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Unitary Quasi-finite Representations of W_∞

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Abstract. We classify the unitary quasi-finite highest-weight modules over the Lie algebra W_∞ and realize them in terms of unitary highest-weight representations of the Lie algebra of infinite matrices with finitely many nonzero diagonals.

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1. Introduction

The $\mathcal{W}$-infinity algebras naturally arise in various physical theories, such as conformal field theory, the theory of the quantum Hall effect, etc. The $W_{1+\infty}$ algebra, which is the central extension of the Lie algebra $\mathcal{D}$ of differential operators on the circle, is the most fundamental among these algebras.

When we study the representation theory of a Lie algebra of this kind, we encounter the difficulty that, although it admits a $\mathbb{Z}$-gradation, each of the graded subspaces is still infinite-dimensional, and therefore the study of highest-weight modules which satisfy the quasi-finiteness condition that its graded subspaces have finite dimension, becomes a nontrivial problem.

The study of representations of the Lie algebra $W_{1+\infty}$ was initiated in [5], where a characterization of its irreducible quasi-finite highest-weight representations was given, these modules were constructed in terms of irreducible highest-weight representations of the Lie algebra of infinite matrices, and the unitary ones were described. On the basis of this analysis, further studies were made within the framework of vertex algebra theory for the $W_{1+\infty}$ algebra [4, 6], and for its matrix version [3]. The case of orthogonal subalgebras of $W_{1+\infty}$ was studied in [7]. The symplectic subalgebra of $W_{1+\infty}$ was considered in [2] in relation to number theory.

The paper [1] developed a theory of quasi-finite highest-weight representation of the subalgebras $W_{\infty,p}$ of $W_{1+\infty}$, where $W_{\infty,p}$ ($p \in \mathbb{C}[x]$) is the central extension of the Lie algebra $\mathcal{D}p(t\partial_t)$ of differential operators on the circle that are a multiple

⋆Current address: Department of Mathematics, MIT, Cambridge, MA 02139, U.S.A. e-mail: liberati@math.mit.edu of $p(t\partial_t)$. The most important of these subalgebras is $W_\infty = W_{\infty,x}$ that is obtained by taking $p(x) = x$. In this Letter we develop in Section 2 a general approach to these problems, which makes the basic ideas of [5] much clearer. In Section 3, we give a description of parabolic subalgebras of $W_{\infty,p}$, which we use in Section 4 to classify all its irreducible quasi-finite highest-weight modules, recovering the main result of [1]. In Section 5, we describe the relation of W_∞ to the central extension of the Lie algebra of infinite matrices with finitely many nonzero diagonals and, using this relation, we establish the main result of this article in Section 6: the classification and construction of all unitary irreducible quasi-finite modules over W_∞. Surprisingly, the list of unitary modules over W_∞ is much richer than that over $W_{1+\infty}$.

2. Quasi-finite Representations of $\mathbb{Z}$-Graded Lie Algebras

Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie algebra over $\mathbb{C}$:

$$\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j\ , \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j},$$

where $\mathfrak{g}_i$ is not necessarily of finite dimension. Let $\mathfrak{g}_\pm = \oplus_{j>0}\mathfrak{g}_{\pm j}$. A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is called *parabolic* if it contains $\mathfrak{g}_0 \oplus \mathfrak{g}_+$ as a proper subalgebra, that is

$$\mathfrak{p} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{p}_j, \quad \text{where } \mathfrak{p}_j = \mathfrak{g}_j \text{ for } j \geqslant 0, \text{ and } \mathfrak{p}_j \neq 0 \text{ for some } j < 0.$$

We assume the following properties of $\mathfrak{g}$:

(P1) $\mathfrak{g}_0$ is commutative,
(P2) if $a \in \mathfrak{g}_{-k}$ $(k > 0)$ and $[a, \mathfrak{g}_1] = 0$, then $a = 0$.

LEMMA 2.1. *For any parabolic subalgebra* $\mathfrak{p}$ *of* $\mathfrak{g}$, $\mathfrak{p}_{-k} \neq 0$, $k > 0$, *implies* $\mathfrak{p}_{-k+1} \neq 0$.
Proof. If $\mathfrak{p}_{-k+1} = 0$, then $[\mathfrak{p}_{-k}, \mathfrak{g}_1] = 0$, i.e. for all $a \in \mathfrak{p}_{-k}$, $[a, \mathfrak{g}_1] = 0$, and using (P2), we get $a = 0$. □

Given $a \in \mathfrak{g}_{-1}$, $a \neq 0$, we define $\mathfrak{p}^a = \oplus_{j\in\mathbb{Z}}\mathfrak{p}^a_j$, where $\mathfrak{p}^a_j = \mathfrak{g}_j$ for all $j \geqslant 0$, and

$$\mathfrak{p}^a_{-1} = \sum [\ldots[[a, \mathfrak{g}_0], \mathfrak{g}_0], \ldots]\ , \qquad \mathfrak{p}^a_{-k-1} = [\mathfrak{p}^a_{-1}, \mathfrak{p}^a_{-k}].$$

LEMMA 2.2. (a) $\mathfrak{p}^a$ *is the minimal parabolic subalgebra containing a.*
(b) $\mathfrak{g}^a_0 := [\mathfrak{p}^a, \mathfrak{p}^a] \cap \mathfrak{g}_0 = [a, \mathfrak{g}_1]$.

Proof. (a) We have to prove that $\mathfrak{p}^a$ is a subalgebra. First, $[\mathfrak{p}^a_{-k}, \mathfrak{p}^a_{-l}] \subseteq \mathfrak{p}^a_{-l-k}$ $(k, l > 0)$ is proved by induction on k:

$$\begin{aligned}[\mathfrak{p}^a_{-k}, \mathfrak{p}^a_{-l}] &= [[(\mathfrak{p}^a_{-1})^{k-1}, \mathfrak{p}^a_{-1}], (\mathfrak{p}^a_{-1})^l] \\ &= [[(\mathfrak{p}^a_{-1})^{k-1}, (\mathfrak{p}^a_{-1})^l], \mathfrak{p}^a_{-1}] + [(\mathfrak{p}^a_{-1})^{k-1}, [\mathfrak{p}^a_{-1}, (\mathfrak{p}^a_{-1})^l]] \\ &\subseteq [(\mathfrak{p}^a_{-1})^{l+k-1}, \mathfrak{p}^a_{-1}] + [(\mathfrak{p}^a_{-1})^{k-1}, (\mathfrak{p}^a_{-1})^{l+1}] \\ &\subseteq (\mathfrak{p}^a_{-1})^{k+l}.\end{aligned}$$

And $[\mathfrak{p}^a_{-k}, \mathfrak{g}_m] \subseteq \mathfrak{p}^a_{m-k}$ $(m < k)$ also follows by induction on k:

$$\begin{aligned}[\mathfrak{p}^a_{-k}, \mathfrak{g}_m] &= [[(\mathfrak{p}^a_{-1})^{k-1}, \mathfrak{p}^a_{-1}], \mathfrak{g}_m] \\ &= [[(\mathfrak{p}^a_{-1})^{k-1}, \mathfrak{g}_m], \mathfrak{p}^a_{-1}] + [(\mathfrak{p}^a_{-1})^{k-1}, [\mathfrak{p}^a_{-1}, \mathfrak{g}_m]] \\ &\subseteq [\mathfrak{p}^a_{m-k+1}, \mathfrak{p}^a_{-1}] + [(\mathfrak{p}^a_{-1})^{k-1}, \mathfrak{g}_{m-1}] \\ &\subseteq \mathfrak{p}^a_{m-k}.\end{aligned}$$

Finally, it is obviously the minimal one, proving (a).

(b) For any $k > 1$:

$$\begin{aligned}[\mathfrak{p}^a_{-k}, \mathfrak{g}_k] &= [[(\mathfrak{p}^a_{-1})^{k-1}, \mathfrak{g}_k], \mathfrak{p}^a_{-1}] + [(\mathfrak{p}^a_{-1})^{k-1}, [\mathfrak{p}^a_{-1}, \mathfrak{g}_k]] \\ &\subseteq [\mathfrak{g}_1, \mathfrak{p}^a_{-1}] + [(\mathfrak{p}^a_{-1})^{k-1}, \mathfrak{g}_{k-1}].\end{aligned}$$

Therefore, by induction, $\mathfrak{g}^a_0 = [\mathfrak{g}_1, \mathfrak{p}^a_{-1}]$. But

$$\begin{aligned}[\mathfrak{g}_1, \mathfrak{p}^a_{-1}] &= \text{ linear span } \{[\ldots[[a, c_1], c_2], \ldots], x] : c_i \in \mathfrak{g}_0, x \in \mathfrak{g}_1\} \quad \text{(using (P1))} \\ &= \text{ linear span } \{[a, [c_1, \ldots [c_{k-1}, [c_k, x] \ldots] : c_i \in \mathfrak{g}_0, x \in \mathfrak{g}_1\} \\ &= [a, \mathfrak{g}_1].\end{aligned}$$

proving the lemma. □

In the particular case of the central extension of the Lie algebra of matrix differential operators on the circle (see [3], Remark 2.2), we observed the existence of some parabolic subalgebras $\mathfrak{p}$ such that $\mathfrak{p}_{-j} = 0$ for $j >> 0$. Having in mind that example, we give the following definition:

DEFINITION 2.3. (a) A parabolic subalgebra $\mathfrak{p}$ is called *nondegenerate* if $\mathfrak{p}_{-j}$ has finite codimension in $\mathfrak{g}_{-j}$, for all $j > 0$.

(b) An element $a \in \mathfrak{g}_{-1}$ is called *nondegenerate* if $\mathfrak{p}^a$ is nondegenerate.

Now, we begin our study of quasi-finite representations over $\mathfrak{g}$. A $\mathfrak{g}$-module V is called $\mathbb{Z}$*-graded* if $V = \oplus_{j\in\mathbb{Z}} V_j$ and $\mathfrak{g}_i V_j \subset V_{i+j}$. A $\mathbb{Z}$-graded $\mathfrak{g}$-module V is called *quasi-finite* if $\dim V_j < \infty$ for all j.

Given $\lambda \in \mathfrak{g}_0^*$, a *highest-weight module* is a $\mathbb{Z}$-graded $\mathfrak{g}$-module $V(\mathfrak{g}, \lambda)$ generated by a highest-weight vector $v_\lambda \in V(\mathfrak{g}, \lambda)_0$ which satisfies

$$hv_\lambda = \lambda(h)v_\lambda \qquad (h \in \mathfrak{g}_0), \qquad \mathfrak{g}_+ v_\lambda = 0.$$

A nonzero vector $v \in V(\mathfrak{g}, \lambda)$ is called *singular* if $\mathfrak{g}_+ v = 0$.

The *Verma module* over $\mathfrak{g}$ is defined as usual:

$$M(\mathfrak{g}, \lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0 \oplus \mathfrak{g}_+)} \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ is the one-dimensional $(\mathfrak{g}_0 \oplus \mathfrak{g}_+)$-module given by $h \mapsto \lambda(h)$ if $h \in \mathfrak{g}_0$, $\mathfrak{g}_+ \mapsto 0$, and the action of $\mathfrak{g}$ is induced by the left multiplication in $\mathcal{U}(\mathfrak{g})$. Here and further $\mathcal{U}(\mathfrak{q})$ stands for the universal enveloping algebra of the Lie algebra $\mathfrak{q}$. Any highest-weight module $V(\mathfrak{g}, \lambda)$ is a quotient module of $M(\mathfrak{g}, \lambda)$. The irreducible module $L(\mathfrak{g}, \lambda)$ is the quotient of $M(\mathfrak{g}, \lambda)$ by the maximal proper graded submodule. We shall write $M(\lambda)$ and $L(\lambda)$ in place of $M(\mathfrak{g}, \lambda)$ and $L(\mathfrak{g}, \lambda)$ if no ambiguity may arise.

Consider a parabolic subalgebra $\mathfrak{p} = \oplus_{j \in \mathbb{Z}} \mathfrak{p}_j$ of $\mathfrak{g}$ and let $\lambda \in \mathfrak{g}_0^*$ be such that $\lambda|_{\mathfrak{g}_0 \cap [\mathfrak{p}, \mathfrak{p}]} = 0$. Then the $(\mathfrak{g}_0 \oplus \mathfrak{g}_+)$-module $\mathbb{C}_\lambda$ extends to a $\mathfrak{p}$-module by letting $\mathfrak{p}_j$ act as 0 for $j < 0$, and we may construct the highest-weight module

$$M(\mathfrak{g}, \mathfrak{p}, \lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \mathbb{C}_\lambda$$

called the *generalized Verma module*.

We will also require the following condition on $\mathfrak{g}$:

(P3) If $\mathfrak{p}$ is a nondegenerate parabolic subalgebra of $\mathfrak{g}$, then there exists a nondegenerate element a such that $\mathfrak{p}^a \subseteq \mathfrak{p}$.

Remark 2.4. In all the examples considered in [3, 5, 7] and Section 3 of this work, property (P3) is satisfied.

THEOREM 2.5. *The following conditions on* $\lambda \in \mathfrak{g}_0^*$ *are equivalent:*

(1) $M(\lambda)$ *contains a singular vector* $a.v_\lambda$ *in* $M(\lambda)_{-1}$, *where* a *is nondegenerate;*
(2) *There exist a nondegenerate element* $a \in \mathfrak{g}_{-1}$, *such that* $\lambda([\mathfrak{g}_1, a]) = 0$.
(3) $L(\lambda)$ *is quasi-finite;*
(4) *There exist a nondegenerate element* $a \in \mathfrak{g}_{-1}$, *such that* $L(\lambda)$ *is the irreducible quotient of the generalized Verma module* $M(\mathfrak{g}, \mathfrak{p}^a, \lambda)$.

Proof. (1) $\Rightarrow$ (4) : Denote by $a\, v_\lambda$ the singular vector, where $a \in \mathfrak{g}_{-1}$, then (4) holds for this particular a. (4) $\Rightarrow$ (3) is immediate. Finally, $L(\lambda)$ quasi-finite implies $\dim(\mathfrak{g}_{-1}.v_\lambda) < \infty$, then there exist an $a \in \mathfrak{g}_{-1}$ such that $a\, v_\lambda = 0$ in $L(\lambda)$, so $0 = \mathfrak{g}_1.(av_\lambda) = a(\mathfrak{g}_1.v_\lambda) + [\mathfrak{g}_1, a]v_\lambda = \lambda([\mathfrak{g}_1, a])\, v_\lambda$, getting (3) $\Rightarrow$ (2) $\Rightarrow$ (1). □

3. The Lie Algebra $\mathbf{W}_{\infty,\mathbf{p}}$ and its Parabolic Subalgebras

We turn now to a certain family of $\mathbb{Z}$-graded Lie algebras. Let $\mathcal{D}$ be the Lie algebra of regular differential operators on the circle, i.e. the operators on $\mathbb{C}[t, t^{-1}]$ of the form

$$E = e_k(t)\partial_t^k + e_{k-1}(t)\partial_t^{k-1} + \cdots + e_0(t), \text{ where } e_i(t) \in \mathbb{C}[t, t^{-1}],$$

The elements

$$J_k^l = -t^{l+k}(\partial_t)^l \quad (l \in \mathbb{Z}_+,\ k \in \mathbb{Z})$$

form its basis, where ∂_t denotes d/dt. Another basis of $\mathcal{D}$ is

$$L_k^l = -t^k D^l \quad (l \in \mathbb{Z}_+,\ k \in \mathbb{Z}),$$

where $D = t\partial_t$. It is easy to see that

$$J_k^l = -t^k [D]_l. \tag{3.1}$$

Here and further, we use the notation

$$[x]_l = x(x-1)\dots(x-l+1).$$

Fix a linear map T: $\mathbb{C}[w] \to \mathbb{C}$. Then we have the following 2-cocycle on $\mathcal{D}$, where $f(w), g(w) \in \mathbb{C}[w]$ [5]:

$$\Psi_{\mathrm{T}}(z^r f(D), z^s g(D)) = \begin{cases} \mathrm{T}\left(\sum_{-r \leqslant m \leqslant -1} f(w+m)g(w+m+r)\right), & \text{if r=-s} \geqslant 0, \\ 0, \quad \text{if r + s} \neq 0. \end{cases} \tag{3.2}$$

We let $\Psi = \Psi_{\mathrm{T}}$ if T: $\mathbb{C}[w] \to \mathbb{C}$ is the evaluation map at $w = 0$. The central extension of $\mathcal{D}$ by a one-dimensional center $\mathbb{C}C$, corresponding to the 2-cocycle Ψ is denoted by $W_{1+\infty}$. The bracket in $W_{1+\infty}$ is given by

$$[t^r f(D), t^s g(D)]$$

$$= t^{r+s}(f(D+s)g(D) - f(D)g(D+r)) + \Psi(t^r f(D), t^s g(D))C. \tag{3.3}$$

Consider the following family of Lie subalgebras of $\mathcal{D}$ ($p \in \mathbb{C}[x]$):

$$\mathcal{D}_p := \mathcal{D}p(D).$$

Denote by $W_{\infty,p}$ the central extension of $\mathcal{D}_p$ by $\mathbb{C}C$ corresponding to the restriction of the 2-cocycle Ψ. Observe that $W_{\infty,x}$ is the well-known W_∞ subalgebra of $W_{1+\infty}$ and, more generally, using (3.1) we have that W_{∞,x^n} is the central extension of the Lie algebra of differential operators on the circle that annihilate all polynomials of degree less than n.

Letting wt $t^k f(D) = k$, wt $C = 0$ defines the *principal gradation* of $W_{1+\infty}$ and of $W_{\infty,p}$:

$$W_{\infty,p} = \bigoplus_{j\in\mathbb{Z}} (W_{\infty,p})_j,$$

where $(W_{\infty,p})_j = \{t^j f(D)p(D) \mid f(w) \in \mathbb{C}[w]\} + \delta_{j0}\mathbb{C}$.

It is easy to check that the $\mathbb{Z}$-graded Lie algebras $W_{\infty,p}$ satisfy the properties (P1)–(P2).

Remark 3.4. The Lie algebra $W_{\infty,p}$ contains a $\mathbb{Z}$-graded subalgebra isomorphic to the Virasoro algebra if and only if deg $p \leqslant 1$. Indeed, from the commutator:

$$[tf(D)p(D), g(D)p(D)] = t(g(D)p(D) - g(D+1)p(D+1))f(D)p(D),$$

it is immediate that if the relation $[L_1, L_0] = L_1$ is satisfied (for some elements $L_i \in (W_{\infty,p})_i$, $i = 0, 1$), then deg $p \leqslant 1$. The existence of Virasoro subalgebras for deg $p \leqslant 1$ was observed in [5].

Let $\mathfrak{p}$ be a parabolic subalgebra of $W_{\infty,p}$. Observe that for each $j \in \mathbb{N}$ we have

$$\mathfrak{p}_{-j} = \{t^{-j} f(D) \mid f(w) \in I_{-j}\},$$

where I_{-j} is a subspace of $p(w)\mathbb{C}[w]$. Since

$$[f(D)p(D), t^{-k}q(D)] = t^{-k}(f(D-k)p(D-k) - f(D)p(D))q(D),$$

we see that I_{-k} satisfies $A_{p,k} \, . \, I_{-k} \subseteq I_{-k}$ where

$$A_{p,k} = \{f(w-k)p(w-k) - f(w)p(w) \mid f(w) \in \mathbb{C}[w]\}.$$

LEMMA 3.5. (a) *I_{-k} is an ideal for all $k \in \mathbb{N}$ if deg $p \leqslant 2$ (there are examples of parabolic subalgebras where I_{-1} is not an ideal for any deg $p > 2$).*

(b) *If $I_{-k} \neq 0$, then it has finite codimension in* $\mathbb{C}[x]$.

Proof. Observe that if deg $p \leqslant 1$, then $A_{p,k} = \mathbb{C}[w]$ for all $k \geqslant 1$, and if deg $p = 2$, $A_{p,k}$ is a subspace which contains a polynomial of degree l for all $l \geqslant 1$, proving the first part. Now, observe that for $p(x) = [x]_n$, we have $A_{p,1} = \mathbb{C}[w][w-1]_{n-1}$, and take

$$I_{-1} = \mathbb{C}\,[w]_n \oplus \mathbb{C}\; w[w]_n \oplus A_{p,1}[w]_n \oplus A_{p,1}w[w]_n,$$
$$I_{-k} = \mathbb{C}[w][w-k+1]_n \ldots [w-1]_n[w]_n, \quad k > 1.$$

Then, after some computation, it is possible to see that these subspaces define a parabolic subalgebra $\mathfrak{p} = \oplus_{k\in\mathbb{Z}}\mathfrak{p}_k$, where $\mathfrak{p}_{-k} = \{t^{-j} f(D) \mid f(w) \in I_{-j}\}$ for $k \geqslant 1$. Observe that for $n > 2$, I_{-1} is no longer an ideal.

Finally, since $A_{p,k}$ contains a polynomial of degree l for all $l \geqslant m$, for some $m \in \mathbb{N}$, part (b) follows. □

Remark 3.6. Due to Lemma 3.5, we no longer have the situation of parabolic subalgebras described in terms of ideals, as in the Lie algebras considered in [3, 5, 7]. But for the algebra W_∞ the parabolic subalgebras are as in these references.

We shall need the following proposition to study modules over $W_{\infty,p}$ induced from its parabolic subalgebras.

PROPOSITION 3.7. (a) *Any nonzero element $d \in (W_{\infty,p})_{-1}$ is nondegenerate.*
(b) *Any parabolic subalgebra of $W_{\infty,p}$ is nondegenerate.*
(c) *Let $d = t^{-1}b(D) = t^{-1}a(D)p(D) \in (W_{\infty,p})_{-1}$, then*

$$\begin{aligned}(W_{\infty,p})_0^d :&= [(W_{\infty,p})_1, d]\\ &= \operatorname{span}\{p(D-1)g(D)b(D) - p(D)g(D+1)b(D+1)+\\ &\qquad + g(0)p(-1)b(0)C \mid g \in \mathbb{C}[w]\}.\end{aligned}$$

Proof. Let $0 \neq d \in (W_{\infty,p})_{-1}$, then $\mathfrak{p}_{-j}^d \neq 0$ for all $j \geqslant 1$. So, by Lemma 3.5 (b), part (a) follows . Let $\mathfrak{p}$ be any parabolic subalgebra of $W_{\infty,p}$, using Lemma 2.1 we get $\mathfrak{p}_{-1} \neq 0$. Then, using (a) and $\mathfrak{p}^d \subseteq \mathfrak{p}$ (for any nonzero $d \in \mathfrak{p}_{-1}$), we obtain (b). Finally, part (c) follows by Lemma 2.2 (b) and the commutator:

$$\begin{aligned}&[tg(D)p(D), t^{-1}b(D)]\\ &\quad = p(D-1)g(D-1)b(D) - p(D)g(D)b(D+1) + g(0)p(-1)b(0)C.\end{aligned}$$ □

4. Quasi-finite Highest-Weight Modules over $\mathbf{W_{\infty,p}}$

By Proposition 3.7, $W_{\infty,p}$ also satisfies property (P3), hence we can apply Theorem 2.5.

Let $L(\lambda)$ be a quasi-finite highest-weight module over $W_{\infty,p}$. By Theorem 2.5, there exists some monic polynomial $b(w) = a(w)p(w)$ such that $(t^{-1}b(D))v_\lambda = 0$. We shall call such monic polynomial of minimal degree, uniquely determined by the highest-weight λ, the *characteristic* polynomial of $L(\lambda)$.

A functional $\lambda \in (W_{\infty,p})_0^*$ is described by its *labels* $\Delta_l = -\lambda(D^l p(D))$, where $l \in \mathbb{Z}_+$, and the *central charge* $c = \lambda(C)$. We shall consider the generating series

$$\Delta_\lambda(x) = \sum_{l=0}^{\infty} \frac{x^l}{l!} \Delta_l. \tag{4.1}$$

Recall that a *quasi-polynomial* is a linear combination of functions of the form $p(x)e^{\alpha x}$, where $p(x)$ is a polynomial and $\alpha \in \mathbb{C}$. Recall the well-known characterization: a formal power series is a quasi-polynomial if and only if it satisfies a nontrivial linear differential equation with constant coefficients. One has the following characterization of quasi-finite highest-weight modules over $W_{\infty,p}$, which extends that for $W_{\infty,1} = W_{1+\infty}$ obtained in [5].

THEOREM 4.2 [1]. *A $W_{\infty,p}$-module $L(\lambda)$ is quasi-finite if and only if there exist a quasi-polynomial $\phi_\lambda(x)$ with $\phi_\lambda(0) = 0$, such that*

$$\Delta_\lambda(x) = p\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\left(\frac{\phi_\lambda(x)}{\mathrm{e}^x - 1}\right). \tag{4.3}$$

For completeness, we give a proof of this Theorem, which is probably more clear than the one in [1]. As always, the basic idea is the original one in [5].

Proof. From Proposition 3.7 (a) and (c), and Theorem 2.5(2), we have that $L(\lambda)$ is quasi-finite if and only if there exist a polynomial $b(w) = p(w)a(w)$ such that

$$\lambda(p(D-1)g(D)b(D) - p(D)g(D+1)b(D+1) + g(0)p(-1)b(0)C) = 0$$

for any polynomial g or, equivalently,

$$0 = \lambda\big(b(D)p(D-1)\mathrm{e}^{xD} - b(D+1)p(D)\mathrm{e}^{x(D+1)}\big) + g(0)p(-1)b(0)c. \tag{4.4}$$

Now, take $\Gamma_\lambda(x) = \sum_{i=0}^{\infty} b_i x^i$ a solution of

$$\Delta_\lambda(x) = p\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\Gamma_\lambda(x). \tag{4.5}$$

Using $\Delta_\lambda(x) = -\lambda(p(\mathrm{d}/\mathrm{d}x)\mathrm{e}^{xD})$, and the identities

$$(D)\mathrm{e}^{xD} = f\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)(\mathrm{e}^{xD}), \qquad p(D)\mathrm{e}^{x(D+1)} = \mathrm{e}^x p(D)\mathrm{e}^{xD} = \mathrm{e}^x p\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\mathrm{e}^{xD},$$
$$\mathrm{e}^x p\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)f(x) = p\left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right)\mathrm{e}^x f(x)$$

condition (4.4) can be rewritten as

$$\begin{aligned}
0 &= \lambda\big(b(D)p(D-1)\mathrm{e}^{xD} - b(D+1)p(D)\mathrm{e}^{x(D+1)}\big) + p(-1)b(0)c \\
&= \lambda\left(b\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)p\left(\frac{\mathrm{d}}{\mathrm{d}x}-1\right)(\mathrm{e}^{xD}) - b\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\big(p(D)\mathrm{e}^{x(D+1)}\big)\right) + p(-1)b(0)c \\
&= \lambda\left(b\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)p\left(\frac{\mathrm{d}}{\mathrm{d}x}-1\right)(\mathrm{e}^{xD}) - b\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\left(\mathrm{e}^{x}p\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\mathrm{e}^{xD}\right)\right) + p(-1)b(0)c \\
&= -a\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\left(\left[p\left(\frac{\mathrm{d}}{\mathrm{d}x}-1\right) - p\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\mathrm{e}^{x}\right]\Delta_\lambda(x)\right) + p(-1)b(0)c \\
&= a\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\left(\left[p\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\mathrm{e}^{x} - p\left(\frac{\mathrm{d}}{\mathrm{d}x}-1\right)\right]\Delta_\lambda(x) + p(-1)p(0)c\right) \\
&= a\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\left(\left[p\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\mathrm{e}^{x}p\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) - p\left(\frac{\mathrm{d}}{\mathrm{d}x}-1\right)p\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\right]\Gamma_\lambda(x) + p(-1)p(0)c\right) \\
&= a\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)p\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)p\left(\frac{\mathrm{d}}{\mathrm{d}x}-1\right)((\mathrm{e}^{x}-1)\Gamma_\lambda(x) + c).
\end{aligned}$$

Thus, $L(\lambda)$ is quasi-finite if and only if there exists a polynomial $a(w)$ such that

$$a\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)p\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)p\left(\frac{\mathrm{d}}{\mathrm{d}x}-1\right)((\mathrm{e}^{x}-1)\Gamma_\lambda(x) + c) = 0. \tag{4.6}$$

Therefore, $L(\lambda)$ is quasi-finite if and only if $(\mathrm{e}^{x}-1)\Gamma_\lambda(x) + c$ is a quasi-polynomial, proving the theorem. □

DEFINITION 4.7. The quasi-polynomial $\phi_\lambda(x) + c$, where $\phi_\lambda(x)$ is from (4.3) and c is the central charge, can be (uniquely) written in the form

$$\phi_\lambda(x) + c = \sum_{r\in I} p_r(x)\mathrm{e}^{rx}, \tag{4.8}$$

where all r are distinct numbers. The numbers r appearing in (4.8) are called *exponents* of the $W_{\infty,p}$-module $L(\lambda)$, and the polynomial $p_r(x)$ is called the *multiplicity* of r, denoted by mult(r). Note that, by definition, $c = \sum_r p_r(0)$.

COROLLARY 4.9. *Let $L(\lambda)$ be a quasi-finite irreducible highest-weight module over $W_{\infty,p}$, let $b(w) = p(w)a(w)$ be its characteristic polynomial, let $\Gamma_\lambda(x)$ be a solution of (4.5) and let $F_\lambda(x) = (\mathrm{e}^{x}-1)\Gamma_\lambda(x) + c$. Then*

$$a\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)p\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)p\left(\frac{\mathrm{d}}{\mathrm{d}x}-1\right)F(x) = 0$$

is the minimal order homogeneous linear differential equation with constant coefficients of the form

$$f\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)p\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)p\left(\frac{\mathrm{d}}{\mathrm{d}x}-1\right),$$

satisfied by $F(x)$*. Moreover, the exponents appearing in (4.8) are all roots of the polynomial* $p(w-1)p(w)a(w)$.

Now, we will consider the restriction of quasi-finite highest-weight modules over $W_{1+\infty}$ to $W_{\infty,p}$. We will need some notation.

A functional $\lambda \in (W_{1+\infty})_0^*$ is characterized by its labels $\Gamma_m = -\lambda(D^m)$, where $m \in \mathbb{Z}_+$, and the central charge $c = \lambda(C)$, cf. (4.1). Introduce the new generating series:

$$\Gamma_\lambda(x) = \sum_{m=0}^{\infty} \frac{x^m}{m!} \Gamma_m.$$

Observe that $\Gamma_\lambda(x)$ satisfies (4.5). Recall that a $W_{1+\infty}$-module $L(W_{1+\infty}, \lambda)$ is quasi-finite if and only if $\Gamma_\lambda(x) = (\phi(x))/(e^x - 1)$, where $\phi(x)$ is a quasi-polynomial such that $\phi(0) = 0$ [5]. We have the following partial restriction result:

PROPOSITION 4.10. *Any quasi-finite* $W_{\infty,p}$*-module* $L(\lambda)$ *can be obtained as a quotient of the* $W_{\infty,p}$*-submodule generated by the highest-weight vector of a quasi-finite* $W_{1+\infty}$*-module* $L(W_{1+\infty}, \tilde{\lambda})$*, for some quasi-finite functional* $\tilde{\lambda} \in (W_{1+\infty})_0^*$ *such that* $\tilde{\lambda}_{|(W_{\infty,p})_0} = \lambda$.

Proof. Given

$$\Delta_\lambda(x) = p\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\left(\frac{\phi(x)}{\mathrm{e}^x - 1}\right),$$

consider $\tilde{\lambda} \in (W_{1+\infty})_0^*$ determined by $\Gamma_{\tilde{\lambda}}(x) = \phi(x)/(\mathrm{e}^x - 1)$, and the proposition follows. □

Let $\mathcal{O}$ be the algebra of all holomorphic functions on $\mathbb{C}$ with the topology of uniform convergence on compact sets. We consider the vector space $\mathcal{D}^{\mathcal{O}}$ spanned by the differential operators (of infinite order) of the form $t^k f(D)$, where $f \in \mathcal{O}$. The bracket in $\mathcal{D}$ extends to $\mathcal{D}^{\mathcal{O}}$. Then the cocycle Ψ extends to a 2-cocycle on $\mathcal{D}^{\mathcal{O}}$ by formula (3.2). Let $W_{1+\infty}^{\mathcal{O}} = \mathcal{D}^{\mathcal{O}} + \mathbb{C}C$ be the corresponding central extension with the principal gradation as in $W_{1+\infty}$.

Consider the Lie subalgebras of $\mathcal{D}^{\mathcal{O}}$:

$$\mathcal{D}_p^{\mathcal{O}} := \mathcal{D}^{\mathcal{O}}\, p(D). \tag{4.11}$$

We shall denote by $W_{\infty,p}^{\mathcal{O}}$ the central extension of $\mathcal{D}_p^{\mathcal{O}}$ by $\mathbb{C}C$ corresponding to the restriction of the cocycle Ψ. And we shall use de notation $W_\infty^{\mathcal{O}} = W_{\infty,x}^{\mathcal{O}}$ (i.e. the case $p(x) = x$). Observe that $W_{\infty,p}^{\mathcal{O}}$ inherit a $\mathbb{Z}$-gradation from $W_{1+\infty}^{\mathcal{O}}$.

In the following section, we shall need the following proposition:

PROPOSITION 4.12. *Let* V *be a quasi-finite* $W_{\infty,p}$*-module. Then the action of* $W_{\infty,p}$ *on* V *naturally extends to the action of* $(W_{\infty,p}^{\mathcal{O}})_k$ *on* V *for any* $k \neq 0$.

Proof. The proof is analogous to that of Proposition 4.3 in [5]. □

5. Embedding of $\mathbf{W}_\infty$ into $\widehat{\mathbf{gl}}_\infty$.

In the following, *we will suppose that* $p(x) = x$, *i.e. we consider the algebra* W_∞. We shall use the notation $\mathcal{D}_x := \mathcal{D}_p$ and $\mathcal{D}_x^{\mathcal{O}} := \mathcal{D}_p^{\mathcal{O}}$.

Let gl_∞ be the $\mathbb{Z}$-graded Lie algebra of all matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ with finitely many nonzero diagonals (deg $E_{ij} = j - i$). Consider the central extension $\widehat{\mathrm{gl}}_\infty = \mathrm{gl}_\infty + \mathbb{C}C$ defined by the cocycle:

$$\Phi(A, B) = \mathrm{tr}\,([J, A]B), \qquad J = \sum_{i\leqslant 0} E_{ii}.$$

Given $s \in \mathbb{C}$, we will consider the natural action of the Lie algebra $\mathcal{D}_x$ (resp. $\mathcal{D}_x^{\mathcal{O}}$) on $t^s\mathbb{C}[t, t^{-1}]$. Taking the basis $v_j = t^{-j+s}$ $(j \in \mathbb{Z})$ of this space, we obtain a homomorphism of Lie algebras $\varphi_s : \mathcal{D}_x \to \mathrm{gl}_\infty$ (resp. $\varphi_s : \mathcal{D}_x^{\mathcal{O}} \to \mathrm{gl}_\infty$):

$$\varphi_s(t^k f(D)D) = \sum_{j\in\mathbb{Z}} f(-j+s)(-j+s)E_{j-k,j}.$$

This homomorphism preserves gradation and it lifts to a homomorphism $\widehat{\varphi}_s$ of the corresponding central extensions as follows [5] (cf. [1]):

$$\widehat{\varphi}_s(De^{xD}) = \varphi_s(De^{xD}) - \left(\frac{e^{sx}-1}{e^x-1}\right)' C, \qquad \widehat{\varphi}_s(C) = C.$$

Let $s \in \mathbb{Z}$ and denote by $\widehat{\mathrm{gl}}_{\infty,s}$ the Lie subalgebra of $\widehat{\mathrm{gl}}_\infty$ generated by C and $\{E_{ij} | i \neq s \text{ and } j \neq s\}$. Observe that $\widehat{\mathrm{gl}}_{\infty,s}$ is naturally isomorphic to $\widehat{\mathrm{gl}}_\infty$. Let p_s: $\widehat{\mathrm{gl}}_\infty \to \widehat{\mathrm{gl}}_{\infty,s}$ be the projection map. If $s \in \mathbb{Z}$, we redefine $\widehat{\varphi}_s$ by the homomorphism $p \circ \widehat{\varphi}_s : W_\infty \to \widehat{\mathrm{gl}}_{\infty,s}$.

Given $\mathbf{s} = (s_1, \ldots, s_m) \in \mathbb{C}^m$, we have a homomorphism of Lie algebras over $\mathbb{C}$:

$$\widehat{\varphi}_{\mathbf{s}} = \oplus_{i=1}^m \widehat{\varphi}_{s_i} : W_\infty \to \mathfrak{g}_{\mathbf{s}} = \oplus_{i=1}^m \mathfrak{g}_{s_i} \tag{5.1}$$

where $\mathfrak{g}_{s_i} = \widehat{\mathrm{gl}}_\infty$ if $s_i \notin \mathbb{Z}$, and $\mathfrak{g}_{s_i} = \widehat{\mathrm{gl}}_{\infty,s_i}$ if $s_i \in \mathbb{Z}$. The proof of the following proposition is similar to that of Proposition 3.2 in [5].

PROPOSITION 5.2. *The homomorphism* $\widehat{\varphi}_{\mathbf{s}}$ *extends to a homomorphism of Lie algebras over* $\mathbb{C}$, *which is also denoted by* $\widehat{\varphi}_{\mathbf{s}}$:

$$\widehat{\varphi}_{\mathbf{s}} : W_\infty^{\mathcal{O}} \to \mathfrak{g}_{\mathbf{s}}.$$

The homomorphism $\widehat{\varphi}_{\mathbf{s}}$ *is surjective provided that* $s_i - s_j \notin \mathbb{Z}$ *for* $i \neq j$.

Remark 5.3. For $s \in \mathbb{Z}$ the image of $W_\infty^{\mathcal{O}}$ under the homomorphism $\widehat{\varphi}_s$ is $\nu^k(\widehat{\mathrm{gl}}_{\infty,s-k})$ for any $k \in \mathbb{Z}$, where ν is the automorphism defined by

$$\nu(E_{ii}) = E_{i+1,i+1}. \tag{5.4}$$

Hence, we may (and will) assume that $0 \leqslant \mathrm{Re}\, s < 1$ throughout the paper.

6. Unitary Quasi-finite Highest-Weight Modules over $\mathbf{W}_\infty$

The algebra $\mathcal{D}_x$ acts on the space $V = \mathbb{C}[t, t^{-1}]/\mathbb{C}$. One has a nondegenerate Hermitian form on V:

$$B(f, g) = \mathrm{Res}_t\, \overline{f} \mathrm{d}g,$$

where $(\overline{\sum a_i t^i}) = \sum \overline{a_i} t^{-i}$, $a_i \in \mathbb{C}$, (cf. [2]).

Consider the additive map $\omega : \mathcal{D}_x \to \mathcal{D}_x$, defined by:

$$\omega(t^k f(D)D) = t^{-k}\overline{f}(D-k)D$$

where for $f(D) = \sum_i f_i D^i$, we let $\overline{f}(D) = \sum_i \overline{f_i} D^i (f_i \in \mathbb{C})$.

PROPOSITION 6.1. *The map ω is an anti-involution of the Lie algebra $\mathcal{D}_x$, i.e. ω is an additive map such that*

$$\omega^2 = id, \quad \omega(\lambda a) = \bar{\lambda}\omega(a), \quad \text{and} \quad \omega([a, b]) = [\omega(b), \omega(a)], \text{ for } \lambda \in \mathbb{C}, a, b \in \mathcal{D}_x.$$

Furthermore, the operators $\omega(a)$ and a are adjoint operators on V with respect to B, and $\omega((\mathcal{D}_x)_j) = (\mathcal{D}_x)_{-j}$.

Proof. The properties $\omega^2 = \mathrm{id}$, $\omega(\lambda a) = \bar{\lambda}\omega(a)$ are obvious. Now,

$$\begin{aligned}
&\omega([t^k f(D)D, t^l g(D)D]) \\
&\quad = \omega(t^{k+l}(f(D+l)(D+l)g(D) - g(D+k)(D+k)f(D))D) \\
&\quad = t^{-(k+l)}(\bar{f}(D-k)(D-k)\bar{g}(D-k-l) - \bar{g}(D-l)(D-l)\bar{f}(D-k-l))D.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&[\omega(t^l g(D)D), \omega(t^k f(D)D)] = [t^{-l}\bar{g}(D-l)D, t^{-k}\overline{f}(D-k)D] \\
&\quad = t^{-(k+l)}(\bar{g}(D-k-l)(D-k)\bar{f}(D-k) - \bar{f}(D-k-l)(D-l)\bar{g}(D-l))D.
\end{aligned}$$

Hence, ω is an anti-involution. Now we compute

$$B(t^k f(D)D\ t^l, t^n) = B(l f(l)\, t^{k+l}, t^n) = n\, l\, \overline{f}(l)\delta_{k+l,n}.$$

We also have

$$\begin{aligned} &B(t^l, \omega(t^k f(D)D)\ t^n) \\ &\quad = B(t^l, t^{-k}\overline{f}(D-k)D\ t^n) \\ &\quad = B(t^l, n\overline{f}(n-k)t^{n-k}) = n\, l\overline{f}(n-k)\delta_{l,n-k}, \end{aligned}$$

proving the proposition. □

This anti-involution ω extends to the whole algebra $\mathcal{D}_x^{\mathcal{O}}$, defined in (4.11). Observe that

$$\Psi(\omega(A), \omega(B)) = \omega(\Psi(B, A)), \quad A, B \in \mathcal{D}_x^{\mathcal{O}}.$$

Therefore, the anti-involution ω of the Lie algebras $\mathcal{D}_x$ and $\mathcal{D}_x^{\mathcal{O}}$ lifts to an anti-involution of their central extensions W_∞ and $W_\infty^{\mathcal{O}}$, such that $\omega(C) = C$, which we again denote by ω.

In this section we shall classify and construct all *unitary* (irreducible) quasi-finite highest-weight modules over W_∞ with respect to the anti-involution ω. In order to do it, we shall need the following lemma:

LEMMA 6.2. *Let V be a unitary quasi-finite highest-weight module over W_∞ and let $b(w) = wa(w)$ be its first characteristic polynomial. Then $a(w)$ has only simple real roots.*

Proof. Let v_λ be a highest-weight vector of V. Then the first graded subspace V_{-1} has a basis

$$\{(t^{-1}D^{j+1})v_\lambda \mid 0 \leqslant j < \deg a\}.$$

Consider the action of

$$S = -\frac{1}{2}\left(D^2 + \left(\frac{1-\Delta_1}{1+\Delta_0}\right)D\right)$$

on V_{-1}. It is straightforward to check that

$$S^j((t^{-1}D)v_\lambda) = (t^{-1}D^{j+1})v_\lambda \quad \text{for all } j \geqslant 0.$$

It follows that $a(S)((t^{-1}D)v_\lambda) = 0$, and that $\{S^j((t^{-1}D)v_\lambda) \mid 0 \leqslant j < \deg a\}$ is a basis of V_{-1}. We conclude from the above that $a(w)$ is the characteristic polynomial of the operator S on V_{-1}. Since the operator S is self-adjoint, all the roots of $a(w)$ are real.

Now, suppose that $a(w) = (w-r)^m c(w)$ for some polynomial $c(w)$ and $r \in \mathbb{R}$. Then $v = (S-r)^{m-1}c(S)((t^{-1}D)v_\lambda)$ is a nonzero vector in V_{-1}, but

$$(v, v) = (c(S)((t^{-1}D)v_\lambda), (S-r)^{2m-2}c(S)((t^{-1}D)v_\lambda)) = 0 \quad \text{if } m \geqslant 2.$$

Hence, the unitarity condition forces $m = 1$. □

Take $0 < s < 1$. Then under the homomorphism $\widehat{\varphi}_s: W_\infty^{\mathcal{O}} \to \widehat{\mathrm{gl}}_\infty$, the anti-involution ω induces the following anti-involution on $\widehat{\mathrm{gl}}_\infty$:

$$\omega'(E_{ij}) = \left(\frac{s-i}{s-j}\right) E_{ji}, \quad \omega'(C) = C.$$

Indeed

$$\begin{aligned}\varphi_s(\omega(t^k f(D)D)) &= \varphi_s(t^{-k}\bar{f}(D-k)D) \\ &= \sum_{j\in\mathbb{Z}} \bar{f}(s-j-k)(s-j)E_{j+k,j} \\ &= \sum_{j\in\mathbb{Z}} \bar{f}(s-j)(s-j+k)E_{j,j-k} \\ &= \omega'\left(\sum_{j\in\mathbb{Z}} f(s-j)(s-j)E_{j-k,j}\right) \\ &= \omega'(\varphi_s(t^k f(D)D)).\end{aligned}$$

Let $e_i = E_{i,i+1}$ and $f_i = E_{i+1,i}$, then

$$\omega'(e_i) = \lambda_i f_i \quad \text{with } \lambda_i = \frac{s-i}{s-i-1}.$$

Observe that

$$\lambda_i < 0 \quad \text{if and only if} \quad i < s < i+1. \tag{6.3}$$

If we consider the linear automorphism T defined by $T(e_i) = \mu_i e_i = e_i'$, $T(f_i) = \mu_i^{-1} f_i = f_i'$ for some $\mu_i \in \mathbb{C}$, then $\omega'(e_i') = \omega(\mu_i e_i) = \bar{\mu}_i \lambda_i f_i = |\mu_i|^2 \lambda_i f_i'$. Hence, by (6.3), ω' is equivalent to the anti-involution $\tilde{\omega}$ defined by

$$\tilde{\omega}(e_i) = \begin{cases} f_i & \text{if } i \neq 0, \\ -f_i & \text{if } i = 0. \end{cases}$$

After a shift by the automorphism ν defined in (5.4), we may assume $0 < s < 1$, then under the homomorphism $\widehat{\varphi}_s : W_\infty^{\mathcal{O}} \to \widehat{\mathrm{gl}}_\infty$, the anti-involution ω induces an anti-involution on $\widehat{\mathrm{gl}}_\infty$ that is equivalent to the following:

$$E_{ij}^\dagger = E_{ji} \text{ if } i,j > 0 \text{ or } i,j \leqslant 0, \quad E_{ij}^\dagger = -E_{ji} \text{ otherwise, and } C^\dagger = C.$$

As usual, for any $\lambda \in (\widehat{\mathrm{gl}}_\infty)_0^*$ we have the associated irreducible highest-weight $\widehat{\mathrm{gl}}_\infty$-module $L(\widehat{\mathrm{gl}}_\infty, \lambda)$. An element $\lambda \in (\widehat{\mathrm{gl}}_\infty)_0^*$ is determined by its *labels* $\lambda_i = \lambda(E_{ii})$, $i \in \mathbb{Z}$, and *central charge* $c = \lambda(C)$. Let $n_i = \lambda_i - \lambda_{i+1} + \delta_{i,0}c$ $(i \in \mathbb{Z})$. The following classification is taken from [6] and [8]:

PROPOSITION 6.4. *A nontrivial highest-weight $\widehat{\mathrm{gl}}_\infty$-module with highest-weight λ and central charge c is unitary with respect to † if and only if the following properties* *hold:*

$$n_i \in \mathbb{Z}_+ \quad \text{if } i \neq 0 \quad \text{and} \quad c = \sum_i n_i < 0, \tag{6.5a}$$

$$\text{if } n_i \neq 0 \quad \text{and} \quad n_j \neq 0, \quad \text{then } |i-j| \leqslant -c. \tag{6.5b}$$

In the case of $s=0$, the homomorphism $\widehat{\varphi}_0 : W_\infty^{\mathcal{O}} \to \widehat{\mathrm{gl}}_{\infty,0} \simeq \widehat{\mathrm{gl}}_\infty$ induces the standard anti-involution on $\widehat{\mathrm{gl}}_\infty$: $(A)^* = {}^t\bar{A}$. The following is a very well known result:

PROPOSITION 6.6. *A highest-weight $\widehat{\mathrm{gl}}_\infty$-module with highest-weight λ and central charge c is unitary with respect to * if and only if $n_i \in \mathbb{Z}_+$ and $c = \sum_i n_i$.*

Let $\lambda_i \in (\mathfrak{g}_{s_i})_0^*$ such that $L(\mathfrak{g}_{s_i}, \lambda_i)$ is a quasi-finite $\mathfrak{g}_{s_i}$-module. Then the tensor product

$$L(\mathfrak{g}_{\mathbf{s}}, \boldsymbol{\lambda}) := \otimes_i L(\mathfrak{g}_{s_i}, \lambda_i)$$

is an irreducible $\mathfrak{g}_{\mathbf{s}}$-module.

THEOREM 6.7. *Let V be a quasi-finite $\mathfrak{g}_{\mathbf{s}}$-module, viewed as a W_∞-module via the homomorphism $\widehat{\varphi}_{\mathbf{s}}$, where $s_i - s_j \notin \mathbb{Z}$ if $i \neq j$, and $0 \leqslant \mathrm{Re}\, s_i < 1$. Then any W_∞-submodule of V is also a $\mathfrak{g}_{\mathbf{s}}$-submodule. In particular, the W_∞-modules $L(\mathfrak{g}_{\mathbf{s}}, \lambda)$ are irreducible, and in this way we obtain all quasi-finite W_∞-modules $L(\lambda)$ with $\phi_\lambda(x) = \sum_i n_i e^{r_i x}$, $n_i \in \mathbb{C}$, $r_i \in \mathbb{C}$.*

Proof. Consider any W_∞-submodule W of V. By Proposition 4.12, the action of W_∞ can be extended to $(W_\infty^{\mathcal{O}})_k$ $(k \neq 0)$. Using Proposition 5.2, we see that the subspace W is preserved by $\mathfrak{g}_{\mathbf{s}}$. Therefore, the W_∞-modules $L(\mathfrak{g}_s, \lambda)$ are quasi-finite and irreducible. Then it is easy to calculate the generating series of the highest-weight (see Section 4.6 in [5]): in the case $s \in \mathbb{R}\backslash\mathbb{Z}$, we have

$$\Delta_{s,\lambda}(x) = -\lambda(\hat{\varphi}_s(De^{xD})) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sum_{k\in\mathbb{Z}} e^{(s-k)x} n_k - c}{e^x - 1}\right), \tag{6.8}$$

in the case of $s = 0$, we have

$$\Delta_{0,\lambda}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sum_{j>0} e^{-jx} n_j + \sum_{j<0} e^{(-j+1)x} n_j + e^{2x}\lambda_0 - e^{-x}\lambda_1}{e^x - 1}\right), \tag{6.9}$$

and the last part of the theorem follows from Equations (6.8)–(6.9). □

In fact, as in Theorem 4.6 in [5], it is possible to construct all irreducible quasi-finite W_∞-module in terms of representations of $\widehat{\mathrm{gl}}(\infty, R_m)$ (or a subalgebra of it), where $\widehat{\mathrm{gl}}(\infty, R_m)$ is the central extension of the Lie algebra of infinite matrices with finitely many nonzero diagonals and coefficients in the algebra of truncated polynomials $R_m := \mathbb{C}[u]/(u^{m+1})$. More precisely, we consider the homomorphism

$\varphi_s^{[m]} : \mathcal{D}^{(1)} \to \mathrm{gl}(\infty, R_m)$ given by

$$\varphi_s^{[m]}(t^k f(D)D) = \sum_{i=0}^{m} \sum_{j\in\mathbb{Z}} \frac{(f^{(i)}(s-j)(s-j) + if^{(i-1)}(s-j))}{i!} u^i E_{j-k,j}.$$

In the case where $s \in \mathbb{R}\backslash\mathbb{Z}$, we take $\mathfrak{g}_s^{[m]} = \widehat{\mathrm{gl}}(\infty, R_m)$ and, for $s \in \mathbb{Z}$, we have to remove the generators $E_{r,s}$ and $u^m E_{-s,r}$ for all $r \in \mathbb{Z}$. All quasi-finite irreducible $L(\lambda)$ can be obtained using representations of the Lie algebra $\mathfrak{g}_{\mathbf{s}}^{[\mathbf{m}]} = \oplus_i \mathfrak{g}_{s_i}^{[m_i]}$ via the homomorphism $\varphi_{\mathbf{s}}^{[\mathbf{m}]} = \oplus_i \varphi_{s_i}^{[m_i]}$, and as in [5] the coefficients in $\mathbf{m}$ are given by the degree of the (polynomial) multiplicities of $\phi_\lambda(x)$.

LEMMA 6.10. *Only those highest-weight representations of* $\widehat{\mathrm{gl}}(\infty, R_m)$ *that factor through* $\widehat{\mathrm{gl}}(\infty, \mathbb{C})$ *are unitary.*

Proof. Indeed, let v be a highest-weight vector. Fix $i \in \mathbb{Z}$ and let $e = E_{i,i+1}$, $f = E_{i+1,i}$, $h = E_{ii}$. Now take the maximal j such that $(u^j f)v \neq 0$. We have to show that j=0. In the contrary case, $(u^j f)v$ is a vector of norm 0: $((u^j f)v, (u^j f)v) = \pm(v, (u^j e)(u^j f)v) = \pm(v, (u^{2j} h)v) = 0$, since otherwise $(u^{2j} h)v \neq 0$, hence $(u^{2j} f)v \neq 0$ (by applying e to it). Hence, we get a nonzero vector of zero norm, unless the module is actually a $\widehat{\mathrm{gl}}(\infty, \mathbb{C})$-module. □

Therefore, using Lemma 6.10, Lemma 6.2 and Corollary 4.9, we have

LEMMA 6.11. *If* $L(\lambda)$ *is a unitary quasi-finite* W_∞*-module, then* $\phi_\lambda(x) = \sum_i n_i e^{r_i x}$*, with* $n_i \in \mathbb{C}$*,* $r_i \in \mathbb{R}$.

Now we can formulate the main result of this section, that follows (in the same way as Theorem 5.2 in [5]), from Theorem 6.7, Propositions 6.4 and 6.6, and Lemma 6.11:

THEOREM 6.12. (a) *Let* $L(\lambda)$ *be a nontrivial quasi-finite* W_∞*-module. For each* $0 \leqslant \alpha < 1$*, let* E_α *denote the set of exponents of* $L(\lambda)$ *that are congruent to* α mod $\mathbb{Z}$*. Then* $L(\lambda)$ *is unitary if and only if the following three conditions are satisfied*:

(1) *All exponents are real numbers.*
(2). *The multiplicities of the exponents* $r_i \in E_0$ *are positive integers.*
(3) *For exponents* $r_i \in E_\alpha$ $(0 < \alpha < 1)$*, all multiplicities* $mult(r_i)$ *are integers and only one of them is negative,* $m_\alpha := -\sum_{r_i \in E_\alpha}(mult(r_i))$ *is a positive integer, and* $r_i - r_j \leqslant m_\alpha$ *for all* $r_i, r_j \in E_\alpha$.

(b) *Any unitary quasi-finite* W_∞*-module* $L(\lambda)$ *is obtained by taking tensor product of unitary irreducible quasi-finite highest-weight modules over* $\mathfrak{g}_{s_i}$*,* $i = 1, \ldots, m$*, and restricting to* W_∞ *via the embedding* $\hat{\varphi}_{\mathbf{s}}$*, where* $\mathbf{s} = (s_1, \ldots, s_m) \in \mathbb{R}^m$*,* $0 \leqslant s_i < 1$ *and* $s_i - s_j \notin \mathbb{Z}$ *if* $i \neq j$.

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