from $\mathbf{A}$ into $\mathbf{K}(\mathbf{F}(\mathbf{A}))$. Moreover, the map $\rho_{\mathbf{L}}\colon \mathbf{K}(\mathbf{L})^- \to \mathbf{L}$ given by $\rho_{\mathbf{L}}(a, e) = a$ is an isomorphism from $\mathbf{F}(\mathbf{K}(\mathbf{L}))$ onto $\mathbf{L}$.

Therefore, we can define

$$\phi\colon \mathbf{Id}_{\mathcal{K}} \to \mathbf{K} \circ \mathbf{F} \tag{4.1}$$

by $\phi(\mathbf{A}) = \phi_{\mathbf{A}}$, and

$$\rho\colon \mathbf{F} \circ \mathbf{K} \to \mathbf{Id}_{\mathcal{I}} \tag{4.2}$$

by $\rho(\mathbf{K}(\mathbf{L})^-) = \rho_{\mathbf{L}}$. A simple verification shows that the operators ϕ and ρ define natural transformations and, since ρ is an isomorphism, (4.2) is an equivalence.

For $\mathbf{A} \in \mathbb{K}$, if $a \in \mathbf{F}(\mathbf{A}) = \mathbf{A}^-$, then $\mathbf{F}(\phi_{\mathbf{A}})(a) = (a, e)$ and $\rho_{\mathbf{F}(\mathbf{A})}((a, e)) = a$. Hence, we have

$$\mathbf{F} \xrightarrow{\mathbf{F}\phi} \mathbf{FKF} \xrightarrow{\rho\mathbf{F}} \mathbf{F} = Id_{\mathbf{F}}. \tag{4.3}$$

For all integral $\mathbf{L} \in \mathbb{CRL}$, if $(a, e) \in \mathbf{F}(\mathbf{K}(\mathbf{L})) = \mathbf{K}(\mathbf{L})^-$, then

$$\phi_{\mathbf{K}(\mathbf{L})}(a, e) = ((a, e) \sqcap (e, e), (e, a) \sqcap (e, e)) = ((a, e), (e, e)),$$

and $\mathbf{K}(\rho_{\mathbf{L}})((a, e), (e, e)) = (a, e)$. Hence,

$$\mathbf{K} \xrightarrow{\phi\mathbf{K}} \mathbf{KFK} \xrightarrow{\mathbf{K}\rho} \mathbf{K} = Id_{\mathbf{K}}. \tag{4.4}$$

Recalling now [17, Section IV.1, Theorem 2 (v)], from (4.1)–(4.4) we obtain the following theorem.

Theorem 4.1. *The functor* $\mathbf{F}\colon \mathcal{K} \to \mathcal{I}$ *is the left adjoint to* $\mathbf{K}\colon \mathcal{I} \to \mathcal{K}$*. The natural transformation* ϕ *is the unit of the adjunction, and the counit is the natural equivalence* ρ*.*

If we denote by $\mathcal{T}$ the full subcategory of $\mathcal{K}$ whose objects are the total K-lattices, and by $\mathbf{F_t}$ the restriction of $\mathbf{F}$ to $\mathcal{T}$, we have the following.

Corollary 4.2. *The quadruple* $(\mathbf{F_t}, \mathbf{K}, \phi, \rho)$ *defines an equivalence between the categories* $\mathcal{I}$ *and* $\mathcal{T}$*.*

5. Representable K-lattices

A residuated lattice is *representable* if it is a subdirect product of linearly ordered residuated lattices. Given a subvariety $\mathbb{V} \subseteq \mathbb{CRL}$, it is shown in [22] that the representable residuated lattices in $\mathbb{V}$ form a subvariety of $\mathbb{V}$ characterized by the equations

$$e \wedge (x \vee y) = (e \wedge x) \vee (e \wedge y) \tag{5.1}$$

and

$$e \wedge ((x \to y) \vee (y \to x)) = e. \tag{5.2}$$

We will characterize representable K-lattices. To achieve this aim, we will investigate the possible structure of totally ordered K-lattices. Obviously, the trivial K-lattice whose only element is e is totally ordered.

We introduce the following K-lattices: let $\mathbf{B}$ be the two-element boolean algebra with underlying set $\{0, 1\}$. Then $\mathbf{K}(\mathbf{B})$ is a K-lattice with universe $\{(0, 1), (1, 1), (1, 0), (0, 0)\}$, where $e = (1, 1)$. We define $\mathbf{P}_3$, as the unique non-trivial proper subalgebra of $\mathbf{K}(\mathbf{B})$, i.e., the subalgebra with universe $P_3 = \{(0, 1), (1, 1), (1, 0)\}$. In Figure 1 one can see the lattice reduct of the algebra $\mathbf{K}(\mathbf{B})$.

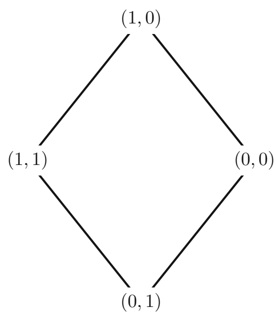

FIGURE 1

Lemma 5.1. *Every three-element K-lattice is isomorphic to* $\mathbf{P}_3$.

Proof. Assume that $\mathbf{A}$ is a K-lattice with three elements. If there is an element $a \in A \setminus A^-$, then $A^+ \setminus \{e\} \neq \emptyset$ since $e \neq a \vee e \in A^+$. Besides, because of the involution, the negative and the positive cones of $\mathbf{A}$ must be symmetric with respect to e. If $A = \{a, e, b\}$, then $a < e < b$ with e the neutral element. So the operations $\wedge$ and $\vee$ are uniquely determined by the order and the involution is given by

x	a	e	b
$\sim x$	b	e	a

Because of equation M_4, it only remains to see the behavior of $\cdot$.

Based on R_1, we can assert that $b^2 = b \cdot b \geq b \cdot e = b$ and $a^2 = a \cdot a \leq a \cdot e = a$. Thus, $b^2 = b$ and $a^2 = a$. Now we study $a \cdot b$. If $a \cdot b = b$, then from M_4 and RL_1, we get $a = {\sim} b = {\sim}(a \cdot b) = {\sim}({\sim} b \cdot b) = b \to {\sim}{\sim} b = b \to b \geq e$, which is absurd. If $a \cdot b = e$, then $a \cdot (a \cdot b) = a \cdot e = a$ while $(a \cdot a) \cdot b = a \cdot b = e$, another absurdity. Therefore, $a \cdot b = a$.

So the product $\cdot$ in $\mathbf{A}$ is uniquely determined by

$\cdot$	a	e	b
a	a	a	a
e	a	e	b
b	a	b	b

and there is only one possible structure of three-element K-lattice. It is easy to see that $\mathbf{A} \cong \mathbf{P}_3$.

□

Theorem 5.2. $\mathbf{P}_3$ *is the only nontrivial K-lattice in which every element is comparable with* e.

Proof. Assume that $\mathbf{A}$ is a K-lattice such that $A = A^+ \cup A^-$. According to Theorem 3.7, there is an integral commutative residuated lattice $\mathbf{L}$ such that $\mathbf{A}$ is isomorphic to a subalgebra of $\mathbf{K}(\mathbf{L})$. We identify $\mathbf{A}$ with this subalgebra of pairs. Observe that our hypothesis means that every pair of elements in A is either of the form (a, e) or (e, a) with $a \in L$. If there are $a, b \in L$ such that $a \neq e$, $b \neq e$ and (a, e), (b, e) are in A, then $(e, a), (e, b) \in A$. Therefore, $(a, e) \cdot (e, b) = (a, a \to b) \in A$. Since $a \neq e$, our hypothesis forces $a \to b = e$; thus, $a \leq b$. Similarly, $(b, e) \cdot (e, a) = (b, b \to a) \in A$ implies $b \leq a$. Hence, there is at most one element in $A^+ \setminus \{(e, e)\}$, namely (e, a) for some $a \in L$. The final result follows from Lemma 5.1. □

As an immediate corollary we have the following.

Theorem 5.3. *The K-lattice* $\mathbf{P}_3$ *is the only nontrivial K-lattice that is totally ordered.*

Corollary 5.4. *The subvariety of representable K-lattices is generated by* $\mathbf{P}_3$ *and it is characterized by Equation* (5.2).

Remark 5.5. It follows from Corollary 2.4 that $\mathbf{P}_3$ is a simple algebra; therefore, representable K-lattices are semisimple.

6. K-expansions

Throughout this section, $\mathbf{L}$ will denote an integral commutative residuated lattice. We say that a K-lattice $\mathbf{A}$ is a *K-expansion of* $\mathbf{L}$ provided $\mathbf{A}^-$ is isomorphic to $\mathbf{L}$. We say that a subalgebra $\mathbf{S}$ of $\mathbf{K}(\mathbf{L})$ is *admissible* provided it is a twist-product obtained from $\mathbf{L}$, that is, if $\mathbf{S}$ contains the elements (x, e) for all $x \in L$. Obviously, in such case the elements of the form (e, x) are also in $\mathbf{S}$ for all $x \in L$. The smallest admissible subalgebra of $\mathbf{K}(\mathbf{L})$ is the subalgebra generated by $\{(x, e) : x \in L\}$, which we shall denote by $\mathbf{S}_0(\mathbf{L})$, and the largest admissible subalgebra is $\mathbf{K}(\mathbf{L})$.

By Theorem 3.1, $\mathbf{K}(\mathbf{L})$ is a K-expansion of $\mathbf{L}$, and it follows from Theorem 3.7 and Remark 3.8 that the K-expansions of $\mathbf{L}$ are in correspondence with the admissible subalgebras of $\mathbf{K}(\mathbf{L})$. Therefore, a natural problem is to characterize these admissible subalgebras. We aim to give a partial solution to this problem for some classes of residuated lattices. We start with a general result.

Lemma 6.1. *The following properties are satisfied for each admissible subalgebra* $\mathbf{S}$ *of* $\mathbf{K}(\mathbf{L})$*:*

(i) *Let $x, y, s, t \in L$ be such that $x \leq s$ and $y \leq t$. If $(x, y) \in S$, then $(s, y) \in S$ and $(x, t) \in S$.*
(ii) *For all $x, y \in L$, $(x \to y, x) \in S$.*
(iii) *For all $x, y, z \in L$, if $(x, y) \in S$, then $(z, (x \to z) \to y) \in S$.*

Proof. (i): Since $(s, y) = (x, y) \sqcup (s, e)$ and $(x, t) = (x, y) \sqcap (e, t)$, we have (i).

(ii): Observe that $(x, e) \to (y, e) = (x \to y, x)$.

(iii): Note that by (ii), $(x \to z, x) \in S$. Hence,

$$(x, y) \cdot (x \to z, x) = (x \cdot (x \to z), (x \to z) \to y) \in S,$$

and then $(z, e) \sqcup (x \cdot (x \to z), (x \to z) \to y) = (z, (x \to z) \to y) \in S$. $\square$

By a *bounded residuated lattice* we mean an algebra $\mathbf{B} = \langle B, \cdot, \to, \vee, \wedge, e, 0\rangle$ of type $\langle 2, 2, 2, 2, 0, 0\rangle$ such that $\langle B, \cdot, \to, \vee, \wedge, e\rangle$ is a commutative residuated lattice and 0 is the smallest element with respect to the lattice order. In this case, $\mathbf{B}$ also has an upper bound given by $\top := 0 \to 0$. If in addition $\mathbf{B}$ is integral, then $\top = e$. The class of all bounded residuated lattice is a variety that will be denoted by $\mathbb{BCRL}$.

On a bounded residuated lattice $\mathbf{B}$, we define a unary operation $\neg$ by the prescription

$$\neg x = x \to 0, \text{ for all } x \in B. \tag{6.1}$$

If $\mathbf{B}$ is bounded with smallest element 0, then $\mathbf{K}(\mathbf{B})$ is also bounded with smallest element $(0, \top)$, and for each $(x, y) \in K(\mathbf{B})$, $\neg(x, y) = (\neg x \wedge y, x)$.

In light of Theorem 3.7, the last equality implies that for each bounded K-lattice $\mathbf{A}$, we have that $\neg x \leq {\sim} x$ for all $x \in A$.

Lemma 6.2. *The following properties hold true in any* $\mathbf{B} \in \mathbb{BCRL}$*:*

(i) $x \leq y$ *implies* $\neg y \leq \neg x$.
(ii) $\neg x = \neg\neg\neg x$.
(iii) $x \leq \neg\neg x$.
(iv) $\neg\neg(\neg\neg x \wedge \neg\neg y) = \neg\neg x \wedge \neg\neg y$.
(v) $x \to \neg y = y \to \neg x$.
(vi) $x \to \neg y = \neg\neg x \to \neg y$.
(vii) $\neg\neg(x \to \neg\neg y) = x \to \neg\neg y$.
(viii) $\neg\neg(\neg\neg x \to \neg\neg y) = \neg\neg x \to \neg\neg y$.
(ix) $\neg(x \cdot y) = x \to \neg y$.
(x) $\neg(x \vee y) = \neg x \wedge \neg y$.

Proof. (i)–(v): Properties (i),(ii), (iii), and (v) easily follow from the definition (for a proof see [12, Lemma 2.8]), and (iv) follows immediately from the previous ones.

Using some of the ideas of [10] we prove the remaining properties.

(vi): This is a consequence of items (ii) and (v), since

$$x \to \neg y = y \to \neg x = y \to \neg\neg\neg x = \neg\neg x \to \neg y.$$

(vii): By (iii), we have $x \to \neg\neg y \leq \neg\neg(x \to \neg\neg y)$. On the other hand, observing RL_3 in Lemma 2.1 together with (vi), we get

$$\begin{aligned}\neg\neg(x \to \neg\neg y) \to (x \to \neg\neg y) &= x \to (\neg\neg(x \to \neg\neg y) \to \neg\neg y)\\ &= x \to ((x \to \neg\neg y) \to \neg\neg y) = (x \to \neg\neg y) \to (x \to \neg\neg y) \geq e;\end{aligned}$$

thus, RL_1 implies $x \to \neg\neg y \geq \neg\neg(x \to \neg\neg y)$.

(viii): This is an easy consequence of (vii).

(ix): This is an application of RL_3.

(x): This follows from the fact that $(x \vee y) \to z = (x \to z) \wedge (y \to z)$ holds for all $x, y, z \in B$. □

Let $\mathbf{B} \in \mathbb{BCRL}$. An element $x \in B$ is said to be *regular* provided $\neg\neg x = x$, and the set of regular elements of $\mathbf{B}$ will be denoted by $Reg(\mathbf{B})$. By an *involutive residuated lattice* we mean a bounded commutative residuated lattice $\mathbf{B}$ such that $B = Reg(\mathbf{B})$, i.e., such that $\mathbf{B}$ satisfies the equation $\neg\neg\, x = x$. In other words, B is involutive if and only if 0 is a dualizing element.

Notice that if $\mathbf{B}$ is bounded with lowest element 0, then $(x, y) \in Reg(\mathbf{K}(\mathbf{B}))$ if and only if $x \cdot y = 0$. Indeed, take $x, y \in B$. If $x \cdot y = 0$, one gets $y \leq \neg x$ and $x \leq \neg y$. Then

$$\neg\neg(x, y) = \neg(\neg x \wedge y, x) = \neg(y, x) = (\neg y \wedge x, y) = (x, y).$$

On the other hand, since the second coordinate of $\neg\neg(x, y)$ is $\neg x \wedge y$, if the equation $\neg\neg(x, y) = (x, y)$ holds, then $y = y \wedge \neg x$, and this implies $x \cdot y = 0$.

In particular, $(e, e) \notin Reg(\mathbf{K}(\mathbf{B}))$. Hence, no (nontrivial) bounded K-lattice is involutive.

Suppose now that $\mathbf{B} \in \mathbb{BCRL}$ is integral, and for each $\diamond \in \{\vee, \wedge, \to, \cdot\}$, define $x \diamond_R y = \neg\neg(x \diamond y)$ for $x, y \in Reg(\mathbf{B})$. It is easy to see that

$$\mathbf{Reg}(\mathbf{B}) = (Reg(\mathbf{B}), \vee_R, \wedge_R, \to_R, \cdot_R, e, 0)$$

is an integral involutive residuated lattice.

Remark 6.3. Let $\mathbf{B} \in \mathbb{BCRL}$. If e is not regular, then there is no unit for the product $\cdot_R$ defined on $Reg(\mathbf{B})$. If $\mathbf{B}$ is integral, then (iii) in Lemma 6.2 guaranties that e is regular.

Since (2.1) implies that the product as well as the lattice operations of an integral bounded commutative residuated lattices are compatible with the congruences of its underlying BCK-algebra structure, the following result is a consequence of [10, Lemma 3.3 and Theorem 4.6].

Theorem 6.4. *For each integral bounded commutative residuated lattice* $\mathbf{B}$*, the following are equivalent statements:*

(i) $\neg\neg\colon \mathbf{B} \to \mathbf{Reg}(\mathbf{B})$ *is a homomorphism.*
(ii) $\mathbf{B}$ *satisfies the equation* $\neg\neg(\neg\neg\, x \to x) = e$.
(iii) $\mathbf{B}$ *satisfies the equation* $\neg\neg(x \to y) = x \to \neg\neg y$.

By a *Glivenko residuated lattice* we mean an integral bounded commutative residuated lattice satisfying any of the equivalent conditions stated in the above theorem. The variety of Glivenko residuated lattices will be denoted by $\mathbb{G}$.

Integral involutive residuated lattices are trivially Glivenko. Heyting algebras are Glivenko. More generally, integral bounded commutative residuated lattices that satisfy the hoop equation

$$x \wedge y = x \cdot (x \to y) \tag{6.2}$$

are Glivenko (see the proof of [9, Lemma 1.3]).

Let $\mathbf{B} \in \mathbb{G}$. It follows from (i) in Theorem 6.4 and (iv) in Lemma 6.2 that

$$\neg\neg(x \wedge y) = \neg\neg\, x \wedge_R \neg\neg y = \neg\neg\, x \wedge \neg\neg y, \tag{6.3}$$

and from (i) in Theorem 6.4 and (viii) in Lemma 6.2 that

$$\neg\neg(x \to y) = \neg\neg\, x \to_R \neg\neg y = \neg\neg x \to \neg\neg y. \tag{6.4}$$

In what follows, $\mathbf{B}$ will denote a Glivenko residuated lattice. A lattice filter F of $\mathbf{B}$ is said to be *regular* provided $\neg\neg x \in F$ implies $x \in F$.

Lemma 6.5. *For each lattice filter* F *of* $\mathbf{B}$*,* $\overline{F} = F \cup \{x \in B : \neg\neg x \in F\}$ *is a regular lattice filter.*

Proof. Observe that $x \in \overline{F}$ if and only if $\neg\neg x \in F$. Clearly, $e \in \overline{F}$. Suppose $x \in \overline{F}$ and $x \leq y$. Then $\neg\neg x \leq \neg\neg y$, and since $\neg\neg x \in F$, $\neg\neg y \in F$; therefore, $y \in \overline{F}$. Finally, suppose $x, y \in \overline{F}$. Then by (6.3), $\neg\neg(x \wedge y) = \neg\neg x \wedge \neg\neg y \in F$, and $x \wedge y \in \overline{F}$. $\square$

Lemma 6.6. *For each lattice filter F of* $\mathbf{B}$*,*

$$S_F = \{(x, y) \in K(\mathbf{B}) : \neg x \to \neg\neg y \in F\}$$

is the universe of an admissible subalgebra $\mathbf{S}_F$ *of* $\mathbf{K}(\mathbf{B})$ *such that for all* $x, y \in B$*, if* $(\neg\neg x, \neg\neg y) \in S_F$*, then* $(x, y) \in S_F$*. Moreover,* $\mathbf{S}_F = \mathbf{S}_{\overline{F}}$*.*

Proof. Clearly, $S_F \subseteq S_{\overline{F}}$. On the other hand, if $(x, y) \in S_{\overline{F}}$, then by (vii) in Lemma 6.2, we get $\neg x \to \neg\neg y = \neg\neg(\neg x \to \neg\neg y) \in F$, so $(x, y) \in S_F$. Thus, $\mathbf{S}_F = \mathbf{S}_{\overline{F}}$. Also, by (ii) in Lemma 6.2, we get that $(\neg\neg x, \neg\neg y) \in S_F$ implies $(x, y) \in S_F$.

Since $\neg x \to \neg\neg e = e$, we have that for every $x \in B$, the pairs (x, e) and (e, x) are in S_F. It follows from (ix) in Lemma 6.2 that $\neg x \to \neg\neg y = \neg y \to \neg\neg x$. Hence, S_F is closed under $\sim$. Therefore, in the light of M_3 and M_4, to complete the proof we need to show that S_F is closed under $\cdot$ and $\sqcup$.

Suppose (x, y) and (s, t) are both in S_F. From the definition of S_F (1.1), item RL_2 in Lemma 2.1, and the fact that lattice filters are closed under $\wedge$, this is equivalent to the existence of $f \in F$ such that $f \cdot \neg x \leq \neg\neg y$ and $f \cdot \neg s \leq \neg\neg t$.

By (ix) in Lemma 6.2, $x \cdot \neg(x \cdot s) = x \cdot (x \to \neg s) \leq \neg s$. Hence,

$$f \cdot x \cdot \neg(x \cdot s) \leq f \cdot \neg s \leq \neg\neg t,$$

and then taking into account (iii) in Theorem 6.4, we have

$$f \cdot \neg(x \cdot s) \leq x \to \neg\neg t = \neg\neg(x \to t).$$

Analogously, we obtain that $f \cdot \neg(x \cdot s) \leq \neg\neg(s \to y)$. Hence, by (6.3),

$$f \cdot \neg(x \cdot s) \leq \neg\neg(x \to t) \wedge \neg\neg(s \to y) = \neg\neg((x \to t) \wedge (s \to y)),$$

implying that $(x, y) \cdot (s, t) \in S_F$. Therefore, S_F is closed under $\cdot$.

By (x) in Lemma 6.2 and (6.3), we have

$$f \cdot \neg(x \vee s) = f \cdot (\neg x \wedge \neg s) \leq (f \cdot \neg x) \wedge (f \cdot \neg s) \leq \neg\neg y \wedge \neg\neg t = \neg\neg(y \wedge t),$$

implying that $(x, y) \sqcup (s, t) \in S_F$. Therefore, S_F is also closed under $\sqcup$. □

Lemma 6.7. *Let* $\mathbf{S}$ *be an admissible subalgebra of* $\mathbf{K}(\mathbf{B})$*. Then*

$$F_{\mathbf{S}} = \{x \in B : (0, x) \in S\}$$

is a lattice filter of $\mathbf{B}$*.*

Proof. Since $\mathbf{S}$ is an admissible subalgebra of $\mathbf{B}$, we have that for each $x \in B$, the elements (x, e) and (e, x) are in S. In particular $(0, e) \in S$; thus, $e \in F_{\mathbf{S}}$. Clearly, if $x, y \in F_{\mathbf{S}}$, we have that $(0, x)$ and $(0, y)$ are in S and $(0, x) \sqcup (0, y) = (0, x \wedge y)$ is in S; hence, $x \wedge y \in F_{\mathbf{S}}$. Finally, if $x \in F_{\mathbf{S}}$ and $y \geq x$, then $(0, x) \sqcap (e, y) = (0, y)$ is an element of S. □

We say that an admissible subalgebra $\mathbf{S}$ of $\mathbf{K}(\mathbf{B})$ is *regular* provided that $(\neg\neg x, \neg\neg y) \in S$ implies $(x, y) \in S$.

Lemma 6.8. *For each admissible subalgebra* $\mathbf{S}$ *of* $\mathbf{K}(\mathbf{B})$*, one has* $S \subseteq S_{F_\mathbf{S}}$*, and the equality holds if and only if* $\mathbf{S}$ *is regular.*

Proof. Let $(x, y) \in S$. Take $z = 0$ in Lemma 6.1(iii); then $(0, \neg x \to y) \in S$, and this means that $\neg x \to y \in F_\mathbf{S}$. As $\neg x \to y \leq \neg x \to \neg\neg y$, so $\neg x \to \neg\neg y \in F_\mathbf{S}$, which means that $(x, y) \in S_{F_\mathbf{S}}$. Therefore, $S \subseteq S_{F_\mathbf{S}}$. By Lemma 6.6, $S_{F_\mathbf{S}}$ is regular; hence, if $S = S_{F_\mathbf{S}}$, then $\mathbf{S}$ has to be regular. Suppose that $\mathbf{S}$ is regular, and let $(x, y) \in S_{F_\mathbf{S}}$. This means that $\neg x \to \neg\neg y \in F_\mathbf{S}$, which in turn means that $(0, \neg\, x \to \neg\neg y) \in S$. Then

$$((\neg x, e) \to (0, \neg\, x \to \neg\neg y)) \sqcap (e, \neg\neg y) = (\neg\neg x, \neg\neg y) \in S,$$

and since $\mathbf{S}$ is regular, we have $S_{F_\mathbf{S}} \subseteq S$. □

Lemma 6.9. *For each regular lattice filter* F *of* $\mathbf{B}$*, one has* $F = F_{\mathbf{S}_F}$*.*

Proof. $x \in F$ if and only if $\neg\, 0 \to \neg\neg\, x = \neg\neg\, x \in F$ if and only if $(0, x) \in S_F$ if and only if $x \in F_{\mathbf{S}_\mathbf{F}}$. □

From Lemmas 6.8 and 6.9 we obtain the following two theorems.

Theorem 6.10. *For each* $\mathbf{B} \in \mathbb{G}$*, the correspondence* $F \mapsto \mathbf{S}_\mathbf{F}$ *defines a bijection from the set of regular lattice filters of* $\mathbf{B}$ *onto the set of regular admissible subalgebras of* $\mathbf{K}(\mathbf{B})$*. The inverse mapping is given by the correspondence* $\mathbf{S} \mapsto F_\mathbf{S}$*.*

Theorem 6.11. *For each integral involutive residuated lattice* $\mathbf{B}$*, the correspondence* $F \mapsto \mathbf{S}_\mathbf{F}$ *defines a bijection from the set of lattice filters of* $\mathbf{B}$ *onto the set of admissible subalgebras of* $\mathbf{K}(\mathbf{B})$*. The inverse mapping is given by the correspondence* $\mathbf{S} \mapsto F_\mathbf{S}$*.*

Lemma 6.12. *Let* $\mathbf{S}$ *be an admissible subalgebra of* $\mathbf{K}(\mathbf{B})$*, and let* F' *be a regular lattice filter of* $\mathbf{B}$ *such that* $S \subseteq S_{F'}$*. Then* $S_{F_S} \subseteq S_{F'}$*.*

Proof. Assume $(x, y) \in S_{F_S}$. Then $\neg x \to \neg\neg y \in F_S$ and $(0, \neg x \to \neg\neg y) \in S$. By our hypothesis, we get $(0, \neg x \to \neg\neg y) \in S_{F'}$ and $\neg 0 \to (\neg x \to \neg\neg y) \in F'$. The integrality of $\mathbf{B}$ yields that

$$\neg 0 \to (\neg x \to \neg\neg y) = e \to (\neg x \to \neg\neg y) = \neg x \to \neg\neg y$$

is in F', and therefore $(x, y) \in S_{F'}$. □

An element x in a bounded residuated lattice $\mathbf{B}$ is said to be *dense* provided $\neg x = 0$, and the set of dense elements of $\mathbf{B}$ will be denoted by $D(\mathbf{B})$. It is easy to corroborate that when the bounded residuated lattice $\mathbf{B}$ is integral, then $D(\mathbf{B})$ is an i-filter of $\mathbf{B}$. Therefore, it is also a lattice filter of $\mathbf{B}$. Observe that if $\mathbf{B} \in \mathbb{G}$ and $x \in B$ is such that $\neg\neg x \in D(\mathbf{B})$, then $\neg\neg\neg x = 0$, and from (ii) in Lemma 6.2, we conclude that $D(\mathbf{B})$ is a regular filter of $\mathbf{B}$. As a matter of fact, for each regular filter F of $\mathbf{B}$ and for each $x \in D(\mathbf{B})$, since $\neg\neg x = e \in F$, one has that $D(\mathbf{B}) \subseteq F$. Besides, due to (ii) in Theorem 6.4, for each $x \in B$, $\neg\neg x \to x \in D(\mathbf{B})$; thus, any i-filter containing $D(\mathbf{B})$ is regular, considered as a lattice filter. It follows that $S_{D(\mathbf{B})}$ is a regular admissible subalgebra of $\mathbf{B}$, and that for each regular admissible subalgebra S of $\mathbf{B}$, one has $S_{D(\mathbf{B})} \subseteq S$. Moreover, it is easy to check that $(x, y) \in K(\mathbf{B})$ belongs to $S_{D(\mathbf{B})}$ if and only if $\neg x \cdot \neg y = 0$. Hence, $\{(\neg x, \neg y) : (x, y) \in S_{D(\mathbf{B})}\} \subseteq Reg(\mathbf{K}(\mathbf{B}))$. This fact together with Lemma 6.12 yields the following.

Theorem 6.13. *With the previous notation, one has* $S_{D(\mathbf{B})} \cong S_{F_{S_0(\mathbf{B})}}$.

Let $\mathbf{A}$ be a bounded K-lattice, and let $\phi_{\mathbf{A}} \colon \mathbf{A} \to \mathbf{K}(\mathbf{A}^-)$ be the embedding defined in Theorem 3.7. Then the image $\phi_{\mathbf{A}}(A)$ is the universe of a regular admissible subalgebra of $\mathbf{K}(\mathbf{A}^-)$ if and only if $\mathbf{A}$ satisfies the following condition:

6.14. For all $x, y \in A^-$, if there is $z \in A$ such that

$$\neg_e \neg_e x = z \wedge e \text{ and } \neg_e \neg_e y = \sim z \wedge e,$$

then there is $w \in A$ such that $x = w \wedge e$ and $y = \sim w \wedge e.$, where $\neg_e$ denotes the negation in the negative cone.

Hence, Corollary 6.10 shows that for each $\mathbf{B} \in \mathbb{G}$, the correspondence $F \mapsto \mathbf{S_F}$ defines a bijection from the set of regular lattice filters of $\mathbf{B}$ onto the set of K-expansions of $\mathbf{B}$ that satisfies condition 6.14. On the other hand, Corollary 6.11 shows that for each integral involutive lattice $\mathbf{B}$, the correspondence $F \mapsto \mathbf{S_F}$ defines a bijection from the set of lattice filters of $\mathbf{B}$ onto the set of all K-expansions of $\mathbf{B}$.

By a *pseudocomplemented residuated lattice* we mean a bounded integral commutative residuated lattice $\mathbf{B}$ that satisfies the equation $x \wedge \neg x = 0$.

Lemma 6.15. *Let* $\mathbf{B}$ *be a distributive pseudocomplemented residuated lattice. Then all admissible subalgebras of* $\mathbf{K}(\mathbf{B})$ *are regular.*

Proof. Let $\mathbf{S}$ be an admissible subalgebra of $\mathbf{B}$ such that $(\neg\neg x, \neg\neg y) \in S$. Since by (ii) in Lemma 6.1, $(\neg x, x) \in S$, we have that $(\neg x, x) \sqcap (\neg\neg x, \neg\neg y) = (0, x \vee \neg\neg y) \in S$. Then $(0, x \vee \neg\neg\, y) \sqcup (y, \neg y) = (y, \neg y \wedge x) \in S$, and $(y, \neg y \wedge x) \sqcap (e, x) = (y, x) \in S$. $\square$

Lemma 6.16. *Every pseudocomplemented residuated lattice is Glivenko.*

Proof. Let $\mathbf{B} \in \mathbb{BCRL}$ be integral. For each $a \in B$, taking into account RL_3 and RL_4, we have

$$\begin{aligned}(\neg a \vee a) \to (\neg\neg a \to a) &= \neg a \to (\neg\neg a \to a)\\ &= \neg\neg a \to (\neg a \to a) \geq \neg\neg a \to \neg\neg a = e.\end{aligned}$$

Therefore, $\neg\neg(\neg a \vee a) \leq \neg\neg(\neg\neg a \to a)$. Finally, observe that all pseudocomplemented residuated lattices satisfy the equation $\neg\neg(\neg x \vee x) = e$. $\square$

From Lemmas 6.10, 6.15, and 6.16 we obtain the following theorem.

Theorem 6.17. *For each pseudocomplemented distributive residuated lattice* $\mathbf{B}$*, the correspondence* $F \mapsto \mathbf{S_F}$ *defines a bijection from the set of regular filters of* $\mathbf{B}$ *onto the set of all admissible subalgebras of* $\mathbf{K}(\mathbf{B})$*.*

A filter of a Heyting algebra is called *boolean* if it contains all the dense elements [21]. The next result should be compared with [4, Section 3].

Corollary 6.18. *For each Heyting algebra* $\mathbf{H}$*, the correspondence* $F \mapsto \mathbf{S_F}$ *defines a bijection from the set of boolean filters of* $\mathbf{H}$ *onto the set of all admissible subalgebras of* $\mathbf{K}(\mathbf{H})$*.*

7. Open problems and further research

We believe that the key to understanding K-lattices is the study of twist-products obtained from an arbitrary commutative integral residuated lattice $\mathbf{L}$. This is equivalent to the investigation of admissible subalgebras of $\mathbf{K}(\mathbf{L})$. We list in order of complexity some of the open questions that could help to achieve such an aim.

(1) Determine if in the statement of Lemma 6.15, the distributivity can be relaxed.
(2) Determine if there is a Glivenko residuated lattice $\mathbf{B}$ such that $\mathbf{K}(\mathbf{B})$ has an admissible subalgebra which is not regular.
(3) Characterize admissible subalgebras of the full twist-product $\mathbf{K}(\mathbf{B})$ for $\mathbf{B}$ an arbitrary bounded integral commutative residuated lattice.
(4) Characterize admissible subalgebras of the full twist-product $\mathbf{K}(\mathbf{L})$ for $\mathbf{L}$ an arbitrary integral commutative residuated lattice.

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