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A POSTERIORI ERROR ESTIMATES FOR ELLIPTIC PROBLEMS WITH DIRAC MEASURE TERMS IN WEIGHTED SPACES

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Abstract. In this article we develop *a posteriori* error estimates for second order linear elliptic problems with point sources in two- and three-dimensional domains. We prove a global upper bound and a local lower bound for the error measured in a weighted Sobolev space. The weight considered is a (positive) power of the distance to the support of the Dirac delta source term, and belongs to the Muckenhoupt's class A_2. The theory hinges on local approximation properties of either Clément or Scott–Zhang interpolation operators, without need of modifications, and makes use of weighted estimates for fractional integrals and maximal functions. Numerical experiments with an adaptive algorithm yield optimal meshes and very good effectivity indices.

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1. Introduction

The main goal of this article is to develop *a posteriori* error estimates for elliptic second order partial differential equations on two- and three-dimensional domains with point sources. Elliptic problems with Dirac measure source terms arise in modeling different applications as, for instance, the electric field generated by a point charge, the acoustic monopoles or pollutant transport and degradation in an aquatic media where, due to the different scales involved, the pollution source is modeled as supported on a single point [3]. Other applications involve the coupling between reaction-diffusion problems taking place in domains of different dimension, which arise in tissue perfusion models [11].

In spite of the fact that the solution of one such problem typically does not belong to H^1, it can be numerically approximated by standard finite elements, but there is no obvious choice for the norm to measure the error. Babuška [5], Scott [28] and Casas [8] obtained *a priori* estimates for the error measured in L^2 and in fractional Sobolev norms H^s, for s in some subinterval of $(0, 1)$, depending on the dimension of the underlying domain. Eriksson [13] showed optimal order error estimates in the L^1 and $W^{1,1}$ norms, for adequately refined meshes; he also obtained pointwise estimates far from the singularity and the boundary. In a recent article, by using graded

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meshes, Apel *et al.* [1] obtained L^2 error estimates of almost optimal order on convex polygonal domains. More recently, D'Angelo [10] proved the well-posedness of Poisson problem with singular sources on weighted Sobolev spaces, over three-dimensional domains, obtaining also stability and optimal estimates for *a priori* designed meshes, in the spirit of [1,13]. D'Angelo measures the error in $H^1_\alpha = H^1_{\mathrm{d}^{2\alpha}}$, where $\mathrm{d}(x) = \operatorname{dist}(x, \Lambda)$, $\alpha \in (0,1)$, and Λ is the support of the singular source term, which is a smooth curve; his results carry over immediately to two dimensional domains with point sources. Regularity estimates in weighted norms have also been used in the design of graded meshes for elliptic problems with corner singularities; see [2,6,20] and references therein.

A posteriori error estimates on two dimensional domains have been obtained by Araya *et al.* [3,4] for the error measured in L^p $(1 < p < \infty)$ and $W^{1,p}$ $(p_0 < p < 2)$ for certain value of p_0, and by Gaspoz *et al.* [16] for the error measured in H^s $(1/2 < s < 1)$. Recall that the usual test and ansatz space for elliptic problems is the Sobolev space $H^1_0 = W^{1,2}_0$. Point sources do not belong to the dual space of H^1_0, because H^1_0 is not immersed into the space of continuous functions, but in two dimensions very little is missing, since functions in $W^{1,p'}_0$ and H^s_0 are continuous if $p' > 2$ and $s > 1$. This fact was exploited in [3,4] and [16].

In this article we develop residual type *a posteriori* error estimators for the weighted Sobolev norm $\|\cdot\|_{H^1_\alpha}$; the same notion of error estimated *a priori* by D'Angelo in [10]. The space H^1_α that we consider here is also "larger" than H^1 and seems to be more appropriate than the $W^{1,p}$ and the H^s spaces, because the weight weakens the norm only around the singularity, letting it behave like the usual $W^{1,2} = H^1$ norm far from the location of the support of the Dirac's delta.

We consider the following linear elliptic problem on a Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with a polygonal/polyhedral boundary $\partial\Omega$:

$$\begin{cases} -\nabla \cdot \big(\mathcal{A}\nabla u\big) + \boldsymbol{b} \cdot \nabla u + cu = \delta_{x_0} & \text{in } \Omega \\ \qquad\qquad\qquad\qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mathcal{A} \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ is piecewise $W^{1,\infty}$ and uniformly symmetric positive definite (SPD) over Ω, *i.e.*, there exist constants $0 < \gamma_1 \le \gamma_2$ such that

$$\gamma_1 |\xi|^2 \le \xi^T \mathcal{A}(x) \xi \le \gamma_2 |\xi|^2, \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^n, \tag{1.2}$$

$\boldsymbol{b} \in W^{1,\infty}(\Omega; \mathbb{R}^n)$, $c \in L^\infty(\Omega)$, and δ_{x_0} is the Dirac delta distribution supported at an inner point x_0 of Ω. We assume that $c - \frac{1}{2}\operatorname{div}(\boldsymbol{b}) \ge 0$.

The main results of this article, stated precisely in Theorems 5.1 and 5.3, are a global upper bound for the error, measured in $H^1_\alpha(\Omega)$ for $\alpha \in \mathbb{I} \subset (\frac{n}{2} - 1, \frac{n}{2})$ (see (2.15)), in terms of the *a posteriori* estimators and a local lower bound up to some oscillation term, which we roughly state as follows:

> Given a shape-regular triangulation $\mathcal{T}$, we let U be the Galerkin approximation of the exact solution u with continuous finite elements of arbitrary (fixed) degree, and prove that the *a posteriori local error estimators* η_T satisfy
>
> $$\|U - u\|_{H^1_\alpha(\Omega)} \le \tilde{C}_{\mathcal{U}} \Big(\sum_{T \in \mathcal{T}} \eta_T^2 \Big)^{1/2} \quad \text{and} \quad \tilde{C}_{\mathcal{L}}\, \eta_T \le \|U - u\|_{H^1_\alpha(\omega_T)} + \operatorname{osc}_T,\ \forall T \in \mathcal{T},$$
>
> with constants $\tilde{C}_{\mathcal{U}}$, $\tilde{C}_{\mathcal{L}}$ that depend only on mesh regularity, the domain Ω, the problem coefficients and α, and can be chosen independent of α on compact subintervals of $\mathbb{I}$. The set ω_T is the patch of all neighbours of T in $\mathcal{T}$, and osc_T is an oscillation term, which is generically of higher order than η_T.

As we have mentioned earlier, *a posteriori* error estimates for elliptic problems with point sources have already been obtained. The main advantages and novelties of the present work are the following:

- The equivalence of the error and estimator is valid for a large class of linear elliptic problems. Previous works [4,16] considered Poisson problem, and a diffusion-advection-reaction equation was studied in [3], assuming that an inf-sup condition holds in the context of $W^{1,p}$ spaces. We prove the necessary continuous inf-sup condition for linear elliptic problems in the weighted spaces considered here (see Thm. 2.3).

- In contrast to the norms used in [3,4,16], when considering the weighted spaces a discrete inf-sup condition can be proved (see Sect. 3), allowing us to conclude convergence of adaptive methods by resorting to the general theory developed in [22,29].
- The proposed weight only weakens the norm around x_0, but behaves as the usual H^1 norm in subsets at a positive distance to x_0. Whence the convergence alluded to in the previous item implies the convergence to zero of the H^1 error over such sets.
- Our estimates are valid in two and three dimensions, whereas the results from [3,4,16] cannot be immediately extended to the three dimensional case.

In [4] the solution is seen as an element of $W^{1,p}(\Omega)$, for some $p<2$, and the test functions belong to $W^{1,p'}(\Omega)$, with $1/p+1/p'=1$ and thus $p'>2$. By Sobolev embeddings the test functions are continuous, whence the usual proof for the upper bound can be done resorting to the Lagrange interpolant. The same happens in [16], where the solution is seen as an element of $H^{1-s}(\Omega)$ and the test functions belong to $H^{1+s}(\Omega)$ for $0<s<1/2$. In this article we see the solution as an element of the weighted Sobolev space $H^1_\alpha(\Omega)=\{v:\int_\Omega(v^2+|\nabla v|^2)\,\mathrm{d}_{x_0}^{2\alpha}<\infty\}$, with $\mathrm{d}_{x_0}(x)=|x_0-x|$ and $\frac{n}{2}-1<\alpha<\frac{n}{2}$, n being the dimension of the underlying domain Ω. Even though $\delta_{x_0}(v)$ is well defined for all test functions $v\in H^1_{-\alpha}(\Omega)$, they are not necessarily continuous, and thus, we are not able to use Lagrange interpolation. Instead, we resort to Clément, or Scott–Zhang operator, whose well known properties are sufficient for our purposes. In contrast to [7], where weighted spaces appear due to dimension reduction in an axisymmetrical problem, we do not need to modify the interpolation operators, but just use their local approximation and stability properties stated in (4.8) and (4.9).

The rest of this article is organized as follows. In Section 2 we define the weighted spaces and discuss the well-posedness of the problem. In Section 3 we specify the finite element spaces, and the discrete solution, proving stability of the discrete formulation. In Section 4 we prove Poincaré type and interpolation results on simplices, these will be instrumental for proving the main results in Section 5. We end the article with some numerical simulations in Section 6 illustrating the behavior of an adaptive algorithm based on the obtained *a posteriori* estimators.

2. Weighted spaces and weak formulation

Let $\Omega\subset\mathbb{R}^n$ be a bounded polygonal ($n=2$) or polyhedral ($n=3$) domain with Lipschitz boundary and x_0 an inner point of Ω. For $\beta\in(-\frac{n}{2},\frac{n}{2})$, we denote by $L^2(\Omega,\mathrm{d}_{x_0}^{2\beta})$ the space of measurable functions u such that

$$\|u\|_{L^2_\beta(\Omega)}:=\|u\|_{L^2(\Omega,\mathrm{d}_{x_0}^{2\beta})}:=\left(\int_\Omega|u(x)|^2\,\mathrm{d}_{x_0}(x)^{2\beta}\mathrm{d}x\right)^{\frac{1}{2}}<\infty,$$

where $\mathrm{d}_{x_0}(x)=|x-x_0|$ is the euclidean distance from x to x_0. We will write $L^2_\beta(\Omega)$ to denote $L^2(\Omega,\mathrm{d}_{x_0}^{2\beta})$ and observe that it is a Hilbert space equipped with the scalar product

$$\langle u,v\rangle_{\Omega,\beta}:=\int_\Omega u(x)v(x)\,\mathrm{d}_{x_0}(x)^{2\beta}\mathrm{d}x.$$

We also define the weighted Sobolev space $H^1_\beta(\Omega)$ of weakly differentiable functions u such that $\|u\|_{H^1_\beta(\Omega)}<\infty$, with

$$\|u\|_{H^1_\beta(\Omega)}:=\|u\|_{L^2_\beta(\Omega)}+\|\nabla u\|_{L^2_\beta(\Omega)}.$$

We immediately observe that, if $0<\alpha<\frac{n}{2}$, then $H^1_{-\alpha}(\Omega)\subset H^1(\Omega)\subset H^1_\alpha(\Omega)$ with continuity. Since the source term of (1.1) does not belong to the dual of $H^1_0(\Omega)$, we intend to use appropriate subspaces of $H^1_{-\alpha}(\Omega)$ and $H^1_\alpha(\Omega)$ for the test and ansatz space, respectively. We need to prove that this leads to a stable formulation, and we thus recall some known facts about weighted spaces.

The theory of weighted L^p spaces over n-dimensional domains is well developed and much attention has been payed to the class of Muckenhoupt weights A_p [23]. In our context of Hilbert spaces over two-dimensional and three-dimensional domains only the Muckenhoupt class A_2 matters, which is defined as the set of weights w such that their A_2 constant $\sup_B \left(\frac{1}{|B|}\int_B w(x)\,\mathrm{d}x\right)\left(\frac{1}{|B|}\int_B w(x)^{-1}\,\mathrm{d}x\right)$ is finite, where the supremum is taken over all balls B in $\mathbb{R}^n$. It is easy to prove that the weight function $\mathrm{d}_{x_0}^{2\beta}$ belongs to A_2 if and only if $-\frac{n}{2} < \beta < \frac{n}{2}$. For β in this range, the results from [18, 19, 21] imply that smooth functions are dense in $H^1_\beta(\Omega)$, and also a Rellich–Kondrachov theorem and a Poincaré inequality hold in $H^1_\beta(\Omega)$.

In the following we recall some results which are instrumental to state the weak formulation of (1.1).

By [20], Lemma 7.1.3, if $\frac{n}{2} - 1 < \alpha < \frac{n}{2}$ there exists a unique linear continuous map $\delta_{x_0} : H^1_{-\alpha}(\Omega) \to \mathbb{R}$ such that $\delta_{x_0}(\varphi) = \varphi(x_0)$ for any $\varphi \in C^1(\bar{\Omega})$. More precisely, there exists a constant C depending on α and Ω, such that

$$|\delta_{x_0}(\varphi)| \le C\|\varphi\|_{H^1_{-\alpha}(\Omega)}, \qquad \forall \varphi \in H^1_{-\alpha}(\Omega); \tag{2.1}$$

see also Theorem 4.7 below.

Since we are considering Dirichlet boundary conditions, we define

$$W_\beta := \{u \in H^1_\beta(\Omega) : u_{|\partial\Omega} = 0\},$$

and since $\mathrm{d}_{x_0}^{2\beta}$ belongs to A_2, from [15], Theorem 1.3 it follows that Poincaré inequality holds in W_β and therefore $\|u\|_{W_\beta} := \|\nabla u\|_{L^2_\beta(\Omega)}$ is a norm in W_β equivalent to the inherited norm $\|u\|_{H^1_\beta(\Omega)}$. More precisely, for $-\frac{n}{2} < \beta < \frac{n}{2}$, there exists a constant $C_{P,\beta}$, depending on the diameter of Ω, such that

$$\|u\|_{W_\beta} \le \|u\|_{H^1_\beta(\Omega)} \le C_{P,\beta}\,\|u\|_{W_\beta}, \qquad u \in W_\beta, \tag{2.2}$$

where $C_{P,\beta}$ blows up as $|\beta|$ approaches $\frac{n}{2}$.

Given $\frac{n}{2} - 1 < \alpha < \frac{n}{2}$, the considerations above yield $W_{-\alpha} \subset H^1_0(\Omega) \subset W_\alpha$ and $\delta_{x_0} \in (W_{-\alpha})'$. We thus say that u is a weak solution of (1.1) if

$$u \in W_\alpha : \qquad a(u, v) = \delta_{x_0}(v), \quad \forall\, v \in W_{-\alpha}, \tag{2.3}$$

where $a : W_\alpha \times W_{-\alpha} \to \mathbb{R}$ is the bilinear form given by

$$a(u, v) = \int_\Omega \mathcal{A}\nabla u \cdot \nabla v + \boldsymbol{b} \cdot \nabla u\, v + c\, u\, v, \tag{2.4}$$

which is clearly well-defined and bounded in $W_\alpha \times W_{-\alpha}$ due to Hölder inequality. At this point it is not clear that the bilinear form $a(\cdot,\cdot)$ satisfies an inf-sup condition on $W_\alpha \times W_{-\alpha}$. Therefore, existence and uniqueness of solution to (2.3) must be proved.

Problem (2.3) is a particular case of the following problem: Given $F \in (W_{-\alpha})'$,

$$\text{Find } u \in W_\alpha \quad \text{such that} \quad a(u, v) = F(v), \quad \forall\, v \in W_{-\alpha}. \tag{2.5}$$

The rest of this section will be devoted to proving existence and uniqueness of solutions to (2.5). We will proceed by splitting it into two subproblems, taking advantage of the following facts:

- An inf-sup condition holds on $W_\alpha \times W_{-\alpha}$ for the purely second order part $\int_\Omega \mathcal{A}\nabla u \cdot \nabla v$ of $a(\cdot,\cdot)$.
- The full bilinear form $a(\cdot,\cdot)$ is coercive on $H^1_0(\Omega) \times H^1_0(\Omega)$.

Observe that one solution to (2.5) is given by $u = \bar{u} + \bar{w}$, if

$$\bar{u} \in W_\alpha : \qquad \int_\Omega \mathcal{A}\nabla \bar{u} \cdot \nabla v = F(v), \quad \forall\, v \in W_{-\alpha}, \tag{2.6}$$

and

$$\bar{w} \in H^1_0(\Omega) : \qquad a(\bar{w}, v) = l(v) := -\int_\Omega (\boldsymbol{b} \cdot \nabla \bar{u} + c\bar{u})v, \quad \forall\, v \in H^1_0(\Omega). \tag{2.7}$$

In fact, since $W_{-\alpha} \subset H_0^1(\Omega)$, from (2.6) and (2.7) we have, for $v \in W_{-\alpha}$,

$$a(u,v) = a(\bar{u},v) + a(\bar{w},v) = F(v) + \int_\Omega (\boldsymbol{b} \cdot \nabla \bar{u} + c\bar{u})v + a(\bar{w},v) = F(v).$$

The next two propositions state the well-posedness of (2.6) and (2.7), yielding existence of a solution to (2.5).

Proposition 2.1. *If $\alpha \in (0, \frac{n}{2})$ and $F \in (W_{-\alpha})'$ then problem (2.6) has a unique solution $\bar{u} \in W_\alpha$ which satisfies*

$$\|\bar{u}\|_{W_\alpha} \leq \frac{2}{\gamma_1} \|F\|_{(W_{-\alpha})'}, \tag{2.8}$$

where γ_1 is given by (1.2).

To prove this proposition we will use the following decomposition of $[L^2_\beta(\Omega)]^n$, for $\beta \in (-\frac{n}{2}, \frac{n}{2})$:

For each $\boldsymbol{\tau} \in [L^2_\beta(\Omega)]^n$, there exists a unique pair $(\boldsymbol{\sigma}, v) \in [L^2_\beta(\Omega)]^n \times W_\beta$ such that

$$\boldsymbol{\tau} = \nabla v + \boldsymbol{\sigma}, \quad \langle \mathcal{A}\boldsymbol{\sigma}, \nabla z \rangle_\Omega = 0 \quad \forall\, z \in W_{-\beta},$$
$$\|\nabla v\|_{L^2_\beta(\Omega)} \leq 2\, \|\boldsymbol{\tau}\|_{L^2_\beta(\Omega)}\,, \quad \|\boldsymbol{\sigma}\|_{L^2_\beta(\Omega)} \leq \|\boldsymbol{\tau}\|_{L^2_\beta(\Omega)}\,.$$

Here, $\langle \cdot, \cdot \rangle_\Omega$ denotes the usual $[L^2(\Omega)]^n$ inner product.

This is an immediate generalization of [10], Lemma 2.1, which states the same result for $\mathcal{A} = I$. Its proof follows exactly the same lines, using that $\mathcal{A}$ is uniformly SPD over Ω, and is thus omitted.

Proof of Proposition 2.1. Let $0 < \alpha < \frac{n}{2}$. By Hölder inequality the bilinear form $A[w,v] := \int_\Omega \mathcal{A}\nabla w \cdot \nabla v$ is bounded in $W_\alpha \times W_{-\alpha}$. We use the decomposition of $[L^2_{-\alpha}(\Omega)]^n$ stated above to prove that $A[\cdot,\cdot]$ satisfies an inf-sup condition. Given $w \in W_\alpha$, let $\boldsymbol{\tau} := \nabla w\, \mathrm{d}_{x_0}^{2\alpha} \in [L^2_{-\alpha}(\Omega)]^n$. Thus, there exist $\boldsymbol{\sigma} \in [L^2_{-\alpha}(\Omega)]^n$ and $v \in W_{-\alpha}$ such that $\boldsymbol{\tau} = \nabla v + \boldsymbol{\sigma}$, $\langle \mathcal{A}\nabla z, \boldsymbol{\sigma} \rangle_\Omega = 0$, $\forall\, z \in W_\alpha$ and $2\, \|w\|_{W_\alpha} = 2\|\boldsymbol{\tau}\|_{L^2_{-\alpha}(\Omega)} \geq \|v\|_{W_{-\alpha}}$. Then,

$$\langle \mathcal{A}\nabla w, \nabla v \rangle_\Omega = \langle \mathcal{A}\nabla w, \boldsymbol{\tau} \rangle_\Omega - \langle \mathcal{A}\nabla w, \boldsymbol{\sigma} \rangle_\Omega = \langle \mathcal{A}\nabla w, \nabla w\, \mathrm{d}_{x_0}^{2\alpha} \rangle_\Omega \geq \gamma_1\, \|w\|^2_{W_\alpha} \geq \frac{\gamma_1}{2}\, \|w\|_{W_\alpha}\, \|v\|_{W_{-\alpha}}\,,$$

where γ_1 is given by (1.2). The same estimate still holds if we swap w and v and change the sign of α. So, the following inf-sup conditions are valid:

$$\inf_{w \in W_\alpha} \sup_{v \in W_{-\alpha}} \frac{\int_\Omega \mathcal{A}\nabla w \cdot \nabla v}{\|w\|_{W_\alpha}\, \|v\|_{W_{-\alpha}}} \geq \frac{\gamma_1}{2} \quad \text{and} \quad \inf_{v \in W_{-\alpha}} \sup_{w \in W_\alpha} \frac{\int_\Omega \mathcal{A}\nabla w \cdot \nabla v}{\|w\|_{W_\alpha}\, \|v\|_{W_{-\alpha}}} \geq \frac{\gamma_1}{2}.$$

Finally, the generalized Lax–Milgram theorem due to Nečas [25], Theorem 3.3 leads to existence and uniqueness of a solution $\bar{u}$ to problem (2.6) which satisfies (2.8). $\square$

Proposition 2.2. *Let $\alpha \in (0,1)$. Given $F \in (W_{-\alpha})'$, let $\bar{u} \in W_\alpha$ be the unique solution to (2.6). Then, problem (2.7) admits a unique solution $\bar{w} \in H_0^1(\Omega)$ which satisfies*

$$\|\bar{w}\|_{H_0^1(\Omega)} \leq \tilde{c}_\alpha \|F\|_{(W_{-\alpha})'}, \tag{2.9}$$

where $\tilde{c}_\alpha > 0$ is a constant depending on Ω, the problem coefficients $\{\mathcal{A}, \boldsymbol{b}, c\}$ and blows up when α approaches 1.

In the proof of this proposition we will use the embedding $H^1(\Omega) \hookrightarrow L^2_{-\alpha}(\Omega)$, which holds for $\alpha \in (0,1)$. In fact, taking $1 < p < \frac{1}{\alpha}$ if $n = 2$ and $p = \frac{3}{2}$ if $n = 3$, using Hölder inequality and the Sobolev embedding $H^1(\Omega) \hookrightarrow L^{2q}(\Omega)$ where $\frac{1}{p} + \frac{1}{q} = 1$ we have that

$$\|v\|_{L^2_{-\alpha}(\Omega)} = \left(\int_\Omega v^2\, \mathrm{d}_{x_0}^{-2\alpha}\right)^{\frac{1}{2}} \leq \left(\int_\Omega v^{2q}\right)^{\frac{1}{2q}} \left(\int_\Omega \mathrm{d}_{x_0}^{-2\alpha p}\right)^{\frac{1}{2p}} \leq c_\alpha \|v\|_{H^1(\Omega)}, \qquad \forall\, v \in H^1(\Omega), \tag{2.10}$$

where c_α depends on Ω and α, and blows up when α approaches 1.

Proof of Proposition 2.2. Let $\alpha \in (0,1)$ and $F \in (W_{-\alpha})'$. Let $\bar{u} \in W_\alpha$ be the solution to (2.6). Since we have assumed $c - \frac{1}{2}\operatorname{div}(\boldsymbol{b}) \geq 0$, the bilinear form $a : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ given by (2.4) is continuous and coercive, and thus, by Lax–Milgram theorem, problem (2.7) admits a unique solution $\bar{w} \in H^1_0(\Omega)$ which satisfies

$$\|\bar{w}\|_{H^1_0(\Omega)} \leq \frac{1}{\gamma_1}\|l\|_{H^{-1}(\Omega)}, \tag{2.11}$$

provided $l \in H^{-1}(\Omega) := (H^1_0(\Omega))'$, where $l(v) := -\int_\Omega(\boldsymbol{b}\cdot\nabla\bar{u} + c\bar{u})v \,\mathrm{d}x$, for $v \in H^1_0(\Omega)$.

In order to prove that $l \in H^{-1}(\Omega)$ and bound $\|l\|_{H^{-1}(\Omega)}$, let $v \in H^1_0(\Omega)$, and observe that using (2.10) and (2.2) we have that

$$\begin{aligned}|l(v)| = \left|\int_\Omega(\boldsymbol{b}\cdot\nabla\bar{u} + c\bar{u})v\right| &\leq \left(\|\boldsymbol{b}\|_{L^\infty(\Omega)}\|\nabla\bar{u}\|_{L^2_\alpha(\Omega)} + \|c\|_{L^\infty(\Omega)}\|\bar{u}\|_{L^2_\alpha(\Omega)}\right)\|v\|_{L^2_{-\alpha}(\Omega)} \\ &\leq \max\left\{\|\boldsymbol{b}\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)}\right\}\left(\|\bar{u}\|_{L^2_\alpha(\Omega)} + \|\nabla\bar{u}\|_{L^2_\alpha(\Omega)}\right)c_\alpha\|v\|_{H^1(\Omega)} \\ &\leq C_{P,\alpha}c_\alpha \max\left\{\|\boldsymbol{b}\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)}\right\}\|\bar{u}\|_{W_\alpha}\|v\|_{H^1(\Omega)}.\end{aligned} \tag{2.12}$$

Taking into account (2.8), it follows that

$$\|l\|_{H^{-1}(\Omega)} \leq \frac{2C_{P,\alpha}c_\alpha}{\gamma_1}\max\left\{\|\boldsymbol{b}\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)}\right\}\|F\|_{(W_{-\alpha})'},$$

which by (2.11) leads to the desired bound (2.9). □

As a consequence of Propositions 2.1 and 2.2 we conclude the well-posedness of problem (2.5).

Theorem 2.3. *Let $0 < \alpha < \frac{n}{2}$, if $\boldsymbol{b} = 0$ and $c = 0$, and $0 < \alpha < 1$, otherwise. For each $F \in (W_{-\alpha})'$, there exists a unique solution $u \in W_\alpha$ of problem (2.5) which satisfies*

$$\|u\|_{W_\alpha} \leq C_*\|F\|_{(W_{-\alpha})'}, \tag{2.13}$$

where the constant $C_* > 0$ depends on the domain Ω, the problem coefficients $\{\mathcal{A}, \boldsymbol{b}, c\}$ and α. If $\boldsymbol{b} = 0$ and $c = 0$ then $C_* = 2/\gamma_1$, otherwise, C_* blows up when α approaches 1.

Moreover, the following inf-sup condition holds:

$$\inf_{w\in W_\alpha}\sup_{v\in W_{-\alpha}}\frac{a(w,v)}{\|w\|_{W_\alpha}\|v\|_{W_{-\alpha}}} \geq \frac{1}{C_*}. \tag{2.14}$$

Remark 2.4. The constant C_* depends on $\frac{1}{\gamma_1}\max\{\|b\|_\infty, \|c\|_\infty\}$ and the stability just obtained is not uniform for advection dominated problems. The study of this class of problems falls beyond the scope of this article, and will be subject of future research. It is clear that the main issue in this direction will be to prove an inf-sup condition with a constant C_* independent of the smallness of γ_1.

Proof of Theorem 2.3. If $\boldsymbol{b} = 0$ and $c = 0$, problem (2.5) coincides with (2.6). Therefore, existence, uniqueness and the bound (2.13) follow for $0 < \alpha < \frac{n}{2}$, from Proposition 2.1.

If $\boldsymbol{b} \neq 0$ or $c \neq 0$, assume that $0 < \alpha < 1$ and let $\bar{u}$, $\bar{w}$ denote the solutions of problems (2.6) and (2.7), respectively. Then $u := \bar{u} + \bar{w}$ is a solution of problem (2.5), and (2.13) holds due to (2.8) and (2.9). It remains to prove that this solution is unique. This is not so obvious because we have not proved an inf-sup condition for the full bilinear form $a(\cdot,\cdot)$ but only for the purely second order part.

Let $u, \tilde{u}$ be solutions of (2.5) and define $e := u - \tilde{u}$. Then,

$$e \in W_\alpha : \qquad a(e,v) = 0, \quad \forall\, v \in W_{-\alpha},$$

and also

$$e \in W_\alpha : \qquad \int_\Omega \mathcal{A}\nabla e \cdot \nabla v = L(v) := -\int_\Omega (\boldsymbol{b}\cdot\nabla e + ce)v, \quad \forall\, v \in W_{-\alpha}.$$

Since $e \in W_\alpha$, we have that $L \in H^{-1}(\Omega)$ (*cf.* (2.12)), and there exists a unique $\tilde{e} \in H_0^1(\Omega)$ satisfying $\int_\Omega \mathcal{A}\nabla\tilde{e}\cdot\nabla v = L(v)$, for all $v \in H_0^1(\Omega)$. Since $W_{-\alpha} \subset H_0^1(\Omega) \subset W_\alpha$, it follows from Proposition 2.1 that $\tilde{e}$ is the unique solution to

$$\tilde{e} \in W_\alpha : \qquad \int_\Omega \mathcal{A}\nabla\tilde{e}\cdot\nabla v = L(v), \quad \forall v \in W_{-\alpha}.$$

Therefore $e = \tilde{e} \in H_0^1(\Omega)$ and

$$e \in H_0^1(\Omega) : \quad a(e,v) = 0, \quad \forall\, v \in H_0^1(\Omega),$$

which implies that $e = 0$ by the coercivity of $a(\cdot,\cdot)$ in $H_0^1(\Omega)$.

Finally, existence and uniqueness of solution to problem (2.5) for each $F \in (W_{-\alpha})'$ and the bound (2.13) imply that the inf-sup condition (2.14) holds (*cf.* [25], Thm. 3.3). □

We end this section recalling that $F := \delta_{x_0}$ belongs to $(W_{-\alpha})'$ if $\alpha \in (\frac{n}{2}-1, \frac{n}{2})$ and thus, Theorem 2.3 implies that:

$$\text{Problem (2.3) is well-posed provided} \quad \alpha \in \mathbb{I} := \begin{cases} \left(\frac{n}{2}-1, \frac{n}{2}\right) & \text{if} \quad \boldsymbol{b} = 0,\ c = 0, \\ \left(\frac{n}{2}-1, 1\right) & \text{otherwise.} \end{cases} \tag{2.15}$$

3. Finite element discretization

In this section we define the finite element spaces that we consider, and let the discrete solution U be the usual Galerkin approximation of the weak solution u. We then show that the discretization is stable by proving an inf-sup condition which is independent of the mesh, which can be graded, but must be shape-regular.

Let $\mathcal{T}$ be a conforming triangulation of the domain $\Omega \subset \mathbb{R}^n$. That is, a partition of Ω into n-simplices such that if two elements intersect, they do so at a full vertex/edge/face of both elements. We define the mesh regularity constant

$$\kappa := \sup_{T\in\mathcal{T}} \frac{\operatorname{diam}(T)}{\rho_T},$$

where $\operatorname{diam}(T)$ is the diameter of T, and ρ_T is the radius of the largest ball contained in it. Also, the diameter of any element $T \in \mathcal{T}$ is equivalent to the local mesh-size $h_T := |T|^{1/n}$, with equivalence constants depending on κ.

On the other hand, we denote the subset of $\mathcal{T}$ consisting of an element T and its neighbors by $\mathcal{N}_T$ and the union of the elements in $\mathcal{N}_T$ by ω_T. More precisely, for $T \in \mathcal{T}$,

$$\mathcal{N}_T := \{T' \in \mathcal{T} \mid\ T \cap T' \neq \emptyset\}, \qquad \omega_T := \bigcup_{T'\in\mathcal{N}_T} T'.$$

We denote by $\mathcal{E}_\Omega$ to the set of sides (edges for $n = 2$ and faces for $n = 3$) of the elements in $\mathcal{T}$ which are inside Ω and by $\mathcal{E}_{\partial\Omega}$ to the set of sides which lie on the boundary of Ω. We define ω_S as the union of the two elements sharing S, if $S \in \mathcal{E}_\Omega$, and as the unique element T_S satisfying $S \subset \partial T_S$ if $S \in \mathcal{E}_{\partial\Omega}$.

For the discretization we consider Lagrange finite elements of degree $\ell \in \mathbb{N}$, more precisely, we let

$$\mathbb{V}_\mathcal{T}^\ell := \left\{ V \in H_0^1(\Omega) \mid\ V_{|_T} \in \mathcal{P}_\ell(T),\ \forall\, T \in \mathcal{T} \right\},$$

and observe that $\mathbb{V}_\mathcal{T}^\ell \subset W_\beta$, for $\beta \in (-\frac{n}{2}, \frac{n}{2})$. The discrete counterpart of (2.3) reads:

$$\text{Find } U \in \mathbb{V}_\mathcal{T}^\ell \text{ such that} \quad a(U,V) = \delta_{x_0}(V), \quad \forall\, V \in \mathbb{V}_\mathcal{T}^\ell. \tag{3.1}$$

Clearly, this discrete problem has a unique solution for each mesh; the system matrix is not affected by the right-hand side and is invertible because the assumptions on the problem coefficients guarantee the coercivity of the bilinear form $a(\cdot,\cdot)$ in $\mathbb{V}_{\mathcal{T}}^{\ell} \times \mathbb{V}_{\mathcal{T}}^{\ell}$.

Unlike [3,4,16] we also prove here a stability result for a general right-hand side $F \in (W_{-\alpha})'$; see Theorem 3.1. By the theory of [22, 29] this allows us to conclude that adaptive algorithms with the *a posteriori* estimates developed here yield convergence. Recall also that the discrete inf-sup is usually not used for the derivation of *a posteriori* estimates, only the continuous one needs to be used.

Theorem 3.1 (Stability of discrete solutions)**.** *Let $0 < \alpha < \frac{n}{2}$, if $\boldsymbol{b} = 0$ and $c = 0$, and $0 < \alpha < 1$, otherwise. Let us consider the following problem for $F \in (W_{-\alpha})'$:*

$$\textit{Find} \quad U \in \mathbb{V}_{\mathcal{T}}^{\ell}: \qquad a(U,V) = F(V), \qquad \forall\, V \in \mathbb{V}_{\mathcal{T}}^{\ell}. \tag{3.2}$$

There exists a constant $C^* > 0$, which depends on the domain Ω, the problem coefficients $\{\mathcal{A}, \boldsymbol{b}, c\}$, the mesh regularity constant κ, the polynomial degree ℓ, and α, such that the solution $U \in \mathbb{V}_{\mathcal{T}}^{\ell}$ of (3.2) satisfies

$$\|U\|_{W_\alpha} \leq C^* \|F\|_{(W_{-\alpha})'}.$$

The constant C^* blows up as α approaches $\frac{n}{2}$ or 1, respectively.

In the proof of this theorem we will use the space

$$\mathcal{M}_{\mathcal{T}}^{\ell-1} := \Big\{ \boldsymbol{\lambda} \in \big[L^2(\Omega)\big]^n \mid \boldsymbol{\lambda}_{|T} \in \mathcal{P}_{\ell-1}^n(T),\ \forall\, T \in \mathcal{T} \Big\} \supset \nabla \mathbb{V}_{\mathcal{T}}^{\ell},$$

and apply the following decomposition:

Let $\beta \in (-\frac{n}{2}, \frac{n}{2})$. For each $\boldsymbol{\lambda} \in \mathcal{M}_{\mathcal{T}}^{\ell-1}$, there exists a unique couple $(\boldsymbol{\sigma}, V) \in \mathcal{M}_{\mathcal{T}}^{\ell-1} \times \mathbb{V}_{\mathcal{T}}^{\ell}$ such that

$$\boldsymbol{\lambda} = \nabla V + \boldsymbol{\sigma}, \quad \langle \mathcal{A}\boldsymbol{\sigma}, \nabla Z \rangle_\Omega = 0 \quad \forall\, Z \in \mathbb{V}_{\mathcal{T}}^{\ell},$$

$$\|\nabla V\|_{\mathcal{T},\beta} \leq 2\|\boldsymbol{\lambda}\|_{\mathcal{T},\beta}, \quad \|\boldsymbol{\sigma}\|_{\mathcal{T},\beta} \leq \|\boldsymbol{\lambda}\|_{\mathcal{T},\beta},$$

where $\|\boldsymbol{\lambda}\|_{\mathcal{T},\beta} := \left(\sum_{T\in\mathcal{T}} D_T^{2\beta} \|\boldsymbol{\lambda}\|_{L^2(T)}^2 \right)^{\frac{1}{2}}$, for all $\boldsymbol{\lambda} \in \mathcal{M}_{\mathcal{T}}^{\ell-1}$, and $D_T := \max_{x\in T} \mathrm{d}_{x_0}(x)$.

This is an immediate generalization of [10], Lemma 3.3, with a similar proof, again, taking into account that $\mathcal{A}$ is uniformly SPD. D'Angelo proposed the discrete norm $\|\cdot\|_{\mathcal{T},\beta}$ used in the decomposition, and proved in [10], Lemma 3.2 that it is equivalent to $\|\cdot\|_{L^2_\beta(\Omega)}$, for $\beta \in (-\frac{n}{2}, \frac{n}{2})$, with equivalence constants depending only on κ, the polynomial degree ℓ and $|\beta|$. The proof is based on the fact that for $t \in (0, \frac{n}{2})$ fixed, there exists a constant c_t, depending on κ, ℓ and t, such that, if $|\beta| \leq t$, then

$$\frac{1}{c_t} \|V\|_{L^2_\beta(T)} \leq D_T^\beta \|V\|_{L^2(T)} \leq c_t \|V\|_{L^2_\beta(T)}, \quad \forall\, T \in \mathcal{T},\ \forall\, V \in \mathcal{P}_\ell(T). \tag{3.3}$$

This last local equivalence will also be used in Proposition 4.6.

Proof of Theorem 3.1. Notice that the solution U of problem (3.2) can be split as $U = \bar{U} + \bar{W}$ with

$$\bar{U} \in \mathbb{V}_{\mathcal{T}}^{\ell}: \qquad \int_\Omega \mathcal{A}\nabla\bar{U} \cdot \nabla V = F(V), \quad \forall\, V \in \mathbb{V}_{\mathcal{T}}^{\ell}, \tag{3.4}$$

and

$$\bar{W} \in \mathbb{V}_{\mathcal{T}}^{\ell}: \qquad a(\bar{W}, V) = -\int_\Omega (\boldsymbol{b}\cdot\nabla\bar{U} + c\bar{U})V, \quad \forall\, V \in \mathbb{V}_{\mathcal{T}}^{\ell}. \tag{3.5}$$

Therefore, we just need to bound the solutions $\bar{U}$ and $\bar{W}$ to problems (3.4) and (3.5) in W_α by $\|F\|_{(W_{-\alpha})'}$. Using the aforementioned decomposition and proceeding as in the proof of Proposition 2.1 we arrive at

$$\inf_{W\in\mathbb{V}_{\mathcal{T}}^{\ell}} \sup_{V\in\mathbb{V}_{\mathcal{T}}^{\ell}} \frac{\int_\Omega \mathcal{A}\nabla W\cdot\nabla V}{\|W\|_{W_\alpha}\|V\|_{W_{-\alpha}}} \geq \tilde{\gamma}_1 \quad \text{and} \quad \inf_{V\in\mathbb{V}_{\mathcal{T}}^{\ell}} \sup_{W\in\mathbb{V}_{\mathcal{T}}^{\ell}} \frac{\int_\Omega \mathcal{A}\nabla W\cdot\nabla V}{\|W\|_{W_\alpha}\|V\|_{W_{-\alpha}}} \geq \tilde{\gamma}_1,$$

which holds for $0<\alpha<\frac{n}{2}$, with $\tilde{\gamma}_1$ depending on γ_1 from (1.2), κ, ℓ and α, and vanishing as α approaches $\frac{n}{2}$. Therefore,

$$\|\bar{U}\|_{W_\alpha} \leq \frac{1}{\tilde{\gamma}_1}\|F\|_{(W_{-\alpha})'}. \tag{3.6}$$

On the one hand, if $\boldsymbol{b}=0$ and $c=0$, $\bar{W}=0$.

On the other hand, if $\boldsymbol{b}\neq 0$ or $c\neq 0$, we use that the continuity and coercivity of the bilinear form a are inherited from the continuous space to the discrete one, and thus the solution $\bar{W}$ of problem (3.5) satisfies

$$\|\bar{W}\|_{H_0^1(\Omega)} \leq \frac{1}{\gamma_1}\|\bar{L}\|_{H^{-1}(\Omega)},$$

where $\bar{L}(V) := -\int_\Omega(\boldsymbol{b}\cdot\nabla\bar{U}+c\,\bar{U})V$. In view of (2.12), $\|\bar{L}\|_{H^{-1}(\Omega)}$ is controlled by $\|\bar{U}\|_{W_\alpha}$, and from (3.6) the claim follows. □

4. Some results in weighted spaces on simplices

In this section we state and prove some properly scaled bounds which are valid on the elements of the triangulation, with constants depending only on mesh regularity. These bounds include a local Poincaré inequality, a bound for $\|\delta_{x_0}\|_{(W_{-\alpha})'}$, and bounds for the error in Clément and Scott–Zhang interpolation operators. Most of these bounds are known for the usual Sobolev norms, without weights.

This section is independent of the elliptic operator or the precise problem at hand. The results stated here might be useful in other applications involving point sources.

From now on, we will write $a\lesssim b$ to indicate that $a\leq Cb$ with $C>0$ a constant depending on the shape regularity κ of the mesh and possibly on the domain $\Omega\subset\mathbb{R}^n$, which is assumed polygonal ($n=2$) or polyhedral ($n=3$) with a Lipschitz boundary. Also $a\simeq b$ will indicate that $a\lesssim b$ and $b\lesssim a$.

4.1. Classification of simplices

In order to prove our results we classify the elements according to their relationship to x_0. We categorize the elements of $\mathcal{T}$ into two disjoint classes, defined as follows:

$$\mathcal{T}^{\text{near}} := \{T\in\mathcal{T} \mid x_0\in\omega_T\} \qquad \text{and} \qquad \mathcal{T}^{\text{far}} := \mathcal{T}\setminus\mathcal{T}^{\text{near}}.$$

Recall that for $T\in\mathcal{T}$, $D_T=\max_{x\in T}\mathrm{d}_{x_0}(x)$, and let d_T be defined by $d_T:=\min_{x\in T}\mathrm{d}_{x_0}(x)$. Now, we establish a relationship between the classical local norms $\|\cdot\|_{L^2(T)}$ and the weighted ones $\|\cdot\|_{L^2_\beta(T)}$.

Lemma 4.1. *The following statements hold:*

(i) *If $-\frac{n}{2}<\beta<\frac{n}{2}$ and $T\in\mathcal{T}^{\text{far}}$, then $h_T\lesssim d_T\simeq D_T$ and*

$$\|v\|_{L^2_\beta(T)} \simeq D_T^\beta\|v\|_{L^2(T)}, \qquad \forall\, v\in L^2(T), \tag{4.1}$$

$$\|v\|_{L^2_\beta(\partial T)} \simeq D_T^\beta\|v\|_{L^2(\partial T)}, \qquad \forall\, v\in L^2(\partial T). \tag{4.2}$$

(ii) *If* $0 \leq \alpha < \frac{n}{2}$ *and* $T \in \mathcal{T}^{\text{near}}$, *then* $h_T \simeq D_T$ *and*

$$\|v\|_{L^2_{-\alpha}(T)} \gtrsim h_T^{-\alpha}\|v\|_{L^2(T)}, \qquad \forall\, v \in L^2_{-\alpha}(T), \tag{4.3}$$

$$\|v\|_{L^2_{\alpha}(T)} \lesssim h_T^{\alpha}\|v\|_{L^2(T)}, \qquad \forall\, v \in L^2(T). \tag{4.4}$$

To prove this lemma we will use the following result, which states that a neigborhood of size $\simeq h_T$ of an element T is always contained in ω_T. This result will also be used in the proof of the lower bound (see Thm. 5.3).

Lemma 4.2. *There exists a constant* $c_{\kappa,\Omega} > 0$ *depending on mesh regularity* κ *and the Lipschitz property of* $\partial\Omega$ *such that, if* $T \in \mathcal{T}$, $x \in T$ *and* $y \in \Omega \setminus \omega_T$, *then* $|x-y| \geq c_{\kappa,\Omega} h_T$. *In other words,* $B(x, c_{\kappa,\Omega} h_T) \cap \Omega \subset \omega_T$ *for all* $x \in T$ *and all* $T \in \mathcal{T}$.

Proof. Let $T \in \mathcal{T}$, let ϕ_i, $i = 1, \ldots, n+1$, be the canonical basis functions of $\mathbb{V}^1_{\mathcal{T}}$ corresponding to each vertex of T, and let $\psi = \sum_{i=1}^{n+1} \phi_i$. Then $\|\nabla\psi\|_{L^\infty(\Omega)} \lesssim 1/h_T$, and therefore

$$|\psi(x) - \psi(y)| \leq \frac{1}{c_{\kappa,\Omega} h_T}|x-y|, \qquad \text{for all } x, y \in \Omega,$$

where $c_{\kappa,\Omega}$ depends only on mesh regularity and the Lipschitz property of $\partial\Omega$. Since $\psi(x) = 1$ if $x \in T$ and $\psi(y) = 0$ for $y \notin \omega_T$ the claim follows. $\square$

Proof of Lemma 4.1. Let $T \in \mathcal{T}^{\text{far}}$, then $x_0 \notin \omega_T$, and $d_T = \min_T \mathrm{d}_{x_0} = |x_0 - x|$ for some $x \in T$, whence $h_T \lesssim d_T$ by Lemma 4.2. Therefore, $D_T \lesssim d_T + h_T \lesssim d_T$ and thus $d_T \simeq D_T$, which implies (4.1). Since $d_T \leq \min_{\partial T} \mathrm{d}_{x_0} \leq \max_{\partial T} \mathrm{d}_{x_0} \leq D_T$, (4.2) holds.

Let $T \in \mathcal{T}^{\text{near}}$. Then $x_0 \in \omega_T$, and thus $D_T \leq \operatorname{diam}(\omega_T) \lesssim h_T$. Besides, if x_1, x_2 are two vertices of T,

$$h_T \simeq |x_1 - x_2| \leq |x_1 - x_0| + |x_0 - x_2| \leq 2D_T.$$

Therefore $h_T \simeq D_T$, and thus (4.3) and (4.4) hold. $\square$

4.2. Local Poincaré inequality and interpolation estimates

The usual scaling arguments used to prove Poincaré inequalities on simplices do not lead to a uniform constant for all the elements in the mesh. We thus need to resort to real analysis tools from the theory of weighted inequalities [15, 24]. We start by recalling some definitions and important properties.

Let $0 < \gamma < n$, the fractional integral $I_\gamma(f)$ and the fractional maximal function f^*_γ of a measurable function $f : \mathbb{R}^n \to \mathbb{R}$ are defined, for $x \in \mathbb{R}^n$ by

$$I_\gamma(f)(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}}\,\mathrm{d}y, \quad f^*_\gamma(x) := \sup_B \frac{1}{|B|^{1-\gamma/n}} \int_B |f(y)|\,\mathrm{d}y, \tag{4.5}$$

where the supremum is taken over all balls B with center at x.

These two concepts are related through the following result, proved by Muckenhoupt and Wheeden (*cf.* [24], Thm. 1), for any $n \in \mathbb{N}$.

Lemma 4.3. *Let* $0 < \gamma < n$, $w \in A_\infty = \cup_{q \geq 1} A_q$, *and* $1 < p < \infty$. *Then, there exists a constant* $c > 0$ *such that*

$$\left(\int_{\mathbb{R}^n} |I_\gamma(f)|^p w\right)^{\frac{1}{p}} \leq c \left(\int_{\mathbb{R}^n} |f^*_\gamma|^p w\right)^{\frac{1}{p}},$$

for all measurable functions f.

From [15], Lemma 1.1 and using the same arguments of the proof of [15], Theorem 1.2 the next result follows, for the particular case $\gamma = 1$.

Lemma 4.4. *Let $w \in A_p$, for some p, $1 < p < \infty$. Then, there exists a constant $c > 0$, depending only on the A_p constant of w, such that*

$$\left(\int_{\mathbb{R}^n} |f_1^*|^p w \right)^{\frac{1}{p}} \leq cR \left(\int_{B_R} |f|^p w \right)^{\frac{1}{p}},$$

for all ball B_R of radius $R > 0$, and for all f measurable and supported in B_R.

As a consequence of these results we obtain the following scaled Poincaré inequality.

Theorem 4.5 (Poincaré inequality)**.** *Let $\beta \in (-\frac{n}{2}, \frac{n}{2})$. There exists a constant $C_P > 0$ depending on β and the mesh regularity κ such that, for all $v \in H^1_\beta(\Omega)$,*

$$\|v - v_T\|_{L^2_\beta(T)} \leq C_P h_T \|\nabla v\|_{L^2_\beta(T)}, \qquad \forall T \in \mathcal{T},$$

where $v_T := \frac{1}{|T|} \int_T v$. The constant C_P blows up when $|\beta|$ approaches $\frac{n}{2}$.

As we mentioned earlier, the usual scaling arguments do not yield a uniform constant C_P, and we thus resort to arguments from [15], where weighted Poincaré inequalities are proved on balls, with a uniform constant depending only on the A_p constant of the weight.

Proof. Let $v \in C^1(\bar{\Omega})$ and $T \in \mathcal{T}$. Since T is convex, by [17], Lemma 7.16, p. 162 we have that

$$|v(x) - v_T| \leq \frac{\operatorname{diam}(T)^n}{n\, h_T^n} \int_T \frac{|\nabla v(z)|}{|x - z|^{n-1}}\, dz,$$

for every $x \in T$. Let B_R be a ball containing T such that $R \lesssim h_T$, and define $f := |\nabla v| \chi_T$, where χ_T is the characteristic function of T. Then recalling the definition (4.5), $\int_T \frac{|\nabla v(z)|}{|x-z|^{n-1}}\, dz = I_1(f)(x)$ and thus by mesh regularity

$$|v(x) - v_T| \lesssim I_1(f)(x), \qquad \text{a.e. } x \in T. \tag{4.6}$$

Since $\mathrm{d}_{x_0}^{2\beta} \in A_2 \subset A_\infty$, due to Lemmas 4.3 and 4.4 it follows that

$$\|I_1(f)\|_{L^2_\beta(\mathbb{R}^n)} \leq cR\|f\|_{L^2_\beta(B_R)} = cR\|\nabla v\|_{L^2_\beta(T)}, \tag{4.7}$$

for some constant $c > 0$, depending only on β, through the A_2 constant of $\mathrm{d}_{x_0}^{2\beta}$, which blows up as $|\beta|$ approaches $n/2$. The bounds (4.6) and (4.7) yield the result for smooth functions v. The assertion of the theorem follows by density arguments. ☐

We will now show some interpolation estimates in weighted spaces, which hinge on the Poincaré inequality from Theorem 4.5, and are instrumental for proving the reliability of the error estimators. Let $\mathcal{P} : H^1_0(\Omega) \to \mathbb{V}^1_{\mathcal{T}}$ be either the Clément or the Scott–Zhang interpolation operator. It is well-known [9,30] that, for all $v \in H^1(\Omega)$,

$$\|v - \mathcal{P}v\|_{L^2(T)} \lesssim h_T \|\nabla v\|_{L^2(\omega_T)}, \quad \forall T \in \mathcal{T}, \tag{4.8}$$

$$\|\nabla(v - \mathcal{P}v)\|_{L^2(T)} \lesssim \|\nabla v\|_{L^2(\omega_T)}, \quad \forall T \in \mathcal{T}. \tag{4.9}$$

Since $H^1_{-\alpha}(\Omega) \subset H^1(\Omega)$ for $\alpha > 0$, $\mathcal{P}$ is also well defined for functions in $H^1_{-\alpha}(\Omega)$. Moreover, the above estimates hold in weighted norms, as we show in the following proposition.

Proposition 4.6 (Interpolation estimates)**.** *Let $\mathcal{P}$ denote either the Clément or the Scott–Zhang interpolation operator. Let $t \in (0, \frac{n}{2})$ and $0 \leq \alpha \leq t$. Then, there exists a constant $C_I > 0$ depending on the mesh regularity κ and t such that, for all $v \in H^1_{-\alpha}(\Omega)$,*

$$\|v - \mathcal{P}v\|_{L^2_{-\alpha}(T)} \leq C_I h_T \|\nabla v\|_{L^2_{-\alpha}(\omega_T)}, \quad \forall T \in \mathcal{T}, \tag{4.10}$$

$$\|\nabla(v - \mathcal{P}v)\|_{L^2_{-\alpha}(T)} \leq C_I \|\nabla v\|_{L^2_{-\alpha}(\omega_T)}, \quad \forall T \in \mathcal{T}. \tag{4.11}$$

The constant C_I blows up as t approaches $n/2$.

Proof. Let $v \in H^1_{-\alpha}(\Omega)$. Let $T \in \mathcal{T}$ and $v_T := \frac{1}{|T|}\int_T v$. Then, by (3.3)

$$\begin{aligned}\|v-\mathcal{P}v\|_{L^2_{-\alpha}(T)} &\le \|v-v_T\|_{L^2_{-\alpha}(T)} + c_t D_T^{-\alpha}\|v_T-\mathcal{P}v\|_{L^2(T)}\\ &\le \|v-v_T\|_{L^2_{-\alpha}(T)} + c_t D_T^{-\alpha}\left(\|v_T-v\|_{L^2(T)} + \|v-\mathcal{P}v\|_{L^2(T)}\right)\\ &\lesssim \|v-v_T\|_{L^2_{-\alpha}(T)} + c_t h_T D_T^{-\alpha}\|\nabla v\|_{L^2(\omega_T)},\end{aligned}$$

where the last inequality follows from the classic Poincaré inequality and (4.8). From Theorem 4.5 and the fact that $\mathrm{d}_{x_0}(x) \lesssim D_T$ for all $x \in \omega_T$ (4.10) holds.

Observe now that due to (3.3) and (4.9),

$$\|\nabla \mathcal{P}v\|_{L^2_{-\alpha}(T)} \le c_t D_T^{-\alpha}\|\nabla\mathcal{P}v\|_{L^2(T)} \lesssim c_t D_T^{-\alpha}\|\nabla v\|_{L^2(\omega_T)} \lesssim c_t\|\nabla v\|_{L^2_{-\alpha}(\omega_T)},$$

where we have used again that $\mathrm{d}_{x_0} \lesssim D_T$ in ω_T. The assertion (4.11) follows. □

4.3. A local bound for δ_{x_0}

In this section we present a local bound for δ_{x_0}, which is useful to establish the reliability of the *a posteriori* error estimators (*cf.* Thm. 5.1 below). It is a local version of (2.1), and the proof could be done by scaling. We present an alternative proof, following the lines of [11], Theorem 4.2, in order to show how the constants depend on α.

Theorem 4.7 (A precise bound of δ_{x_0})**.** *Let $\frac{n}{2}-1 < \alpha < \frac{n}{2}$ and $T \in \mathcal{T}$ such that $x_0 \in T$. Then*

$$|\delta_{x_0}(v)| \lesssim h_T^{\alpha-\frac{n}{2}}\|v\|_{L^2_{-\alpha}(T)} + C_\alpha h_T^{\alpha+\frac{2-n}{2}}\|\nabla v\|_{L^2_{-\alpha}(T)}, \qquad \forall\, v \in H^1_{-\alpha}(T), \tag{4.12}$$

where $C_\alpha := \frac{\alpha^{\frac{\alpha-1}{2}}}{(\alpha+1)^{\frac{\alpha+1}{2}}}$ if $n=2$ and $C_\alpha := \frac{(2\alpha-1)^{\frac{\alpha-2}{3}}}{(2\alpha+2)^{\frac{\alpha+1}{3}}}$ if $n=3$.

Note that the constant C_α blows up as α approaches $\frac{n}{2}-1$. This was expected because δ_{x_0} does not belong to the dual space of $H^1_{-\alpha}(\Omega)$, for $\alpha = \frac{n}{2}-1$, but only for $\frac{n}{2}-1 < \alpha < \frac{n}{2}$.

Proof. Assume $n=3$ and let $T \in \mathcal{T}$ such that $x_0 \in T$. By mesh regularity, there exist constants θ_0, θ_1, ϕ_0, ϕ_1 and c_0, depending only on κ, such that a sector S_T with center at x_0 described in local spherical coordinates by

$$\{(r,\theta,\phi) \mid 0 \le r \le c_0 h_T,\ \theta_0 \le \theta \le \theta_1,\ \phi_0 \le \phi \le \phi_1\},$$

is contained in T. Let $\varphi \in C^1(T)$. Then, by using local spherical coordinates centered at x_0 we have for every $r \in (0, c_0 h_T)$, $\theta \in (\theta_0,\theta_1)$ and $\phi \in (\phi_0,\phi_1)$,

$$\varphi(0,0,0) = \varphi(r,\theta,\phi) - \int_0^r \frac{\partial\varphi}{\partial r}(t,\theta,\phi)\,\mathrm{d}t,$$

so that, using the inequality $(a+b)^2 \le 2a^2+2b^2$, and integrating on S_T we get

$$Ch_T^3\varphi(0,0,0)^2 \le \int_{\phi_0}^{\phi_1}\int_{\theta_0}^{\theta_1}\left[\int_0^{c_0h_T}\varphi(r,\theta,\phi)^2 r^2\sin(\theta)\mathrm{d}r + \int_0^{c_0h_T}\left(\int_0^r\frac{\partial\varphi}{\partial r}(t,\theta,\phi)\mathrm{d}t\right)^2 r^2\sin(\theta)\mathrm{d}r\right]\mathrm{d}\theta\mathrm{d}\phi,$$

where $C = \frac{(\phi_1-\phi_0)(cos(\theta_0)-cos(\theta_1))c_0^3}{6}$. To bound the second term we will use the weighted Hardy inequality (see Thm. 4.8 below), the weight functions being $w_1(t) = t^2$, $w_2(t) = t^{2-2\alpha}$ and the positive function

$f(t) = |\partial\varphi/\partial r(t,\theta)|$. Since $\alpha > \frac{1}{2}$, we have

$$D_\alpha := \sup_{r\in(0,c_0h_T)} \left(\int_r^{c_0h_T} w_1(t)\,\mathrm{d}t\right)^{\frac{1}{2}} \left(\int_0^r w_2(t)^{-1}\,\mathrm{d}t\right)^{\frac{1}{2}}$$
$$= \sup_{r\in(0,c_0h_T)} \left[\frac{(c_0h_T)^3 - r^3}{3}\frac{r^{2\alpha-1}}{2\alpha-1}\right]^{\frac{1}{2}} = h_T^{1+\alpha}\frac{c_0^{1+\alpha}(2\alpha-1)^{\frac{\alpha-2}{3}}}{(2\alpha+2)^{\frac{\alpha+1}{3}}} < \infty.$$

Hence, by Theorem 4.8, the following inequality is valid

$$\int_0^{c_0h_T}\left(\int_0^r \frac{\partial\varphi}{\partial r}(t,\theta,\phi)\mathrm{d}t\right)^2 r^2\mathrm{d}r \le 4D_\alpha^2\int_0^{c_0h_T}\left|\frac{\partial\varphi}{\partial r}(t,\theta,\phi)\right|^2 r^{2-2\alpha}\mathrm{d}r.$$

Therefore, using the identity $r = \mathrm{d}_{x_0}(x)$ in S_T and $1 \le \mathrm{d}_{x_0}(x)^{-2\alpha}(c_0h_T)^{2\alpha}$, for all $x \in S_T$, we obtain

$$Ch_T^3\varphi(0,0,0)^2 \le c_0^{2\alpha}h_T^{2\alpha}\|\varphi\|^2_{L^2_{-\alpha}(T)} + 4D_\alpha^2\|\nabla\varphi\|^2_{L^2_{-\alpha}(T)},$$

and thus

$$|\varphi(0,0,0)| \lesssim h_T^{\alpha-\frac{3}{2}}\|\varphi\|_{L^2_{-\alpha}(T)} + C_\alpha h_T^{\alpha-\frac{1}{2}}\|\nabla\varphi\|_{L^2_{-\alpha}(T)},$$

where $C_\alpha = \frac{(2\alpha-1)^{\frac{\alpha-2}{3}}}{(2+2\alpha)^{\frac{\alpha+1}{3}}}$. The assertion follows by the density of $C^1(T)$ in $H^1_{-\alpha}(T)$.

For the case $n = 2$, the proof follows the same lines, considering a circular sector described by polar coordinates inside the triangle and the weight functions being $w_1(t) = t$, $w_2(t) = t^{1-2\alpha}$. □

We end this section by stating a Hardy inequality [27] that was used in the proof of the previous result.

Theorem 4.8 (Weighted Hardy inequality)**.** *Let $0 < R \le \infty$ and let w_1, w_2 be weight functions defined on $(0,\infty)$ such that $D := \sup_{r\in(0,R)}\left(\int_r^R w_1(t)\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^r w_2(t)^{-1}\mathrm{d}t\right)^{\frac{1}{2}} < \infty$. Then,*

$$\int_0^R\left(\int_0^r f(t)\mathrm{d}t\right)^2 w_1(r)\,\mathrm{d}r \le 4D^2\int_0^R f(r)^2\,w_2(r)\,\mathrm{d}r,$$

for all positive functions f on $(0,\infty)$.

5. *A posteriori* error estimates

In this section we first present the *a posteriori error estimators* for the adaptive approximation of problem (2.3) and then prove their *reliability* and *efficiency*.

The *residual* $\mathcal{R}(V)$ of $V \in \mathbb{V}_{\mathcal{T}}^{\ell}$ is given by

$$\mathcal{R}(V) : W_{-\alpha} \to \mathbb{R}, \qquad \langle\mathcal{R}(V), v\rangle := a(V,v) - \delta_{x_0}(v), \quad \forall\, v \in W_{-\alpha}.$$

Let $U \in \mathbb{V}_{\mathcal{T}}^{\ell}$ be the solution of the discrete problem (3.1). Integrating by parts on each $T \in \mathcal{T}$ we have that

$$\langle\mathcal{R}(U), v\rangle = \sum_{T\in\mathcal{T}}\left(\int_T Rv + \int_{\partial T} Jv\right) - \delta_{x_0}(v), \qquad \forall\, v \in W_{-\alpha}, \tag{5.1}$$

where R denotes the *element residual* given by

$$R_{|_T} := -\nabla\cdot[\mathcal{A}\nabla U] + \mathbf{b}\cdot\nabla U + cU, \qquad \forall\, T \in \mathcal{T},$$

and J the *jump residual* given by

$$J_{|S} := \frac{1}{2}\left[(\mathcal{A}\nabla U)_{|T_1}\cdot \boldsymbol{n}_1 + (\mathcal{A}\nabla U)_{|T_2}\cdot \boldsymbol{n}_2\right], \text{ if } S\in\mathcal{E}_\Omega, \quad J_{|S} = 0, \text{ if } S\in\mathcal{E}_{\partial\Omega}.$$

Here, T_1 and T_2 denote the elements of $\mathcal{T}$ sharing S, and $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are the outward unit normals of T_1 and T_2 on S, respectively.

We define the *a posteriori local error estimator* η_T by

$$\eta_T^2 := \begin{cases} h_T^2 D_T^{2\alpha}\,\|R\|_{L^2(T)}^2 + h_T D_T^{2\alpha}\,\|J\|_{L^2(\partial T)}^2 + h_T^{2\alpha+2-n}, & \text{if } x_0\in T \\ h_T^2 D_T^{2\alpha}\,\|R\|_{L^2(T)}^2 + h_T D_T^{2\alpha}\,\|J\|_{L^2(\partial T)}^2, & \text{if } x_0\notin T \end{cases} \tag{5.2}$$

and the *global error estimator* η by $\eta := \left(\sum_{T\in\mathcal{T}}\eta_T^2\right)^{\frac{1}{2}}$.

Notice that by Lemma 4.1, $D_T^{2\alpha}\,\|R\|_{L^2(T)}^2$ and $D_T^{2\alpha}\,\|J\|_{L^2(\partial T)}^2$ are equivalent to $\|R\|_{L^2_\alpha(T)}^2$ and $\|J\|_{L^2_\alpha(\partial T)}^2$, respectively, if $T\in\mathcal{T}^{\mathrm{far}}$. This is consistent with the norm $\|\cdot\|_{W_\alpha}$ used to measure the error.

An alternative definition of η_T in (5.2) would be obtained replacing $D_T^{2\alpha}\,\|R\|_{L^2(T)}^2$ and $D_T^{2\alpha}\,\|J\|_{L^2(\partial T)}^2$ by $\|R\|_{L^2_\alpha(T)}^2$ and $\|J\|_{L^2_\alpha(\partial T)}^2$, respectively. In this case, the equivalence between error and estimator could only hold for $\alpha < \frac{n-1}{2}$, due to the fact that $\mathrm{d}_{x_0}^{-2\alpha}$ is not integrable over ∂T for $\alpha \ge \frac{n-1}{2}$ if $x_0\in\partial T$. Our definition allows us to prove the equivalence between the estimator and the error in W_α for α in the whole interval $\mathbb{I}$ from (2.15); the range of α for which it is known that problem (2.3) is well posed.

5.1. Reliability

We first prove the reliability of the global error estimator.

Theorem 5.1 (Global upper bound)**.** *Let $\alpha\in\mathbb{I}$ and let $u\in W_\alpha$ be the solution of problem (2.3) and let $U\in\mathbb{V}_{\mathcal{T}}^\ell$ be the solution of the discrete problem (3.1). Then, there exists a constant $C_{\mathcal{U}}>0$ depending on the diameter of Ω, the mesh regularity κ and the parameter α such that*

$$\|U-u\|_{H^1_\alpha(\Omega)} \le C_* C_{\mathcal{U}}\,\eta,$$

where C_* is the continuous inf-sup constant from (2.14). The effective constant $C_*C_{\mathcal{U}}$ of this upper bound blows up when α approaches an endpoint of $\mathbb{I}$.

The proof follows the usual steps for proving the reliability of residual-type *a posteriori* error estimators, making use, as in [4], of the continuous inf-sup condition, instead of the usual coercivity. It is strongly based on the weighted estimates and the properties of the quasi-interpolation operator $\mathcal{P}$ stated in the previous section. Recall that $\mathcal{P}$ can be either the Clément or the Scott–Zhang interpolation operator.

Proof. Let $u\in W_\alpha$ be the solution of problem (2.3) and $U\in\mathbb{V}_{\mathcal{T}}^\ell$ be the solution of the discrete problem (3.1). Using the inf-sup condition (2.14) we have that

$$\frac{1}{C_*}\|U-u\|_{W_\alpha} \le \sup_{v\in W_{-\alpha}}\frac{a(U-u,v)}{\|v\|_{W_{-\alpha}}} = \sup_{v\in W_{-\alpha}}\frac{\langle\mathcal{R}(U),v\rangle}{\|v\|_{W_{-\alpha}}} = \|\mathcal{R}(U)\|_{(W_{-\alpha})'}. \tag{5.3}$$

Now, let $v\in W_{-\alpha}$ and let $V=\mathcal{P}v$, with $\mathcal{P}$ either the Clément or the Scott–Zhang interpolation operator. Then, by (3.1), (5.1) and Hölder inequality it follows that

$$|\langle\mathcal{R}(U),v\rangle| = |\langle\mathcal{R}(U),v-V\rangle| \le \sum_{T\in\mathcal{T}}\Big(\|R\|_{L^2(T)}\,\|v-V\|_{L^2(T)} + \|J\|_{L^2(\partial T)}\,\|v-V\|_{L^2(\partial T)}\Big) + |\delta_{x_0}(v-V)|.$$

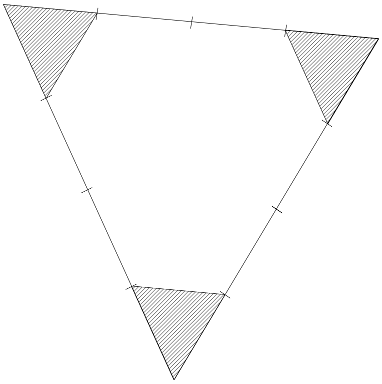

FIGURE 1. Simplex T and equivalent (shaded) sub-simplices, obtained after dividing the edges into four equal segments. T_* is the one which is farthest from x_0 in order to guarantee that $D_T \lesssim d_{T_*}$.

Applying a scaled trace theorem and the interpolation estimates (4.8) and (4.9), for the addition in the right hand side of the last inequality, we have that

$$\begin{aligned}\sum_{T\in\mathcal{T}} \Big(\|R\|_{L^2(T)} \|v-V\|_{L^2(T)} + \|J\|_{L^2(\partial T)} \|v-V\|_{L^2(\partial T)} \Big) \\ \lesssim \sum_{T\in\mathcal{T}} \Big(\|R\|_{L^2(T)} h_T \|\nabla v\|_{L^2(\omega_T)} + \|J\|_{L^2(\partial T)} h_T^{\frac{1}{2}} \|\nabla v\|_{L^2(\omega_T)} \Big) \\ \lesssim \sum_{T\in\mathcal{T}} \Big(h_T D_T^{\alpha} \|R\|_{L^2(T)} + h_T^{\frac{1}{2}} D_T^{\alpha} \|J\|_{L^2(\partial T)} \Big) \|\nabla v\|_{L^2_{-\alpha}(\omega_T)} ,\end{aligned}$$

and using the local bound for the Dirac delta (4.12), and the weighted interpolation estimates (4.10) and (4.11),

$$|\delta_{x_0}(v-V)| \lesssim C_I C_\alpha h_{T_0}^{\alpha+\frac{2-n}{2}} \|\nabla v\|_{L^2_{-\alpha}(\omega_{T_0})} ,$$

where T_0 is any element containing x_0. Thus, recalling the definition of the error estimators (5.2),

$$|\langle \mathcal{R}(U), v\rangle| \lesssim C_I C_\alpha \eta \|v\|_{W_{-\alpha}}.$$

Therefore, the last estimation, (5.3) and (2.2) yield the desired assertion. □

5.2. Efficiency

The proof of the lower bound follows the usual steps using a bubble function to test the residual. We first construct bubble functions and then prove the necessary estimates in Lemma 5.2.

Bubble function for the interior residual estimate

Given $T \in \mathcal{T}$, the goal is to construct a bubble function with its support in T of size $\simeq h_T^n$ and at distance $\gtrsim D_T$ of x_0. To do this, we divide each edge of T into four equal segments and consider the simplices which are determined by one vertex of T and the segments that touch it (see Fig. 1). We then let T_* be the one of these simplices that is farthest from x_0, so that

$$h_T \lesssim d_{T_*} := \min_{x\in T_*} \mathrm{d}_{x_0}(x).$$

Since $D_T \simeq d_T \le d_{T_*}$ for $T \in \mathcal{T}^{\mathrm{far}}$, and $D_T \simeq h_T \lesssim d_{T_*}$ for $T \in \mathcal{T}^{\mathrm{near}}$ (*cf.* Lem. 4.1), we conclude that

$$D_T \lesssim d_{T_*}, \quad \forall T \in \mathcal{T}.$$

Besides, by translating and scaling a fixed bubble function $\hat{\varphi}$ to the sub-element T_* we obtain $\varphi_T \in C_0^\infty(\mathbb{R}^n)$ with

$$\delta_{x_0}(\varphi_T) = \varphi_T(x_0) = 0, \qquad \mathrm{supp}(\varphi_T) \subset T_*, \qquad \|\varphi_T\|_{L^\infty(T)} = 1. \tag{5.4}$$

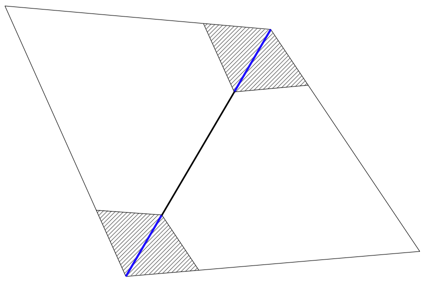

FIGURE 2. Triangles T, T' sharing a common side S. The patch $T_* \cup T'_*$ is one of the shaded regions, the one farther from x_0, and $S_* = T_* \cap T'_*$. Therefore $D_T \lesssim d_{T_*}$ and $D_{T'} \lesssim d_{T'_*}$.

Bubble function for the jump residual estimate

Given $S \in \mathcal{E}_\Omega$, we denote T, T' the two elements sharing S. The goal is now to construct a bubble function with its support in ω_S of size $\simeq h_T^n$ and at distance $\gtrsim D_T$ of x_0. We proceed as before, dividing the edges of T and T' into four equal segments. We then consider the simplices determined by the vertices of S and the segments that touch them. This determines n patches of adjacent simplices. We then choose $T_* \subset T$ and $T'_* \subset T'$ such that $T_* \cap T'_* =: S_* \neq \emptyset$ and

$$h_T \lesssim d_{T_*} \qquad \text{and} \quad h_{T'} \lesssim d_{T'_*},$$

the situation for $n = 2$ is depicted in Figure 2.

By construction, we have

$$D_T \lesssim d_{T_*} \quad \text{and} \quad D_{T'} \lesssim d_{T'_*}.$$

In fact, if $T \in \mathcal{T}^{\text{near}}$, $D_T \simeq h_T \lesssim d_{T_*}$, and if $T \in \mathcal{T}^{\text{far}}$, $D_T \simeq d_T \leq d_{T_*}$. Analogously, the estimate for T' holds.

By translating and scaling a fixed bubble function $\hat{\varphi}$ to S_* we obtain $\varphi_S \in C_0^\infty(\mathbb{R}^n)$ such that

$$\delta_{x_0}(\varphi_S) = \varphi_S(x_0) = 0, \quad \operatorname{supp}(\varphi_S) \subset T_* \cup T'_* \subset \omega_S, \quad \|\varphi_S\|_{L^\infty(\omega_S)} = 1. \tag{5.5}$$

The following result summarizes the properties of the just defined bubble functions φ_T and φ_S that we need to prove the efficiency of the local error estimators.

Lemma 5.2. *Let $0 < \alpha < \frac{n}{2}$ and $T \in \mathcal{T}$. If φ_T is the bubble function satisfying (5.4), then,*

$$\|p\varphi_T\|_{L^2_{-\alpha}(T)} \lesssim D_T^{-\alpha} \|p\|_{L^2(T)}, \tag{5.6}$$

$$h_T \|\nabla(p\varphi_T)\|_{L^2_{-\alpha}(T)} \lesssim D_T^{-\alpha} \|p\|_{L^2(T)}, \tag{5.7}$$

for all $p \in \mathcal{P}_{\ell-1}(T)$. On the other hand, if $S \in \mathcal{E}_\Omega$ is a side of T and φ_S is the bubble function satisfying (5.5), then,

$$h_T^{-\frac{1}{2}} \|p\varphi_S\|_{L^2_{-\alpha}(\omega_S)} \lesssim D_T^{-\alpha} \|p\|_{L^2(S)}, \tag{5.8}$$

$$h_T^{\frac{1}{2}} \|\nabla(p\varphi_S)\|_{L^2_{-\alpha}(\omega_S)} \lesssim D_T^{-\alpha} \|p\|_{L^2(S)}, \tag{5.9}$$

for all $p \in \mathcal{P}_{\ell-1}(S)$, where we extend p to ω_S as constant along the direction of one side of each element of $\mathcal{T}$ contained in ω_S.

Proof. (1) Using that $\|\varphi_T\|_{L^\infty(T)} = 1$ and $\operatorname{supp}(\varphi_T) \subset T_*$, it follows that $\|p\varphi_T\|^2_{L^2_{-\alpha}(T)} = \int_{T_*} p^2 \varphi_T^2 \, \mathrm{d}_{x_0}^{-2\alpha} \leq d_{T_*}^{-2\alpha} \|p\|^2_{L^2(T)}$. Taking into account that $D_T \lesssim d_{T_*}$, (5.6) holds.

(2) The usual scaling arguments yield

$$\|\nabla(p\varphi_T)\|_{L^2(T)} \simeq h_T^{-1} \|p\|_{L^2(T)}, \quad \forall\, p \in \mathcal{P}_{\ell-1}(T),$$

and thus

$$\|\nabla(p\varphi_T)\|^2_{L^2_{-\alpha}(T)} = \int_{T_*} |\nabla(p\varphi_T)|^2 \, \mathrm{d}_{x_0}^{-2\alpha} \lesssim d_{T_*}^{-2\alpha} \|\nabla(p\varphi_T)\|^2_{L^2(T)} \lesssim d_{T_*}^{-2\alpha} h_T^{-2} \, \|p\|^2_{L^2(T)} \, .$$

In consequence, (5.7) follows from $D_T \lesssim d_{T_*}$.

(3) Let $T \in \mathcal{T}$ be such that $S \subset T \subset \omega_S$. Since $\|\varphi_S\|_{L^\infty(\omega_S)} = 1$ and $\operatorname{supp}(\varphi_S) \subset T_* \cup T'_*$, we have

$$\|p\varphi_S\|^2_{L^2_{-\alpha}(T)} = \int_{T_*} p^2 \varphi_S^2 \, \mathrm{d}_{x_0}^{-2\alpha} \le d_{T_*}^{-2\alpha} \int_{T_*} p^2 \lesssim d_{T_*}^{-2\alpha} h_T \int_{S_*} p^2 \le d_{T_*}^{-2\alpha} h_T \, \|p\|^2_{L^2(S)} \, ,$$

and therefore, (5.8) holds, using that $D_T \lesssim d_{T_*}$.

(4) Let $T \in \mathcal{T}$ be such that $S \subset T \subset \omega_S$. The usual scaling arguments yield

$$\|\nabla(p\varphi_S)\|_{L^2(T)} \simeq h_T^{-1} \|p\|_{L^2(T \cap \operatorname{supp}(\varphi_S))}, \quad \forall \, p \in \mathcal{P}_{\ell-1}(T).$$

Let us denote by T_* the element which is contained in T (*cf.* Fig. 2). Hence

$$\begin{aligned} \|\nabla(p\varphi_S)\|^2_{L^2_{-\alpha}(T)} &= \int_{T_*} |\nabla(p\varphi_S)|^2 \, \mathrm{d}_{x_0}^{-2\alpha} \lesssim d_{T_*}^{-2\alpha} \|\nabla(p\varphi_S)\|^2_{L^2(T)} \\ &\simeq d_{T_*}^{-2\alpha} h_T^{-2} \|p\|^2_{L^2(T_*)} \lesssim d_{T_*}^{-2\alpha} h_T^{-1} \|p\|^2_{L^2(S_*)} \le d_{T_{S_*}}^{-2\alpha} h_T^{-1} \|p\|^2_{L^2(S)}. \end{aligned}$$

Finally, (5.9) follows due to $D_T \lesssim d_{T_*}$.

□

As usually happens for residual based error estimators, the lower bound is local, and holds up to some oscillation terms. In this context, we define the *local oscillation* osc_T by

$$\operatorname{osc}_T := \begin{cases} \left(h_T^2 D_T^{2\alpha} \left\| R - \overline{R} \right\|^2_{L^2(\omega_T)} + h_T D_T^{2\alpha} \left\| J - \overline{J} \right\|^2_{L^2(\mathcal{E}_\Omega \cap (\omega_T)^0)} \right)^{\frac{1}{2}}, & \text{if } x_0 \in T, \\ \left(h_T^2 D_T^{2\alpha} \left\| R - \overline{R} \right\|^2_{L^2(\omega_T)} + h_T D_T^{2\alpha} \left\| J - \overline{J} \right\|^2_{L^2(\partial T)} \right)^{\frac{1}{2}}, & \text{if } x_0 \notin T, \end{cases}$$

where $\overline{R}_{|_{T'}}$ denotes the L^2 projection of R on $\mathcal{P}_{\ell-1}(T')$, for all $T' \in \mathcal{T}$, and for each side S, $\overline{J}_{|_S}$ denotes the L^2 projection of J on $\mathcal{P}_{\ell-1}(S)$. Notice that if $x_0 \in T$ the jump oscillations are considered over all $S \in \mathcal{E}_\Omega$ that touch T, including those contained in ∂T and those not contained in ∂T.

The next result is usually called *local efficiency of the error estimator*, based on the fact that whenever a local estimator is large, so is the corresponding local error, provided the local oscillation is relatively small. Its proof follows the usual techniques taking into account the bounds from the last lemma and the boundedness of the bilinear form, yielding

$$|\langle \mathcal{R}(U), v \rangle| = |a(U, v) - \delta_{x_0}(v)| = |a(U, v) - a(u, v)| \le C_a \|U - u\|_{H^1_\alpha(\omega)} \|v\|_{H^1_{-\alpha}(\omega)},$$

for all $v \in W_{-\alpha}$ with $\operatorname{supp}(v) \subset \omega$, for any $\omega \subset \overline{\Omega}$, where $C_a := \max\{\gamma_2, \|\boldsymbol{b}\|_{L^\infty}, \|c\|_{L^\infty}\}$.

Theorem 5.3 (Local lower bound)**.** *Let $\alpha \in \mathbb{I}$, let $u \in W_\alpha$ be the solution of problem (2.3) and let $U \in \mathbb{V}_{\mathcal{T}}^\ell$ be the solution of the discrete problem (3.1). There exists a constant $C_{\mathcal{L}} > 0$ depending on the mesh regularity κ and the parameter α such that*

$$C_{\mathcal{L}} \eta_T \le C_a \|U - u\|_{H^1_\alpha(\omega_T)} + \operatorname{osc}_T,$$

for all $T \in \mathcal{T}$. The constant $C_{\mathcal{L}}$ goes to zero if α approaches $\frac{n}{2}$.

Proof. (1) Let $T \in \mathcal{T}$. We analize first the residual R. Since

$$\|R\|_{L^2(T)} \leq \|\overline{R}\|_{L^2(T)} + \|R - \overline{R}\|_{L^2(T)}, \tag{5.10}$$

it is sufficient to estimate $\|\overline{R}\|_{L^2(T)}$.

Let φ_T be the bubble function satisfying (5.4). The usual scaling arguments yield

$$\|\overline{R}\|^2_{L^2(T)} \simeq \left\|\overline{R}\varphi_T^{\frac{1}{2}}\right\|^2_{L^2(T)} = \int_T \overline{R}^2\varphi_T = \int_T \overline{R}v = \int_T Rv + \int_T (\overline{R} - R)v, \tag{5.11}$$

where $v := \overline{R}\varphi_T$. Since $\operatorname{supp}(v) \subset T$ and $\delta_{x_0}(v) = 0$, the first integral in the right-hand side of (5.11), using (5.6) and (5.7) satisfies

$$\int_T Rv = \langle \mathcal{R}(U), v\rangle \leq C_a\|U - u\|_{H^1_\alpha(T)}\|v\|_{H^1_{-\alpha}(T)} \lesssim C_a h_T^{-1}\|U - u\|_{H^1_\alpha(T)} D_T^{-\alpha}\left\|\overline{R}\right\|_{L^2(T)},$$

while the second one satisfies $\int_T (\overline{R} - R)v \leq \|\overline{R} - R\|_{L^2(T)}\|v\|_{L^2(T)} \leq \|\overline{R} - R\|_{L^2(T)}\|\overline{R}\|_{L^2(T)}$.

Using the two last inequalities in (5.11) we have that

$$h_T D_T^\alpha\left\|\overline{R}\right\|_{L^2(T)} \lesssim C_a\|U - u\|_{H^1_\alpha(T)} + h_T D_T^\alpha\left\|\overline{R} - R\right\|_{L^2(T)}. \tag{5.12}$$

Finally, from (5.10) and (5.12) it follows that

$$h_T D_T^\alpha\|R\|_{L^2(T)} \lesssim C_a\|U - u\|_{H^1_\alpha(T)} + h_T D_T^\alpha\left\|\overline{R} - R\right\|_{L^2(T)}. \tag{5.13}$$

(2) Secondly, we estimate the jump residual J. Let S be a side of T. As before, it is sufficient to bound the projection $\overline{J}$ of J, since

$$\|J\|_{L^2(S)} \leq \|\overline{J}\|_{L^2(S)} + \|J - \overline{J}\|_{L^2(S)}. \tag{5.14}$$

Let φ_S be the bubble function from (5.5). Then, usual scaling arguments lead to

$$\|\overline{J}\|^2_{L^2(S)} \lesssim \left\|\overline{J}\varphi_S^{\frac{1}{2}}\right\|^2_{L^2(S)} = \int_S \overline{J}^2\varphi_S = \int_S \overline{J}v = \int_S Jv + \int_S (\overline{J} - J)v, \tag{5.15}$$

with $v := \overline{J}\varphi_S$. Extending $\overline{J}$ to ω_S as constant along the direction of one side of each element of $\mathcal{T}$ contained in ω_S, using that $\delta_{x_0}(v) = 0$ and $\operatorname{supp}(v) \subset \omega_S$, the first integral in the right-hand side of (5.15) can be bounded as follows:

$$\begin{aligned} 2\int_S Jv &= \langle \mathcal{R}(U), v\rangle - \int_{\omega_S} Rv \leq C_a\|U - u\|_{H^1_\alpha(\omega_S)}\|v\|_{H^1_{-\alpha}(\omega_S)} + \|R\|_{L^2(\omega_S)}\|v\|_{L^2(\omega_S)} \\ &\lesssim h_T^{-\frac{1}{2}} C_a\|U - u\|_{H^1_\alpha(\omega_S)} D_T^{-\alpha}\left\|\overline{J}\right\|_{L^2(S)} + h_T^{\frac{1}{2}}\|R\|_{L^2(\omega_S)}\left\|\overline{J}\right\|_{L^2(S)}, \end{aligned}$$

where in the last inequality we have used (5.8) and (5.9). The second integral in the right-hand side of (5.15), satisfies $\int_S (\overline{J} - J)v \leq \|\overline{J} - J\|_{L^2(S)}\|v\|_{L^2(S)} \lesssim \|\overline{J} - J\|_{L^2(S)}\|\overline{J}\|_{L^2(S)}$.

The last two estimates and (5.15) yield

$$h_T^{\frac{1}{2}} D_T^\alpha\left\|\overline{J}\right\|_{L^2(S)} \lesssim C_a\|U - u\|_{H^1_\alpha(\omega_S)} + h_T D_T^\alpha\|R\|_{L^2(\omega_S)} + h_T^{\frac{1}{2}} D_T^\alpha\left\|\overline{J} - J\right\|_{L^2(S)}. \tag{5.16}$$

Thus, from (5.14) and (5.16) we have that

$$h_T^{\frac{1}{2}} D_T^\alpha\|J\|_{L^2(S)} \lesssim C_a\|U - u\|_{H^1_\alpha(\omega_S)} + h_T D_T^\alpha\|R\|_{L^2(\omega_S)} + h_T^{\frac{1}{2}} D_T^\alpha\left\|\overline{J} - J\right\|_{L^2(S)}.$$

Adding the last inequality over all the sides $S \subset \partial T$ and using (5.13) we obtain

$$h_T^{\frac{1}{2}} D_T^\alpha\|J\|_{L^2(\partial T)} \lesssim C_a\|U - u\|_{H^1_\alpha(\omega_T)} + \operatorname{osc}_T.$$

(3) Recall that if $x_0 \in T$ the indicator η_T contains also a term $h_T^{\alpha+\frac{2-n}{2}}$, we now prove that

$$h_T^{\alpha+\frac{2-n}{2}} \lesssim \left[\left(\frac{n}{2}-\alpha\right)^{-\frac{1}{2}} C_a \|U-u\|_{H^1_\alpha(\omega_T)} + \mathrm{osc}_T \right].$$

Let $\phi \in C^\infty(\mathbb{R}^n)$ with $\|\phi\|_{L^\infty} = \phi(0) = 1$ and $\mathrm{supp}(\phi) \subset B(0,1)$. Let $C = c_{\kappa,\Omega}$ from Lemma 4.2 so that $B(x_0, Ch_T) \subset \omega_T$, and if $\varphi(x) := \phi\left(\frac{x-x_0}{Ch_T}\right)$ then $\delta_{x_0}(\varphi) = \varphi(x_0) = 1$, $\|\varphi\|_{L^\infty} = 1$, $\|\nabla\varphi\|_{L^\infty} \lesssim \frac{1}{h_T}$ and $\mathrm{supp}(\varphi) \subset B(x_0, Ch_T) \subset \omega_T$. Thus, we also have that $\|\varphi\|_{L^2(\omega_T)} \lesssim h_T^{\frac{n}{2}}$, $\|\nabla\varphi\|_{L^2(\omega_T)} \lesssim h_T^{\frac{n-2}{2}}$, and using a scaled trace theorem, $\|\varphi\|_{L^2(\partial T)} \lesssim h_T^{\frac{n-1}{2}}$. On the other hand, since $\left\|\mathrm{d}_{x_0}^{-\alpha}\right\|_{L^2(\omega_T)} \lesssim \frac{1}{\sqrt{\frac{n}{2}-\alpha}} h_T^{\frac{n}{2}-\alpha}$, we have that $\|\varphi\|_{L^2_{-\alpha}(\omega_T)} \lesssim (\frac{n}{2}-\alpha)^{-\frac{1}{2}} h_T^{\frac{n}{2}-\alpha}$ and $\|\nabla\varphi\|_{L^2_{-\alpha}(\omega_T)} \lesssim (\frac{n}{2}-\alpha)^{-\frac{1}{2}} h_T^{\frac{n-2}{2}-\alpha}$. Therefore,

$$\begin{aligned}
1 &= \delta_{x_0}(\varphi) = a(u,\varphi) = a(u-U,\varphi) + a(U,\varphi) \\
&\le C_a \|U-u\|_{H^1_\alpha(\omega_T)} \|\varphi\|_{H^1_{-\alpha}(\omega_T)} + \sum_{T'\subset\omega_T} \left(\int_{T'} R\varphi + \int_{\partial T'} J\varphi \right) \\
&\le C_a \|U-u\|_{H^1_\alpha(\omega_T)} \|\varphi\|_{H^1_{-\alpha}(\omega_T)} + \sum_{T'\subset\omega_T} \|R\|_{L^2(T')} \|\varphi\|_{L^2(T')} + 2 \sum_{S\subset(\omega_T)^0} \|J\|_{L^2(S)} \|\varphi\|_{L^2(S)} \\
&\lesssim \left(\left(\frac{n}{2}-\alpha\right)^{-\frac{1}{2}} C_a \|U-u\|_{H^1_\alpha(\omega_T)} + \sum_{T'\subset\omega_T} h_{T'} D_{T'}^\alpha \|R\|_{L^2(T')} + \sum_{S\subset(\omega_T)^0} h_{T'}^{\frac{1}{2}} D_{T'}^\alpha \|J\|_{L^2(S)} \right) h_T^{-\alpha+\frac{n-2}{2}}.
\end{aligned}$$

The last inequality with the estimates obtained in steps (1) and (2) complete the proof. □

As an immediate consequence of Theorem 5.3, adding over all elements in the mesh we obtain the efficiency of the global error estimator.

Theorem 5.4 (Global lower bound)**.** *Let $\alpha \in \mathbb{I}$, let $u \in W_\alpha$ be the solution of problem (2.3) and let $U \in \mathbb{V}_{\mathcal{T}}^\ell$ be the solution of the discrete problem (3.1). There exists a constant $C_L > 0$ depending on the mesh regularity κ and the parameter α such that*

$$C_L \eta \le C_a \|U-u\|_{H^1_\alpha(\Omega)} + \mathrm{osc},$$

where osc *is the* global oscillation *defined by* $\mathrm{osc} := \left(\sum_{T\in\mathcal{T}} \mathrm{osc}_T^2\right)^{\frac{1}{2}}$*, and the constant C_L goes to zero if α approaches $\frac{n}{2}$.*

Remark 5.5 (Convergence of adaptive algorithms)**.** The general convergence theory from [22,29] states that if the discretization of a linear problem is stable, the *a posteriori* error estimators constitute an upper bound for the error and if there holds a discrete local lower bound, up to oscillation terms, then any adaptive algorithm marking at least the element with the largest indicator will converge. Our indicators fulfill all those assumptions, yielding convergence to zero of the error measured in W_α; and also in $H^1(\Omega_0)$ for any $\Omega_0 \subset \Omega$ such that $\mathrm{dist}(x_0, \Omega_0) > 0$, because $W_\alpha \hookrightarrow H^1(\Omega_0)$. For the discrete lower bound it is enough to observe that discrete bubble functions φ_T and φ_S can be constructed on sufficiently refined meshes, so that they satisfy (5.4), (5.5) and thus also Lemma 5.2.

It is worth mentioning that there are presently no results of optimal complexity for adaptive methods applied to problems involving different ansatz and test spaces (see [26] and references therein). The quasi-orthogonality property used in the current proofs is not readily available in this situation. Optimality is thus an open issue for the problem studied in this article, even though it is observed in the experiments that we report in the next section.

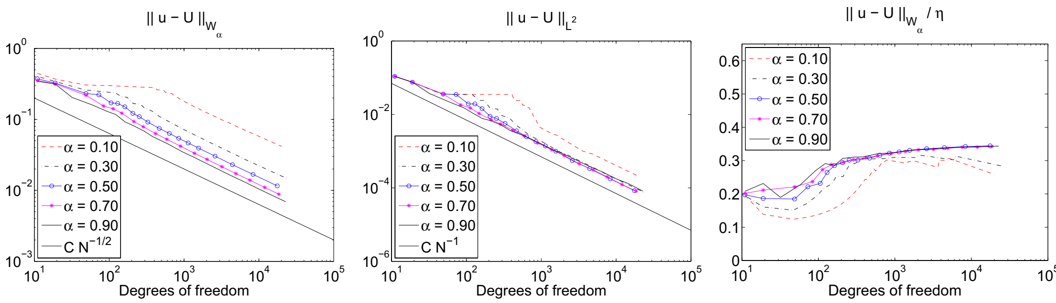

FIGURE 3. Exact errors and effectivity indices for Example 6.1. We plot the W_α (left) and the $L^2(\Omega)$ (middle) norm of the error $u-U$ *versus* the number of Degrees of Freedom (DOFs) in logarithmic scales, for different values of α. We observe the optimal decay $(\#\mathcal{T})^{-1/2}$ and $(\#\mathcal{T})^{-1}$, respectively. We also plot the effectivity index $\|u-U\|_{W_\alpha}/\eta$ and observe that it remains between 0.12 and 0.35 for all the considered values of α, showing the robustness of the estimator with respect to α.

6. Numerical experiments

In this section we report some numerical experiments that document the behavior of the adaptive algorithm based on our *a posteriori* estimators for the error in W_α norm. We implemented a loop of the usual form

$$\text{SOLVE} \longrightarrow \text{ESTIMATE} \longrightarrow \text{MARK} \longrightarrow \text{REFINE}.$$

The step SOLVE consisted in solving the discrete system for the current mesh, the step ESTIMATE consisted in computing the *a posteriori* error estimators η_T for a given value of α. In the step MARK we selected in $\mathcal{M}$ for refinement those elements $T\in\mathcal{T}$ with largest estimators η_T until $\sum_{T\in\mathcal{M}}\eta_T^2 \geq 0.5\sum_{T\in\mathcal{T}}\eta_T^2$, *i.e.*, we used the *Dörfler strategy* with parameter 0.5. The step REFINE consisted in performing two bisections to each marked element, and refining some extra elements in order to keep conformity of the meshes, using the *newest-vertex bisection*. We used a custom implementation in MATLAB.

We present two examples on two-dimensional domains, using piecewise linear finite elements. The first one considering a known solution on an L-shaped domain, and the second one based on the computation of an unknown solution on a rectangle, with variable coefficients, simulating a wiggling flow on a canal.

Example 6.1. We consider the boundary value problem $-\Delta u = \delta_{(0.5,0.5)}$ in the L-shaped domain $\Omega = (-1,1)^2\setminus [0,1)\times(-1,0] \subset \mathbb{R}^2$ with exact solution $u(x) = -\frac{1}{2\pi}\log|x-(0.5,0.5)| + |x|^{2/3}\sin(2\theta/3)$, ($\theta$ the angle measured from 0 to $3\pi/2$ in Ω), and Dirichlet boundary conditions.

The first goal of this example is to test the behavior of the adaptive method guided by the *a posteriori* estimators η_T for different values of α, in a problem with two singularities. One produced by the Dirac delta on the right-hand side and another one produced by the reentrant corner. Our theory predicts that $\eta := \left(\sum_{T\in\mathcal{T}}\eta_T^2\right)^{1/2}$ is equivalent to the error in W_α norm provided $0<\alpha<1$.

In Figure 3 we show the decay of the W_α and the $L^2(\Omega)$ norm of the error $u-U$, *versus* the number of Degrees of Freedom (DOFs) in logarithmic scales, for $\alpha = 0.1, 0.3, 0.5, 0.7, 0.9$. We observe the optimal decay $(\#\mathcal{T})^{-1/2}$ and $(\#\mathcal{T})^{-1}$, respectively. These are consistent with the decay rates proved by D'Angelo [10] and Apel *et al.* [1], respectively, for properly *a priori* graded meshes. As is usual with adaptive methods, the optimal cardinality is obtained automatically, without any fine tuning or additional requirement on the meshes.

We also plot the effectivity index $\|u-U\|_{W_\alpha}/\eta$ and observe that it remains between 0.12 and 0.35 for all the considered values of α, showing the robustness of the estimator with respect to α, with no degeneracy as α approaches the endpoints of $\mathbb{I}$. This is better than expected according to our theory.

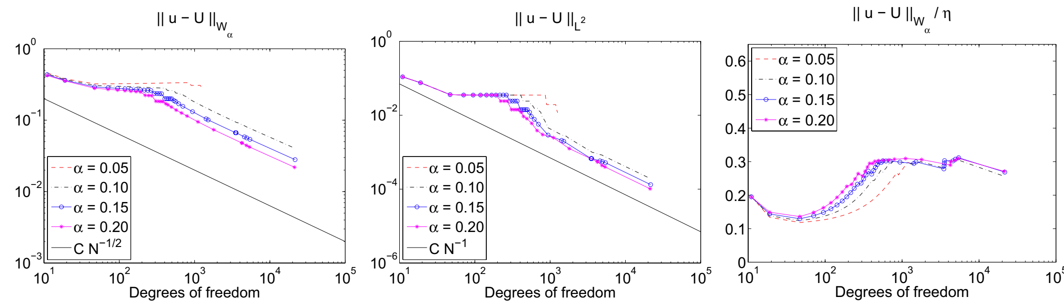

FIGURE 4. Exact errors and effectivity indices for Example 6.1 and α very small. We plot the W_α (left) and the $L^2(\Omega)$ (middle) norm of the error $u - U$ *versus* the number of Degrees of Freedom (DOFs) in logarithmic scales, for different values of α. We observe the optimal decay for all the considered values, except for the smallest value $\alpha = 0.05$. In this extreme situation the algorithm refines purely around $(0.5, 0.5)$ and the elements become excessively small, leading to a nearly singular system matrix (to the working precision) not allowing computation beyond a mesh with 2544 elements and 1286 DOFs, obtained after 53 iterations. The effectivity index $\|u - U\|_{W_\alpha}/\eta$, plotted on the right, remains bounded between 0.11 and 0.32.

In Figure 4 we show the decay of the W_α and the $L^2(\Omega)$ norm of the error $u-U$, for values of α very close to zero. We show the behavior for $\alpha = 0.05, 0.1, 0.15, 0.2$ and observe the optimal decays for the cases $\alpha \geq 0.1$. The algorithm stopped after 53 iterations in the case $\alpha = 0.05$, with a mesh of 2544 elements and 1286 degrees of freedom (DOFs). The refinement is concentrated solely around the support of the Dirac delta, leading to very small elements, with diameter of order 2^{-53}. The resulting system matrix was singular to working precision. We also show the effectivity indices for these values of α and observe that they do not degenerate as α approaches zero.

The meshes after 4, 8 and 12 iterations for $\alpha = 0.25, 0.5, 0.75$ are plotted in Figure 5. The number of elements of the corresponding meshes is indicated in each picture, and the stronger grading obtained for smaller values of α is not so apparent for these values of α, although the case $\alpha = 0.25$ is much different than the other two cases. It is worth observing that the corner singularity is not noticed for $\alpha = 0.25$ after 8 iterations of the adaptive algorithm, and it is immediately noticed for α bigger (see also Fig. 6).

We also plot meshes with a similar number of elements for values of $\alpha = 0.1, 0.3, 0.5$ in Figure 7. The fact that the singularity introduced by the Dirac delta is less severe when the error is measured in W_α for bigger α is noticeable in this picture. The refinement is thus more spread in this case.

The second goal of this example is to compare our estimator with the existing ones for the L^p, $W^{1,p}$ and H^s norms from [4, 16]. For the L-shaped domain being considered, the estimators for the L^p and the $W^{1,p}$ error constitute an upper and lower bound if $3 < p < \infty$ and if $3/2 < p < 2$, respectively, and those from [16] are equivalent to the error in H^s if $1/2 < s < 1$. We ran the adaptive algorithm once for each estimator and computed the $L^2(\Omega)$ and the $H^1(\Omega_0)$ norm of the error, for $\Omega_0 = \{x \in \Omega : \|x-(0.5, 0.5)\|_\infty > 1/4\}$. The results are reported in Figure 8.

When using the estimators for the L^p norm, we chose the parameter $p = 4$. For the estimators corresponding to the $W^{1,p}$, H^s and the W_α norms, we chose p, s and α as the midpoints of the respective intervals of validity of the error-estimator equivalence, *i.e.*, $p = 7/4$, $s = 3/4$ and $\alpha = 1/2$. We plot the $L^2(\Omega)$, the $H^1(\Omega_0)$ and the $L^\infty(\Omega_0)$ norm of the error *versus* the number of degrees of freedom in logarithmic scales. We observe that the algorithm guided by our estimators performs better than the others in the three comparisons.

As a final remark, it is worth observing that not only our estimator behaves better computationally, but the adaptive algorithm guided by the W_α estimators is guaranteed to converge (see Rem. 5.5), whereas convergence is not proved for the other estimators.

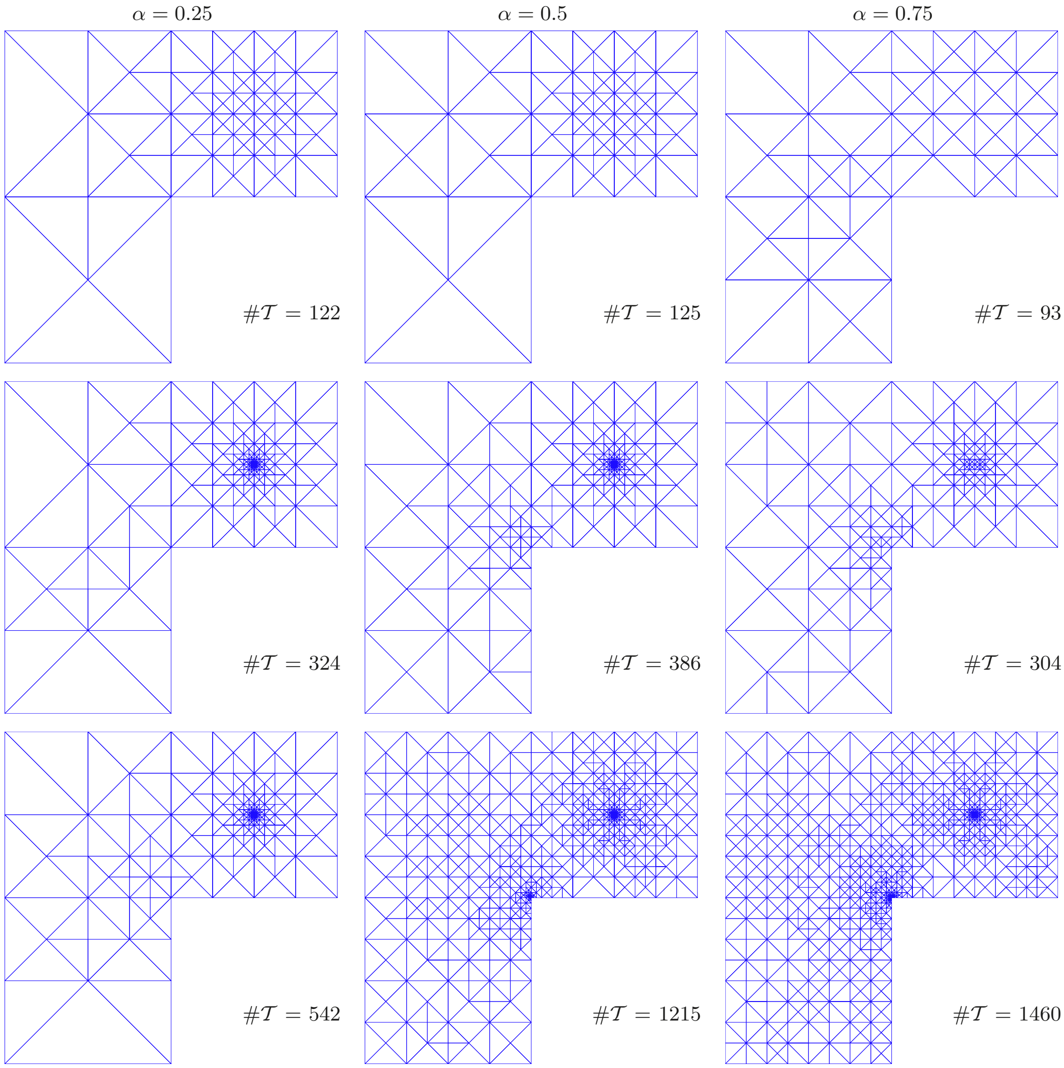

FIGURE 5. Meshes for Example 6.1. We show the meshes after 4 (top), 8 (middle) and 12 (bottom) iterations for $\alpha = 0.25$ (left), $\alpha = 0.5$ (middle) and $\alpha = 0.75$ (right). The number of elements of the corresponding meshes is indicated in each picture, and the stronger grading obtained for smaller values of α is not so clearly visible. It is worth observing that the corner singularity is not noticed at all for $\alpha = 0.25$ after 8 iterations of the adapive algorithm, and barely after 12 iterations, but it is immediately noticed for α big. In the latter case the refinement is more spread throughout the domain, due to the smaller relative importance of the singularity introduced by δ_{x_0}.

Example 6.2. In this example we let $\Omega = (0,3) \times (0,1)$ and consider the problem

$$\begin{aligned} -0.02\,\Delta u + \begin{bmatrix} 2 \\ \sin(5x_1) \end{bmatrix} \cdot \nabla u + 0.1 u = \delta_{(0.2,0.4)} \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega \cap \{x_1 < 3\}, \\ \frac{\partial u}{\partial n} = 0 \quad & \text{on } \partial\Omega \cap \{x_1 = 3\}, \end{aligned}$$

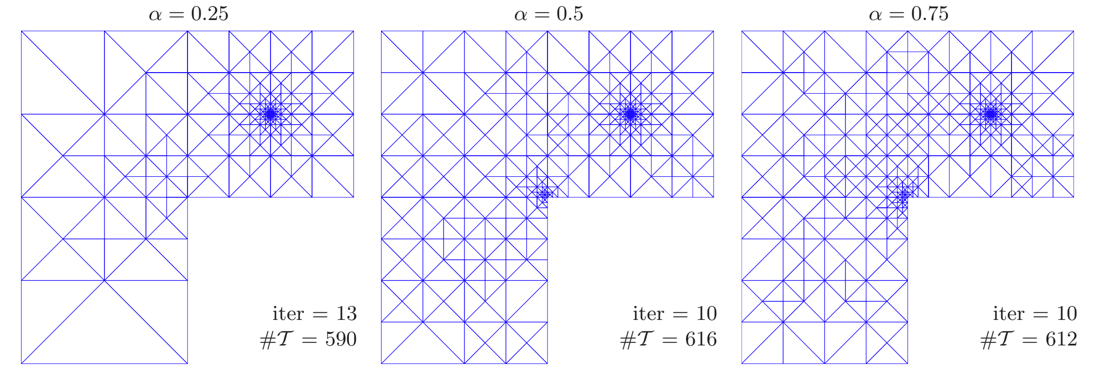

FIGURE 6. Meshes for Example 6.1 with similar number of elements. We show meshes for different values of α and similar number of elements. There is no significant difference for the values of $\alpha = 0.5, 0.75$.

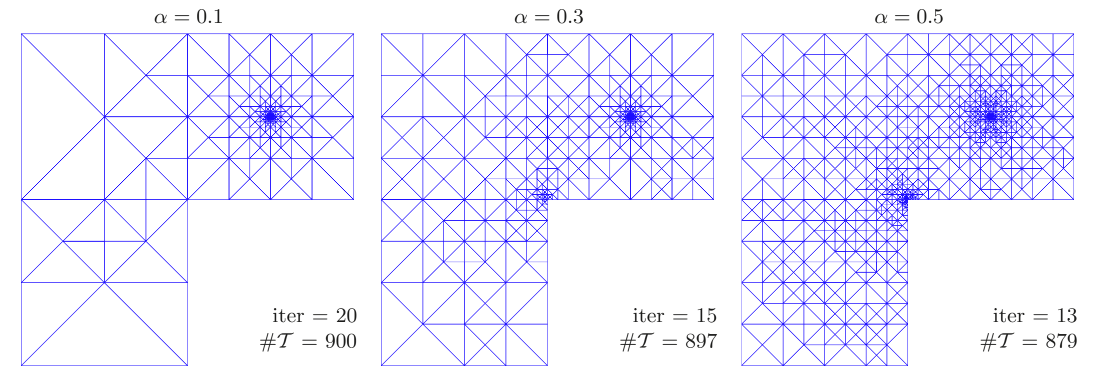

FIGURE 7. Meshes for Example 6.1 with similar number of elements. We show meshes for different values of α close to zero and similar number of elements. We can observe that for smaller values of α the meshes are more strongly graded at $(0.5, 0.5)$ where the Dirac delta is supported. For α big the algorithm notices early the presence of the corner singularity.

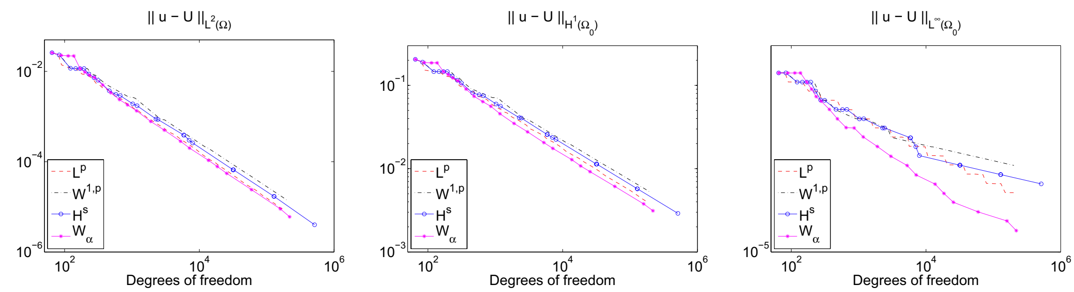

FIGURE 8. Comparison of an adaptive algorithm guided by the error estimators for the L^p, $W^{1,p}$, H^s and W_α norms. We plot the error measured in $L^2(\Omega)$ (left), $H^1(\Omega_0)$ (middle) and $L^\infty(\Omega_0)$ (right) *versus* #DOFs. The algorithm guided by the W_α estimators performs better than the others. Although the advantage in using the W_α estimators is not so impressive for the first two errors, it is really striking when looking at the local L^∞ error in the right picture.

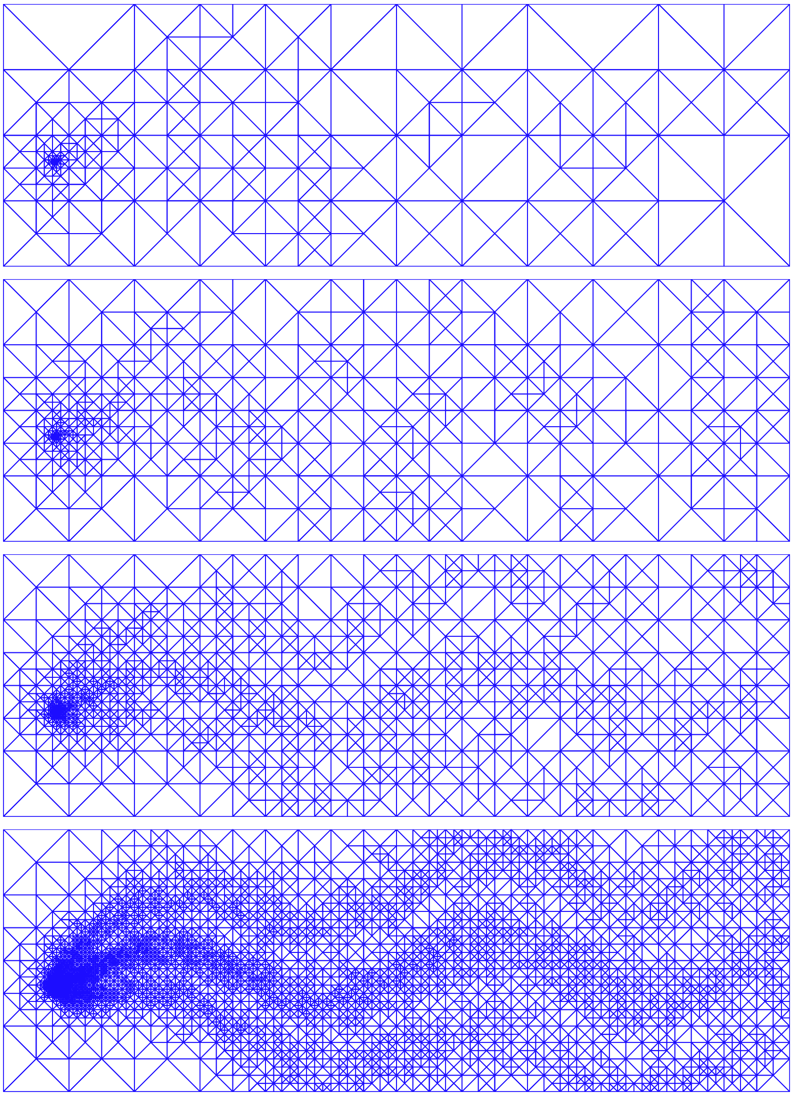

FIGURE 9. Meshes for Example 6.2. We show the meshes obtained by the adaptive loop after 10, 13, 16 and 19 iterations, with 398, 918, 2409 and 10608 elements, respectively.

which is a diffusion-advection-reaction equation, typical from pollutant transport and degradation in aquatic media.

We solved this problem with the same adaptive algorithm described in the previous example, with $\alpha = 0.5$. We started from a uniform coarse initial mesh consisting of 12 elements and 11 vertices. Some mild oscillations were observed at the first iterations but were cured by adaptivity.

A sequence of meshes is presented in Figure 9. The solution in the final mesh, with 22256 elements and 11212 DOFs can be observed in Figure 10.

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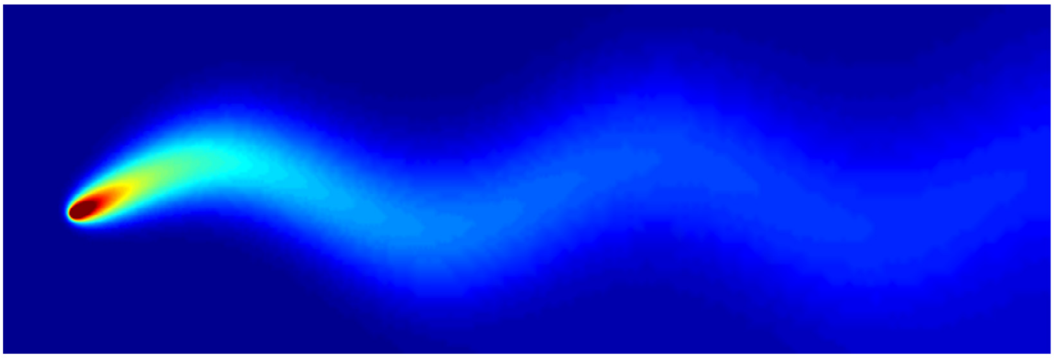

FIGURE 10. Final solution for Example 6.2, obtained by the adaptive loop after 20 iterations, on a mesh with 22256 elements and 11212 DOFs. The error estimator for this mesh is 0.024, which is a 2.2% of the estimator for the initial coarsest mesh.

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