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Weighted Best Local $||\cdot||-$Approximation in Orlicz Spaces†

H. Cuenya, S. Favier, F. Levis and C. Ridolfi

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Abstract

In this paper we prove the existence of best multipoint local $||\cdot||-$approximation to a function f from an $N-$dimensional space S_N for a suitable integer N. This problem is considered in an arbitrary Orlicz space for both the Luxemburg and the Orlicz norms when some bits of data are more important than others. For this purpose, we introduce the concept of $||\cdot||-$balanced integer.

Keywords: best local approximation, $||\cdot||$-approximations, balanced integers.

MSC: Primary 41A10; Secondary 41A30.

§1. Introduction

The notion of best local approximation of a function around a point has been introduced by Chui, Shisha and Smith in [3] although its origin goes as far as the paper of Walsh [9]. The case of more than one point, with same size neighborhoods, were treated in [1] and in [8] with the L^p norms, and in [5] and [4] with the Luxemburg norm in an Orlicz space. In [2], the authors introduced the balanced neighborhood concept and they studied the best local approximation in several points with different size neighborhoods, in L^p spaces. In [6] the last problem was considered for $\phi-$approximation in Orlicz spaces.

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In this article, we begin by studying the best local approximation in Orlicz spaces with Luxemburg norm on $\mathbb{R}$. Later we observe that our results remain valid for the Orlicz norm and several variables. We introduce a concept of balanced integer which extends that given in [2]. The results are obtained with fewer requirements on the function ϕ than those asked in [6] and they provide a generalization of the balanced part given in [2].

We now introduce some notations. Let $X \subset \mathbb{R}$ be a bounded open set, and μ be the Lebesgue measure on X. Denote by $\mathcal{M} = \mathcal{M}(X)$ the set of all the equivalence classes of Lebesgue measurable real valued functions. Let Φ be the set of convex functions $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, with $\phi(x) > 0$ for $x > 0$, and $\phi(0) = 0$.

For $\phi \in \Phi$ define

$$L^\phi(X) = \left\{ f \in \mathcal{M} : \int_X \phi\left(\alpha|f(x)|\right) dx < \infty, \text{ for some } \alpha > 0 \right\}.$$

The function space L^ϕ is called an Orlicz space, and it can be endowed with the following norm

$$\|f\|_\phi = \inf\left\{ \lambda > 0 : \int_X \phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\},$$

called the Luxemburg norm. Sometimes we write $\|f\|_{L^\phi(W)}$ instead of $\|f\chi_W\|_\phi$, where χ_W denote the characteristic function of the set $W \subset X$. In the space L^ϕ usually the Orlicz norm $\|\cdot\|_{(\phi)}$ (see (4.10)) is also considered. The space L^ϕ with both norms is a Banach space; we refer to [7] for a detailed study of Orlicz spaces.

We recall that a function $\phi \in \Phi$ satisfies the Δ_2-condition if there exists a constant $k > 0$ such that $\phi(2x) \leq k\phi(x)$, for $x \geq 0$.

We assume in this article that $\phi \in \Phi$ and it satisfies the Δ_2-condition.

Given $\{x_1, ..., x_n\}$ contained in X, we define for small $\delta > 0$ a net of sets $V_k = V_k(\delta) := x_k + \varepsilon_k(\delta)A_k(\delta) \subset X$, $1 \leq k \leq n$, where $\varepsilon_k = \varepsilon_k(\delta) \searrow 0$, as $\delta \to 0$, and where the sets $A_k = A_k(\delta)$ are measurable and uniformly bounded with $\mu(A_k) = 1$ for all $\delta > 0$.

Let $W \subset X$. For an arbitrary norm $||\cdot||$ in L^ϕ, a function $f \in L^\phi$, and a subspace $S \subset L^\phi$ we consider $g \in S$ such that

$$\|(f-g)\mathcal{X}_W\| \leq \|(f-h)\mathcal{X}_W\|,$$

for all $h \in S$. Whenever it exists, such a function g is called a best $||\cdot||-$approximation of f from S on W. It is well known that a $||\cdot||-$approximation always exists if the dimension of S is finite.

Let $V = \bigcup V_k$ and denote by g_δ a best $||\cdot||-$approximation of f from S on V. When the net $\{g_\delta\}_{\delta>0}$ has a limit in S, as $\delta \to 0$, then this limit is called *the best local* $||\cdot||-$*approximation of* *f from S on* $\{x_1,..,x_n\}$. In Section 3 we introduce the concept of balanced integers and give some properties that we will use later. In Section 4 we prove that the net $\{g_\delta\}_{\delta>0}$ is uniformly bounded when $||\cdot|| = ||\cdot||_\phi$ and S is of finite dimension, and if in addition the dimension of S is balanced, the best local $||\cdot||_\phi-$approximation can be obtained by Hermite interpolation. Analogous results can be obtained for the norm $||\cdot||_{(\phi)}$ and several variables.

Now we make an assumption on the ordered $n-$tuple $\langle\varepsilon_k\rangle := (\varepsilon_1,...,\varepsilon_n)$ which will guarantee that the terms of the form

$$v_k(\alpha) := \|\mathcal{X}_{V_k}\|_\phi\,\varepsilon_k^\alpha = \frac{\varepsilon_k^\alpha}{\phi^{-1}\left(\frac{1}{\varepsilon_k}\right)}, \quad \alpha \text{ nonnegative integer,} \tag{1.1}$$

can be compared with each other as functions of δ. Namely, for any nonnegative integers α and β, and any pair $j,k, 1\le j,k\le n$, we assume

$$\text{either} \quad v_k(\alpha) = O(v_j(\beta)), \quad \text{or} \quad v_j(\beta) = o(v_k(\alpha)). \tag{1.2}$$

Let $\langle i_k\rangle$ be an ordered n-tuple of nonnegative integers. We say that $v_j(i_j)$ is *maximal* if $v_k(i_k) = O(v_j(i_j))$ for all $1\le k\le n$. We denote it by

$$v_j(i_j) = \max\{v_k(i_k)\}.$$

We observe that

$$\sum_{k=1}^n v_k(i_k) \ = O(\max\{v_k(i_k)\}).$$

Let $S_N\subset PC^m(X)$ be a linear subspace of dimension N, and $f\in PC^m(X)$, where $PC^m(X)$ is the class of functions in $L^\phi(X)$ with $m-1$ continuous derivatives and with bounded piecewise continuous m^{th} derivative on X. The space S_N is assumed to be fully interpolating at the points x_j, that is, if $\langle_k\rangle$ is an ordered n-tuple of nonnegative integers with $i_k\le m$ and $\sum_{k=1}^n i_k = N$, then there exists a unique $g\in S_N$ such that $g^{(j)}(x_k) = a_{j,k}$, $0\le j\le i_k-1$, $1\le k\le n$, where $\{a_{j,k}\}$ is an arbitrary set of real numbers.

§2. Preliminary Results

We set the next auxiliary lemmas, which will be used to obtain the main results following the pattern used in [2] for the L^p case and in [6] for the $\phi-$approximation case. The next lemma provides an order of the error $\|f-g\|_{L^\phi(V)}$ for $g\in S_N$ which satisfies $g^{(j)}(x_k) = f^{(j)}(x_k)$, $0\le j\le i_k-1$, $1\le k\le n$.

In the sequel, given a polynomial P, we set

$$I_k(\lambda, P) = \int_{A_k} \varepsilon_k \phi\left(\frac{|P(y)|}{\lambda}\right) dy.$$

Lemma 2.1. *Let $\langle i_k \rangle$ be an ordered n-tuple of nonnegative integers. Suppose $h \in PC^m(X)$, where $m = \max\{i_k\}$, and $h^{(j)}(x_k) = 0$, $0 \le j \le i_k - 1$, $1 \le k \le n$. Then*

$$\|h\|_{L^\phi(V)} \;=\; O\left(\max\left\{v_k(i_k)\right\}\right).$$

Proof. Approximating h by the Taylor polynomial at x_k, we have

$$h(x) = O((x - x_k)^{i_k}), \quad x \in V_k.$$

Thus, there exists a constant $M > 0$ such that

$$\|h\|_{L^\phi(V_k)} \le M \inf\left\{\lambda > 0 : \quad I_k(\lambda, \varepsilon_k^{i_k} y^{i_k}) \le 1\right\}.$$

Since the sets A_k are uniformly bounded in δ, for $\lambda_k = M_k v_k(i_k)$, with $M_k = \max\{\left|y^{i_k}\right| : \; y \in A_k\}$, we have $I_k(\lambda_k, \varepsilon_k^{i_k} y^{i_k}) \le 1$. So

$$\|h\|_{L^\phi(V_k)} = O\left(v_k(i_k)\right).$$

Then $\|h\|_{L^\phi(V)} = O\left(\max\left\{v_k(i_k)\right\}\right).$ ■

In this work we also need the following auxiliary lemma.

Lemma 2.2. *Given a constant $M > 0$, there exist two positive constants M' and M'' such that*

$$M' \le \frac{\phi^{-1}(1/\varepsilon)}{\phi^{-1}(M/\varepsilon)} \le M'', \quad \textit{for all} \quad \varepsilon > 0. \tag{2.1}$$

Proof. Substituting $\frac{1}{\varepsilon}$ by $\phi(x)$ in (2.1), we have to prove that $M' \le \frac{x}{\phi^{-1}(M\phi(x))} \le M''$, for all $x > 0$. Since ϕ is a convex function there exists a constant $K > 0$ such that $\phi(Kx) \le M\phi(x)$ for $x \ge 0$. As ϕ^{-1} is an increasing function we find the upper bound in (2.1), with $M'' = 1/K$. On the other hand, for $M > 1$ we also use the convexity of ϕ to find the lower bound, as required. ■

Let Π^n be the space of polynomials of degree at most n. We now present the Lemma 3 stated in [2], which will be used in the sequel.

Lemma 2.3. *Let $1 \le p \le \infty$ and let Λ be a family of uniformly bounded measurable subsets of the real line with measure 1. Then there exists a constant M (depending on n and p) such that for all the polynomials $P \in \Pi^n$, and all $A \in \Lambda$,*

$$|c_k| \le M \ \ \|P\|_{L_p(A)}, \quad 0 \le k \le n,$$

where $P(x) = \sum_{k=0}^{n} c_k x^k$.

The way employed in this paper to obtain the main result makes us to state the following lemma, which it was not used in [2] nor in [6].

Lemma 2.4. *Let Λ be a family of uniformly bounded measurable subsets of the real line with measure 1. Given r, $0 < r < 1$, there exists a constant $s > 0$ such that*

$$\mu\left(|P|^{-1}\left(\left[\frac{\|P\|_{\infty,A}}{s}, \|P\|_{\infty,A}\right]\right) \cap A\right) \ge r, \tag{2.2}$$

for all $A \in \Lambda$, and for all $P \in \Pi^n$.

Proof. The statement is obvious for constant polynomials. Suppose that $A \subset [a,b]$, for all $A \in \Lambda$. For $0 \ne P(x) = \sum_{k=0}^{n} c_k x^k$, we denote $Q(x) = \frac{P(x)}{\max_k |c_k|}$. By the continuity of the measure, there is $\beta = \beta(A,Q) > 0$ such that

$$r = \mu\left(\left\{x \in A : |Q(x)| > \frac{\|Q\|_{\infty,A}}{\beta}\right\}\right). \tag{2.3}$$

From the equivalence of the norms on Π^n, there exist two constants M and M' such that

$$0 < M \le \|Q\|_{\infty,[a,b]} \le M', \tag{2.4}$$

for all $P \in \Pi^n$, so from (2.3) we obtain

$$r \ge \mu\left(\left\{x \in A : |Q(x)| > \frac{M'}{\beta}\right\}\right). \tag{2.5}$$

Suppose that $\{\beta\}$ is not bounded. Then there are subsequences $\{A_j\} \subset \Lambda$ and $\{Q_j\} \subset \Pi^n$ such that $\beta_j = \beta(A_j, Q_j) \to \infty$, as $j \to \infty$. From (2.4), there is a subsequence of $\{Q_j\}$, that we denote in the same way, uniformly convergent to a polynomial $Q_0 \in \Pi^n$ on $[a,b]$. From (2.4), $Q_0 \neq 0$. Therefore, if $0 < \alpha < 1 - r$, there exists $0 < s < ||Q_0||_{\infty,[a,b]}$ that verifies

$$\mu\left(\{x \in [a,b] : |Q_0(x)| > s\}\right) \geq b - a - \alpha > 0. \tag{2.6}$$

Denote $C = \{x \in [a,b] : |Q_0(x)| > s\}$. Clearly, there exists a nonnegative integer n_1 such that

$$\frac{M'}{\beta_j} < \frac{s}{2} \quad \text{and} \quad ||Q_0(x)| - |Q_j(x)|| < \frac{s}{2}, \quad j \geq n_1,\ x \in [a,b].$$

Then we get

$$C \cap A_j \subset \left\{ x \in A_j : |Q_j(x)| > \frac{M'}{\beta_j} \right\}, \quad j \geq n_1. \tag{2.7}$$

Since $\mu(A_j) = 1$, from (2.5), (2.6), and (2.7) it follows that

$$r \geq \mu(C \cap A_j) = \mu(A_j) - \mu(A_j \backslash C) \geq 1 - \alpha > r,$$

which is a contradiction. Therefore, the set $\{\beta\}$ is bounded. So, from (2.3) we obtain (2.2) with $s = \sup\{\beta\}$. ∎

§3. Balanced Neighborhood in L^ϕ

We begin with the following definition.

Definition 3.1. An $n-$tuple $\langle i_k \rangle$ of nonnegative integers is said to be $|| \cdot ||_\phi-$balanced if for each $i_j > 0$,

$$\frac{1}{v_j(i_j - 1)} \max\{v_k(i_k)\} = o(1).$$

If $\langle i_k \rangle$ is $|| \cdot ||_\phi-$balanced, we say that $\sum_{k=1}^n i_k$ is a $|| \cdot ||_\phi-$balanced integer.

As we have mentioned in the Introduction, this definition generalizes the concept of balanced integer given in [2]. The following lemma allow us to state, for the Luxemburg norm, an algorithm to compute all the $|| \cdot ||_\phi-$balanced n-tuples.

Lemma 3.2. *Let $\langle i_k \rangle$ and $\langle i'_k \rangle$ be two $|| \cdot ||_\phi-$balanced n-tuples with $\sum_{k=1}^n i_k < \sum_{k=1}^n i'_k$. Set $A := \{j : \ v_j(i_j) = \max\{v_k(i_k)\}\}$ and $B := \{j : \ j \notin A\}$. Then* *a) If $j \in A$, then $i'_j \geq i_j + 1$.*

b) If $j \in B$, then $i'_j \geq i_j$.

Proof. From the definiton of maximal, there is a constant $\eta > 0$ such that

$$\eta \ \max\{v_k(i'_k)\} \geq v_j(i'_j), \quad 1 \leq j \leq n.$$

a) Suppose to the contrary that $i'_j \leq i_j$ for some $j \in A$. If there is $l \in B$ such that $i'_l \geq i_l + 1$ then

$$\eta \frac{\max\{v_k(i'_k)\}}{v_l(i'_l - 1)} \geq \frac{v_j(i'_j)}{v_l(i'_l - 1)} \geq \frac{v_j(i_j)}{v_l(i_l)} \to \infty \quad \text{as} \quad \delta \to 0,$$

and $\langle i'_k \rangle$ cannot be $||\cdot||_\phi$−balanced.
If either $B = \emptyset$ or for any $l \in B$, $i'_l \leq i_l$, then there exists $s \in A$ such that $i'_s \geq i_s + 1$. In this case,

$$\eta \frac{\max\{v_k(i'_k)\}}{v_s(i'_s - 1)} \geq \frac{v_j(i'_j)}{v_s(i'_s - 1)} \geq \frac{v_j(i_j)}{v_s(i'_s - 1)} \geq \frac{v_j(i_j)}{v_s(i_s)} \nrightarrow 0 \quad \text{as} \quad \delta \to 0,$$

and again $\langle i'_k \rangle$ cannot be $||\cdot||_\phi$−balanced.
b) It is obvious when $i_j = 0$ for all $j \in B$. Now suppose that $i'_j < i_j$ for some $j \in B$. Then, for $l \in A$,

$$\eta \frac{\max\{v_k(i'_k)\}}{v_l(i'_l - 1)} \geq \frac{v_j(i'_j)}{v_l(i'_l - 1)} \geq \frac{v_j(i_j - 1)}{v_l(i'_l - 1)} \geq \frac{v_j(i_j - 1)}{v_l(i_l)} \to \infty \quad \text{as} \quad \delta \to 0,$$

where the last inequality holds because, by a), $i'_l - 1 \geq i_l$. Therefore, $\langle i'_k \rangle$ cannot be $||\cdot||_\phi$−balanced. ■

Given a $||\cdot||_\phi$−balanced n−tuple $\langle i_k \rangle$, it easy to see that the n-tuple $\langle i'_k \rangle$ defined by $i'_j = i_j + 1$, $j \in A$, and $i'_j = i_j$, $j \in B$, is $||\cdot||_\phi$−balanced.

Algorithm. Begin with the $||\cdot||_\phi$−balanced n−tuple $\langle i_k^{(0)} \rangle = \langle 0 \rangle$ which corresponds to the $||\cdot||_\phi$−balanced integer 0. Then, given $\langle i_k^{(m)} \rangle$ for $m \geq 0$, set $A = \{l : \ v_l(i_l^{(m)}) = \max\{v_k(i_k^{(m)})\}\}$. To build the next $||\cdot||_\phi$−balanced n−tuple $\langle i_k^{(m+1)} \rangle$ we take $i_k^{(m+1)} = i_k^{(m)} + 1$, for $k \in A$, and $i_k^{(m+1)} = i_k^{(m)}$, for $k \notin A$.

Remark 3.3. We observe that to each $||\cdot||_\phi$−balanced integer there corresponds exactly one $||\cdot||_\phi$−balanced n−tuple. We also note that an integer N is $||\cdot||_\phi$−balanced if only if $N = \sum_{k=1}^n i_k$ for some $\langle i_k \rangle$ generated by this algorithm.

Denote $\overline{N}$ and $\underline{N}$ the smallest $||\cdot||_\phi-$balanced integer greater than or equal to N and the largest $||\cdot||_\phi-$balanced integer less than or equal to N, respectively. We write $\sum \bar{i}_k = \overline{N}$ and $\sum \underline{i}_k = \underline{N}$, where $\langle \bar{i}_k \rangle$ and $\langle \underline{i}_k \rangle$ are $n-$tuples $||\cdot||_\phi-$balanced.

Lemma 3.4. *The following statements are satisfied:*

a) If $\underline{i}_j + 1 = \bar{i}_j$, *then* $\max\{v_k(\underline{i}_k)\} = O(v_j(\bar{i}_j - 1))$*;*

b) If $\underline{i}_j = \bar{i}_j$, *then* $\max\{v_k(\underline{i}_k)\} = o(v_j(\bar{i}_j - 1))$*;*

Next we give an example of balanced integers.

Example 3.5. Define $\phi(x) = \frac{x^2}{\ln(e+x)}$, $x \geq 0$. It can be seen that ϕ satisfies the Δ_2-condition ([7], pp. 30). We will prove that $\langle \varepsilon_k \rangle = (\delta, \delta^2)$ satisfies the conditions (1.2) and that every integer is $||\cdot||_\phi-$balanced.
To this purpose we first prove the following functional equation for the function ϕ^{-1}:

$$\phi^{-1}(x) = x^{1/2}[\ln(e + \phi^{-1}(x))]^{1/2}. \tag{3.1}$$

Set $g(x) = \frac{x^{1/2}}{\phi^{-1}(x)}$. Since $x = \phi^{-1}(\phi(x)) = \frac{x}{[\ln(e+x)]^{1/2} g(\phi(x))}$, we obtain $g(x) = \frac{1}{[\ln(e+\phi^{-1}(x))]^{1/2}}$, i.e., (3.1).
Clearly, from (3.1) we get

$$\lim_{x\longrightarrow\infty} \frac{x}{(\phi^{-1}(x))^2} = 0, \quad \text{and} \quad \lim_{x\longrightarrow\infty} \frac{\phi^{-1}(x)}{x} = 0. \tag{3.2}$$

From (3.2) there exists a constant $M > e$ such that

$$x^{1/2} \leq \phi^{-1}(x) \leq x \quad \text{and} \quad e + x^2 < x^3, \quad x \geq M.$$

Thus

$$\frac{1}{6} = \frac{\ln(x^{1/2})}{\ln(x^3)} \leq \frac{\ln(e + \phi^{-1}(x))}{\ln(e + \phi^{-1}(x^2))} \leq 1, \quad x \geq M. \tag{3.3}$$

Now, we are ready to show that $\langle \varepsilon_k \rangle$ satisfies (1.2). From (3.1), we have

$$\frac{v_2(\alpha)}{v_1(\beta)} = \delta^{2\alpha-\beta+\frac{1}{2}} \left(\frac{\ln\left(e + \phi^{-1}\left(\delta^{-1}\right)\right)}{\ln\left(e + \phi^{-1}\left(\delta^{-2}\right)\right)} \right)^{\frac{1}{2}}.$$

So, (3.3) implies $v_2(\alpha) = o(v_1(\beta))$ if $2\alpha - \beta > -\frac{1}{2}$, or $v_1(\beta) = o(v_2(\alpha))$ if $2\alpha - \beta < -\frac{1}{2}$. Similarly, we can prove that $v_1(\alpha) = o(v_2(\beta))$ if $2\beta - \alpha < -\frac{1}{2}$, or $v_2(\beta) = o(v_1(\alpha))$ if $2\beta - \alpha > -\frac{1}{2}$. On the other hand, by (3.1),

$$\frac{v_k(\alpha)}{v_k(\beta)} = \delta^{k(\alpha-\beta)}, \quad 1 \le k \le 2,$$

and consequently $v_k(\alpha) = O(v_k(\beta))$ if $\alpha \ge \beta$, or $v_k(\beta) = o(v_k(\alpha))$ otherwise. Therefore $\langle \varepsilon_k \rangle$ satisfies (1.2).
Finally, using the above analysis, we observe that the set $A := \{j : \; v_j(i_j) = \max\{v_k(i_k)\}\}$ is unitary for all $\langle i_k \rangle$ generated by the algorithm. Therefore Remark 3.3 implies that all nonnegative integers are $||\cdot||_\phi-$balanced.

§4. Best Local Approximation in Orlicz Spaces

We now present the first important result concerning the behavior of a net $\{g_\delta\}_{\delta>0}$ of best $||\cdot||_\phi-$approximations from S_N, as $\delta \to 0$.

Theorem 4.1. *Let N be a positive integer and $m = \max\{\bar{i}_k\}$. If $f \in PC^m(X)$, $S_N \subseteq PC^m(X)$, and $\{g_\delta\}_{\delta>0}$ is a net of best $||\cdot||_\phi-$approximations of f from S_N on V, then $\{g_\delta\}_{\delta>0}$ is uniformly bounded on X.*

Proof. If g_δ is not uniformly bounded in δ, there is a sequence $\{\delta_r\}$ such that

$$\|g_{\delta_r}\|_\phi \to \infty, \text{ as } r \to \infty. \tag{4.1}$$

Let g be a fixed function in S_N such that $f^{(j)}(x_k) = g^{(j)}(x_k)$, $0 \le j \le \underline{i}_k - 1$, $1 \le k \le n$. Without loss of generality we assume $g_{\delta_r} \ne g$ for all r. We define

$$h_{\delta_r} = \frac{g - g_{\delta_r}}{\|g - g_{\delta_r}\|_\phi}.$$

Since g_δ is a best $\|\cdot\|_\phi$-approximation, from Lemma 2.1 and (4.1) we get

$$\|h_{\delta_r}\|_{L^\phi(V)} \le \frac{2\,\|f - g\|_{L^\phi(V)}}{\|g - g_{\delta_r}\|_\phi} = o\left(\max\left\{v_l(\underline{i}_l)\right\}\right). \tag{4.2}$$

Expanding h_{δ_r} by the Taylor polynomial at x_k up to the order $\bar{i}_k - 1$, we obtain

$$h_{\delta_r}(x) = \sum_{j=0}^{\bar{i}_k-1} \frac{h_{\delta_r}^{(j)}(x_k)}{j!}(x-x_k)^j + R_{\delta_r}(x),$$

where $R_{\delta_r}(x) = O((x-x_k)^{\bar{i}_k})$, $1 \le k \le n$.

Consider the norm on S_N defined by $\|h\| := \operatorname{ess\,sup}_{x\in X}(|h(x)| + ... + |h^{(\bar{i}_k)}(x)|)$. Since $\|h_{\delta_r}\|_\phi = 1$, the equivalence of the norms in S_N implies that $R_{\delta_r}(x) = O((x-x_k)^{\bar{i}_k})$ uniformly in δ_r. Let $M > 0$ be such that $A_k \subset [-M, M]$, $1 \le k \le n$. A straightforward computation shows that $||(x-x_k)^{\bar{i}_k}||_{L^\phi(V_k)} \le M^{\bar{i}_k} v_k(\bar{i}_k)$, and consequently

$$\|R_{\delta_r}\|_{L^\phi(V_k)} = O\left(v_k(\bar{i}_k)\right),$$

uniformly in δ_r. Therefore, from (4.2) we have

$$\left\| \sum_{j=0}^{\bar{i}_k-1} \frac{h_{\delta_r}^{(j)}(x_k)}{j!}(x-x_k)^j \right\|_{L^\phi(V_k)} \le o\left(\max\{v_l(\underline{i}_l)\}\right) + O\left(v_k(\bar{i}_k)\right). \tag{4.3}$$

Now, we consider the polynomials net

$$P_{\delta_r,k}(y) = \sum_{j=0}^{\bar{i}_k-1} \frac{h_{\delta_r}^{(j)}(x_k)}{j!}\varepsilon_k^j y^j, \quad 1 \le k \le n.$$

The change of variable $x - x_k = \varepsilon_k y$, $y \in A_k$, yields

$$\left\| \sum_{j=0}^{\bar{i}_k-1} \frac{h_{\delta_r}^{(j)}(x_k)}{j!}(x-x_k)^j \right\|_{L^\phi(V_k)} = \inf\{\lambda > 0 : \ I_k(\lambda, P_{\delta_r,k}) \le 1\}. \tag{4.4}$$

Using Lemma 2.4 we can find a number s, independent on δ_r and k, such that the sets $B_{\delta_r,k} := |P_{\delta_r,k}|^{-1}\left(\left[\frac{\|P_{\delta_r,k}\|_{\infty,A_k}}{s}, \|P_{\delta_r,k}\|_{\infty,A_k}\right]\right) \cap A_k$ satisfy $\mu(B_{\delta_r,k}) \ge \frac{1}{2}$, for all δ_r and k. Denote $\lambda_{\delta_r} := \frac{\|P_{\delta_r,k}\|_{\infty,A_k}}{s\phi^{-1}\left(\frac{2}{\varepsilon_k}\right)}$. Since $\int_{B_{\delta_r,k}} \varepsilon_k \phi\left(\frac{|P_{\delta_r,k}|}{\lambda_{\delta_r}}\right) dy \ge 1$, we have $I_k(\lambda_{\delta_r}, P_{\delta_r,k}) \ge 1$. Thus, by (4.4) we get

$$\frac{\|P_{\delta_r,k}\|_{\infty,A_k}}{s\phi^{-1}\left(\frac{2}{\varepsilon_k}\right)} \le \left\| \sum_{j=0}^{\bar{i}_k-1} \frac{h_{\delta_r}^{(j)}(x_k)}{j!}(x-x_k)^j \right\|_{L^\phi(V_k)}.$$

According to Lemma 2.2 and Lemma 2.3 there exists a constant $M^* > 0$ such that

$$\frac{|h_{\delta_r}^{(j)}(x_k)\varepsilon_k^j|}{s\phi^{-1}\left(\frac{1}{\varepsilon_k}\right)} \leq M^* \left\| \sum_{j=0}^{\bar{i}_k-1} \frac{h_{\delta_r}^{(j)}(x_k)}{j!}(x-x_k)^j \right\|_{L^\phi(V_k)}, \tag{4.5}$$

for $0 \leq j \leq \bar{i}_k - 1,\ 1 \leq k \leq n$. So, (4.3) and (4.5) imply

$$|h_{\delta_r}^{(j)}(x_k)| \leq \frac{1}{v_k(\bar{i}_k-1)}\left(o\left(\max\left\{v_l(\underline{i}_l)\right\}\right) + O\left(v_k(\bar{i}_k)\right)\right),$$

and consequently by Lemma 3.4 we get

$$|h_{\delta_r}^{(j)}(x_k)| = o(1), \quad \text{as } \delta_r \to 0, \tag{4.6}$$

for $0 \leq j \leq \bar{i}_k - 1,\ 1 \leq k \leq n$.
Finally, considering the norm $\|h\| = \sum_{k=1}^n \sum_{j=0}^{\bar{i}_k-1} |h^{(j)}(x_k)|$ on S_N, and the equivalence of the norms in S_N, we obtain $\|h_{\delta_r}\|_\phi \to 0$, as $\delta_r \to 0$, which is a contradiction. Thus g_δ must be uniformly bounded in δ and the proof of the theorem is complete. ■

Lemma 4.2. *Let $\langle i_k \rangle$ be a $||\cdot||_\phi-$balanced $n-$tuple, $0 < N = \sum_{k=1}^n i_k$ and $m = \max\{i_k\}$. If $f \in PC^m(X)$, $S_N \subseteq PC^m(X)$, and $\{g_\delta\}_{\delta>0}$ is a net of best $||\cdot||_\phi-$approximations of f from S_N on V, then*

$$\frac{|(f-g_\delta)^{(j)}(x_k)\varepsilon_k^j|}{\phi^{-1}(\frac{1}{\varepsilon_k})} = O\left(\max\left\{v_k(i_k)\right\}\right), \tag{4.7}$$

$0 \leq j \leq i_k - 1$, $1 \leq k \leq n$.

Proof. For each k with $i_k > 0$, consider the Taylor polynomial of $f - g_\delta$ at x_k of degree $i_k - 1$. Thus

$$(f-g_\delta)(x) \ = \ \sum_{j=0}^{i_k-1} \frac{(f-g_\delta)^{(j)}(x_k)}{j!}(x-x_k)^j + R_\delta(x), \tag{4.8}$$

where $R_\delta(x) = O((x-x_k)^{i_k})$. From Theorem 4.1 and the equivalence of the norms in S_N we can show that $R_\delta(x) = O((x-x_k)^{i_k})$ uniformly in δ. Thus, since the sets A_k are bounded uniformly in δ we obtain for each k

$$\|R_\delta\|_{L^\phi(V_k)} = O\left(\max\left\{v_l(i_l)\right\}\right).$$

Let g be a fixed function in S_N such that $f^{(j)}(x_k) = g^{(j)}(x_k)$, $0 \leq j \leq i_k - 1$, $1 \leq k \leq n$. Then $\|f - g_\delta\|_{L^\phi(V)} \leq \|f - g\|_{L^\phi(V)}$. From (4.8) and Lemma 2.1 follows that

$$\left\| \sum_{j=0}^{i_k-1} \frac{(f-g_\delta)^{(j)}(x_k)}{j!}(x-x_k)^j \right\|_{L^\phi(V_k)} = O\left(\max\left\{v_l(i_l)\right\}\right). \tag{4.9}$$

Now, we can use a similar analysis to that in the proof of Theorem 4.1 with $f - g_\delta$ instead of h_{δ_r}, to obtain as in (4.5)

$$\frac{|(f-g_\delta)^{(j)}(x_k)\varepsilon_k^j|}{s\phi^{-1}(\frac{1}{\varepsilon_k})} \leq M \left\| \sum_{j=0}^{i_k-1} \frac{(f-\mathrm{g}_\delta)^{(j)}(x_k)}{j!}(x-x_k)^j \right\|_{L^\phi(V_k)},$$

$0 \leq j \leq i_k - 1$, $1 \leq k \leq n$, for a constant $M > 0$. Using the statement (4.9) we obtain (4.7) as required. ∎

Now we present the second main result.

Theorem 4.3. *Let $\langle i_k \rangle$ be an $n-$tuple $||\cdot||_\phi-$balanced and let $0 < N = \sum i_k$. If $m = max\{i_k\}$, $f \in PC^m(X)$ and $S_N \subset PC^m(X)$, then the best local $||\cdot||_\phi-$approximation of f from S_N is the unique $g \in S_N$ defined by the N interpolation conditions*

$$f^{(j)}(x_k) = g^{(j)}(x_k),$$

$0 \leq j \leq i_k - 1,\ 1 \leq k \leq n.$

Proof. From the $||\cdot||_\phi-$balanced definition and Lemma 4.2, it follows that

$$g_\delta^{(j)}(x_k) = f^{(j)}(x_k) + o(1),$$

$0 \leq j \leq i_k - 1,\ 1 \leq k \leq n$. As g_δ is uniquely determined via a fixed linear transformation with rank N from the N values $g_\delta^{(j)}(x_k)$, $0 \leq j \leq i_k - 1$, $1 \leq k \leq n$, then g_δ must converge to the generalized Hermite interpolator g, i.e., $g \in S_N$ such that

$$g^{(j)}(x_k) = f^{(j)}(x_k),$$

$0 \leq j \leq i_k - 1$, $1 \leq k \leq n$. This g is by definition the best local approximation of f from S_N. ∎

Under the assumption $\frac{\phi(x)}{x} \to 0$, as $x \to 0$, and $\frac{\phi(x)}{x} \to \infty$, as $x \to \infty$, we can consider the Orlicz norm

$$||f||_{(\phi)} = \inf_{k>0} \frac{1}{k}\left(1 + \int_X \phi(k|f(x)|)dx\right). \tag{4.10}$$

This norm is equivalent to the Luxemburg norm in $L_\phi(X)$, furthermore

$$\|f\|_\phi \le \|f\|_{(\phi)} \le 2\|f\|_\phi, \text{ for all } f \in L^\phi(A) \text{ and } A \subset X. \tag{4.11}$$

As a consequence of (4.11) it is easy to see that Lemmas 2.1, 4.2, and Theorems 4.1, 4.3 remain valid for Orlicz norm.

We can also generalize the results of one variable to several variables. In $\mathbb{R}^d$ we say that $\langle i_k \rangle$ is balanced if $i_j > 0$ implies that $\max\left\{\frac{\varepsilon_k^{i_k}}{\phi^{-1}(1/\varepsilon_k^d)}\right\} = o\left(\frac{\varepsilon_j^{i_j-1}}{\phi^{-1}(1/\varepsilon_j^d)}\right)$, and in that case $\sum_k card\{\alpha : |\alpha| \le i_k - 1\}$ will be called a balanced integer, where $\alpha = \langle \alpha_1, ..., \alpha_d \rangle$ is a d-dimensional multi-index and $|\alpha| = \alpha_1 + ... + \alpha_d$.

Now, we state without proof the theorem for several variables. The necessary changes can be seen in [2].

Theorem 4.4. *Let $\|\cdot\|$ be the Luxemburg or Orlicz norm on a bounded open set $X \subset \mathbb{R}^d$ and let $\{x_1, ..., x_n\}$ contained in X. If N is a balanced integer with balanced $\langle i_k \rangle$ and $S_N \subset PC^m(X)$ with $m = \max\{i_k\}$, then the best local $\|\cdot\|-$approximation of $f \in PC^m(X)$ from S_N on $\{x_1, ..., x_n\}$ is the unique $g \in S_N$ which satisfies*

$$\frac{\partial^{|\alpha|} g(x_k)}{\partial y_1^{\alpha_1} ... \partial y_d^{\alpha_d}} = \frac{\partial^{|\alpha|} f(x_k)}{\partial y_1^{\alpha_1} ... \partial y_d^{\alpha_d}},$$

$|\alpha| \le i_k - 1$, $1 \le k \le n$.

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S. Favier and C. Ridolfi,
Instituto de Matemática Aplicada San Luis, CONICET,
Universidad Nacional de San Luis,
San Luis, 5700, Argentina.
sfavier@unsl.edu.ar
ridolfi@unsl.edu.ar

H. Cuenya and F. Levis,
Departamento de Matemática,
Facultad de Ciencias Exactas Físico Químicas y Naturales,
Universidad Nacional de Río Cuarto,
Río Cuarto, 5800, Argentina.
hcuenya@exa.unrc.edu.ar
flevis@exa.unrc.edu.ar