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Letter to the Editor

On the use of orthogonal polynomials in the study of anisotropic plates

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1. Introduction

After Walter Ritz had presented in 1908 his now famous variational method, interest was particularly shown by several mathematicians from whom the substantiation of the method has received lengthy treatment [1–5]. On the other hand, investigators in the field of applied sciences generated an immense quantity of papers in which approximate solutions of various problems of mathematical physics where constructed with the aid of the Ritz method. Particularly, this method has been used extensively over the years to study the problem of flexural vibrations of rectangular isotropic, orthotropic and anisotropic plates.

It is well known [3], that boundary conditions containing the function w and derivatives of w of orders not greater than $m-1$, are called *stable* or *geometric* for a differential equation of order $2m$, and those containing derivatives of orders higher than $m-1$ are called *unstable* or *natural*.

When using the Ritz method we choose a sequence of functions w_i which constitute a base in the space V where only the homogeneous stable or geometric boundary conditions are included, there is no need to subject the functions w_i to the natural boundary conditions, [2–3].

The fact that the natural boundary conditions of a system need not be satisfied by the chosen co-ordinate functions is a very important characteristic of the Ritz method, specially when dealing with problems for which such satisfaction is very difficult to achieve. For instance, this is the case of a rectangular anisotropic with one or more free edges.

In the present paper, the use of orthogonal polynomials in the Ritz method for the study of rectangular anisotropic plates is further analyzed.

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2. Beam functions and orthogonal polynomials in the Ritz method

In the analyses of the statical and dynamical behaviour of rectangular plates it has been found that the deflections can be well approximated by a series of beam functions in the separable form of the variables [6]

$$w(x, y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} c_{ij} X_i(x) Y_j(y), \tag{1}$$

where c_{ij} are undetermined coefficients and $X_i(x)$ and $Y_j(y)$ are beam mode shapes functions which satisfy the appropriate boundary conditions on the edges $x = 0, a$ and $y = 0, b$ respectively. These functions are commonly taken as the characteristic functions for the normal modes of vibration of beams with end conditions the same as those assumed for the plate at the corresponding edges.

Let us consider a rectangular anisotropic plate (see Fig. 1) and for instance the edge $x = a$. With the adequate selection of the beam functions, each term of series (1) can satisfy the following boundary conditions:

- Edge rigidly clamped:

$$w(a,\ y) = 0, \quad \frac{\partial w(a,\ y)}{\partial x} = 0. \tag{2}$$

- Edge simply supported:

$$w(a,\ y) = 0, \quad \frac{\partial^2 w(a, y)}{\partial x^2} = 0. \tag{3}$$

- Edge free:

$$\frac{\partial^2 w(a, y)}{\partial x^2} = 0, \quad \frac{\partial^3 w(a, y)}{\partial x^3} = 0. \tag{4}$$

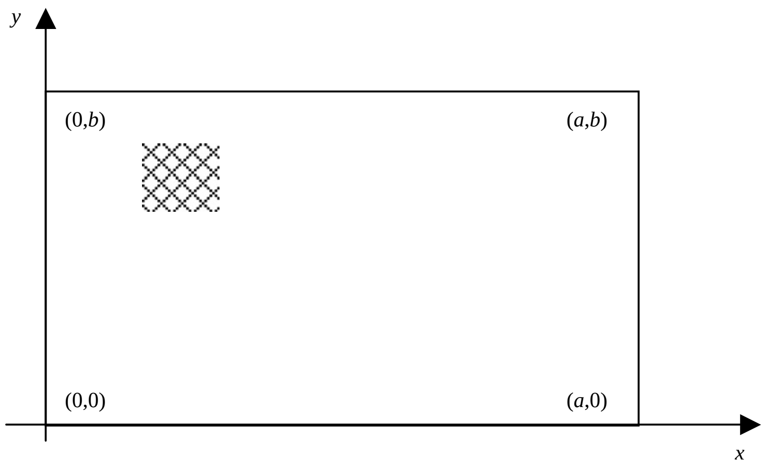

Fig. 1. Anisotropic rectangular plate.

When the edge is rigidly clamped all the boundary conditions are stable. Consequently, the terms of Eq. (1) satisfy exactly all the boundary conditions.

When the edge is simply supported the boundary conditions to be satisfied are

$$w(a, y) = 0, \tag{5}$$

$$D_{11}(a, y)\frac{\partial^2 w(a, y)}{\partial x^2} + D_{12}(a, y)\frac{\partial^2 w(a, y)}{\partial y^2} + 2D_{16}(a, y)\frac{\partial^2 w(a, y)}{\partial x \partial y} = 0. \tag{6}$$

When the plate is isotropic or specially orthotropic [7], the terms of function (1), which satisfy (3), can verify condition (5) and also Eq. (6) because this last condition reduces to $D_{11}(a, y)\ \partial^2 w(a, y)/\partial x^2 = 0$, since the second and third terms in Eq. (6) are equal to zero because $X_i(a)=0\ \forall_i$, and $D_{16}=0$. Unfortunately, when the plate is anisotropic the terms of Eq. (1), cannot satisfy the unstable condition (6).

Finally, when the edge is free there are two unstable boundary conditions:

$$D_{11}(a, y)\frac{\partial^2 w(a, y)}{\partial x^2} + D_{12}(a, y)\frac{\partial^2 w(a, y)}{\partial y^2} + 2D_{16}(a, y)\frac{\partial^2 w(a, y)}{\partial x \partial y} = 0, \tag{7}$$

$$\left(D_{11}(a, y)\frac{\partial^3 w(a, y)}{\partial x^3} + D_{12}(a, y)\frac{\partial^3 w(a, y)}{\partial y^2 \partial x} + 4D_{16}(a, y)\frac{\partial^3 w(a, y)}{\partial^2 x \partial y}\right) + 2\left(D_{26}(a, y)\frac{\partial^3 w(a, y)}{\partial y^3} + 2D_{66}(a, y)\frac{\partial^3 w(a, y)}{\partial x \partial^2 y}\right) = 0, \tag{8}$$

The beam functions cannot verify the unstable boundary conditions (7) and (8). This last condition cannot be verified even if the plate is orthotropic or isotropic.

This situation with the beam functions which do not satisfy the unstable boundary conditions is not irremediable since when dealing with the Ritz method, it is not necessary to subject the co-ordinate functions to this type of boundary conditions. Consequently, these conditions can be ignored in the procedure of construction of the approximating function (1). Unfortunately, the beam functions satisfy for instance conditions (4), and this represents an unnecessary restraint on the system which can led to numerical results with greater errors than those obtained by using similar functions which are not constrained to satisfy conditions (4). This question has been discussed by Bassily and Dickinson [8] in which the use of degenerated beam functions for the study of isotropic plates involving free edges is examined. Later, Dickinson and Di Blasio [9] used the orthogonal polynomials proposed by Bhat [10,11] to generate results for a number of flexural vibrations and buckling problems for rectangular isotropic and orthotropic plates. They use the approximating function

$$w(x, y) \approx \sum_{i=0}^{N}\sum_{j=0}^{M} c_{ij}\phi_i(x)\varphi_j(y), \tag{9}$$

where c_{ij} are arbitrary coefficients which are to be determined and $\{\phi_i(x), \varphi_j(y)\}$ is a set of orthogonal polynomials. All the polynomials satisfy only the geometric (or stable) boundary conditions. They demonstrated that the lack of satisfaction of the natural boundary conditions of the equivalent beam of the mentioned polynomials relaxes the over-restraint encountered in the use of the true beam functions and permits the treatment of plates involving free edges with a degree of accuracy equivalent to that obtained using degenerated beam functions.

3. Numerical results

The mentioned drawbacks of the beam functions are, of course, greater in the case of anisotropic plates. Nevertheless, the orthogonal polynomials are very satisfactory for use in the Ritz method for the study of the statical and dynamical behaviour of anisotropic plates. To illustrate this let us consider a simply supported rectangular anisotropic plate with the stiffness properties characterized by the rigidity coefficients:

$$D_{22}/D_{11} = 1, \quad (D_{12} + 2D_{66})/D_{11} = 1.5, \quad D_{16}/D_{11} = D_{26}/D_{11} = -0.5.$$

The "popular" co-ordinate function solution is given by

$$w(x,y) \approx \sum_{i=0}^{N} \sum_{j=0}^{M} c_{ij} \sin \frac{i\pi x}{a} \sin \frac{i\pi x}{b}. \tag{10}$$

In the case of isotropic and specially orthotropic plates the deflection function (10) exactly satisfies the governing differential equation and the boundary conditions. For anisotropic plates this procedure does not lead to the exact solution. In this case the boundary conditions are

$$w(x,0) = 0, \quad w(a,y) = 0, \quad w(x,b) = 0, \quad w(0,y) = 0, \tag{11}$$

$$D_{22}(x,0)\frac{\partial^2 w(x,0)}{\partial y^2} + D_{12}(x,0)\frac{\partial^2 w(x,0)}{\partial x^2} + 2D_{26}(x,0)\frac{\partial^2 w(x,0)}{\partial x \partial y} = 0, \tag{12}$$

$$D_{11}(a,y)\frac{\partial^2 w(a,y)}{\partial x^2} + D_{12}(a,y)\frac{\partial^2 w(a,y)}{\partial y^2} + 2D_{16}(a,y)\frac{\partial^2 w(a,y)}{\partial x \partial y} = 0, \tag{13}$$

$$D_{22}(x,b)\frac{\partial^2 w(x,b)}{\partial y^2} + D_{12}(x,b)\frac{\partial^2 w(x,b)}{\partial x^2} + 2D_{26}(x,b)\frac{\partial^2 w(x,b)}{\partial x \partial y} = 0, \tag{14}$$

$$D_{11}(0,y)\frac{\partial^2 w(0,y)}{\partial x^2} + D_{12}(0,y)\frac{\partial^2 w(0,y)}{\partial y^2} + 2D_{16}(0,y)\frac{\partial^2 w(0,y)}{\partial x \partial y} = 0, \tag{15}$$

Each term in Eq. (10) satisfy the following conditions:

$$\begin{gathered} D_{22}(x,0)\frac{\partial^2 w(x,0)}{\partial y^2} = 0, \quad D_{11}(a,y)\frac{\partial^2 w(a,y)}{\partial x^2} = 0, \quad D_{22}(x,b)\frac{\partial^2 w(x,b)}{\partial y^2} = 0, \\ D_{11}(0,y)\frac{\partial^2 w(0,y)}{\partial x^2} = 0, \end{gathered} \tag{16}$$

which constitute the mentioned over-restrictions.

Fig. 2 shows a graphical comparison of the fundamental frequency coefficient $\Omega = \sqrt{\rho h/D_{11}}\omega b^2$ obtained with Eqs. (9) and (10). The rapid convergence can be seen when using Eq. (9) in comparison to Eq. (10). This is due to the fact that the functions in Eq. (10) are restricted to satisfy conditions (16) while the orthogonal polynomials are not restricted at all.

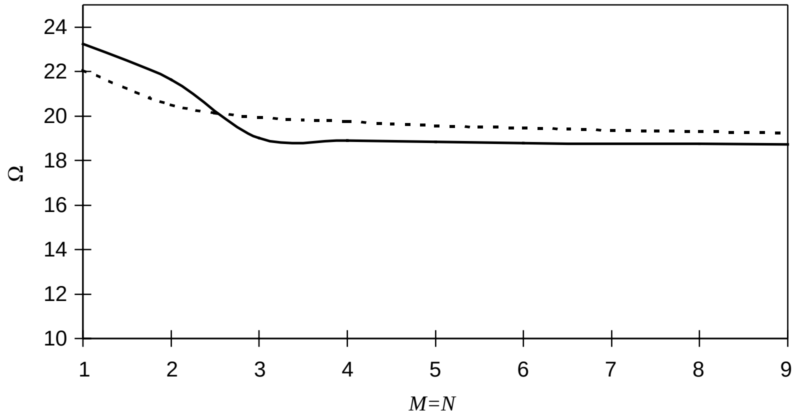

Fig. 2. Fundamental frequency coefficient $\Omega = \sqrt{\rho h/D_{11}}\omega b^2$ of a square anisotropic plate with edges simply supported —, I, values obtained with orthogonal polynomials (Eq. (9)); - - -, II, values obtained with the sine series (10). N and M are the number of terms in both Eqs. (9) and (10).

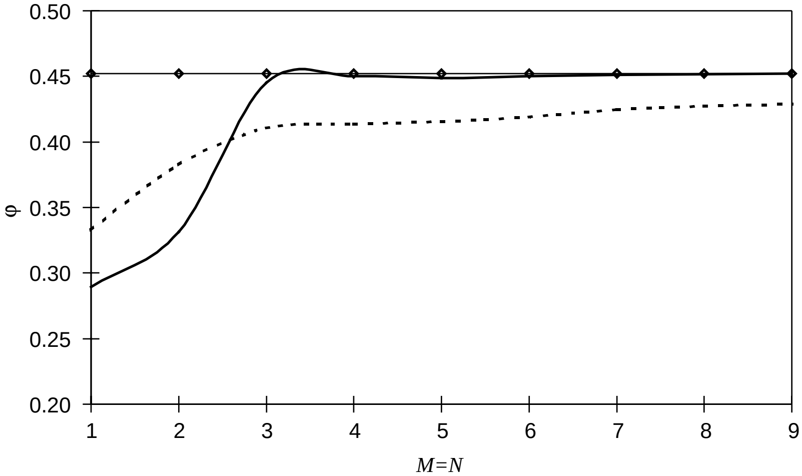

Fig. 3. Maximum deflection $w = \varphi q b^4/D_{11}\ 10^{-2}$ at the center of a square anisotropic plate with edges simply supported: —, I, values obtained with orthogonal polynomials (Eq. (9)); - - -, II, values obtained with the sine series (10). N and M are the number of terms in both Eqs. (9) and (10). ◆, III, exact solution [12] (Ashton).

Fig. 3 presents a comparison of the mentioned solutions for the deflection $w = \varphi q b^4/D_{11}\ 10^{-2}$. Both solutions appear to be converging to the exact solution obtained by Ashton [12], but the convergence obtained with Eq. (10) is much slower. Finally, Fig. 4 shows the moment $M_x=\beta q b^2$. The beam function solution appears to be oscillating about a relative constant value. On the other hand the convergence of the orthogonal polynomials solution is rapid and there is practically no oscillation.

Other combinations of the classical boundary conditions have been taken into account. Figs. 5–7 present the solutions for the fundamental frequency coefficient, the maximum deflection and the center moment. These results illustrate the very rapid convergence achieved by Eq. (9) in which orthogonal polynomials are used.

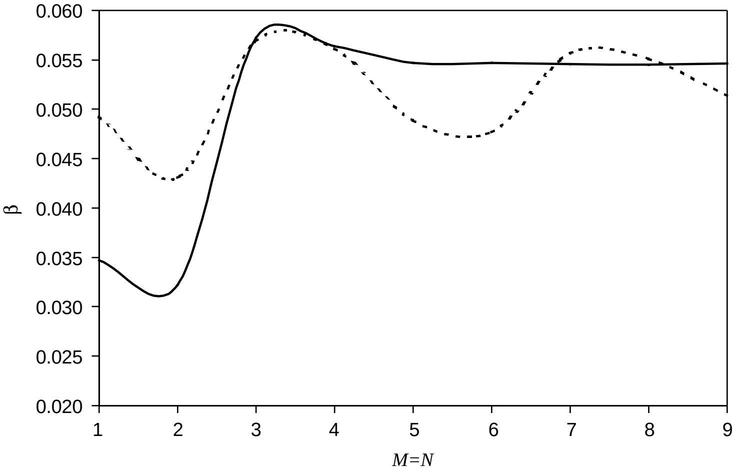

Fig. 4. Center moment $M_x=\beta qb^2$ of a square anisotropic plate with edges simply supported: —, I, values obtained with orthogonal polynomials (Eq. (9)); - - -, II, values obtained with the sine series (10). N and M are the number of terms in both Eqs. (9) and (10).

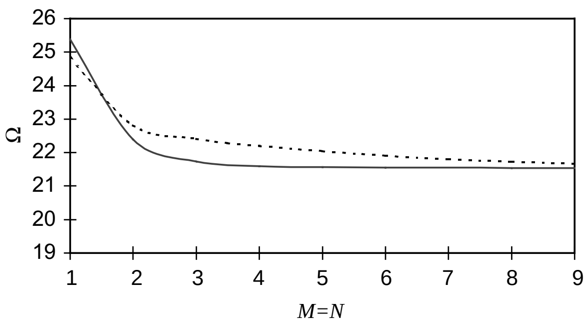

Fig. 5. Fundamental frequency coefficient $\Omega = \sqrt{\rho h/D_{11}}\omega b^2$ of a square anisotropic plate with three edges clamped and one edge free: —, I, values obtained with orthogonal polynomials (Eq. (9)); - - -, II, values obtained with the popular beam functions. N and M are the number of terms in both Eq. (9) and beam functions.

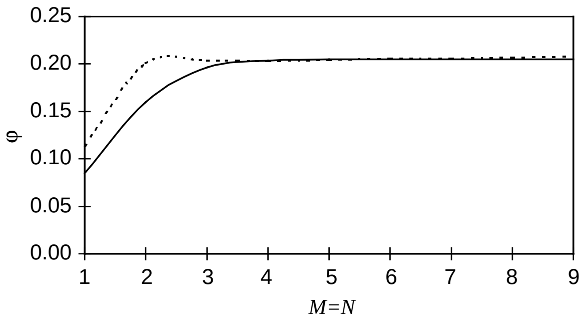

Fig. 6. Center deflection $w = \varphi qb^4/D_{11}\ 10^{-2}$ of a square anisotropic plate with three edges clamped ($x = 0, y = 0$, $y = b$) and one edge free ($x = a$): —, I, values obtained with orthogonal polynomials (Eq. (9)); - - -, II, values obtained with the popular beam functions. N and M are the number of terms in both Eq. (9) and beam functions.

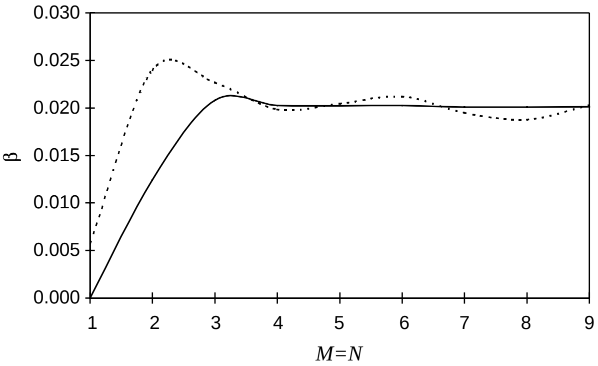

Fig. 7. Center moment $M_x=\beta q b^2$ of a square anisotropic plate with three edges clamped ($x = 0, y = 0, y = b$) and one edge free ($x = a$) —, I, values obtained with orthogonal polynomials (Eq. (9)); - - -, II, values obtained with the popular beam functions. N and M are the number of terms in both Eq. (9) and beam functions.

4. Conclusions

From the results presented in this paper, it would appear that the Ritz method with orthogonal polynomials functions is very satisfactory for the study of anisotropic plates. This is particularly true in the case of bending deflections, free vibration and also in the response which requires derivatives of the deflections.

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